Measure and Integration

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Module No. # 06

Lecture No. # 20

Properties of Integrable Functions and Dominated Convergence Theorem

Welcome to lecture 30 on measure and integration. In the previous lecture, we had started defining what is called the notion of a function to be an integrable function and then we started looking at some of the properties of integrable functions.

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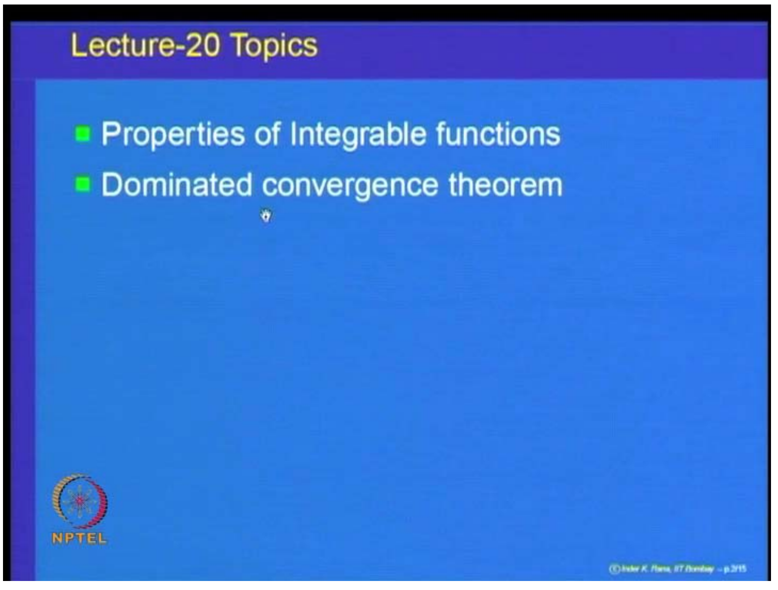

Let us just recall what is an integrable function and then we will start looking at various properties of these integrable functions. We will prove one important theorem called dominated convergence theorem.

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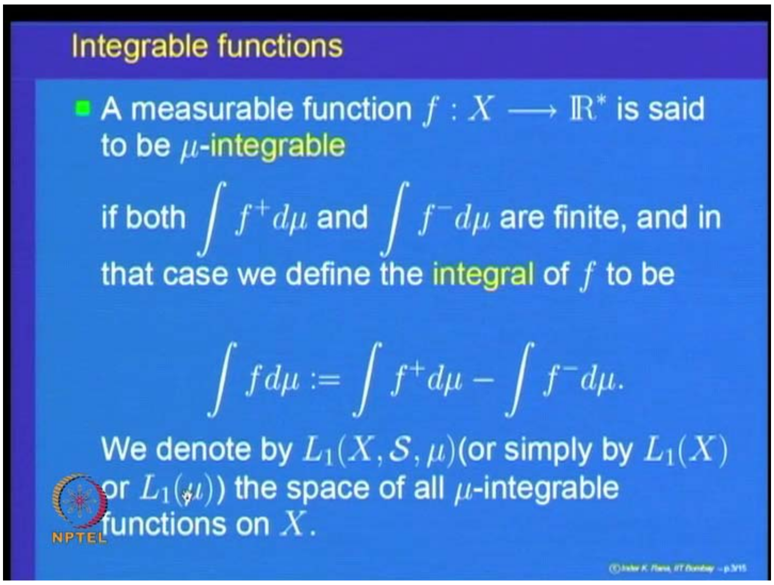

If you recall, we said a measurable function f on X to extended real ((.)) measurable function f is said to be integrable with respect to mu and written as mu integrable if both the integral of the positive part of the function and the negative part of the function are finite. We say f is mu integrable if integral f plus into d mu and integral f minus into d mu both are finite numbers. In that case, we say the integral of f is equal to integral of f plus minus integral of f minus.

Let us just once again emphasize saying that function is mu integrable if and only if both f plus and f minus are having finite integrals and the integral of f is written as integral of f plus minus integral of f minus. The class of all integrable functions on the space X, S, mu is normally denoted by L_1 – capital L lower 1 – of X, S, mu or sometimes we drop S and mu if we are clear from the context what are the sigma algebras or what is the measure; sometimes we just emphasize mu because we know what is X and what is S. These are various notations used for denoting integrable functions – L_1 of X, S, mu or L_1 bracket X or L_1 bracket mu; this is the space of all mu-integrable functions.

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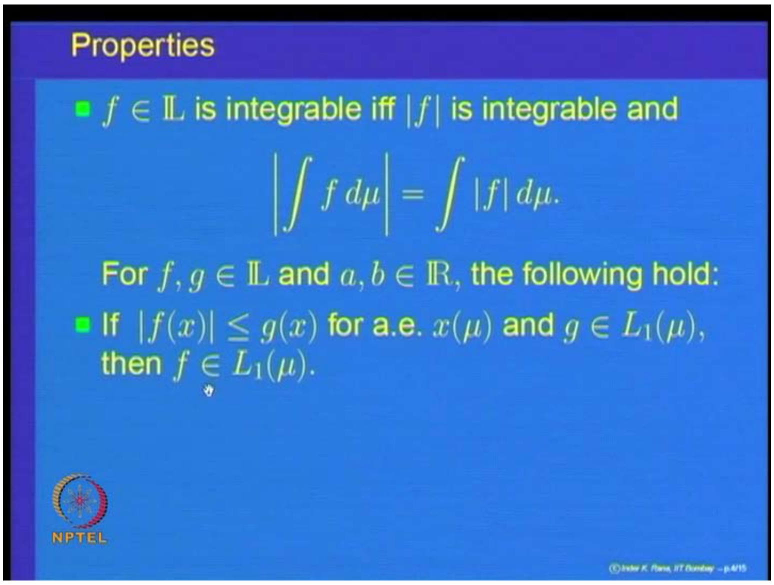

We started looking at the properties of these functions. The first important thing we observed was a function f which is measurable is integrable if and only if mod f which is a nonnegative function is integrable. That means to check whether a measurable function is integrable or not, it is enough to look at the integral of the function mod f and see whether that is finite or not and this is always true; for the integrable function, integral of mod f integral mod of the integral of f into d mu is less than or equal to integral of mod f into d mu.

This is an important criteria; this is an equivalent way of defining integrability of a measurable function, namely mod f is measurable. This is not equal; this is wrong here; it should be less than or equal to; mod of integral f into d mu is less than or equal to; this is a typing mistake here; this should have been less than or equal to integral of mod f into d mu.

Let us recall some of the other properties that we had proved. We said if f and g are measurable functions and mod f is less than or equal to g of x for almost all x with respect to mu and g is integrable, then f is also integrable; that means if a function f of x is dominated by an integrable function, then that measurable function automatically becomes integrable.

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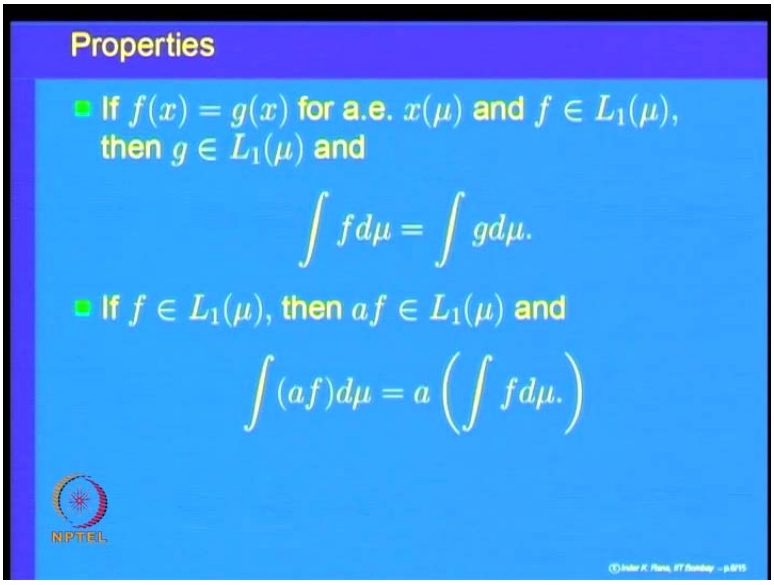

We also proved the following property: if two functions f and g are equal almost everywhere and one of them is integrable, say f is integrable, then the function g is also integrable and integral of f is equal to integral of g. That essentially says that the integrable of the function does not change if the function is changed, if the values of the function are changed almost everywhere.

So, f is equal to g almost everywhere; f and g are measurable functions; one of them, say, f is integrable implies g is integrable and the integrable of the two are equal. We also proved the following property: if f is an integrable function and alpha is any real number, then alpha times f is also integrable and the integral of alpha f is equal to alpha times the integral of f. We continue this study of properties of integrable functions.

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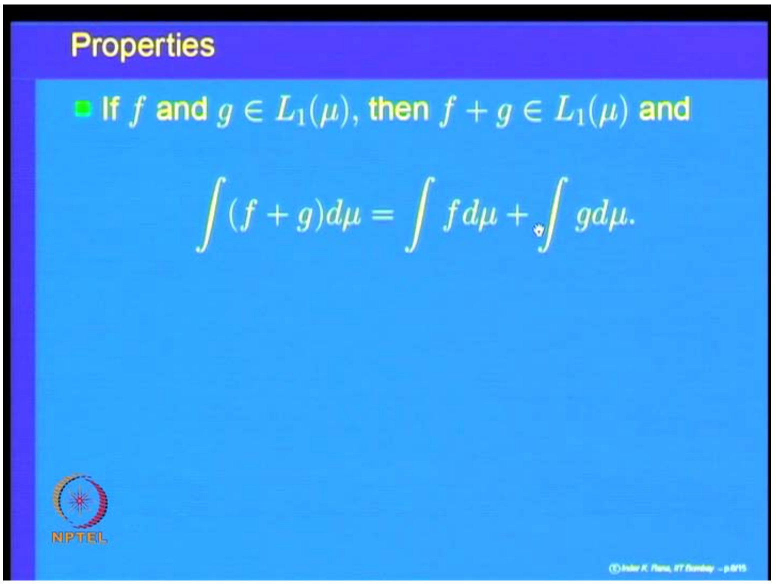

Next we want to check the integrability property; if f and g are integrable functions, then we want to show that f plus g is also integrable and integral of f plus g is equal to integral f plus integral g.

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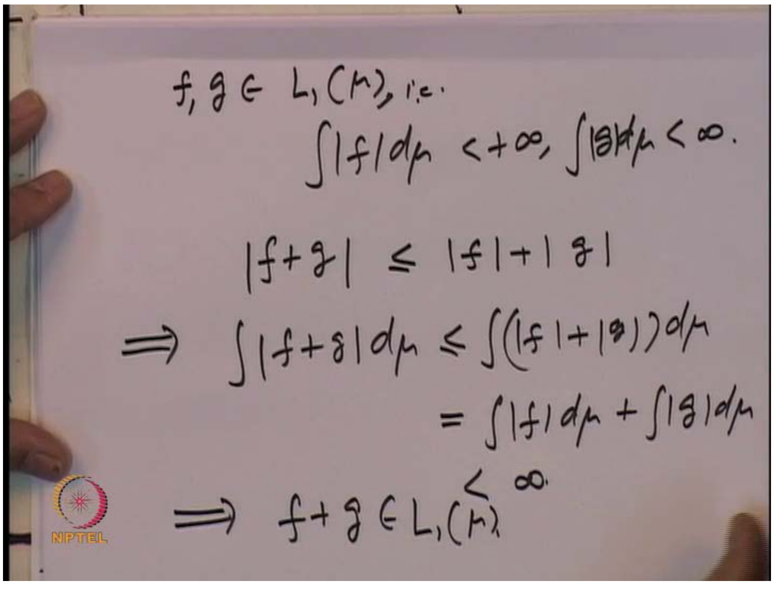

To prove this property, let us look at what we are given. We are given that f and g are integrable functions; that is, integral mod f into d mu is finite and integral of g into d mu is also… absolute value of g with respect to mu is also finite. To check whether the function f plus g is integrable or not, we have to look at the integral of f plus g, the absolute value, and show that integral of absolute value of f plus g is also finite. That follows easily because absolute value of f plus g is always less than or equal to absolute value of f plus absolute value of g.

All are nonnegative measurable functions. So, using the property of the integral for nonnegative measurable functions, this implies that integral of mod f plus g into d mu is less than or equal to integral of mod f plus mod g into d mu and that by linearity is the same as integral mod f into d mu plus integral mod g into d mu. We are given that both of them are finite; so, this is finite (Refer Slide Time: 07:35). It implies that f plus g is integrable.

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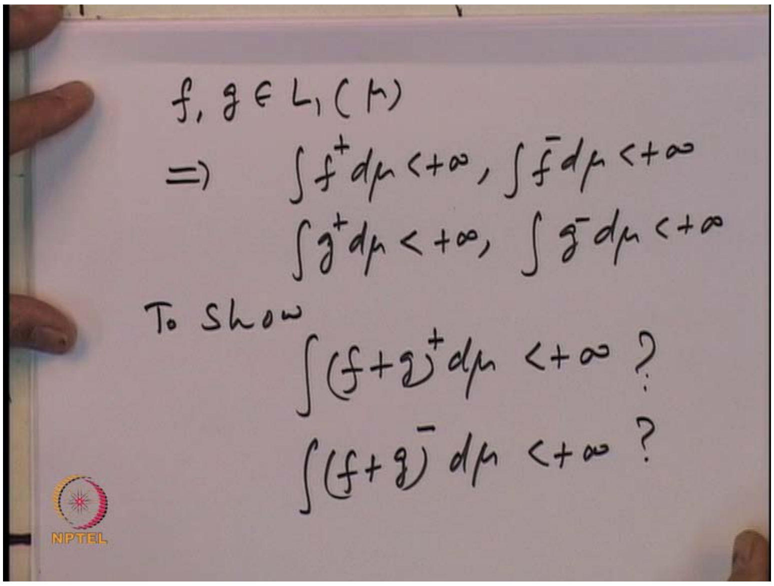

To compute the integral of f plus g, we have to go back to the definition of the integral. f and g are integrable; that implies integral of f plus d mu is finite, integral of f minus d mu is finite, integral of g plus – the positive part of g – is finite, and integral of g minus d mu is finite. We have to show that integral (f plus g) plus into d mu is finite and integral (f plus g) minus into d mu is finite; these two properties we have to show. To show this, somehow we have to relate the positive part of f plus g with the positive part of f and positive part of g and similarly, the negative part of f plus g with the negative part of f and negative part of g; that is done as follows.

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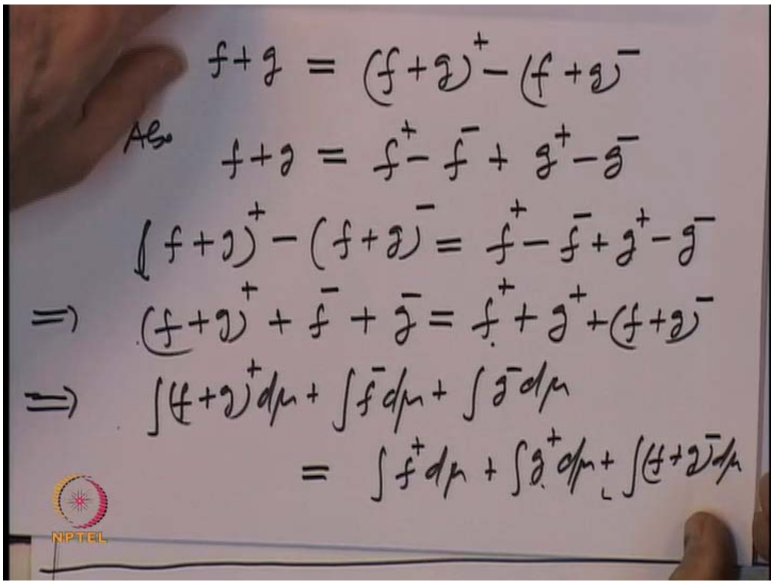

What we do is look at f plus g. By definition, we can write it as f plus g positive part minus f plus g the negative part; that is by the definition of the positive part and the negative part of a function. Also, f plus g we can also write it as shown; decompose f into positive part and into the negative part – that is, f plus minus f minus – and similarly write g as g plus minus g minus.

From these two, it follows that integral of f, sorry not the integral; from this, it follows that f plus g positive part minus f plus g the negative part is equal to f plus minus f minus plus g plus minus g minus; from these two equations, it follows this is so. Now, what we do is all the negative terms we shift on the other side of the equation; this implies that (f plus g) plus plus f minus plus g minus is equal to f plus plus g plus from here and this term on the other side will give me plus (f plus g) minus.

We also rearrange the terms; now, you observe that the left-hand side is a nonnegative function and the right-hand side is a nonnegative function. By the properties of integrals for nonnegative functions, this implies that integral of (f plus g) plus into d mu plus integral of f minus into d mu plus integral of g minus into d mu so, that is the integral of the left-hand side, is equal to integral of f plus into d mu plus integral g plus into d mu plus integral f plus g minus into d mu.

From this equation by using the properties of integral for nonnegative functions, the linearity property, the integral of the left-hand side is equal to integral of the right-hand side (Refer Slide Time: 11:46). Integral of the left-hand side consists of integral of (f plus g) plus plus integral of f minus plus integral of g minus and that is equal to integral of f plus plus integral of g plus plus integral of (f plus g) minus.

Now, we observe that in this equation all the terms are finite quantities or real numbers; that is because f plus g we have already shown is integrable; this first integral of (f plus g) plus is finite, integral f minus is finite and similarly all the terms are actually nonnegative real numbers. We can again manipulate them and shift terms on the left-hand side and right-hand side. What we will do is this term (f plus g) minus on the right-hand side (Refer Slide Time: 12:42) we will bring it on the left-hand side and the terms integral f minus into d mu and integral g minus into d mu we shift it on the right-hand side; that gives us the property.

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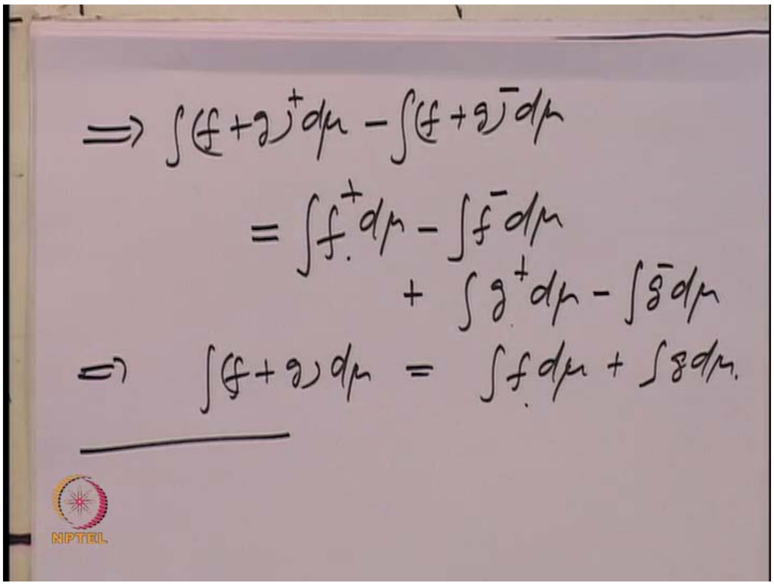

Shifting implies that integral of (f plus g) plus into d mu minus this term will give you integral (f plus g) minus into d mu; this term we have shifted (Refer Slide Time: 13:10). Shift these two terms on the other side. It is equal to integral f plus into d mu; that is this term and bringing this integral of f minus on this side will give you integral f minus into d mu plus integral of g plus into d mu which is already there and integral of g minus from the left-hand side will give you integral of g minus into d mu.

(Refer Slide Time: 13:40) This rearrangement of the terms here once again gives you that integral of (f plus g) plus into d mu and the integral of the negative part of f plus g is equal to integral of f plus minus integral f minus plus integral g plus. Now by the definition, the left-hand side is nothing but integral of f plus g into d mu and the right-hand side is integral f into d mu plus integral g into d mu.

That proves the linearity property of the integral that if f and g are integrable functions, not only f plus g is integrable but integral of f plus g is equal to integral of f plus integral of g into d mu; that is the linearity property of the integral (Refer Slide Time: 14:38). We have proved the basic properties of the integrals, namely the integral of a function which is integrable, of course it is a finite quantity and it is linear; if you take a function f multiplied by a scalar alpha, then alpha times f is integrable and the integral of alpha f is equal to alpha times integral of f. Similarly, if f and g are integrable, then f plus g is integrable and the integral of f plus g is equal to integral of f plus integral of g. Let us look at some more properties of this integral which are going to be useful later on. Let us look at the next property.

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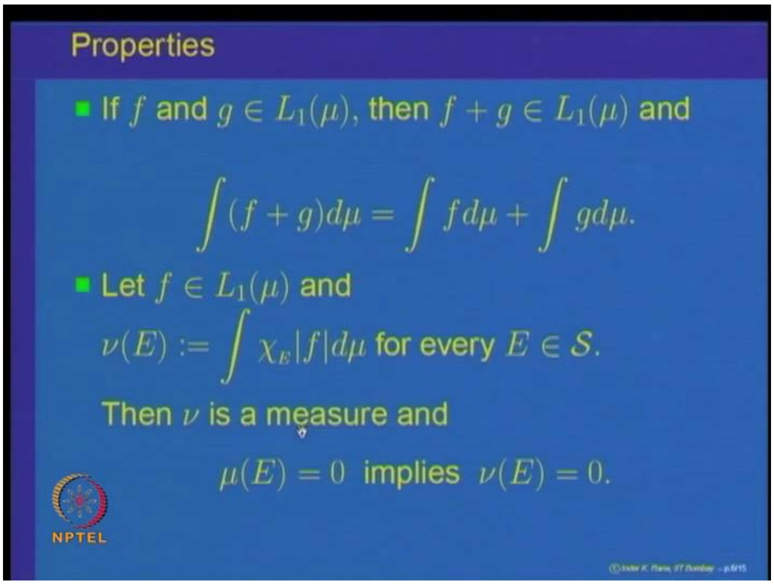

For an integrable function f in L_1 of mu, let us look at we have already shown; if mod f is a nonnegative measurable function and if you multiply it by the indicator function of a set E, then we already shown that this is again a nonnegative measurable function; of course, this function is less than or equal to integral of mod f. So, nu of E is going to be always a finite quantity.

The claim is nu is a measure, in fact a finite measure, and it has the property that mu of E equal to 0 implies nu of E equal to 0. Whenever a set E has got mu measure 0, the measure of nu is also going to be equal to 0. This basically follows from the properties of the integral for nonnegative functions because if f is integrable, then mod f is a nonnegative measurable function and its integral is a finite quantity.

So, nu of E is a finite measure for every E, the function chi_E times mod of f is less than or equal to mod f; this integral is going to be less than or equal to integral of mod f which is finite. Obviously for nonnegative functions, we have already proved this property: if mu of E is 0, then the integral over E is equal to 0; this property follows from our earlier discussions.

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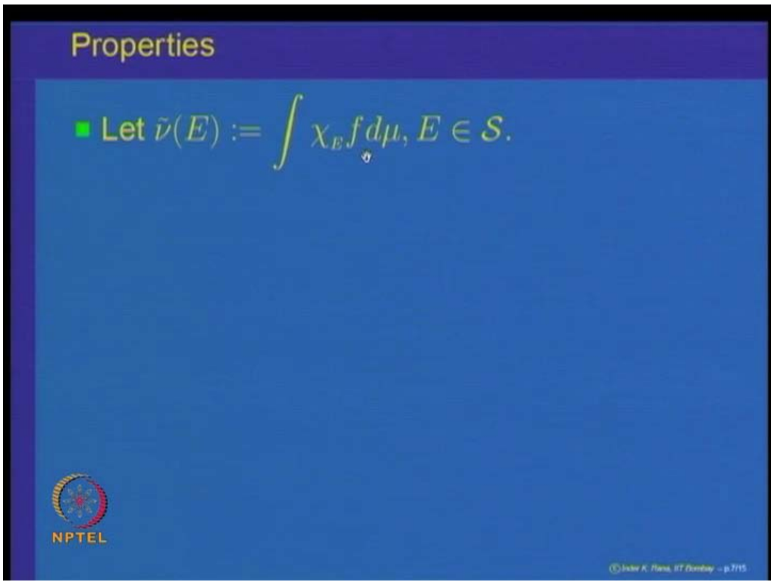

Let us look at the integral for the integrable function – not the integral of mod f but let us look at integral of f times indicator function of E. If you recall, we had already shown that if f is measurable and E is a set in the sigma algebra, then chi_E times f is again a measurable function and just now we observed that this number is going to be a finite number because this is again an integrable function.

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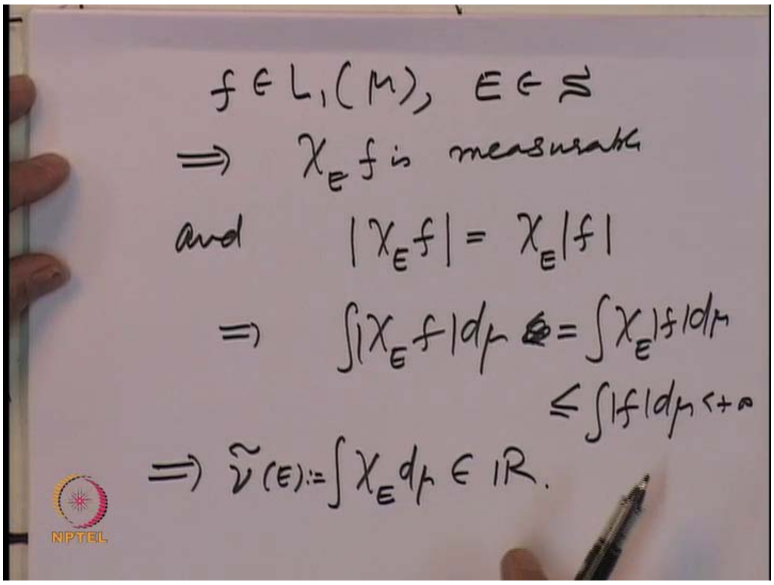

Let us just observe this property once again that if f belongs to L_1 of mu and E is a set in the sigma algebra, then this implies that chi_E times f is a measurable function; that we have already seen; because f is a measurable function and indicator function of E is a measurable function, the product of measurable function is measurable. We observed just now that if you look at the absolute value of chi of E times f, that is the same as indicator function of E because that is negative into absolute value of f.

This implies that the integral of chi_E times f absolute value into d mu is less than actually is equal to integral chi_E of mod f into d mu which is less than or equal to integral mod f into d mu which is finite. What does that imply? This implies that integral chi_E into d mu, this we are denoting it by nu tilde of E, is a real number – a finite real number. That is the observation and we want to claim that mu of E is equal to 0 implies that nu tilde of E is also equal to 0 (Refer Slide Time: 18:57).

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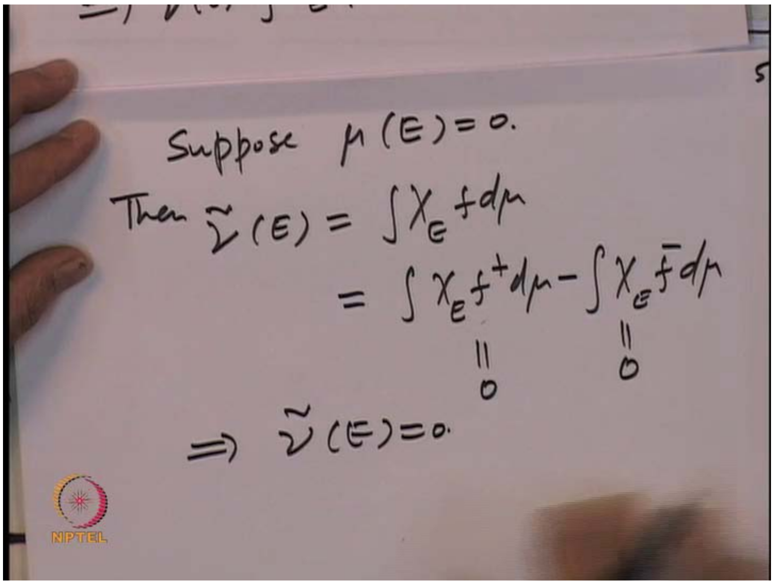

The claim is that the claim is that this property. Let us prove this property. Suppose mu of E is equal to 0, then what is nu tilde of E? nu tilde of E by definition is integral chi_E of f into d mu which is same as the integral of chi_E f plus into d mu minus integral chi_E f minus into d mu. Now, we observe mu of E equal to 0, chi_E of f plus is a nonnegative function and properties of nonnegative functions imply if the set has got measure 0, then the integral of this is equal to 0.

The first integral is equal to 0 and the second integral is equal to 0 by properties of integrals of nonnegative measurable functions. It implies nu tilde of E is equal to 0. What we are saying is mu of E equal to 0 implies nu tilde of E is also equal to 0 (Refer Slide Time: 20:25); that is the property we are proving here. Keep in mind that nu tilde of E is defined as a real number for every E belonging to S but it is not a nonnegative number because f may not be a nonnegative function; so, we cannot say nu tilde of E is a measure. We will look at this property a bit later; it may not be a measure but it has some properties similar to a measure.

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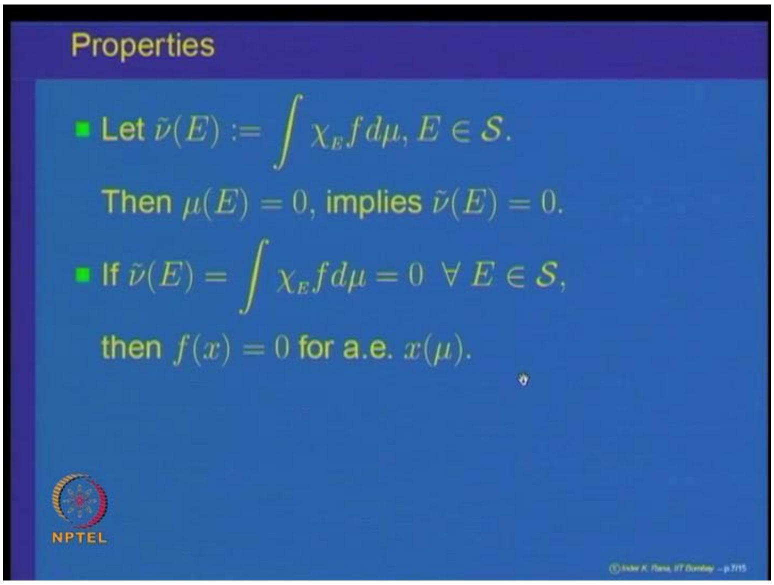

Here is another important property. Let us look at again the same value nu tilde of E which is equal to integral of f over E d mu. Suppose this is equal to 0 for every set E in the sigma algebra, then the claim is this function f must be equal to 0 for almost all x belonging to mu.

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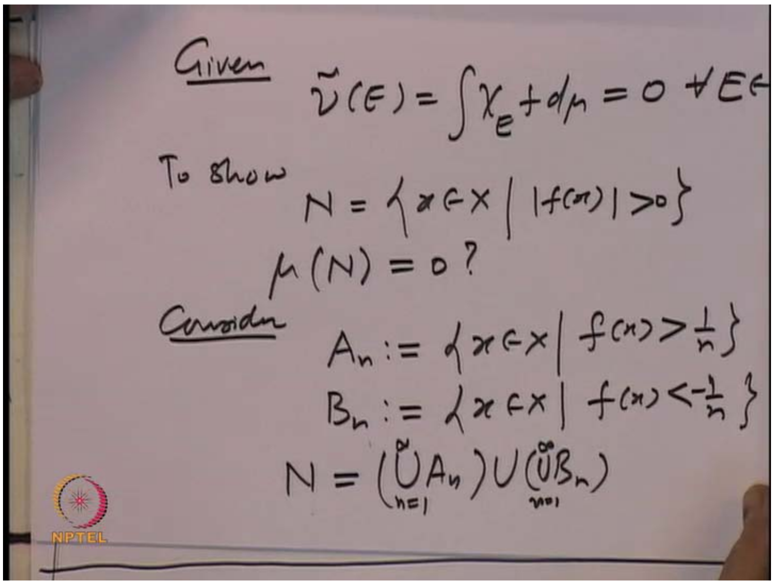

Let us prove this property. Given nu tilde of E which is nothing but integral chi_E times f into d mu is equal to 0 for every E belonging to S; that is what is given to us. We want to show that if you take the set N which is x belonging to X such that mod f of x bigger than 0 if we write the set N, then note that this set N is a set in the sigma algebra and we want to show that mu of N… We want to show f is 0 almost everywhere and this is the set where f is not 0. So, we want to show this is equal to 0 (Refer Slide Time: 22:22). This is the problem we want to show.

Let us look at consider the consider consider the set say for example, let us look at. Let us write, say, A_n to be the set where x belongs to X such that f of x is bigger than 1 over n. Similarly, let us write B_n to be the set of x belonging X where f of x is less than minus 1 by n. Now, the claim is that the set N is nothing but union over A_n union over B_n, n equal to 1 to infinity union of union n equal to 1 to infinity. That means all these sets A_ns and B_ns if you take their unions, that is precise as a set N where N is… What is the set N? N is the set where f of x is not equal to 0.

If f of x is not equal to 0, then either f of x is positive or f of x is negative. If it is positive, then it is going to be bigger than 1 over n for some n. If x is positive and bigger than 1 over n, it is going to belong to 1 over n or if f of x is not 0 and it is negative, that means it is negative so it is going to be less than minus 1 over n for some n; so, it belongs to B_n.

Every point x in N either belongs to A_n or belongs to B_n and obviously if x belongs to A_n or B_n, then f of x is not equal to 0; it belongs to N; so, N is equal to this. N is written as a countable union of sets and all of these are sets in the sigma algebra S. We want to show this union has got measure 0 (Refer Slide Time: 24:41). In case mu of N is not 0, that will mean for some n, either A_n has got positive measure or B_n has got positive measure, because otherwise mu of N will be less than or equal to sigma mu of A_ns plus sigma mu of B_ns, all of them equal to 0.

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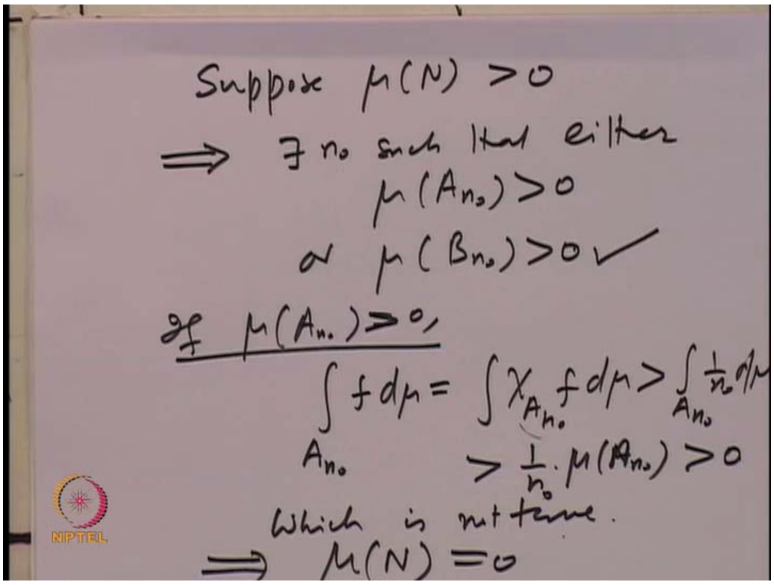

What we are saying is the following: to show that mu of n is equal to 0. Suppose mu of n is bigger than 0, then this condition implies there exists some n_0 such that either mu of A A_{n0} naught is bigger than 0 or mu of B n_0 is bigger than 0, because if not then mu of N will be equal to 0. Let us look at these conditions. The first one holds; if mu of A_{n0} is bigger than 0, then look at the integral; then integral of f over the set A_{n0} let us look at; let us look at integral of f over the set An_0.

That is equal to integral so which is same as integral chi$_{An0}$ times f into d mu. On the set A_{n0}, f is bigger than 1 over n. This is bigger than obviously integral 1 over n_0 times mu of A_{n0}. Let us observe that on the set A_{n0} … Outside A_{n0}, this function is equal to 0 (Refer Slide Time: 26:42); the indicator function of A_{n0} times f is 0; on A_{n0}, f is bigger than 1 over n_0. So, this function is bigger than 1 over n_0 and outside A_{n0} it is 0. This is going to be bigger integral over A_{n0} of 1 over n_0 into d mu; that is what we are saying.

This is nothing but this integral and that is bigger than 0. So, in case mu of A_{n0} is bigger than 0, integral of f over A_{n0} is going to be bigger than 0 which is a contradiction; it is not true because we are given integral of f over every set E is equal to 0, which is not true. If this holds (Refer Slide Time: 27:34), then it is a contradiction. Similarly, if this holds, one can approve it is a contradiction; then the integral of f over B_{n0} will be strictly less than 0 and not equal to 0. In either case, both of these are not possible; so, our assumption that mu of N is bigger than 0 must be wrong and hence mu of N so implies.

This implies that the measure of the set N is equal to 0. N was the set where f of x is bigger than 0 (Refer Slide Time: 28:11). This set has got measure 0. This is what we wanted to prove. We have proved the property that if integral of a function over f is an integrable function ((.)) its integral over E is equal to 0 for every E belonging to S, then f must be equal to 0 almost everywhere (Refer Slide Time: 28:33). This is a very nice property and useful property.

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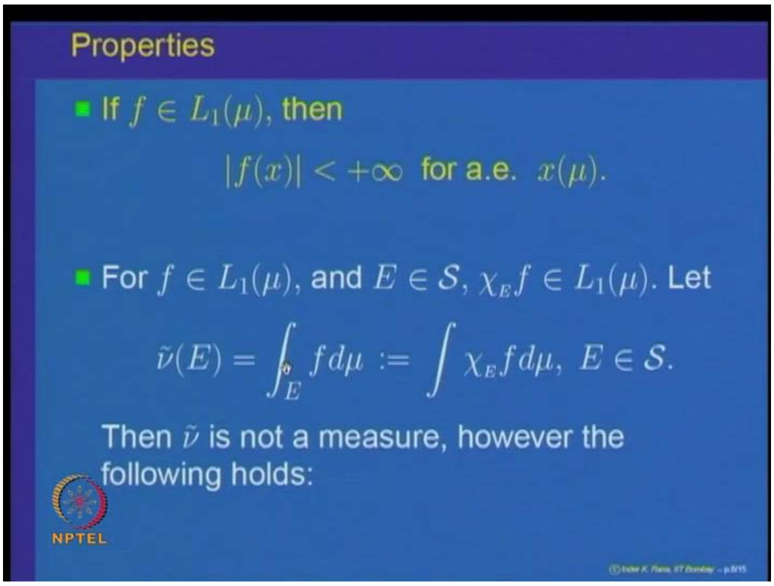

Now, let us look at the additive property of the integral over the sets; this just now we have proved (Refer Slide Time: 28:53). Another property is that if f is integrable, then f must be a finite number for almost all x – a similar argument as before; let us prove that property also.

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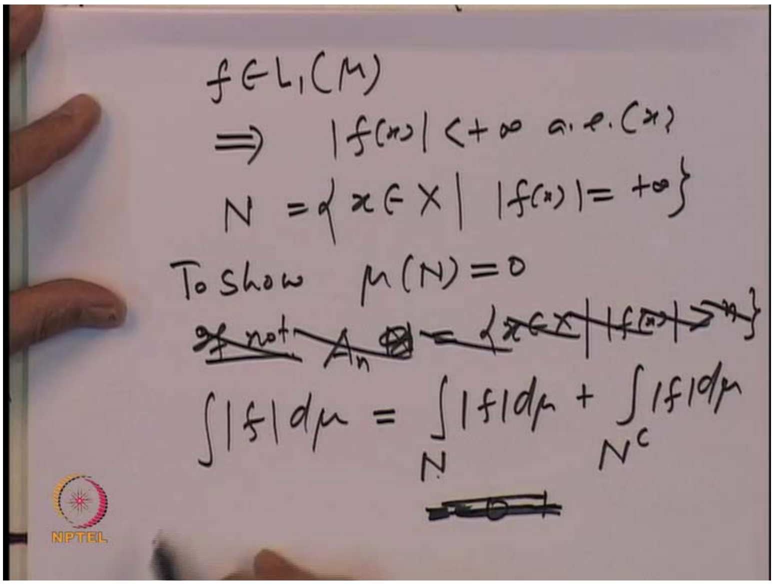

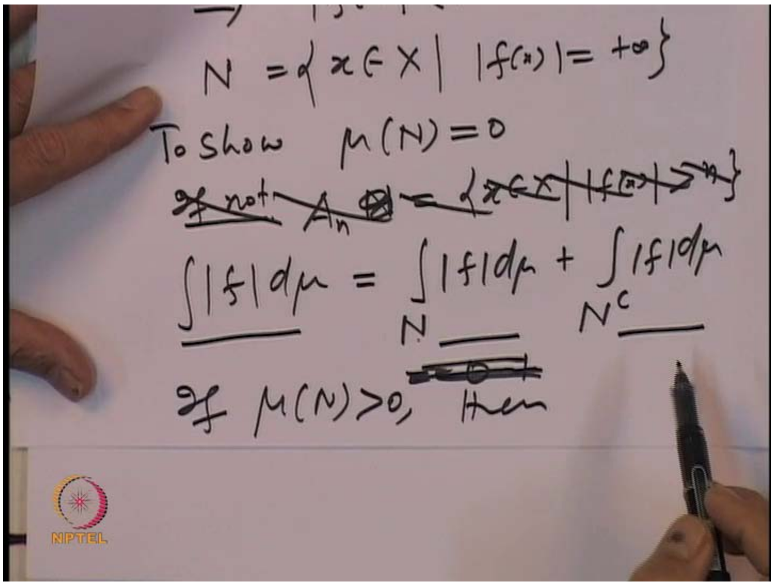

It says if f is integrable, then this implies that mod f of x is finite almost everywhere x. Once again, the idea is let us write the set N to be the set where mod f of x is equal to plus infinity; so, f of x is equal to infinity or equal to minus infinity; we have put them together in the set N. We have to show that the set mu of N is equal to 0. Once again if not, let us write N as x belonging to X such that mod f of x is bigger than say some quantity; it is not equal to 0.

If this is not equal to 0 and N is a set where f of x is equal to plus infinity, let us write f of x bigger than n. Sorry, let us write A_n to be the set where f of x is bigger than n Refer Slide Time: 30:368). Then, each set A_n is in the sigma algebra. Sorry, this is not required because we want to just want to show that f is finite (Refer Slide Time: 30:53). Now, observe that integral of mod f into d mu I can write as integral over N mod f into d mu plus integral over N complement mod f into d mu because N and N complement together make up the whole space and just now we observed that integral of mod f over a set E is a measure; so, integral of mod f over the whole space can be written as integral of mod f over N plus integral of mod f over N complement.

On N, mu of N is equal to 0; so, the first integral is 0. It is equal to 0 plus N sorry yes no let us observe this is so, if so, If mu of N is bigger than 0, then what will happen? Let us assume mu of N is bigger than 0. Then this integral is equal to this integral plus integral over N plus integral over N complement.

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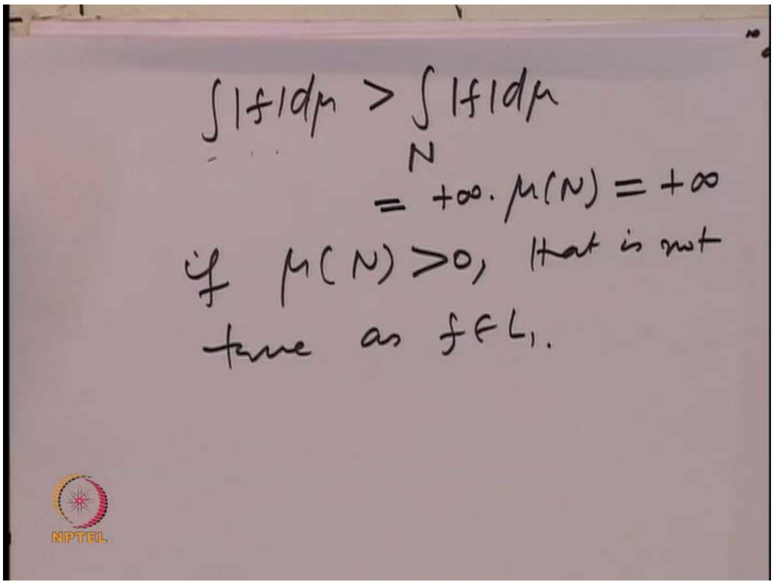

That means integral of mod f into d mu is always bigger than integral over N mod f into d mu. Integral of mod f is integral over N plus integral over N complement; let us just drop the second term (Refer Slide Time: 32:35). So, integral of mod f over the whole space is going to be bigger than integral over N f into d mu. On N, the function takes the value plus infinity; this is going to be equal to plus infinity multiplied with mu of N.

If mu of N is bigger than 0, then this will be equal to plus infinity if mu of N is bigger than 0. That is a contradiction; that is not possible; not true as f belongs to L_1; so, this integral must be a finite quantity. Here, we are saying in that case it will be equal to infinity. That proves that if a function is integrable, then it must be finite almost everywhere.

(Refer Slide Time: 33:30) Let us come back to the question if f is integrable and E belongs to the set S, then the indicator function of E times f is integrable; that we have just now observed. Let us write nu tilde of E as before: the integral of f over E. We observe that this number may not be a nonnegative number; however, it still has a property something similar to that of countable additive property for measures.

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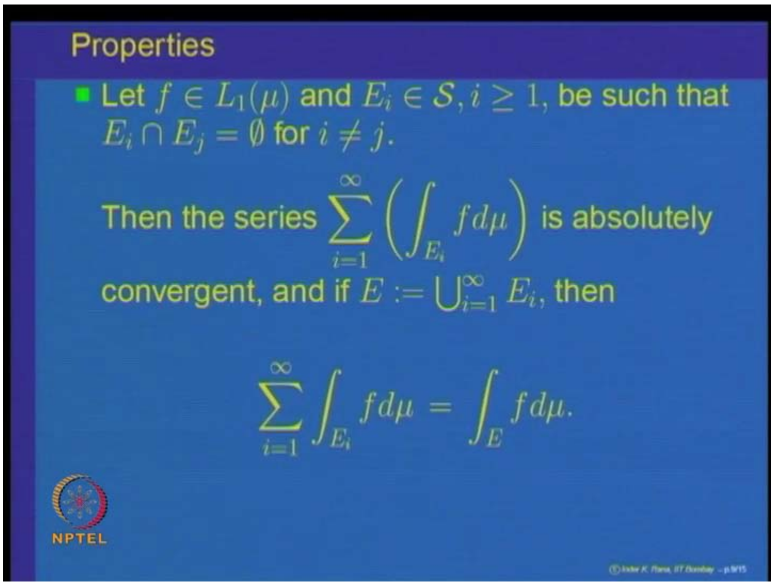

Let us state that property that if you take sets E_ns in the sigma algebra S which are pairwise disjoint and E is the union of the sets, then the claim is that the series which is integral of E_is f into d mu summation 1 to infinity – this series is absolutely convergent. If we write E as the union, then integral of f over E is equal to summation of integral of f over E_is. Essentially, we want to say that the integral of f of an integrable function over a set E can be written as summation integral of E_is where E_is are pairwise disjoint.

We are saying it is always possible for f to be an integrable function. Why are we saying absolutely convergent? Here, one should note that when E is equal to union E_i, it does not matter whether you write as E_1 union E_2 union E_3 and so o or any other order, say, E_2 union E_1; the union does not depend on the order in which you write the sequence E_i. That means in this series, the summation should not depend upon the order of the terms. All are nonnegative; that means we should prove that these are absolutely convergent.

That is what we want to prove: if E_is are pairwise disjoint, then the series integral over E_i of f into d mu summation 1 to infinity is an absolutely convergent series and this integral of f over E is summation of integral over E_is.

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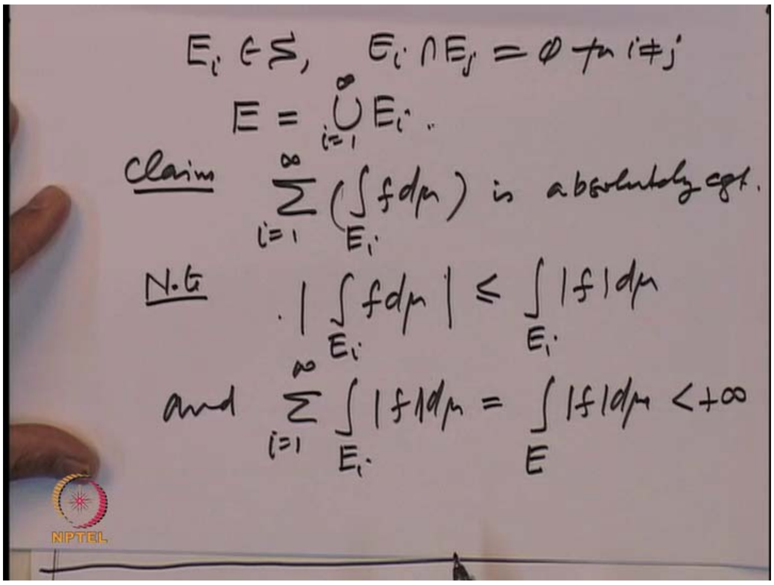

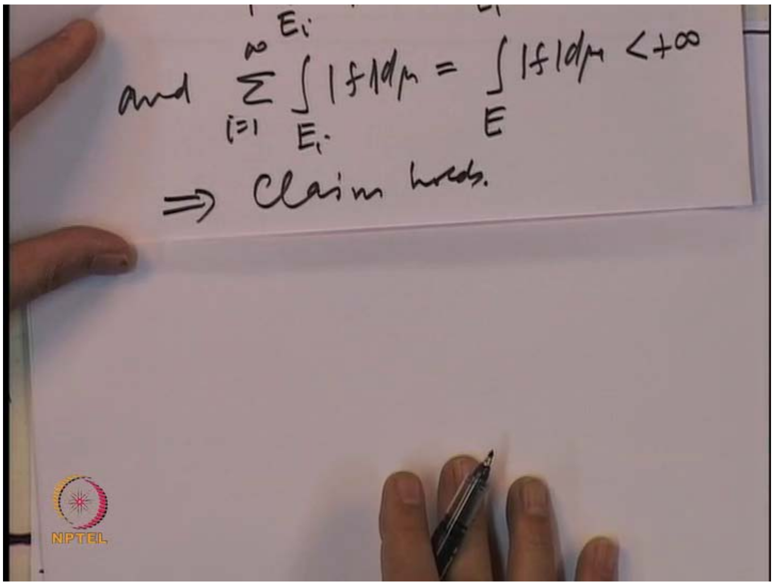

Let us prove this property. We have got E_i is a sequence of sets in the sigma algebra; they are pairwise disjoint and equal to empty set for i not equal to j; E is equal to union of E_is 1 to infinity. The first claim is that the series summation i equal to 1 to infinity of integral over E_i of f into d mu is absolutely convergent. Let us observe what absolute convergent means; that means absolute values of these terms is a series of nonnegative terms that must converge.

For that, let us note that absolute value of integral E over E_i of f into d mu is less than or equal to integral of mod f over E_i into d mu; that is a property of integrable functions – absolute value of the integral is less than or equal to integral of the absolute value. mod f is a nonnegative function and E is a disjoint union of sets; that implies that integral over E_i of mod f into d mu if I sum it up i equal to 1 to infinity, that is same as integral over E of mod f into d mu and f being integrable, that is a finite member.

Here, we have used two things: one, for nonnegative measurable functions, the integral over a set is a measure; so, integral of mod f into d mu over E_i summation 1 to infinity is equal to integral of absolute value of f over E and f being integrable, this is finite. That proves that the series integral f into d mu over E_i is absolutely convergent because this sum is less than or equal to this sum (Refer Slide Time: 38:36). From these two, it implies that the series is absolutely convergent. Once the series is absolutely convergent, its sum is equal to sum of the partial sums. Now, we can easily write this implies that the claim holds; so, the series is absolutely convergent.

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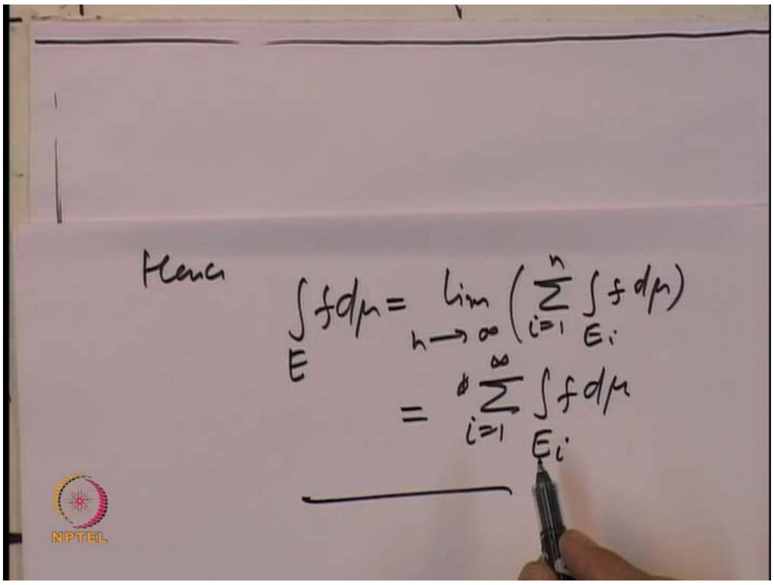

Hence, integral over E f into d mu is equal to limit n going to infinity of the partial sums, i equal to 1 to n integral of f over E_i d mu and that is nothing but partial sums; that is same as saying that sigma i equal to 1 to infinity integral over E_i of f into d mu. That proves that though the integral of an integrable function over a set (Refer Slide Time: 39:45) need not be a measure but we can say this is a countable additivity property of this integral – that integral over E is equal to summation of integrals over E_is whenever E is a union of pairwise disjoint sets E_i (Refer Slide Time: 40:06); this is the property that we have just now proved.

These were some of the properties that we have proved about the integral of integrable functions; now we want to prove an important property; we want to analyze sequences of integrable functions; we want to analyze the property that if f_n is a sequence of integrable functions and it converges to a function f, can we say that f is integrable and can we say integral of f_ns will converge to integral of f?

We have seen that this need not be true even for nonnegative functions, but under some suitable condition we can say that integral of f_ns will converge to integral of f and that is an important theorem called Lebesgue's dominated convergence theorem. Let us prove interchange of integral with the limits and look at the theorem called Lebesgue's dominated convergence theorem.

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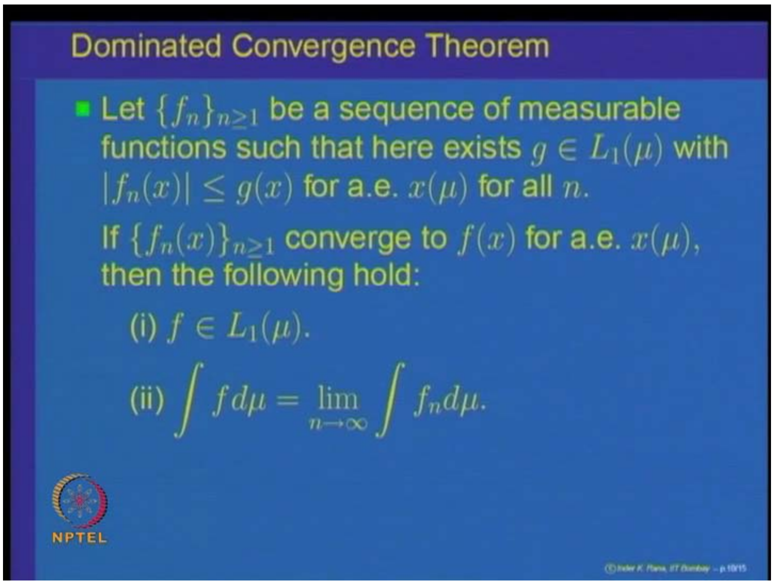

It says let f_n be a sequence of measurable functions such that there exists a function g which is integrable with the property that integral of mod f_ns are less than or equal to g for almost all x in mu for all n. Then the claim is if f_ns converge to f almost everywhere, then the limit function is integrable – (i) (Refer Slide Time: 41:42). Secondly, integral of the limit function is equal to limit of the integrable functions.

Let us observe once again what is given and what is true. We are saying here f_n is a sequence of measurable functions and all the f_ns are dominated by a single function g which is integrable and this dominance could be almost everywhere. All the f_ns are dominated by a single function g and the conclusion is if f_ns converge to f, then f is integrable and integral of f is equal to limit of integral of f_ns. This is what is called dominated convergence theorem; it is an important theorem; let us prove this theorem.

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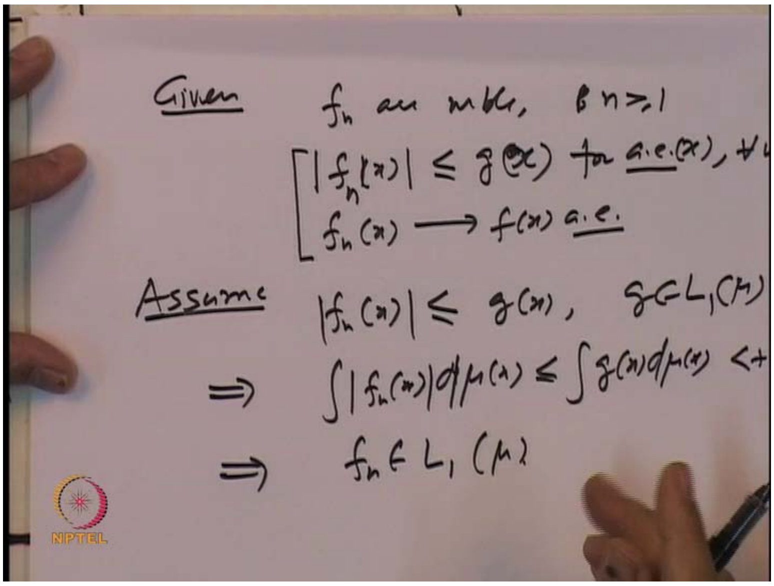

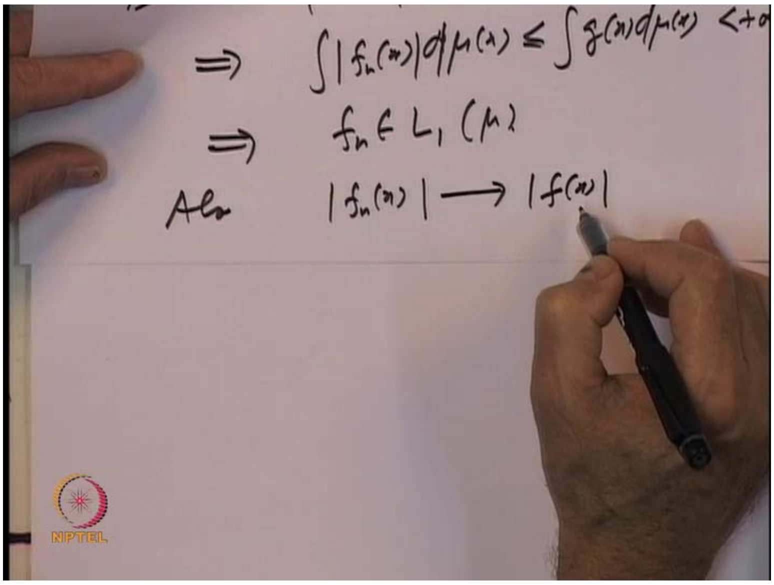

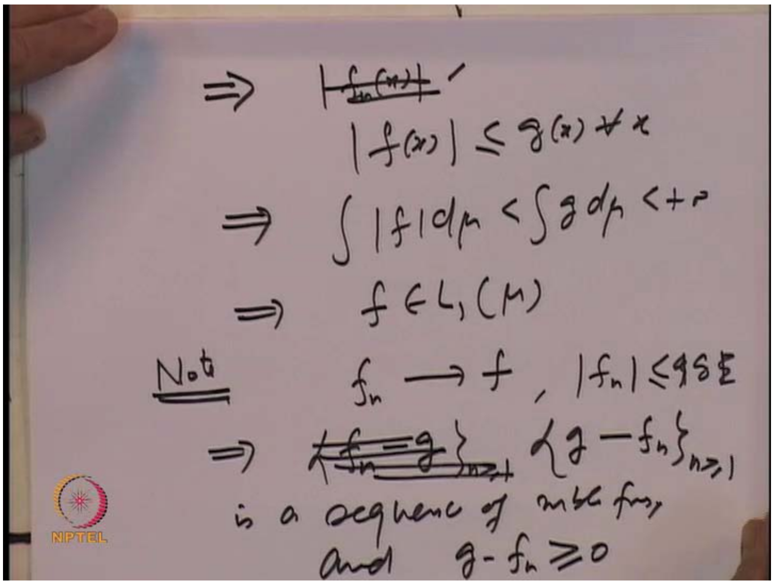

We are given that all the f_ns are measurable for n bigger than or equal to 1; mod f of x is less than or equal to g of x for almost all x and for every n; we are given that f_n of x converges to f of x almost everywhere. To prove the required claim, for the time being let us assume that this almost everywhere is everywhere (Refer Slide Time: 43:23). The proof is not going to change much; we will see that (( )). Let u assume for the time being that mod f_n of x is less than or equal to g of x where g is in L_1 is an integrable function. That implies that integral of mod f_n of x into d mu of x is less than or equal to integral g of x into d mu of x; g is integrable and so that is finite.

That implies that each f_n is an integrable function; also, mod f_n converges to mod f because f_n converges to f. Each mod f_n is less than or equal to g. (Refer Slide Time: 44:31) This implies that mod f_n of x is less than or equal to g and that converges; that implies that mod f of x is also less than or equal to g of x for every x. That once again implies integral mod f into d mu is less than integral g into d mu which is finite; this again implies that f is in L_1 of mu.

Under the given conditions, we have shown that if f_ns are dominated by an integrable function and f_ns converge to f, then f is an integrable function. To look at the limits, let us note f_n converges to f and mod f_n is less than equal to mod g. This implies that look at the sequence f_n minus g; look at this sequence. This is a sequence of measurable functions; of course, mod f_n is less than or equal to g. This will be negative; we want nonnegative and so let us look at g minus f_n; look at this sequence instead (Refer Slide Time: 46:11)

This is a sequence of nonnegative measurable functions because mod f_n is bigger than or equal to g; that is given. So, g minus f_n is a sequence of measurable functions. Let us observe g minus f_n is bigger than or equal to 0 because g is bigger than or equal to f_n. This is a sequence of nonnegative measurable functions.

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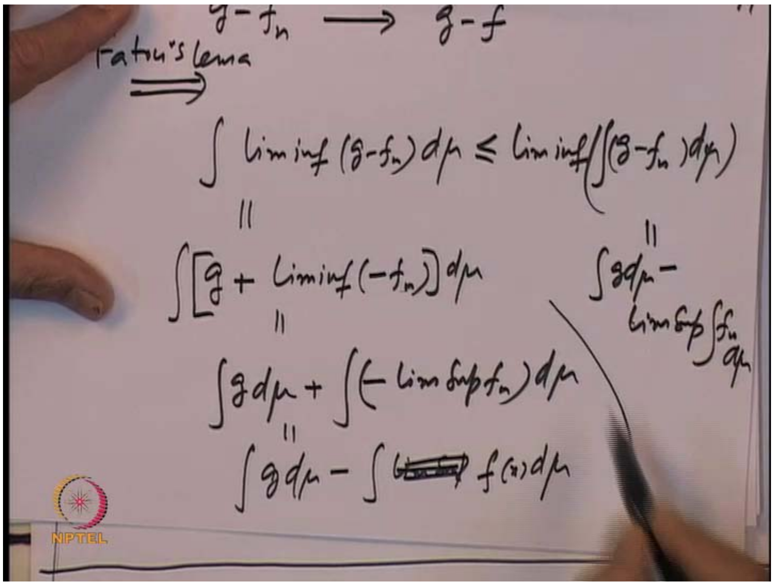

Let us write that g minus f_n is a sequence of nonnegative measurable functions and it converges to g minus f because f_n converges to f. Now, we can apply Fatou's Lemma; this implies by Fatou's Lemma that integral limit inferior of g minus f_n into d mu will be less than or equal to limit inferior of integral of g minus f_n into d mu; that is the application of Fatou's Lemma. Recall we had Fatou's Lemma which was applicable for a sequence of functions which is not necessarily increasing.

Look at this. Now, let us compute both sides. What is the left-hand side? This is equal to integral limit infimum of g minus f_n is g minus limit inferior plus limit inferior of minus f_n; that is the left-hand side d mu. That is equal to integral of g that is equal to integral of g into d mu. What can we say about this (Refer Slide Time: 48:42) f_ns all are integrable function; so, everything is finite. This limit inferior of minus f_n is equal to minus limit superior of f_ns into d mu.

This is a property of limit superior and limit inferior that limit inferior of minus f_n is equal to minus of limit superior; so, this is equal to integral of g into d mu minus integral limit superior of f_ns; limit f_n is convergent and so limit superior is same as f of x into d mu ((.)); that is the left hand side. Let us see what the right-hand side is. Once again, limit inferior of integral and so this is equal to limit inferior of integral; integral of g minus f_n is integral g and that does not depend upon limit; so, it is integral g into d mu and then limit inferior of minus; that will be minus limit superior of integral f_n into d mu. From these two, this is less than or equal to this. What does that imply? (Refer Slide Time: 50:09)

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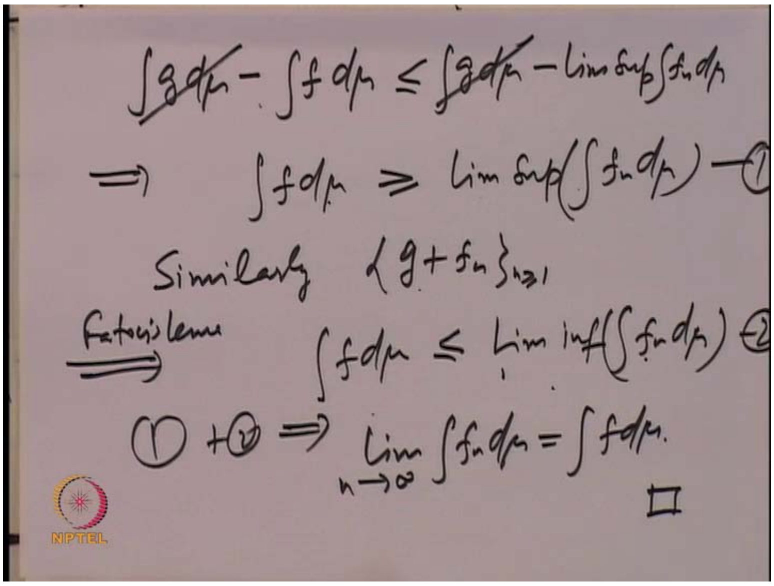

That implies that integral g into d mu minus integral f into d mu, I am just writing integral this (Refer Slide Time: 50:22), is less than or equal to integral g into d mu minus limit superior of integral f and d mu. Everything is finite and so I can cancel out these; negative sign gives you the other way inequality. It implies integral f into d mu is bigger than or equal to limit superior of integral f_n into d mu.

(Refer Slide Time: 50:54) Looking at the sequence g minus f_n, we got that g minus f_n is nonnegative converges to g minus f gives us this (Refer Slide Time: 51:03). Similarly, if I look at the sequence g plus f_n, that is again a sequence of nonnegative measurable functions and application of Fatou's Lemma will give me that integral of f into d mu is less than or equal to limit inferior of integral f_n into d mu; a similar application of Fatou's Lemma to this sequence will give me this (Refer Slide Time: 51:38).

This is (1) and this is (2). (1) plus (2) together imply that integral f into d mu is bigger than limit superior; that is always bigger than limit inferior and that is bigger than integral f into d mu; it implies that limit integral f_n d mu exists, this limit exists and this is equal to integral f into d mu (Refer Slide Time: 52:05).

That proves the dominated convergence theorem. The proof of the dominated convergence theorem is essentially very simple; it is just a straightforward application of Fatou's Lemma because mod f_n is less than or equal to g implies g minus f_n and g plus f_n both are sequences of nonnegative measurable functions; apply Fatou's Lemma and you have the conclusion that integral of f is equal to limit of integral of f_ns. We have proved this under the conditions that f_n of x converges to f of x and f_n of x is dominated by g of x for every x. The modification for this for almost everywhere ((.)) is simple and we will do it next time. Thank you very much.