Measure and Integration

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Module No. # 06

Lecture No. # 19

Monotone Convergence Theorem and Fatou's Lemma

Welcome to lecture number 19 on measure and integration. In the previous lecture, we had started looking at the properties of integral for nonnegative measurable functions. We had looked at the linearity property of the integral for nonnegative measurable functions and then we said we will start looking at the limiting properties of functions which are nonnegative measurable and integrals of them.

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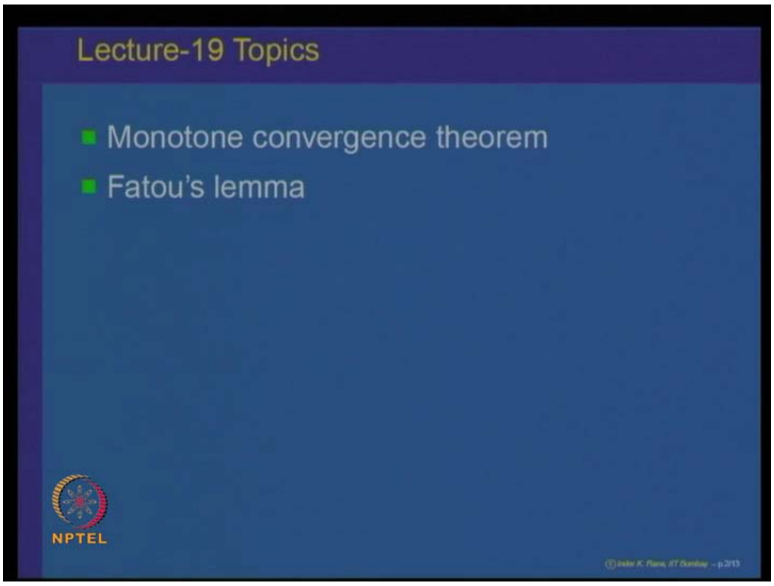

Today, we will prove some important theorems. We will start with proving what is called monotone convergence theorem; then we will prove Fatou's Lemma and then go over to define integral for general functions.

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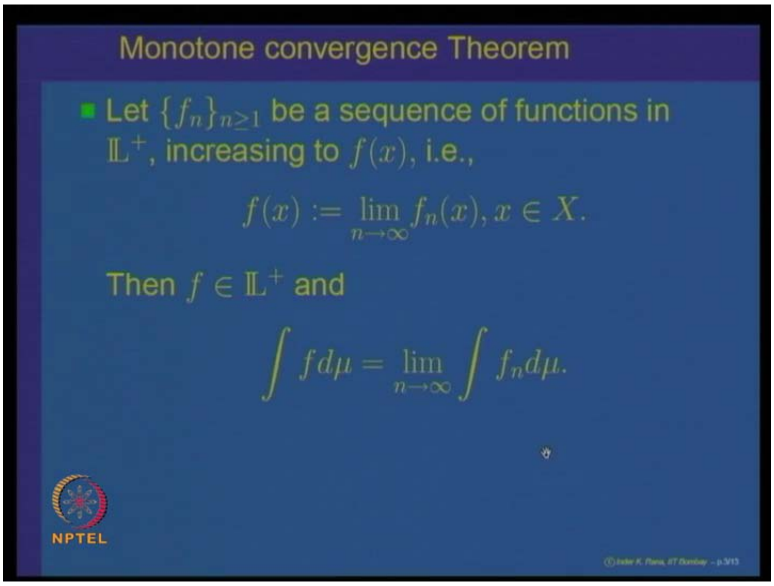

Let us look at what is called as monotone convergence theorem. The monotone convergence theorem says that let f_n be a sequence of functions in class L plus. That means f_n is a sequence of nonnegative measurable functions increasing to a function f of x at every point; that means f of x for every x in X is limit n going to infinity of f_n of x. We are given a sequence of f_n of nonnegative measurable functions which is increasing and the limit is f of x.

Then the claim is the function f belongs to L plus, this we have already observed, and the additional property is that the integral of the limit f into d mu is same as limit of the integrals of f_n into d mu. That means whenever a sequence f_n of nonnegative measurable functions increases to f, then integral of the limit is equal to limit of the integrals. This is the first important theorem about convergence of sequences of nonnegative measurable functions and their integrals. Let us prove this property.

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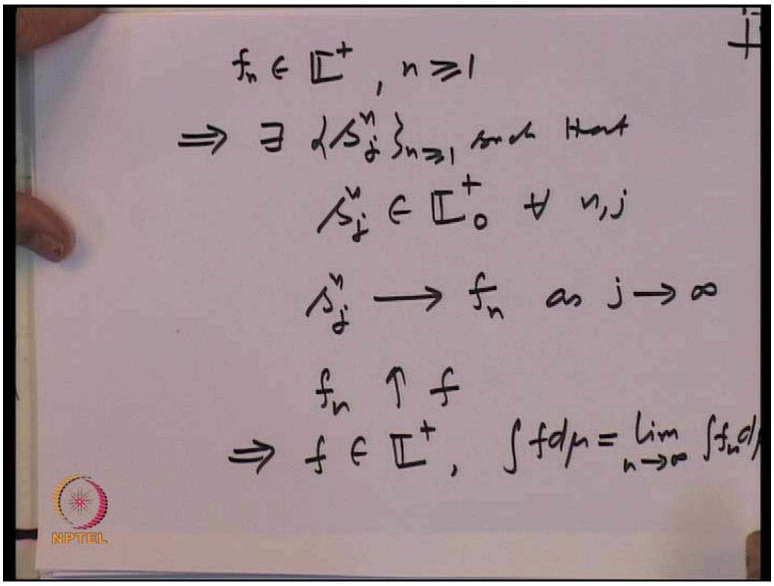

We are given f_n is a sequence; each f_n belongs to L plus; it is a nonnegative measurable function for every n bigger than or equal to 1. That implies there exists a sequence, we will denote it by s_j n, of functions n bigger than or equal to 1 such that s_j n are nonnegative measurable simple functions for every n and for every j and s_j n increases to f of…

Let us fix the notation – which one we are going to vary. Let us say that the upper one will be fixed; so, this is going to f n as j goes to infinity (Refer Slide Time: 03:12). For every n fixed, s_j n is a sequence on nonnegative simple measurable functions increasing to f_ns and f_ns increase to f. We want to show, we have already shown but we will show again, that this implies f belongs to L plus is a nonnegative function and integral f into d mu is equal to limit n going to infinity integral f_n into d mu. To prove this, we are going to use this sequence s_ns and construct a new sequence of nonnegative simple measurable functions out of it.

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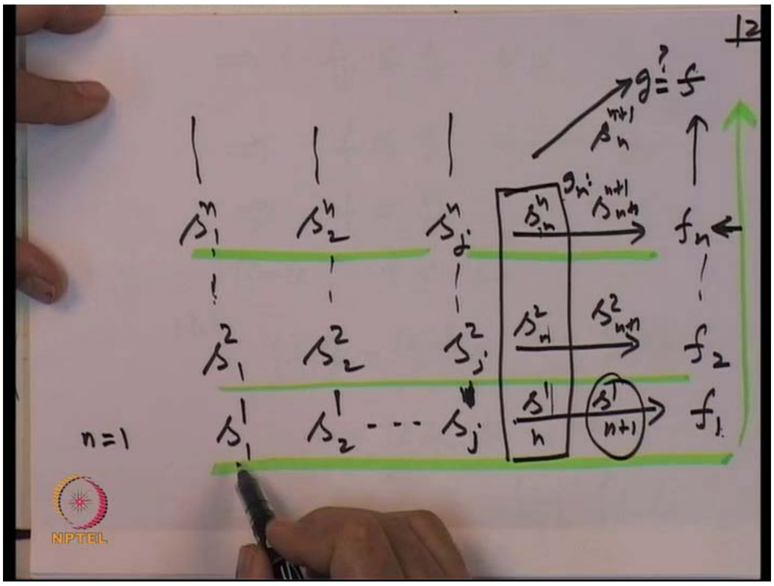

What we will do is the following. Let us write for n is equal to 1 that s_1 1, s_2 1 up to s_j 1 converges to f_1. The upper index is going to give you s_1 2, s_2 2 up to s_j 2 increases to f_2. In general, we will have s_1 n, s_2 n up to s_j n will increase to f_n and so on; this increases to f (Refer Slide Time: 05:00).

Let us observe that as we go from left to right, this is increasing; everywhere left to right, it is increasing (Refer Slide Time: 05:13); down to up, that also is increasing. If you look at every sequence, this is an array of nonnegative simple measurable functions; each row is increasing to the function on the right side and this is increasing upwards.

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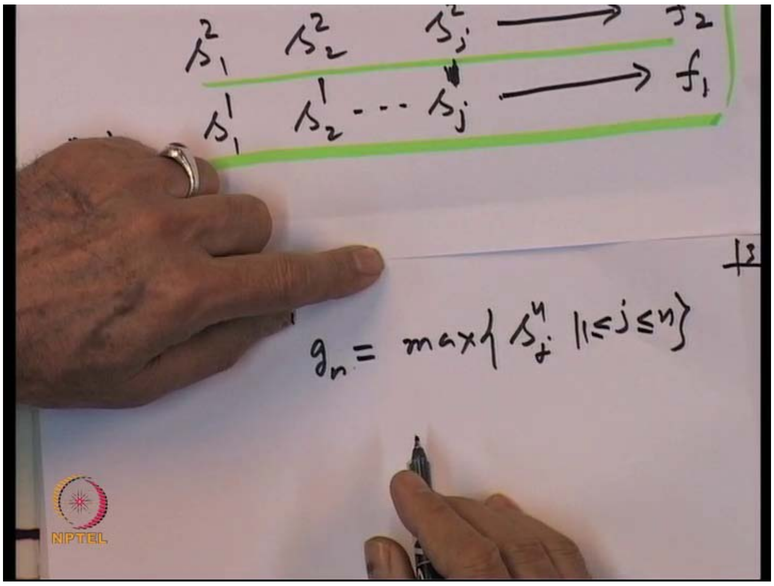

Let us look at the function. Let me define from this a function; let us define g_n to be the function which is maximum of s_j n, j between 1 and n.

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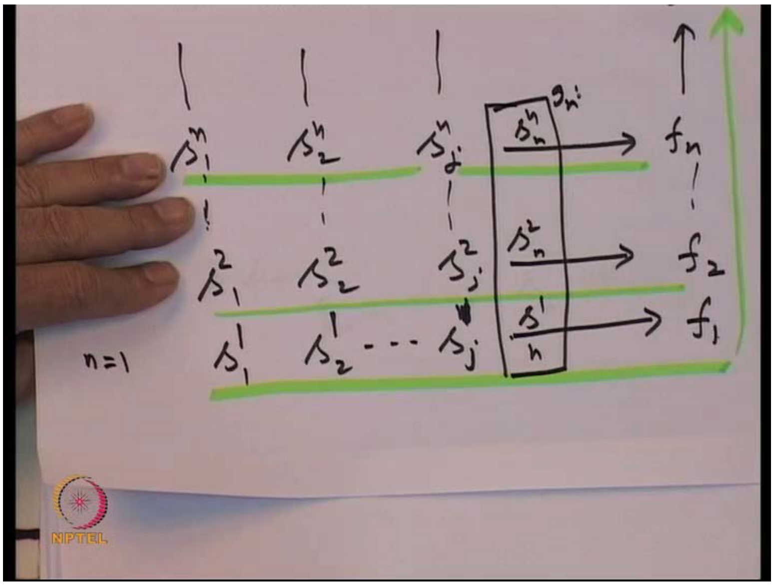

In essence, what I am doing is in this picture, let us say here is s_n 1, here is s_n 2 and here is s_n n. I look at this column; we are looking at the column s_n 1; let us look at this column and call the maximum of this to be g_n.

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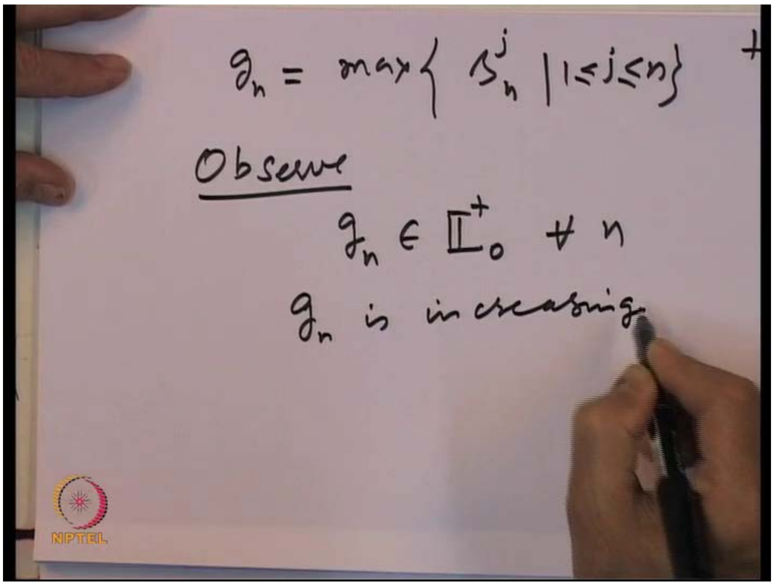

What is g_n? Let me write again. g_n is the maximum; so, define g_n equal to maximum of s_j n, j equal to 1 to n. Let us observe that each g_n is a maximum of nonnegative simple measurable functions; each g_n is a nonnegative simple measurable function for every n. g_n is increasing because at the next stage n plus 1…

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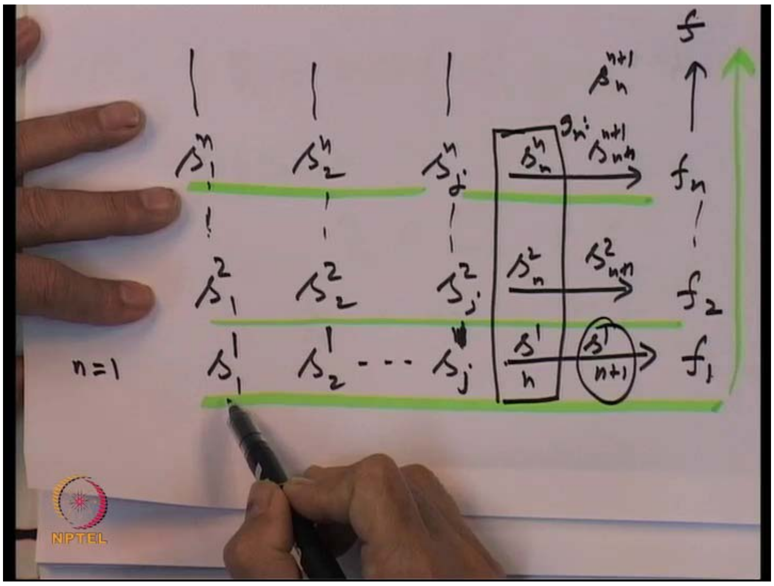

This is going to be bigger in the next stage. If we look at $g_{n \text{ plus } 1}$, that is going to be $s_{n \text{ plus } 1}$ 1, $s_{n \text{ plus } 1}$ 2 and so on $s_{n \text{ plus } 1}$ n plus 1 n and $s_{n \text{ plus } 1}$ n. This one (Refer Slide Time: 08:05) is going to be bigger than everything on the left-hand side and we are looking at the maximum; the maximum of these is going to be bigger than or equal to maximum of these because at each the right-hand side function is bigger than the left-hand side function. So, this is going to give us that at each g_n is an increasing sequence of functions (Refer Slide Time: 08:30).

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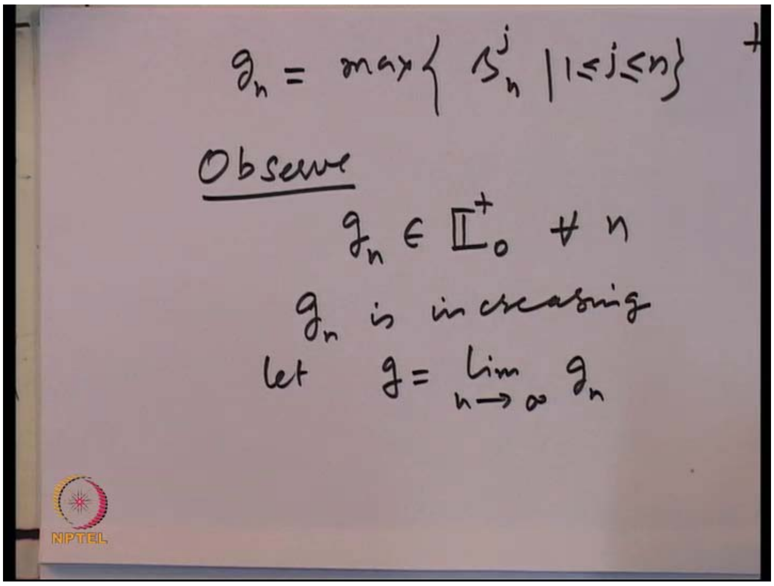

Let us write let g be equal to limit n going to infinity of g_n.

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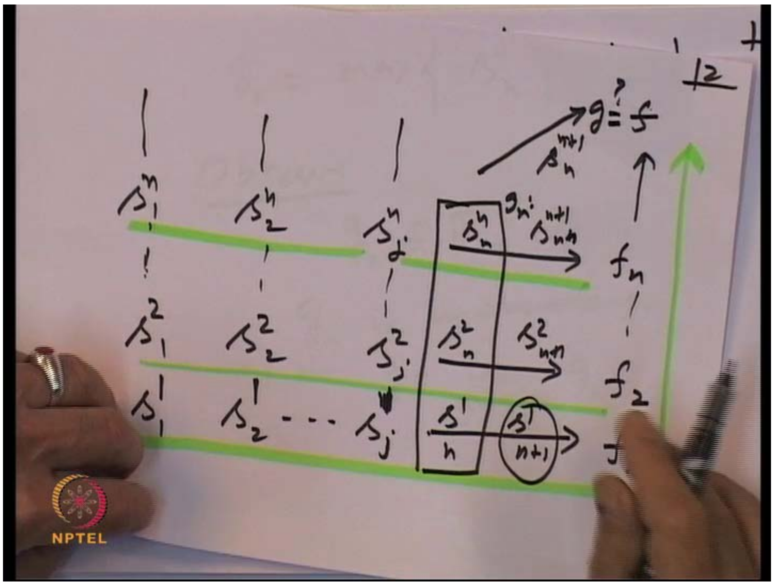

All these g_ns are increasing and they are going to increase to some function g. What we are going to show is g is equal to f; that is what we are going to check.

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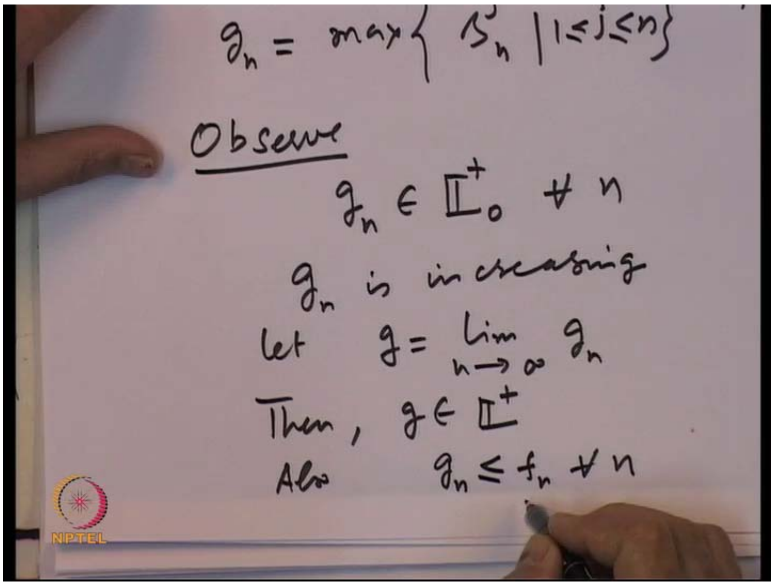

Clearly by definition, g is a nonnegative simple measurable function because it is a limit of an increasing sequence of nonnegative simple measurable functions; so, g belongs to L plus. Also let us observe that each g_n is less than equal to f_n; each g_n is less than or equal to f_n for every n. That is because g_n is the maximum of this (Refer Slide Time: 09:38) and the maximum of each one of them is less than f_1 is less than f_2 is less than f_n. The maximum of these g_ns is going to be less than or equal to this f_n for every n. f_n is increasing to f and so that will imply that g_n is less than or equal to f_n for every f and f_n is less than or equal to f.

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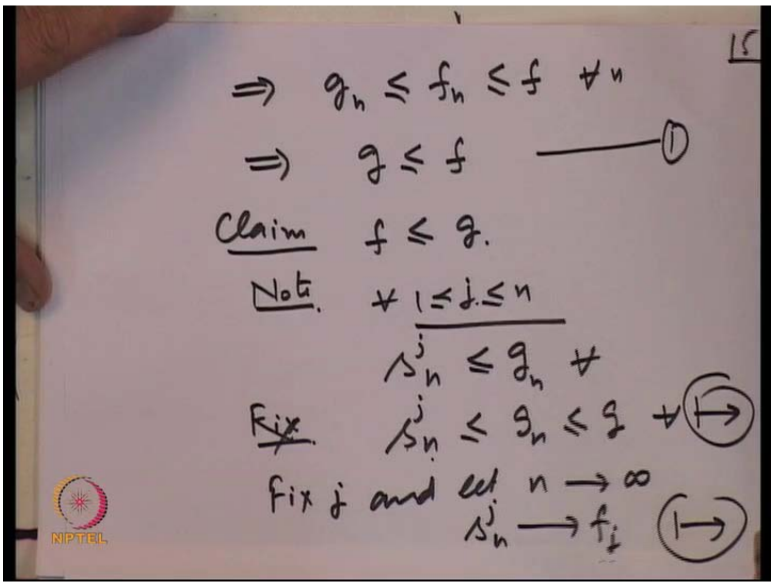

It implies that g_n is less than or equal to f_n and is less than or equal to f for every n. Hence, when g_n is increasing to g, that implies g is less than or equal to f; that is one observation that the function g is less than or equal to f. We claim that the other way round is also true; the claim is that f is also less than or equal to g. Let us note that for every j between 1 and n if I look at s_j n, g_n is the maximum of this and so this is less than or equal to g_n for every n; this is less than or equal to g_n for every n and j between less than this.

If we fix and g_n is less than or equal to so so and this is less than or equal to g; so, s_j n is less than or equal to g_n is less than or equal to g for every j between 1 and n and for every n. Let us now fix j and let n go to infinity; as n goes to infinity, what happens? This converges to f_n. Note that as n goes to infinity, s_j n goes to f_j. From this and this, these two observations, s_j n is less than or equal to g for every…. If you fix j and let n go to infinity, then n is going to cross over j and s_j n as n goes to infinity converges to f_j (Refer Slide Time: 12:09). This implies f_j is less than or equal to g for every j.

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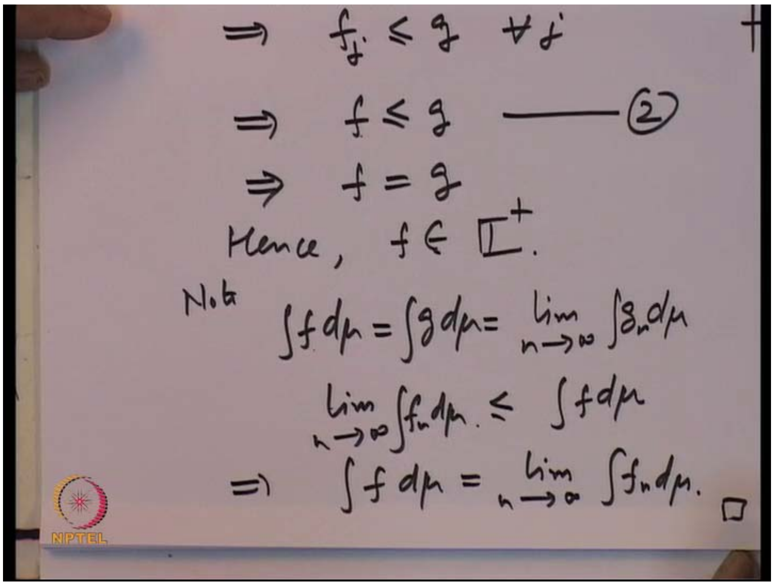

This implies that f_j is less than or equal to g for every j; f_js are increasing and so this implies that f is also less than or equal to g. We have already shown g is less than or equal to f (Refer Slide Time: 12:32). Now, we are saying f is less than or equal to g. This implies that f is equal to g. Hence, one observation from here is hence that g belongs to L plus and so f belongs to L plus. We have once again proved that if f_ns are increasing to a function f and f_ns are nonnegative measurable, then f is also nonnegative measurable.

Now note that integral of f into d mu is same as integral of g into d mu because f is equal to g. This is equal to limit n going to infinity of integral g_n into d mu because g_ns are nonnegative simple measurable increasing to g. By definition this is so, but each g_n is less than equal to f, if you recall (Refer Slide Time: 13:37). Each g_n is less than or equal to f; so, integral of g_n will be less than or equal to integral of f; so, limit of integrals of f_ns will be less than or equal to integral f. This is less than or equal to integral f into d mu (Refer Slide Time: 13:55). You can even introduce in between; g_n is less than or equal to f_n; so, it is less than or equal to limit n going to infinity integral f_n into d mu which is less than or equal to integral f into d mu.

What does this imply? Integral f into d mu is less than or equal to limit f_n integral of f_n into d mu and that is less than or equal to f into d mu. That implies that integral of f into d mu is equal to limit n going to infinity integral f_n into d mu. That proves the theorem completely, namely integral of f d mu is equal to limit n going to infinity integral of f_n d mu. This is a construction which is quite useful; this is the kind of analysis one has to carry out.

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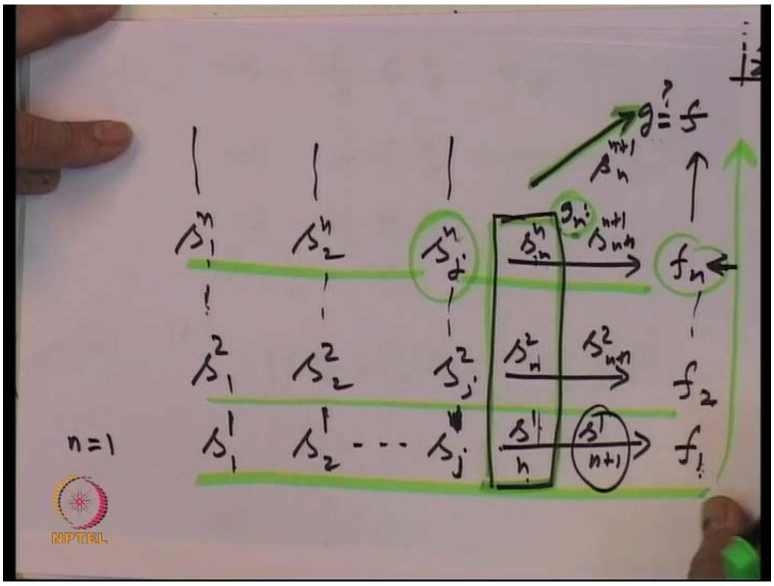

Let us go through the proof again so we that understand what we are doing. Each f_j or f_n is a measurable function. So, I can look at a sequence s_1 1, s_2 1, s_j 1 up to s_n 1 which is going to increase to f_1 (Refer Slide Time: 15:08). Similarly, the upper ((.)) is fixed – 2; so, s_1 2, s_2 2, s_j 2 up to s_n 2 increases to f_2 and so on. Each row is increasing to the function on the right side and the functions f_1, f_2 up to f_ns are increasing to the function f.

What we do is we look at the maximum of this column (Refer Slide Time: 15:38). What is this column? This column is the maximum of the functions s_n 1, s_n 2, up to s_n n. Call this as g_n; this function is called g_n. The observation is each g_n is a maximum of nonnegative simple measurable functions and so it is nonnegative simple measurable. Each g_n is lesser or equal to f_n because we are going up to this corner only (Refer Slide Time: 16:08). So, each g_n is less than or equal to… because s_n 1 is less than f_1, s_n 2 is less than f_2, f_1 is less than f_2 and so on; so, this says g_n will be less than or equal to f_n and each f_n is less than or equal to f.

So, each g_n is less than or equal to f_n is less than f. If we write the limit of g_n to be g, then g is less than or equal to f by this simple construction. Also, for any fixed j let us look at s_j n. Let us look at s_n j where j is fixed and n is going to vary. As n varies, what happens to these functions? For every fixed j, this sequence of functions is going to be s_j n is less than or equal to g_n and g_n is less than or equal to f. g is less than or equal to f. Sorry, s_j n is less than or equal to g_n for j between 1 and n; that will give us that f is also less than or equal to g.

That will prove the theorem that limit of increasing sequence of nonnegative measurable functions.... (Refer Slide Time: 17:37) If f_n is a sequence of nonnegative measurable functions increasing to f, then integral f into d mu is equal to limit of n going to infinity integral f_n into d mu. This is called monotone convergence theorem, monotone because we are looking at monotonically increasing sequences f_ns and convergence because we are looking at the convergence of the integrals – integral f_n into d mu. This proves the monotone convergence theorem.

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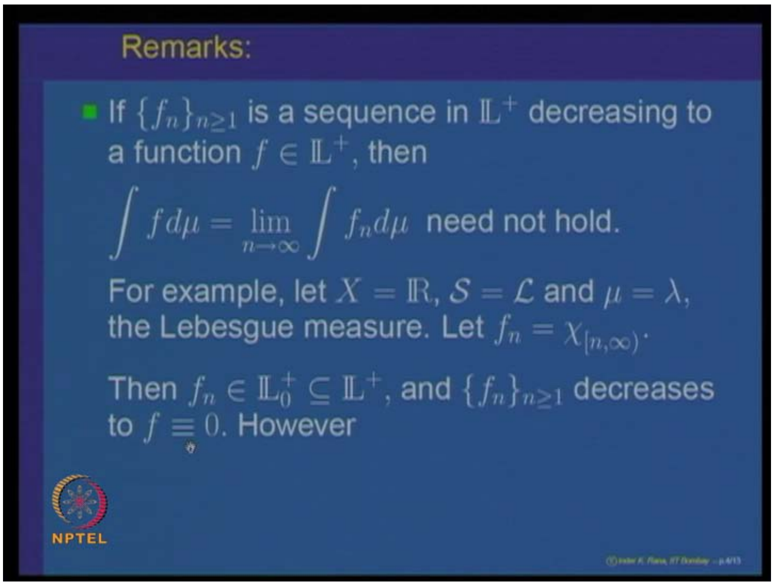

We have proved the monotone convergence theorem for when f_n is an increasing sequence (Refer Slide Time: 18:12). Naturally, the question arises: will the similar result hold if I have a decreasing sequence f_n of nonnegative measurable functions? That result unfortunately is not true. Here is an example which says that if f_n is a sequence of functions which are nonnegative measurable and they decrease to a function f, then integral of f need not be equal to integral of f_n into d mu. The example is on the Lebesgue measurable space.

Look at X to be the real line, the sigma algebra to be the sigma algebra of Lebesgue measurable sets and mu to be the Lebesgue measure. Look at the function f_n which is the indicator function of the interval n to infinity. This is actually a nonnegative simple measurable function – each f_n; f_n is decreasing and decreasing to the identity function identically equal to 0; that is quite obvious to see.

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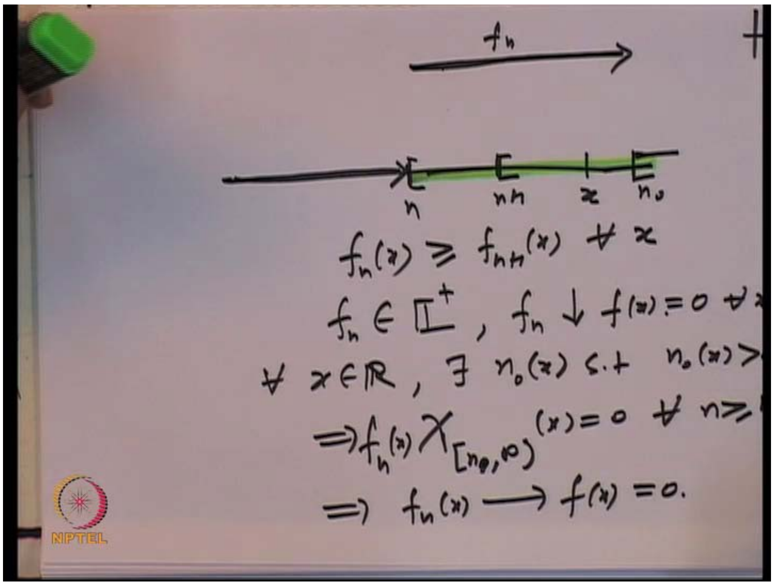

What is f_n? Here is n and we are looking at the interval n to infinity. We are looking at this interval (Refer Slide Time: 19:31). We are looking at the indicator function of n to infinity. So, the function is 0 and it is 1 here. This height is 1; this is the function f_n; it is 0 up to here (Refer Slide Time: 19:46). Then, it starts and goes; that is the function f_n. We take n plus 1; this is n plus 1; so n plus 1 will be 0 here (Refer Slide Time: 20:00) but f_n is equal to 1 here. Clearly, f_n of x is bigger than or equal to $f_{n \text{ plus } 1}$ of x for every x.

So, f_n is a sequence in L plus and f_n is decreasing. The claim is f_n decreases to f of x which is identically equal to 0 for every x. If I take any point x on the real line, then I can find some integer n, say, n_0 which is on the right side of it. So, for every x belonging lo real line ((.)), I can find a point n_0 – a positive integer n_0 (of course, it will depend on x) such that n_0 of x is bigger than x. That will imply that the indicator function of n to infinity at x is going to be equal to 0 for every n bigger than equal to n_0 and that is my f_n of x. So, f_n of x is equal to 0 for every n bigger than n; that means f_n of x converges to f of x which is equal to 0. So, f_n is a sequence of nonnegative measurable functions which is decreasing to f identically 0, but if we look at the integral of each f_n, what is the integral of each f_n?

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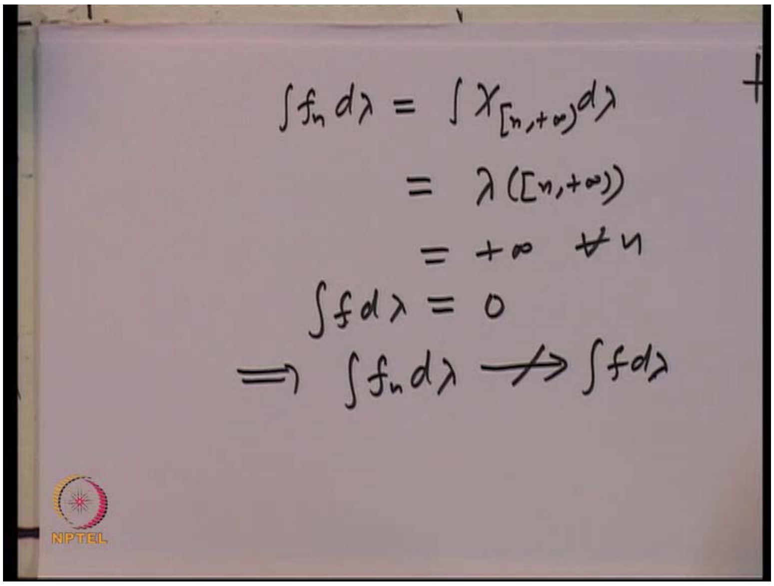

Integral of f_n into d lambda is the integral of the indicator function n to infinity d lambda. That is equal to lambda of n to plus infinity; that is equal to plus infinity for every n. Integral of f_n is equal to plus infinity for every n and integral of f into d lambda is equal to f is 0 and so it is 0. This implies that integral f_n into d lambda does not converge to integral f into d lambda whenever f_n is a decreasing sequence of function nonnegative, simple nonnegative; even simple function we have given example here (Refer Slide Time: 22:24).

For decreasing sequences, this result does not hold; that gives the importance to the monotone convergence theorem; that means whenever a sequence f_n of nonnegative functions is increasing, then integral f is equal to limit integral f_n into d mu. For decreasing, this need not hold; this is what we have shown just now by an example.

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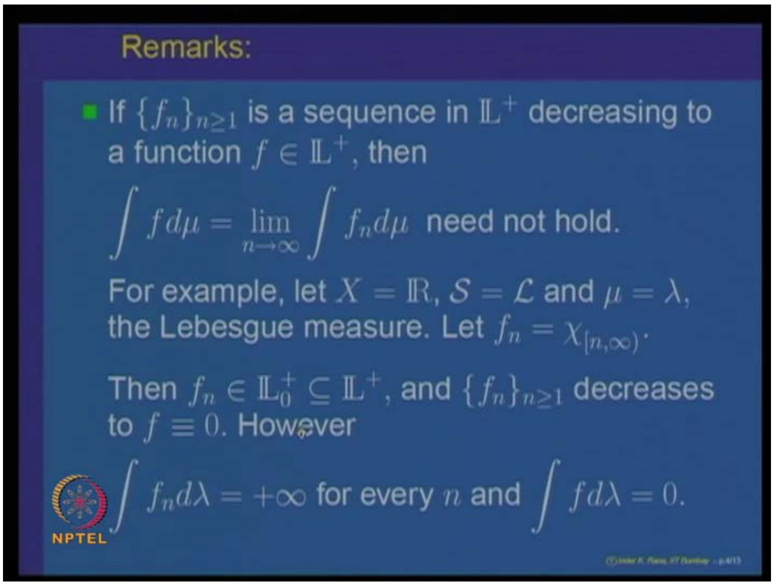

However, one can prove not an inequality but some kind of a inequality for a sequence of nonnegative measurable functions. That is also an important result

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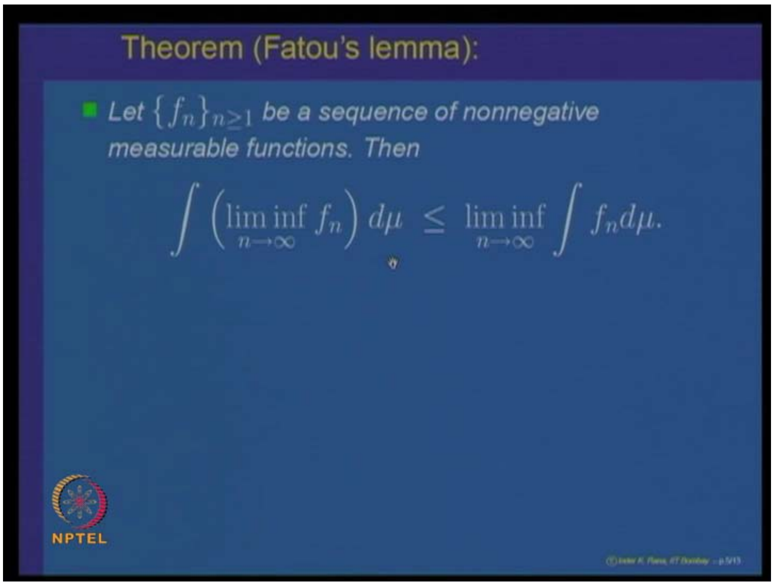

Let us prove the result which is called Fatou's Lemma. It says let f_n be a sequence of nonnegative measurable functions. Then the integral of limit inferior of f_n into d mu is less than or equal to limit inferior of the integrals f_n into d mu. This is only an inequality and it need not be an equality. What we are saying is if f_n is sequence of nonnegative measurable functions, then it is always true that the integral of the limit inferior of f_ns is less than or equal to limit inferior of the integrals.

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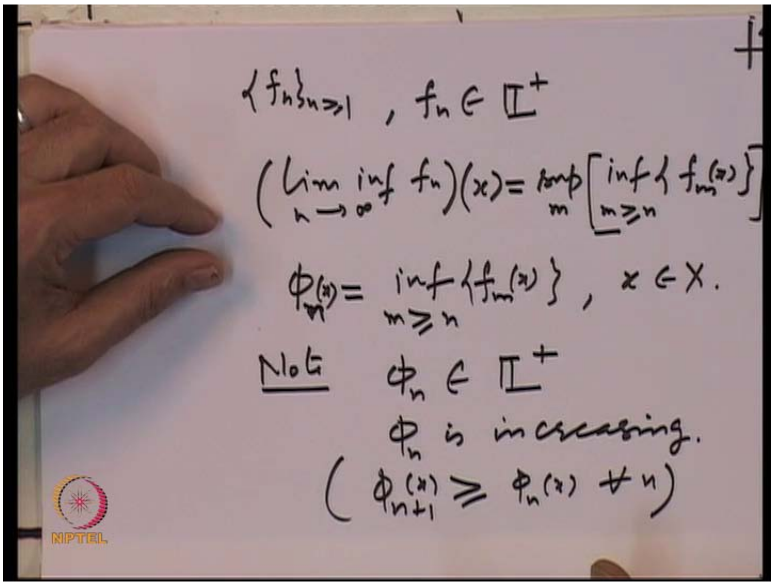

Let us give a proof of this theorem. To prove this theorem, let us just first recall what is f_n. It is a sequence of nonnegative measurable functions; each f_n is a nonnegative measurable function. We want to look at limit inferior of f_n as n goes to infinity. This is a function; let us observe how this function is defined. Limit inferior of f_n at a point x is defined as you take the infimum from some stage onwards; so, it is m bigger than or equal to n of f_n of x.

Look at the numbers f_m of x for m bigger than or equal to n; I am looking at the tail of the sequence f_n of x from m onwards. So, this number infimum will depend on m (Refer Slide Time: 24:45). Let me take the supremum of this over all m. First, take the infimum from some stage onwards and then take the supremum of these infimums. Let us put a bracket here. Let me call phi_m to be the infimum from the stage n onwards – infimum of m bigger than or equal to n of f_n of x; phi_m of x to be defined as the infimum from the stage n onwards of f_m of x.

Then, because it is an infimum of a sequence of functions which are nonnegative measurable, clearly the observation is that each phi_n is also a nonnegative measurable function; so, it is a nonnegative measurable function; that is one. Secondly, we are taking the infimum from some stage n onwards; so, if we increase, the claim is this phi_n is increasing.

(Refer Slide Time: 26:19) phi_n, the infimum from the stage n onwards, is going to be less than or equal to the infimum from the stage n plus 1 onwards because we will have more numbers for which you are taking infimum. When you take infimum of more numbers, then infimum can decrease; so, infimum from the stage n onwards and the infimum from the stage n plus 1 onwards…. That says that the infimum from the stage n plus 1 onwards will be bigger than or equal to the infimum; so, it is increasing; that is, $\text{phi}_{n \text{ plus } 1}$ is bigger than or equal to phi_n of x for every n. It is an increasing sequence of nonnegative measurable functions and its limit is nothing but the limit inferior; so, it is increasing.

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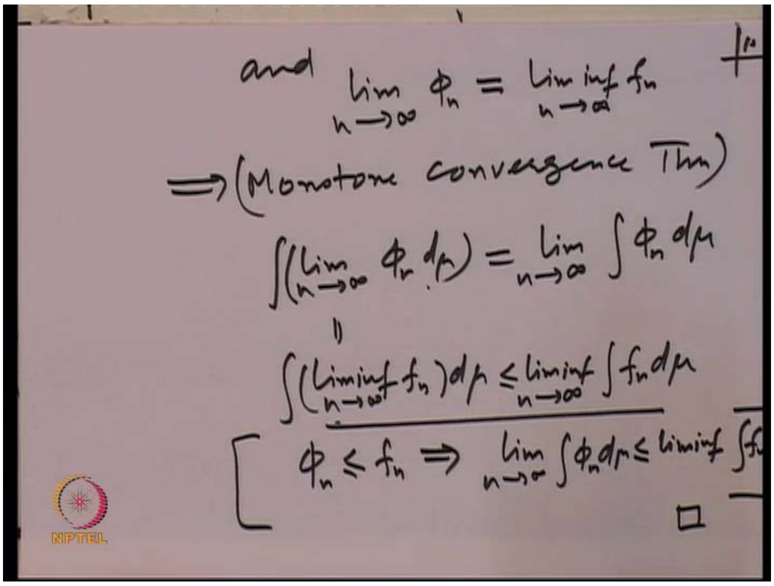

It is increasing and limit n going to infinity of phi_n is equal to limit inferior of f_n, n going to infinity. The stage is set – perfect – for application of monotone convergence theorem (Refer Slide Time: 27:42). phi_n is a sequence of nonnegative measurable functions and phi_ns are increasing. We can apply monotone convergence theorem. It implies that by monotone convergence theorem the integral of limit n going to infinity of phi_n into d mu is equal to limit integral phi_n into d mu, n going to infinity.

The left-hand side is nothing but integral of limit inferior n going to infinity of f_n into d mu; that is equal to limit of integral f_ns. Now, let us look at what is phi_n (Refer Slide

Time: 28:37). phi_n is the infimum from the stage n onwards; each phi_n is less than or equal to f_n; that is the observation from here by the definition of phi_n (Refer Slide Time: 28:50). We have that that each phi_n is less than or equal to f_n; so, integral of phi_n will be less than or equal to integral of f_n; it will be less than or equal to limit inferior of n going to infinity.

What we are observing here is because each phi_n is less than or equal to f_n, this implies (this is what we are using here) that if phi_n is less than or equal to f_n, then the integrals of phi_ns are increasing; so, its limit exists; so, it is limit n going to infinity integral phi_n into d mu is less than or equal to… However, integral of f_ns may not exist. We can say that it will be less than or equal to limit inferior of integral f_ns into d mu; this is what is being used in this conclusion.

That proves the theorem called Fatou's Lemma. So, we have two important results for nonnegative (Refer Slide Time: 29:59). We have got two important results for sequences of nonnegative measurable functions. One of them is called the monotone convergence theorem which says if f_n is an increasing sequence of nonnegative measurable functions increasing to a function f, then integrals of f_ns will converge to integral of f. Keep in mind that the monotone convergence theorem is for a nonnegative sequence of nonnegative measurable functions which is increasing to f.

In case the sequence of nonnegative measurable functions is not increasing, then we have Fatou's Lemma which says that for any sequence f_n of nonnegative measurable functions, the integral of the limit inferior of f_ns is going to be less than or equal to, is always less than or equal to, limit inferior of the integrals. These are the two important theorems which help us to relate the limit of the integrals with the integral of the limits; we will see applications of this in the rest of our course.

With this, we conclude the section on definition of an integral for nonnegative measurable functions. Let us just recall what we have done. We started with defining the integral of nonnegative simple measurable functions, the functions which look like linear combinations of indicator functions sigma a_i indicator functions of A_i and for them we defined the integral to be nothing but summation of a_i times mu of A_i. We showed that it is independent of the representation and we proved various properties of the integral for nonnegative simple measurable functions.

Then we looked at the class of nonnegative measurable functions. Since every nonnegative measurable function is a limit of some sequence of nonnegative simple measurable functions increasing to that function f, we defined integral of the nonnegative simple measurable functions to be nothing but the limit of the integrals of that sequence of nonnegative simple measurable functions increasing to it. We showed this integral is independent of the limit of the sequence s_n you select which increases to f; then we proved various properties including monotone convergence theorem and Fatou's Lemma. Now, let us look at how we can define the integral for a function which is not necessarily nonnegative.

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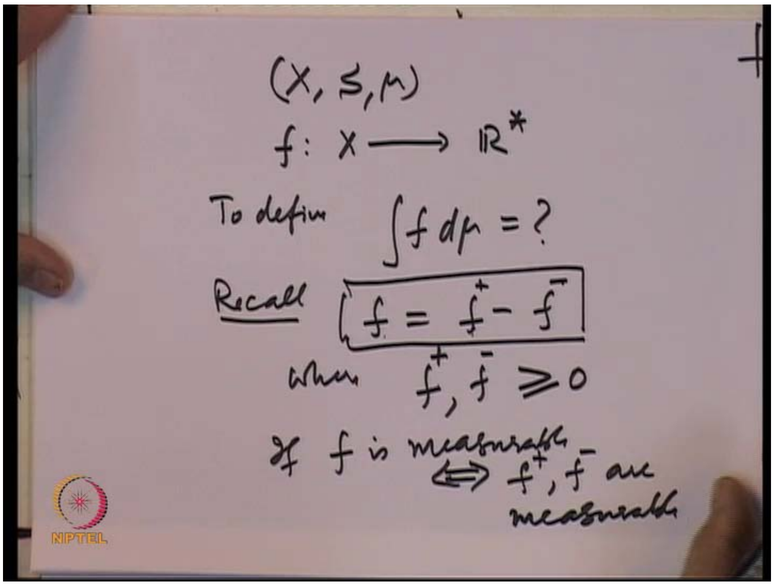

For that, we will do the following. Let us look at a function f. Keep in mind we have got a measure space X as mu which is complete; f is a function which is defined on X taking extended real values; we want to define integral f into d mu – we want to know what it should look like; of course, we would like this integral to have nice properties; we would like to have it to be a function integral to be a linear operation.

Let us recall the function f can be written as f plus minus f minus. We can split it into two parts – the positive part and the negative part of the function – where f plus n f minus are both nonnegative functions and if f is measurable, of course, with respect to the sigma algebra S, that is true if and only if both f plus and f minus are measurable. This is the clue how we should go about it (Refer Slide Time: 34:19); f can be written as a difference of two nonnegative measurable functions if f is measurable and integral of nonnegative functions is defined; so, integral of f plus is defined and integral of f minus is defined.

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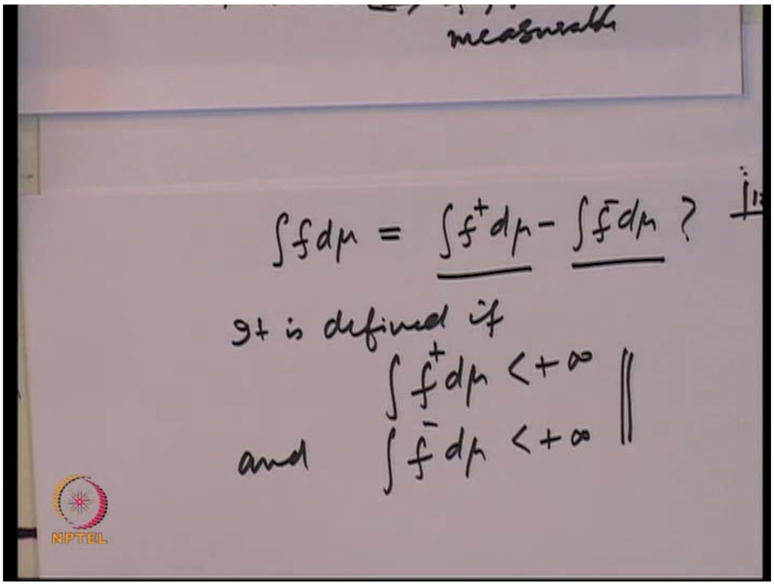

If our integration is going to be linear, it is all but necessary that our integral… We should define integral f into d mu; whatever be the way we define it, it should have the property f plus into d mu minus integral of f minus into d mu; this is what we would like to have. This is defined and this is defined (Refer Slide Time: 34:59). Now, the question is: is the difference defined?

The difference will be defined if both of these quantities are finite numbers. It is defined if f plus into d mu is finite and integral f minus into d mu is finite. That means whenever f is a measurable function, the integral f plus into d mu is finite and integral f minus into d u is finite, we can define its integral to be equal to integral of f plus into d mu minus integral f minus into d mu.

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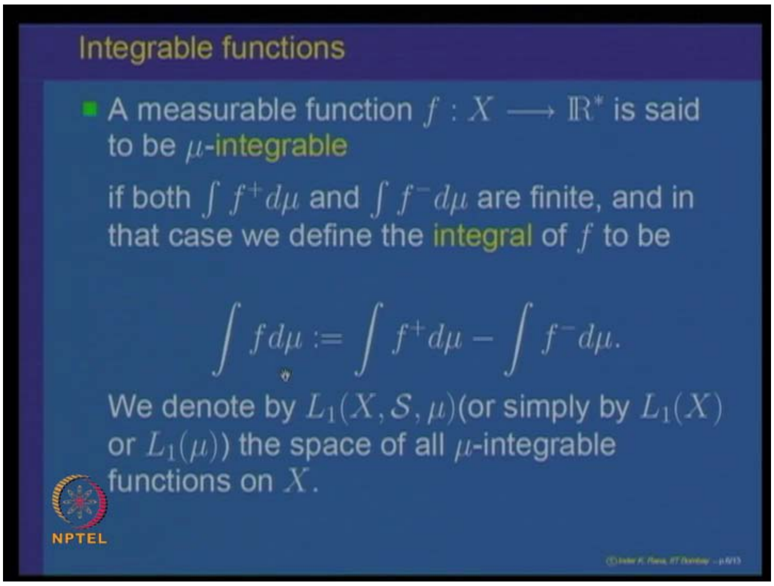

With that, let us define what is called an integrable function. A measurable function f defined on X taking extended real values is said to be mu integrable (of course mu is the measure underlying the space, which is fixed) if both integral of f plus into d mu… f plus is a nonnegative measurable function and f minus is a nonnegative measurable function; so, by our earlier discussion, both these numbers f plus into d mu and f minus into d mu are defined.

If they are both finite, in that case we say that the function f is integrable and its integral is defined as integral f plus minus integral f minus. Integral of f is defined as integral of the positive part of the function minus the integral of the negative part of the function. Whenever a function f is defined on X, we say f is integrable if both the positive part and the negative part have finite integrals and in that case we define the integral of f (we have written as f into d mu) to be integral of f plus minus integral of f minus.

We will denote by the symbol capital L lower 1 of X, S, mu to be the class of all mu-integrable functions. In case it is clear what is X, what is mu and what is S, we can sometimes simply write it as L_1 of X or simply L_1 of mu. If we understand what is the underlying measure space, the space of integrable functions, either it will be explicitly written as L_1 of X, S, mu or sometimes simply as L_1 of X or L_1 of mu. This is the class of all integrable functions; that means all functions f such that integral f plus is finite and integral f minus is finite; in that case, integral of f is defined as integral f plus minus integral of f minus. So for all integrable functions f belonging to L_1 of X, we have integral f into d mu.

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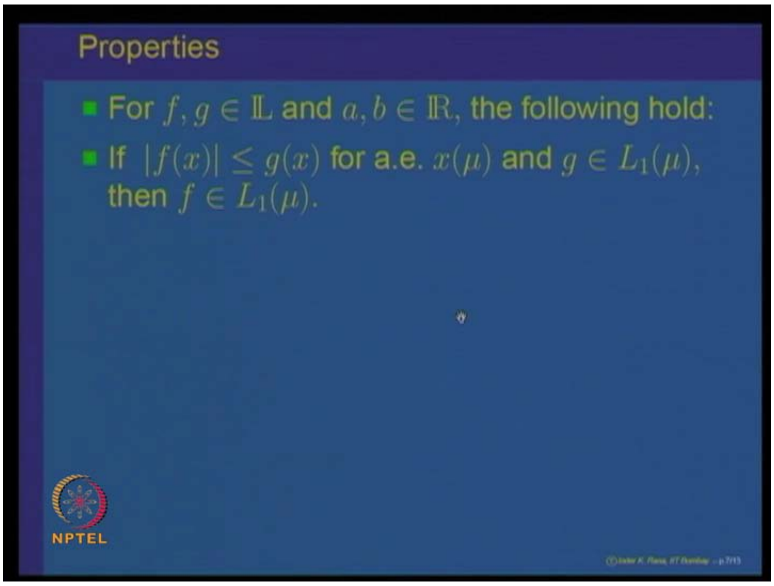

We will now study the properties of this integral. The first property is let us fix functions f and g which are integrable and a and b to be real numbers. If f and g are measurable functions and mod f of x is less than g of x for almost all x and g belongs to L_1, then the claim is f is in L_1.

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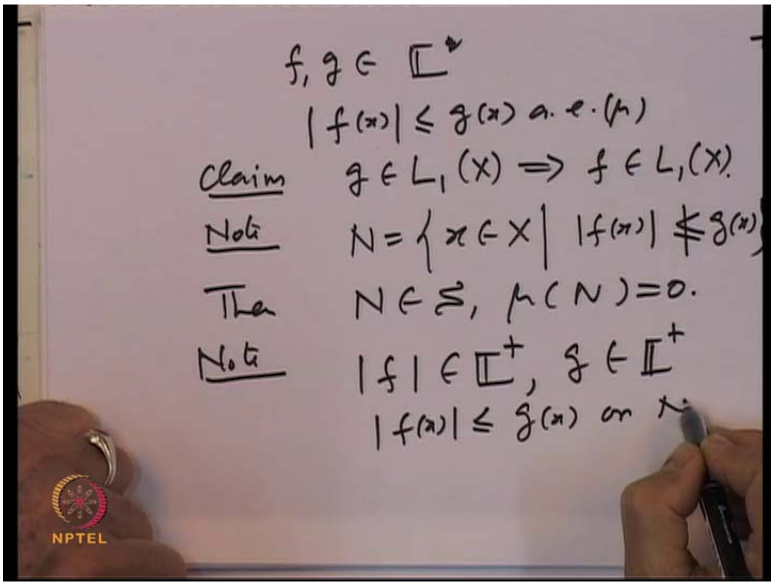

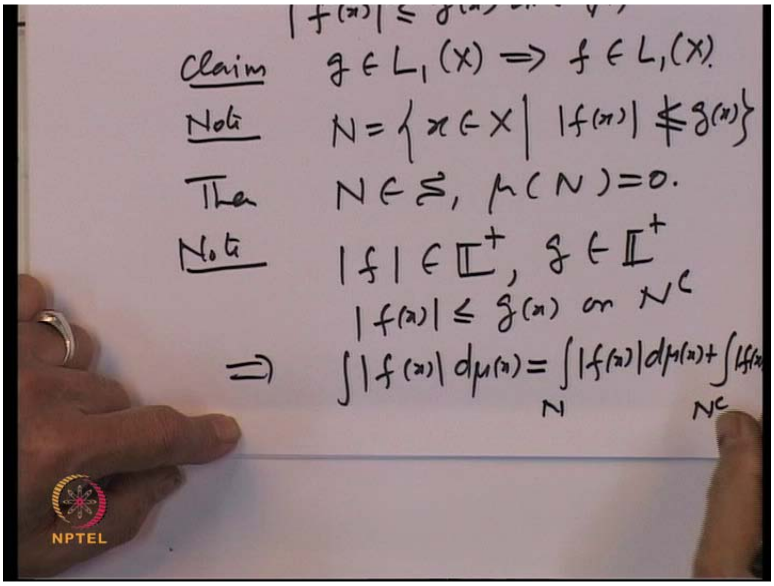

This is a very simple property we want to check. f and g are measurable functions on x and we are given that mod f of x is less than or equal to g of x almost everywhere mu. From here, the first claim is that if g is L_1 of x, then that implies that f is in L_1 of x. To prove that, let us observe. Note that we are given f of x is less than or equal to g of x almost everywhere x. Let us define N to be the set of all points x belonging to X where mod f of x is not less than or equal to g of x; here, this property is not true.

Then, we know that N belongs to the sigma algebra and mu of N is equal to 0 n of n is equal to 0. Now note that because mod of f is nonnegative, mod f belongs to L plus; g is nonnegative measurable and so g belongs to L plus. It is almost everywhere and so mod f of x is less than or equal to g of x on N complement; that is what is given to us. This implies mod f of x into d mu x which we can write as integral over N mod f of x into d mu of x plus integral over N complement mod f of x into d mu. Now, let us observe that the set N has got measure 0; so this part is 0 (Refer Slide Time: 41:18) and on N complement, mod f is less than or equal to g.

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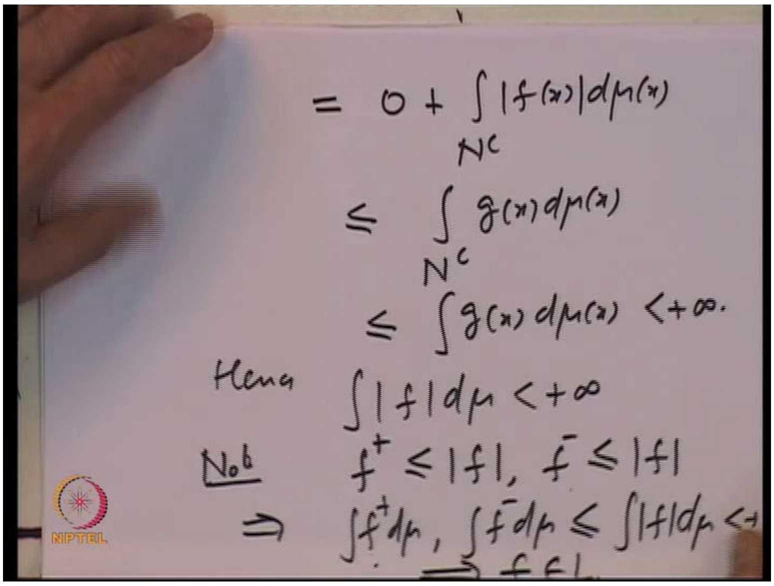

This is equal to 0 plus integral over N complement of mod f of x into d mu of x and on N complement, f is less than or equal to g. This is less than or equal to integral over N complement of g of x into d mu x. That is less than or equal to integral over the whole space g of x into d mu x which is finite. What we have shown is that in case mod f of x is less than or equal to g of x, then we have shown that the integral of mod f is finite (Refer Slide Time: 42:10).

Hence, integral mod f into d mu is finite. Now, let us note that f plus is always less than or equal to mod f and f minus is also less than or equal to mod f. For any function, the positive part is less than or equal to mod f; the negative part also is less than or equal to mod f. That implies that integral f plus into d mu and integral f minus into d mu – both of them are less than or equal to integral mod f which is finite.

We have shown that the integral of f plus and integral of f minus are both finite whenever mod of f is less than or equal to. What we have shown is this property is true (Refer Slide Time: 43:13).

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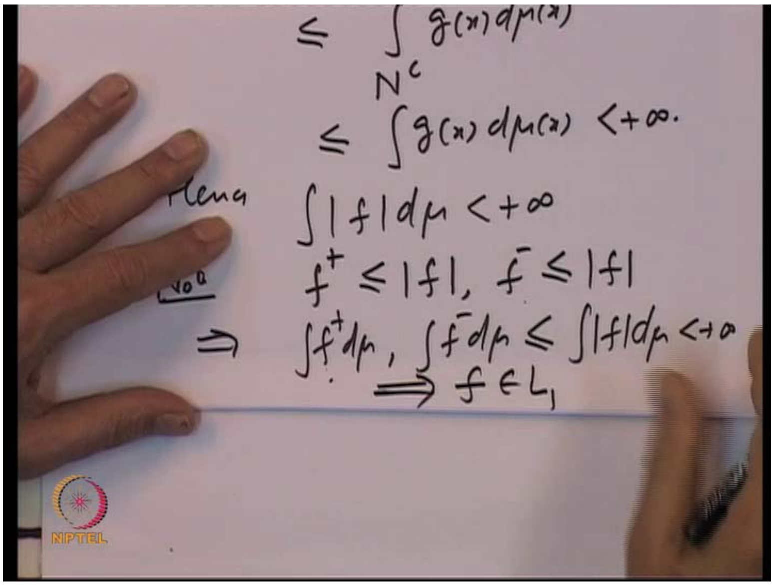

This implies that integral of f plus and integral of f minus both are finite. That implies f belongs to L_1.

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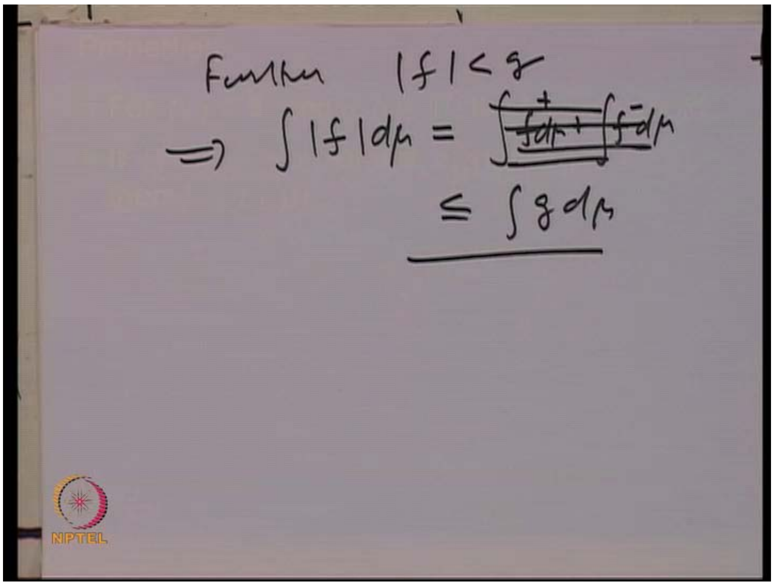

Further, let us calculate what is integral of mod f into d mu. Mod f, if you recall, is nothing but f plus plus f minus. That means this is equal to integral d mu plus integral of f minus into d mu. So, integral of mod f is nothing but integral of f plus plus integral of f minus into d mu. Both of them are finite; that we have already observed. We wanted to check that integral of mod f is less than or equal to integral of integral g d mu, which we have already checked.

We have already checked that integral of f plus which is less than integral of mod f is less than… So mod f is less than integral g implies this and we do not have to do this (Refer Slide Time: 44:36). It is less than or equal to integral g d mu; that follows directly from that mod f. We have shown it is integrable and its integral is finite; so, this is less than or equal to integral of g.

(Refer Slide Time: 44:52) This proves the first property: if f and g are measurable functions and mod f of x is less than or equal to g of x for almost all x and if g is integrable, then f is integrable. What we are saying is if a function f of x which is measurable is dominated by a function g which is integrable, then the function f also becomes integrable.

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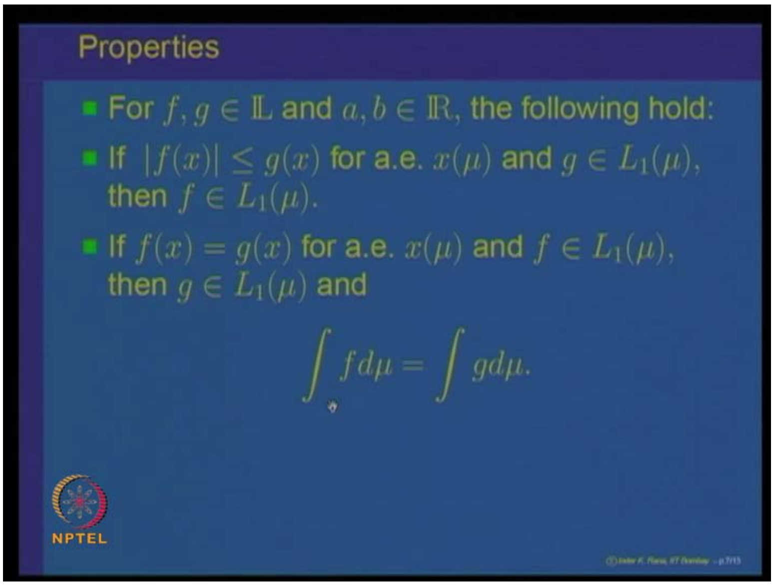

Let us look at the next property that if f and g are equal almost everywhere and f is integrable, then g is integrable and integrals of the two are equal. That property is something similar to what we have just now shown; a similar analysis will work.

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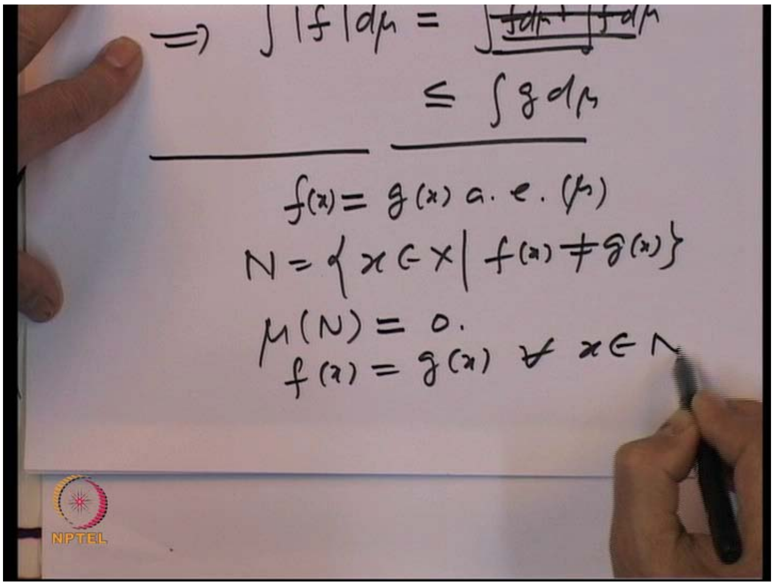

We have two functions f and g; f of x is equal to g of x almost everywhere mu. Let us write the set N – x belonging to X where f of x is not equal to g of x. Then, by the given condition, mu of N not equal to that is equal to 0; f of x is equal to g of x for every x belonging to N complement.

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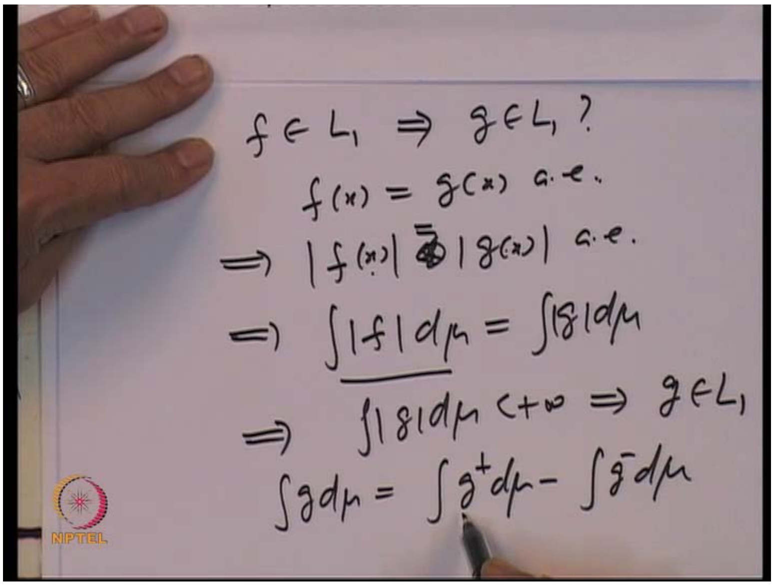

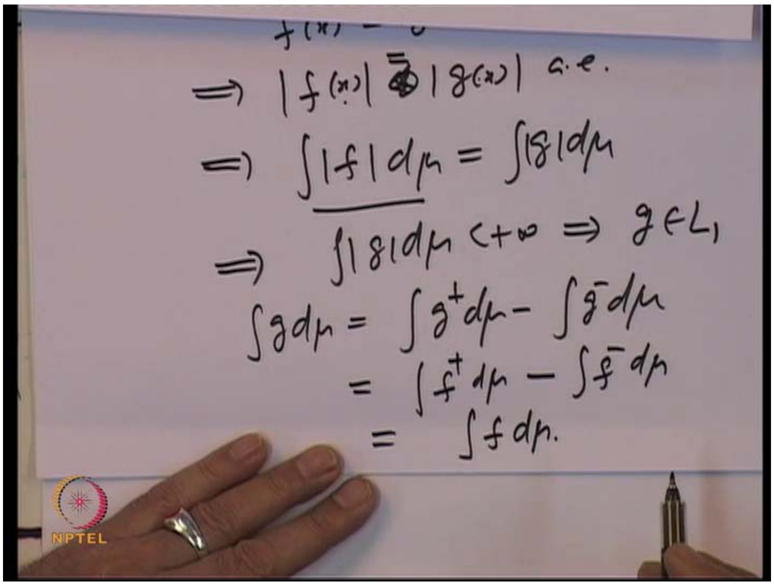

We are given that the function f is belonging to L_1. We want to show that g belongs to L_1 and that is because if f of x is equal to g of x almost everywhere, then that implies that mod f of x is also less than or equal to mod g of x almost everywhere. The sets are not equal; wherever they are equal, mod x is equal to g x because on the N complement that will happen; that is ((.)).

That implies integral of f of x is equal to g of x. Sorry, we should say that f x is equal to g x almost everywhere (Refer Slide Time: 47:10). That implies f of x is equal to g of x; mod f x is equal to mod g of x. Just now we showed that whenever f and g are equal almost everywhere, integral mod f d mu is equal to integral mod g d mu. Either of them finite implies the other is finite. We are given this is finite and so this implies mod g d mu is finite. It implies once again that g is in L_1. g is in L_1 and so integral of g d mu is equal to integral g plus d mu minus integral g minus d mu, but f is equal to g almost everywhere; we ask the reader to verify; that means f plus must be equal to g plus and f minus must be equal to g minus almost everywhere.

Once again, this integral is equal to minus integral of f plus d mu minus integral of f minus d mu which is nothing but equal to integral of f d mu. So, integral g is equal to integral f whenever f and g are equal almost everywhere. (Refer Slide Time: 48:24) These are simple properties of integrable functions that we have looked at. If f is equal to g almost everywhere and one of them is integrable, then the other is integrable and the two integrals are equal.

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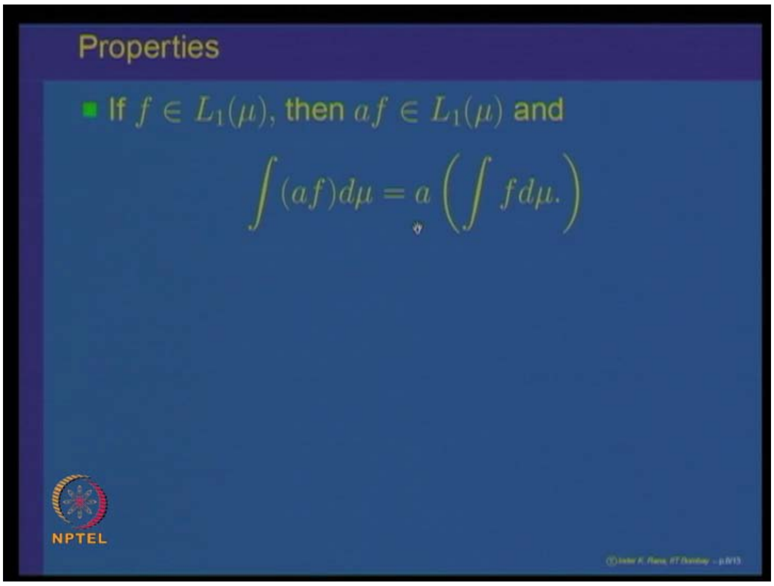

Next, let us check the property of linearity. If f is in L_1, then we want to check that alpha is also in L_1 and alpha f of d mu is equal to alpha times integral of f into d mu.

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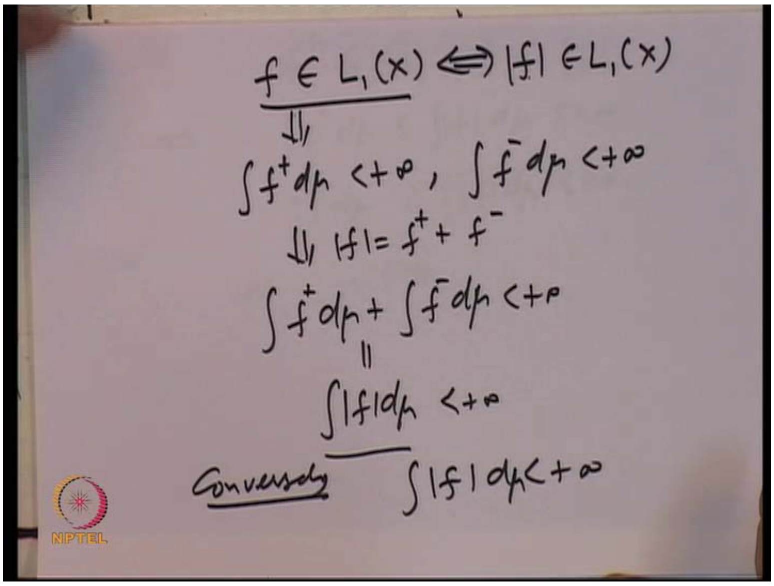

To check that property, let us just observe one thing. Just now we looked at this kind of analysis: if f belongs to L_1 of x, it is same as if and only if mod f belongs to L_1 of x. Why is that? Once again, let us do this because this we are going to use again and again. Saying that f belongs to L_1 this implies integral of f plus d mu is finite and integral of f minus d mu is finite. What is mod f? mod f is equal to f plus plus f minus. That implies integral f plus d mu plus integral f minus d mu is finite and this is equal to integral of mod f d mu. So, f belonging to L_1 implies integral of mod f is finite. Let us look at the converse part – if mod f integral is finite.

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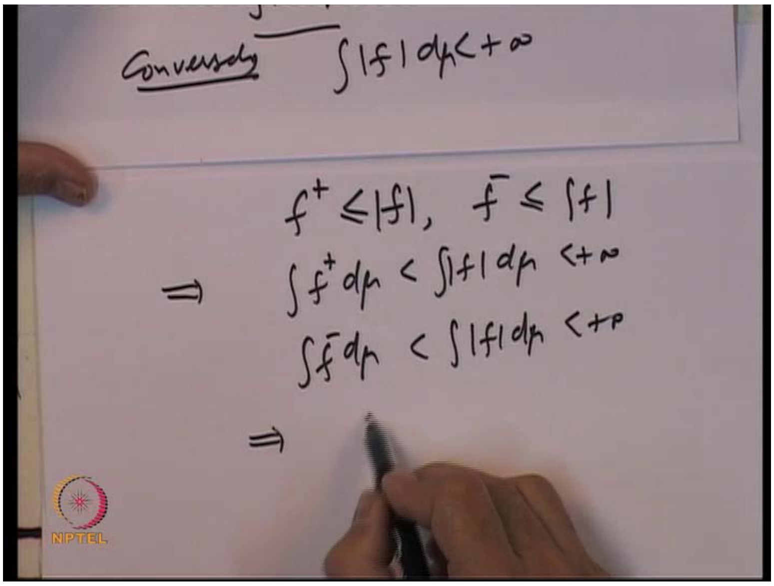

This is given to us: integral of mod f d mu is finite (Refer Slide Time: 50:31). Once again, let us observe that f plus is less than or equal to mod f and f minus is less than or equal to mod f. All are nonnegative measurable functions; that implies integral of f plus d mu is less than integral mod f d mu which is finite and integral f minus d mu is less than integral mod f d mu which is finite; that implies that f belongs to L_1. Saying that (Refer Slide Time: 51:06) a function is integrable is equivalent to saying that mod f which is a nonnegative measurable function has got a finite integral. This property will be used again and again. Let us see how this property is used in our proposition.

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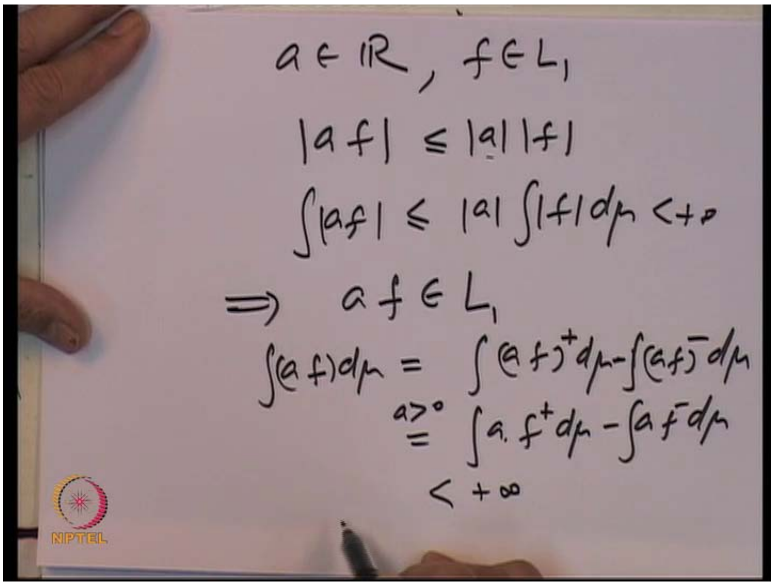

a belongs to the real line and f is L_1. Look at mod of a f. Mod of a f is less than or equal to mod a into mod f. All are nonnegative functions; so, integral of mod a f is less than or equal to integral of this which is mod a times integral mod f d mu which is finite. That implies that a f is an integrable function. Now, not only is it integrable but the integral of a f d mu you can write as integral of a f plus d mu minus integral of a f minus d mu.

Now, the possibility is either a is equal to 0; in this case, a f will be 0 and everything is 0; so, no problem. If a is positive, then this part is same as a times f plus d mu minus a times integral f minus d mu if a is positive. This a comes out because of the property for integral of nonnegative measurable functions; so, this will be finite. In case a is less than 0, this becomes a of minus negative part and again that thing is okay and similarly for a minus. That proves the property that if f is integrable and a is a real number, then a f is integrable and a comes out (Refer Slide Time: 53:02).

We will continue looking at the properties of integrable functions. In the next lecture, we will show that this integral is a linear operation on the space of integrable functions and various other properties of this space of integrable functions and integral on it. We will continue this study in the next lecture. Thank you.