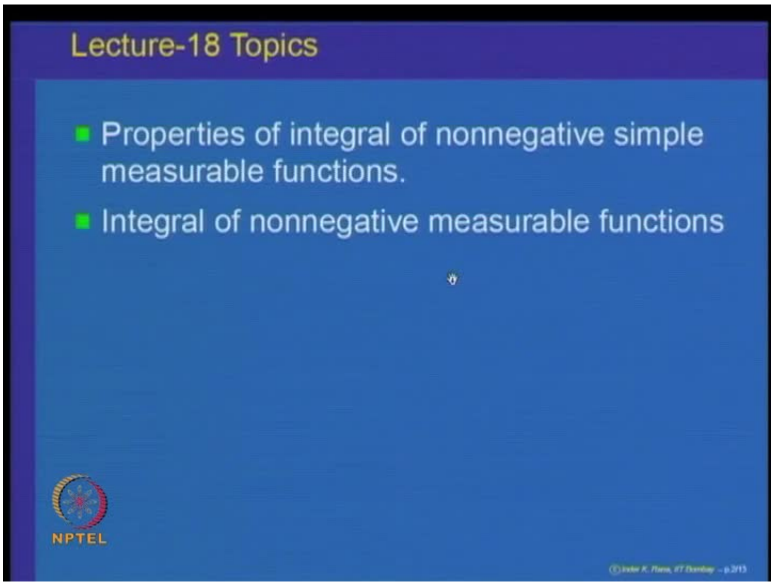
Measure and Integration
Prof. Inder K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay

Module No. # 06
Lecture No. # 18
Properties of Nonnegative Simple Measurable Function

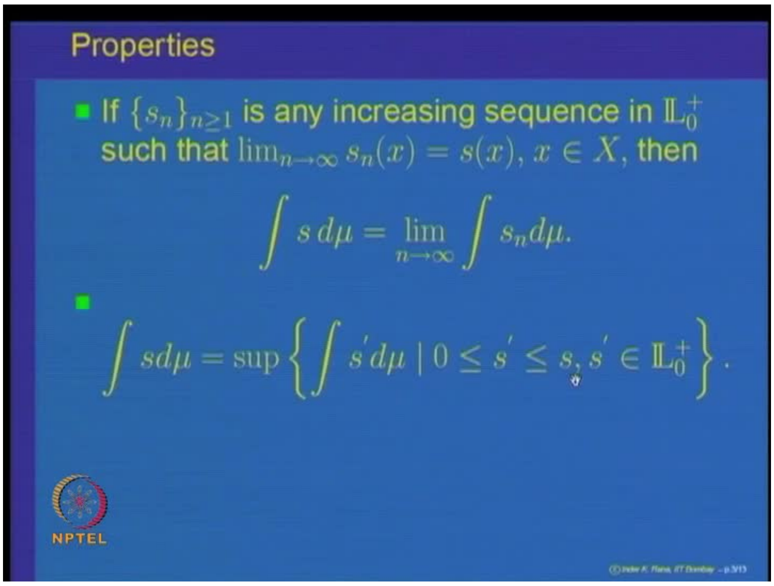
Welcome to lecture number 18 on measure and integration. If you recall in the previous lecture, we have defined the notion of integral for nonnegative simple measurable functions and we had started looking at the properties of this integral. So, we will continue the study of the properties of the integral for nonnegative simple measurable functions. Then, later on we will extend it to this integral to nonnegative measurable functions.

(Refer Slide Time: 00:55)



Topics for today would be properties of integral for nonnegative simple measurable functions; continue the study of that and then define integral for nonnegative simple measurable functions.

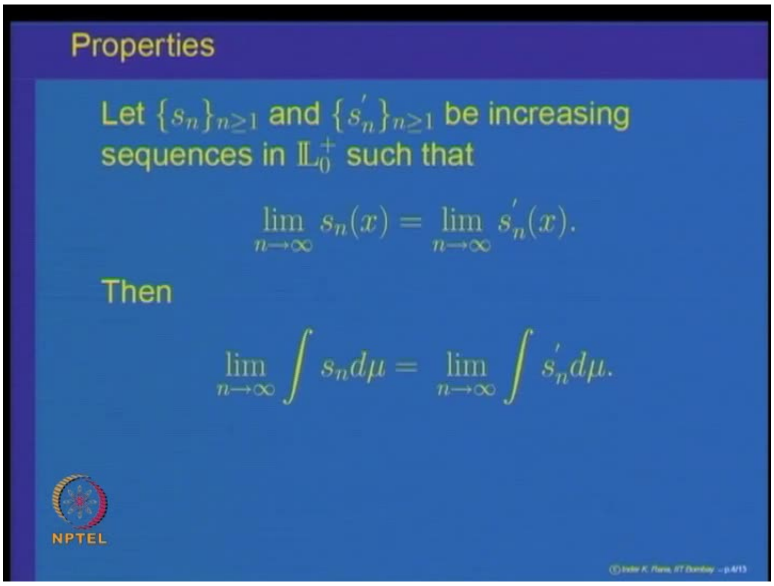
(Refer Slide Time: 01:06)



If you recall, the last property that we have proved in the previous lecture was that if s n is any increasing sequence of nonnegative measurable functions, such that they converge to a simple nonnegative simple function s of x then, integral of s is equal to limit of integral of s n d mu. That means, under increasing limits if the limit is again a nonnegative simple measurable function, then you can interchange the order of integration and the notion of limit. So, integral of s d mu which is s limit of s n's is same as limit of n going to infinity of integral s n's d mu; so integrals of s n's converge to integral of s.

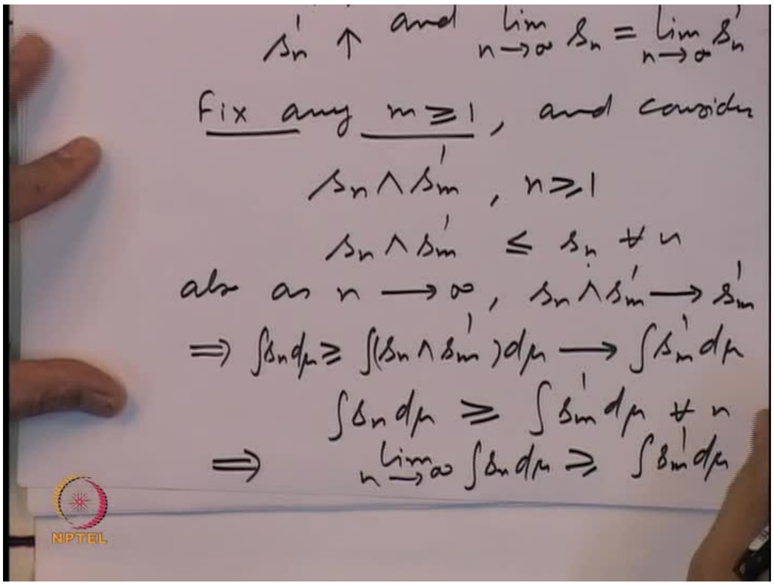
Let us observe one more simple property of this integral for any nonnegative simple measurable function s. The integral sd mu can also be represented as the supremum of the integrals of s prime d mu, where s primes are nonnegative simple measurable functions less than s. This property is obvious because s is less than or equal to s, so the supremum has to be at least integral sd mu. It cannot be more because s prime less than s implies that integral of s prime is less than or equal to integral s. So, the supremum cannot be bigger than or equal to sd mu also, this is obvious property but, we will see an extension of this property later on.

(Refer Slide Time: 02:58)



Next, let us observe another important property about the integral of nonnegative simple measurable functions and that is the following. Suppose, s n and s n dash are two increasing sequences of nonnegative simple measurable functions, both converging to the same limit, so limit of s n x is same as limit of s n dash x. Then the claim is that the limit of integral s n d mu has to be equal to the limit integral s n prime d mu; that means if two sequences of nonnegative simple measurable functions have the same limit, then their integrals also converge to the same values.

(Refer Slide Time: 03:43)



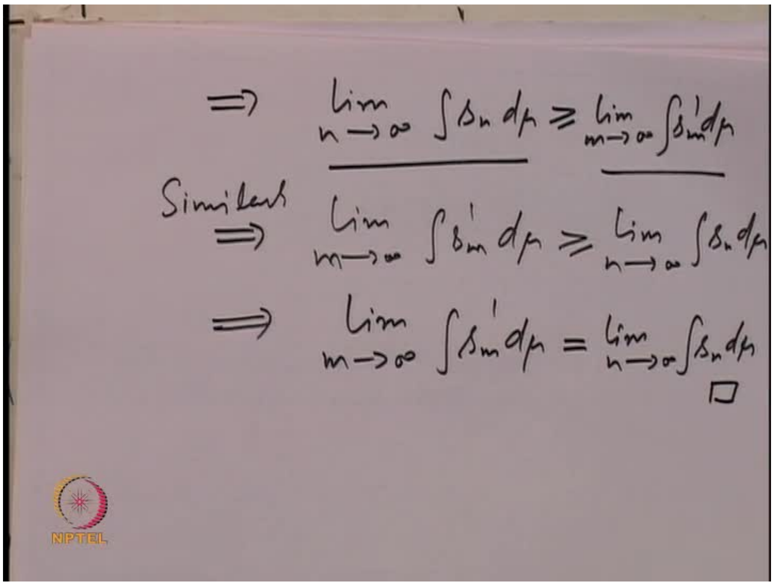
Let us prove this result. We have got s n is a sequence of nonnegative simple measurable functions, s n's are increasing and also s n dash is another sequence which is also increasing. The limit n going to infinity of s n is equal to limit n going to infinity of s n dash so that is given to us.

Now, let us fix any positive integer m bigger than or equal to 1 and consider the sequence s n minimum s m dash; look at the sequence n bigger than or equal to 1. So, look at this functions; these are the functions, which have the property that s n which s m dash is always is the minimum; so, it is less than or equal to s n for every n.

Also, as n goes to infinity look at the sequence s n which s m dash, the minimum of s n and s m dash as n goes to infinity s n is going to increase to a limit. So, it will take over s m dash at some stage that means this is going to converge to s m dash. This is obvious because s n and s m dash both the sequences has the same limit; at some stage s n will crossover s m dash for every m fix, so we have fixed an integer m.

That implies, integral of s n which s m dash d mu will converge to integral of s m dash d mu. So, this is less than or equal to integral of s n d mu and that is because of this (Refer Slide Time: 05:57). So, we have integral s n d mu is bigger than or equal to integral s m dash d mu for every n.

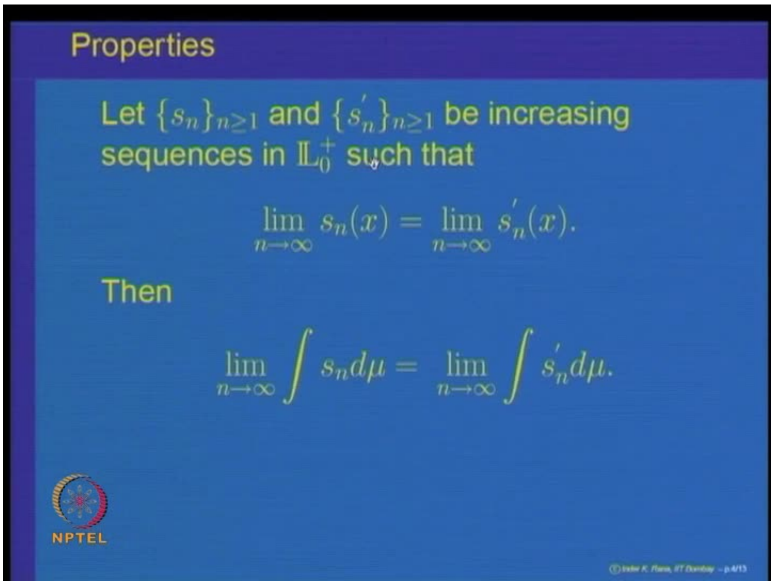
(Refer Slide Time: 06:40)



That means, this implies for all n large enough, so that implies that limit of n going to infinity of integral s n d mu is bigger than or equal to integral s m dash d mu for every m fix, because this is true for every m fix. So, this implies that this is true for every m fix implies, limit n going to infinity integral s n d mu is bigger than or equal to limit m going to infinity of integral s m dash d mu.

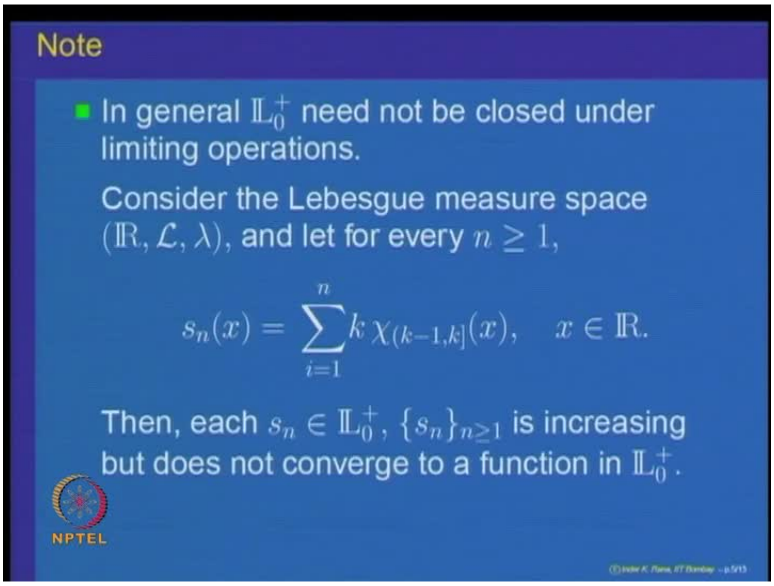
We have shown that limit of s n integral s n's is bigger or equal to limit of integral s m's, because now, you can interchange the two. That implies, similarly, limit n going to infinity integral s n dash d mu is bigger than or equal to limit n going to infinity of integral s n d mu. So that proves that two are equal. So, this will prove that limit m going to infinity of integral s m dash d mu is equal to limit n going to infinity of integral s n d mu.

(Refer Slide Time: 07:53)



That proves the result namely if s n and s n dash are two increasing sequence is of nonnegative simple functions having the same limit, then their integrals also converge to the same limit, this will be used soon.

(Refer Slide Time: 08:08)



Let us look at an observation that in general, the class of nonnegative simple measurable functions need not be closed under limiting operations. We showed that some of nonnegative simple measurable functions are nonnegative simple measurable function. We also just now observed that if a sequence s n is increasing to a sequence in L plus 0 that means, if a sequence of nonnegative simple measurable functions converges to a nonnegative simple measurable function then the integrals converge.

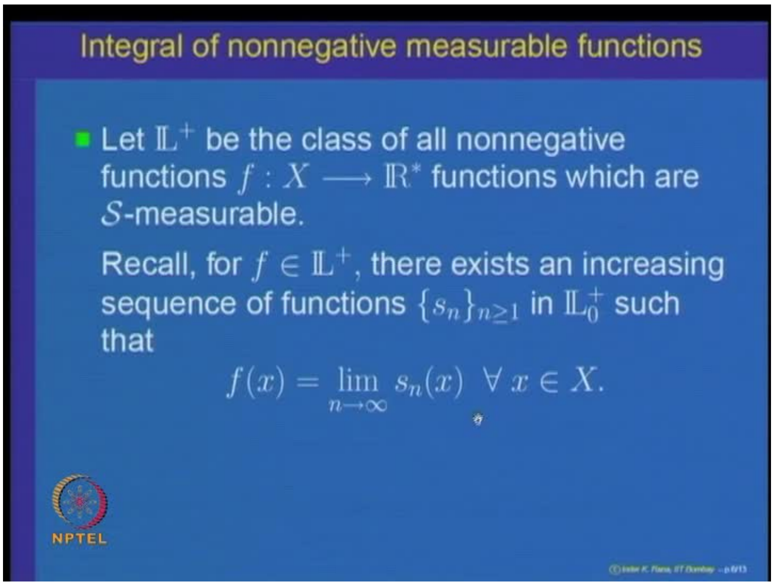
In general for decreasing sequences for example, this need not hold or for even, the limit of nonnegative simple measurable functions may not be a nonnegative simple measurable function.

So, to give an example of that let us consider the Lebesgue measure space R L and lambda; R is the real line, L is the space of the sigma algebra of Lebesgue measurable sets and lambda is the Lebesgue measure. Let us define s n of x for every n to be equal to some of the indicator functions of k minus 1 to k, where k goes from 1 to n, so this is just it takes the constant value k on the interval k minus 1 to k.

As it is cut clear that this is the nonnegative simple measurable function on the real line and as n increases this is going to be an increasing sequence that also is clear. Actually, it increases to a function which is equal to k on every interval k minus 1 to k that the limit function is not going to be a nonnegative simple measurable function.

Of course, it will be a nonnegative measurable function; what we are saying is that the class L plus 0 is not closed under limiting operations. So that stress we should go over to a bigger class of functions namely, the class of nonnegative measurable functions and define integral there also.

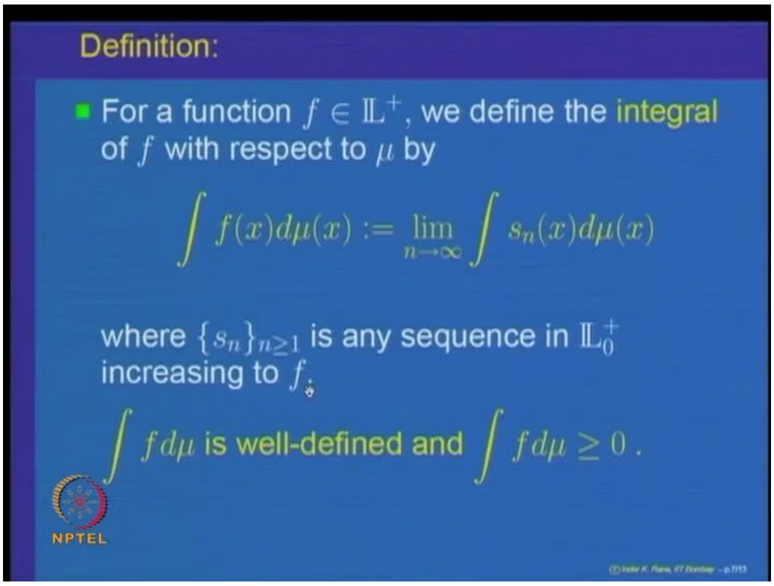
(Refer Slide Time: 10:35)



Let us denote by L plus the class of all nonnegative functions X to R star which are S measurable. Keep in mind, we have got a fixed measure space, which is complete that is X S and mu. Look at nonnegative S measurable functions on the set X. Let us denote this class of functions by L plus and if you recall, we had proved the theorem that for a nonnegative measurable function, there is a sequence of nonnegative simple measurable functions which is increasing to f.

For every f belonging to which is a nonnegative measurable function, the function in the class L plus we know that there exists a sequence s n of nonnegative simple measurable functions s n which increases to f. So, f is a limit of an increasing sequence of nonnegative simple measurable functions. We have defined the concept of integral for nonnegative simple measurable functions s n, it is natural to define integral of f to be nothing but, the limit of integrals of s n.

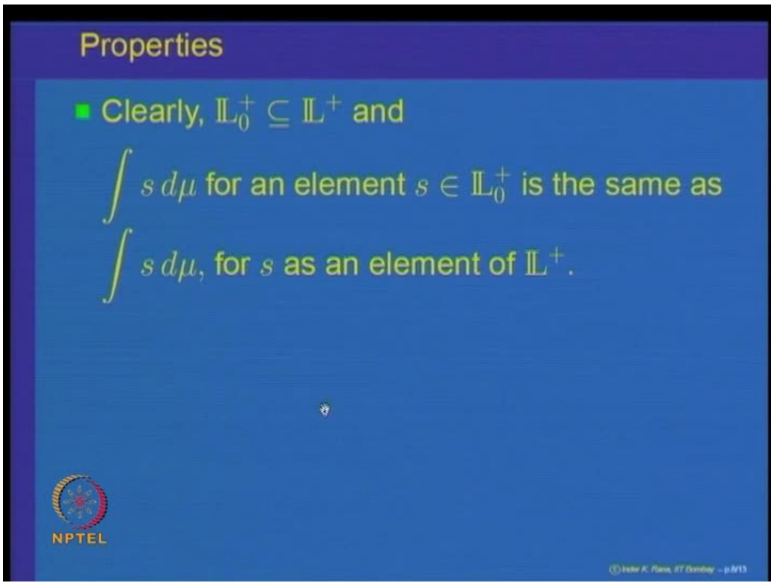
(Refer Slide Time: 12:00)



For a function f in L plus we define it is integral with respect to mu denoted by integral f x d mu x or simply by f d mu to be limit n going to infinity of s n x d mu x, where s n is any sequence in L 0 plus increasing to f. The first obvious claim is that this integral is well defined; it does not depend upon the choice of the sequence s n that we take which increases to the function f. That is because just now we have proved the result that if they are two different sequences s n and s n prime nonnegative simple measurable both increasing to f, then their limits are same.

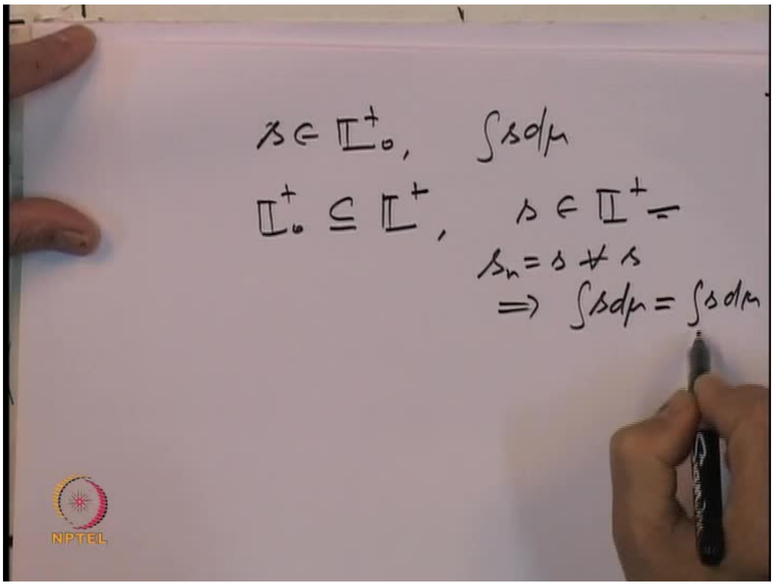
So, whichever sequence we take s n which increases to f, its limit is going to be the same extended real number and that extended real number is called the integral of f d mu. Integral f d mu is well defined that means whatever sequence s n increasing to f is integrals of s n’s is same, so that is called integral of f d mu, because it is limit of integrals of s n’s which are nonnegative simple measurable function. So, each integral s n is a nonnegative simple measurable function, as a result the limit also is nonnegative. Integral of a nonnegative measurable function f, by this process is a well-defined number that it is bigger than or equal to 0.

(Refer Slide Time: 13:52)



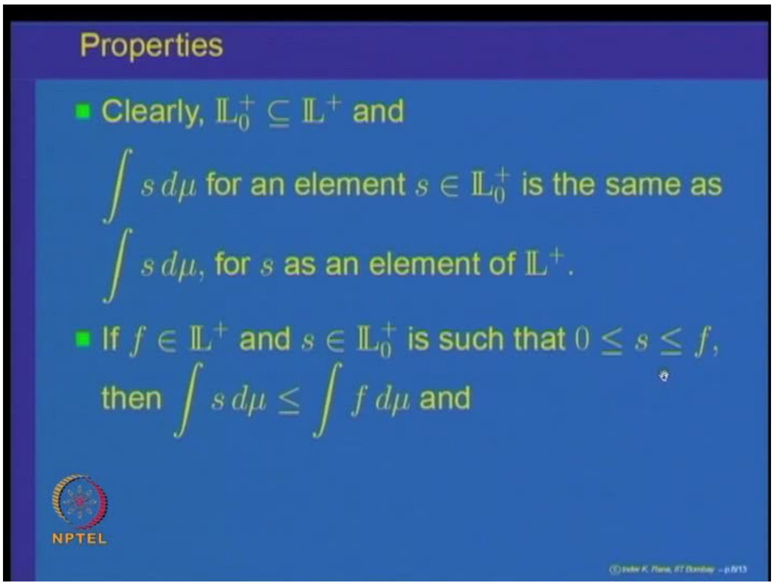
Now, let us look at the properties of this that the class L 0 plus is a subset of L plus that is obvious. We want to claim that integral of s d mu as an element of s is same as an element of L plus that is also obvious because of the following fact.

(Refer Slide Time: 14:21)



Let us take s belonging to L plus 0. Then we have, it is integral s d mu as an integral of a nonnegative simple function and L plus 0. We are now treating it as a subset of L plus, if you treat s as an element in L plus then we can take the constant sequence s n is equal to s for every s. That will imply integral of s d mu as an element in L plus is same as limit of integral s n's, which is same as integral s d mu as an element in L plus 0.

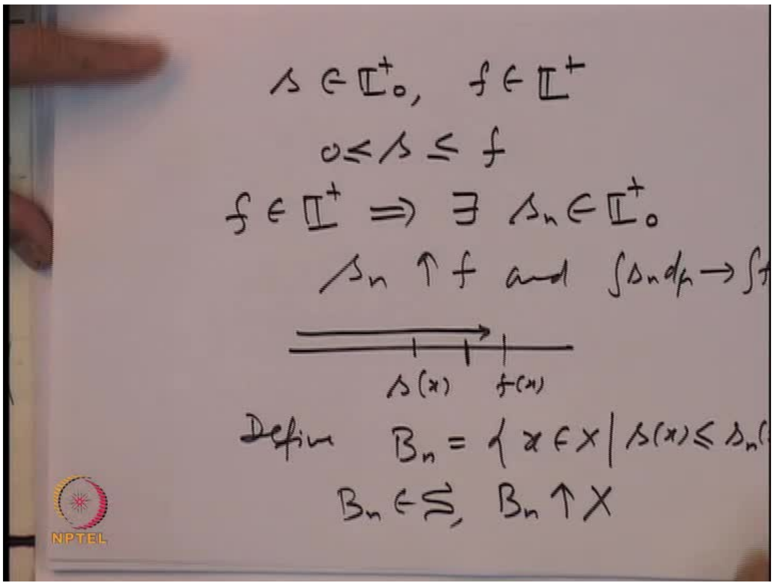
(Refer Slide Time: 15:11)



If you take a nonnegative simple measurable function as an element of L plus and look at the integrals as an element of L plus, then that integral is same as an element of the nonnegative simple measureable. That means the new integral that we have defined is in fact an extension of the notion of integral from nonnegative simple measurable functions to nonnegative measurable functions.

Next, let us look at the property that if f is a function in L plus then s is a function in L 0 plus such that 0 is less than or equal to s less than or equal to f, then integral of s d mu is less than or equal to integral f d mu.

(Refer Slide Time: 16:08)

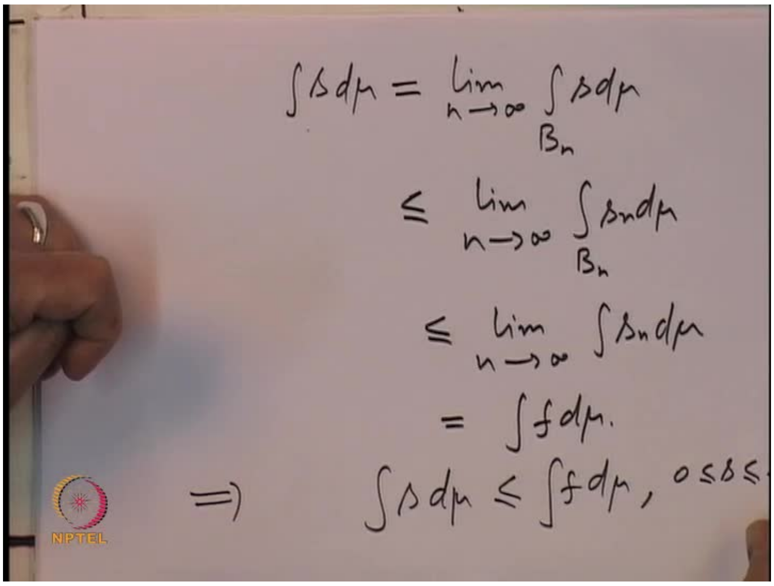


Let us prove this property. We have got s, a nonnegative simple measurable function and f is a nonnegative measurable function and we have given that s is less than or equal to f. Since, f belongs to L plus implies there is a sequence s n of nonnegative simple measurable functions such that s n increases to f. Let us look at s n increases to f and the integral of s n d mu converges to integral f d mu.

Next, let us observe here is s and s of x for any point and here will be some f of x and s n is going to increase to f, so s n x is going to cross over s of x for some n. Let us define B n to be the set of all those points x belonging to x such that s of x is less than or equal to s n of x.

So from observations, this set B n is in the sigma algebra S and because s n is increasing this sequence B n of sets is also increasing to the whole space x, because s n is covering to f of x so B n is going to increase to s of x. These are obvious properties, because if s n is bigger than or equal to s of x then s n plus 1 is also bigger than that means B n is inside B n plus X. As we observed that for every x there will be some m such that s n of x will cross over s of x, every x belongs to some B n, so B n is going to increase to X.

(Refer Slide Time: 18:16)

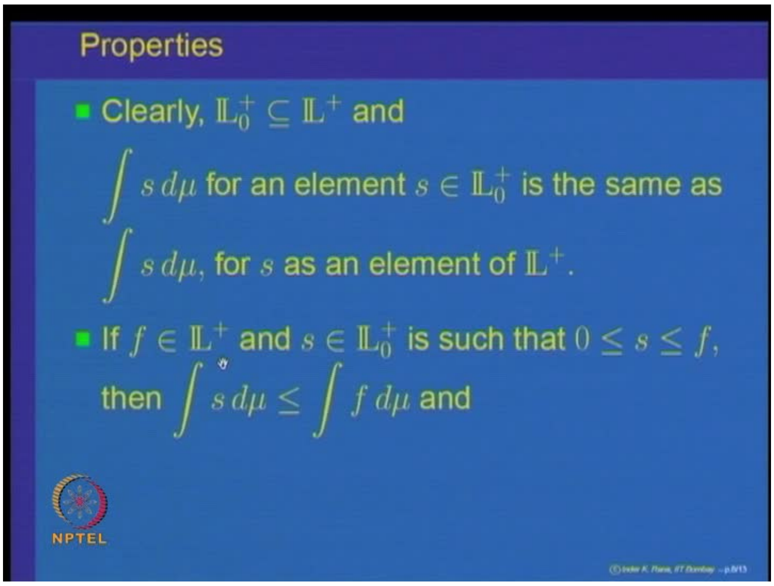


Now, we observed the property that look at integral of the nonnegative simple measurable function s d mu, so that we can write it as limit integral n going to infinity integral over B n of s d mu. This is because s n is an increasing sequence, s n increases to B n is an increasing sequence of sets B n increases to X and the integral over a set is a measure. So, keep in mind that the integral of a nonnegative simple measurable function over a set e gives you a measure.

So that measure mu of that measure at B n will go to that value at X that is same as saying that integral s d mu is limit of integral s over B n d mu. We know that on B n s n is bigger than s, let us use that fact. So, this is less than or equal to limit n going to infinity integral over B n of s n d mu that is a nonnegative simple function. One nonnegative simple measurable function is less than another, then the integral of one will be less than the other (Refer Slide Time: 19:10).

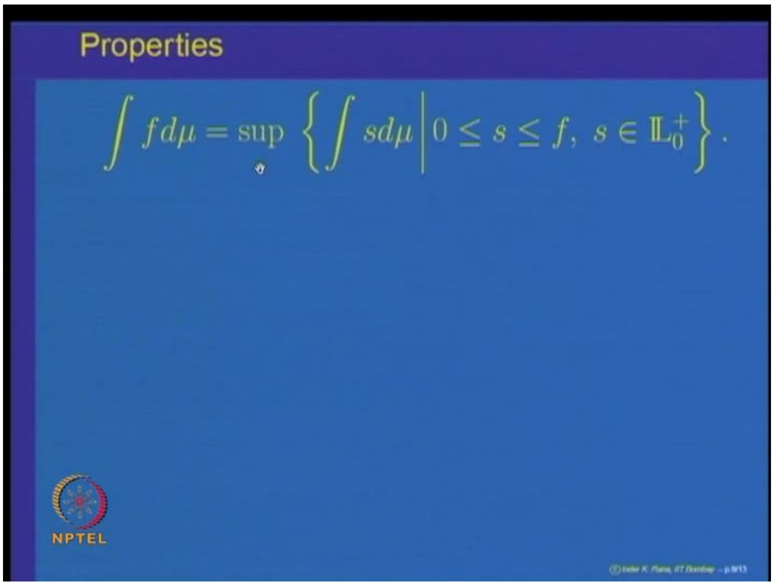
Now, this is integral s n over B n. If you replace that set B n by the whole space this will still be less than or equal to limit n going to infinity integral over the whole space x of s n d mu that is equal to integral f d mu. So that proves integral of s d mu is less than or equal to integral of f d mu, whenever s is less than f and s is nonnegative simple measurable function.

(Refer Slide Time: 20:15)



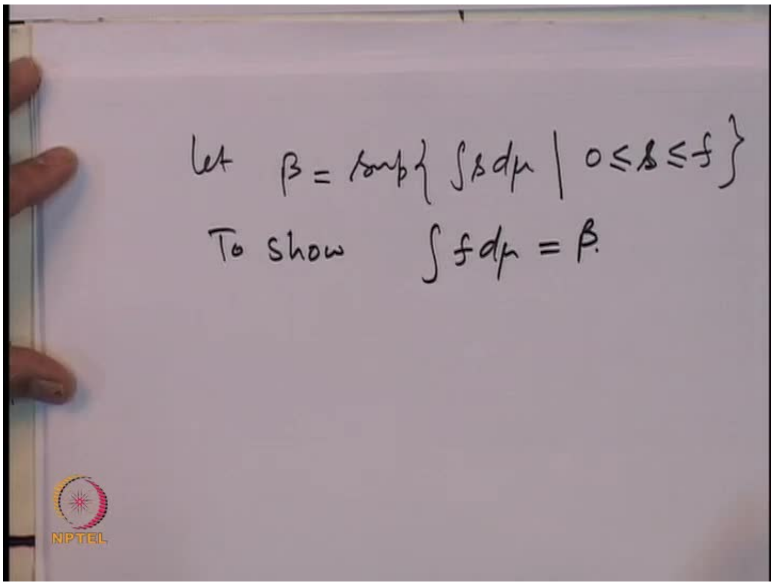
That proves this property that if f is a nonnegative measurable function and s is a nonnegative simple measurable function, such that s is less than or equal to f then the integral of s is less than or equal to integral of f.

(Refer Slide Time: 20:34)



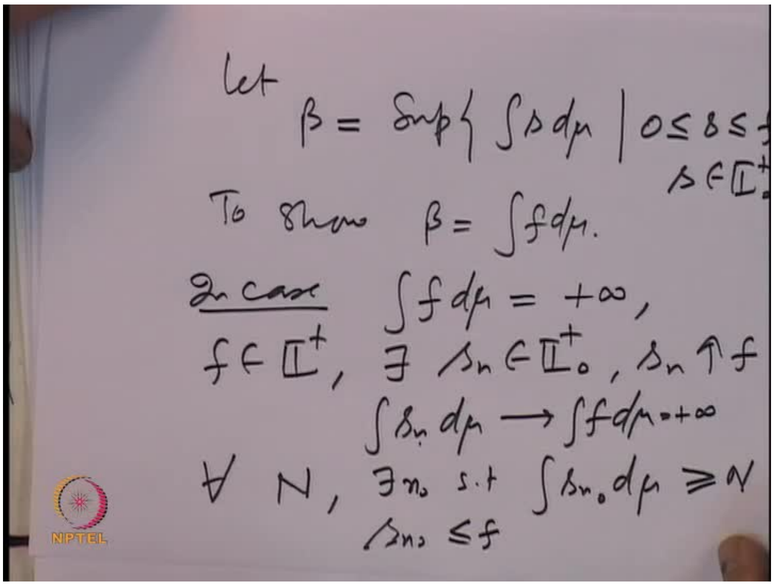
As a consequence of this, let us observe the property that integral of f d mu which we defined as the limit of integrals of s n d mu. For any sequence s n can also be represented as supremum over of integrals sd mu, where s is less than or equal to f and s is a nonnegative simple measurable function.

(Refer Slide Time: 21:05)



Let us prove this property. Let us define beta to be the supremum of integral of nonnegative simple measurable function s d mu, where 0 is less than or equal to the nonnegative simple measurable function s less than or equal to f, let us call this. So, we want to show that integral f d mu is equal to beta. Let us prove that integral of f d mu can also be written as supremum of integral s d mu, where s is less than or equal to f.

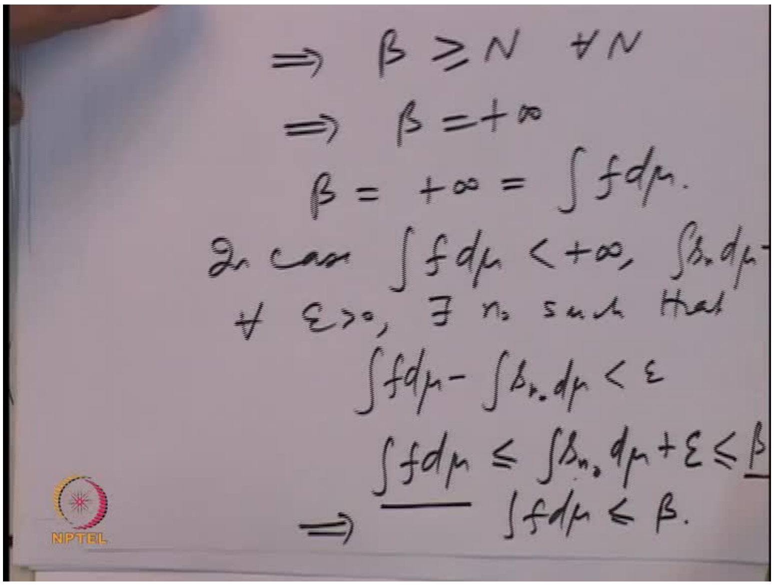
(Refer Slide Time: 21:47)



To prove this property let us define beta B equal to supremum of integral s d mu, where 0 is less than or equal to s is less than f and s is a nonnegative simple measurable function. So, we want to show that beta is equal to integral f d mu. Now, one possibility is in case integral f d mu is equal to plus infinity, then we know that f belongs to L plus.

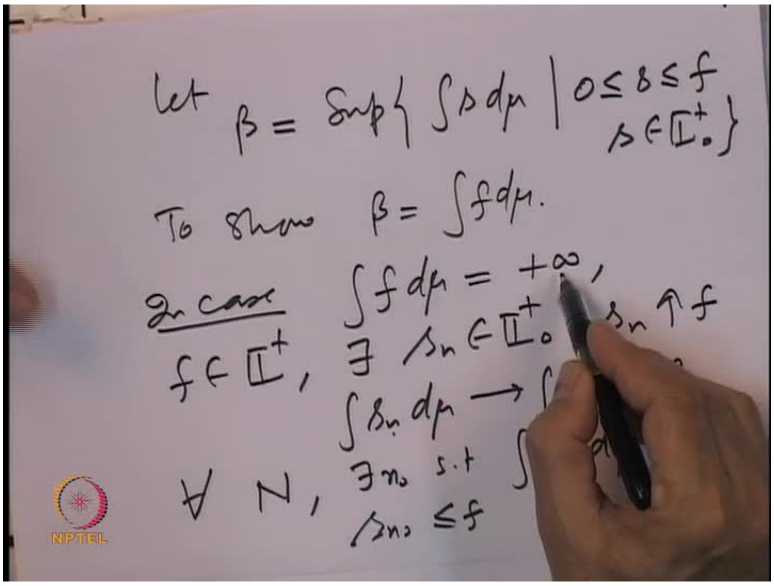
There is a sequence s n in L plus 0 nonnegative simple measurable functions, s n increasing to f and integral s n d mu converging to integral f d mu. But now in this case we know this is equal to plus infinity that means for every positive integer N there exist some n naught such that integral s n naught d mu will be bigger than or equal to N, because this number is going to converts infinity. This must exceed every N so there is n naught and this s n naught is less than or equal to f.

(Refer Slide Time: 23:30)



We have found a nonnegative simple measurable function s n naught less than or equal to f such that its integral is bigger than or equal to N that implies the supremum beta must also be bigger than or equal to N. So, this implies that the supremum beta is bigger than or equal to N for every N and hence that implies beta is equal to plus infinity.

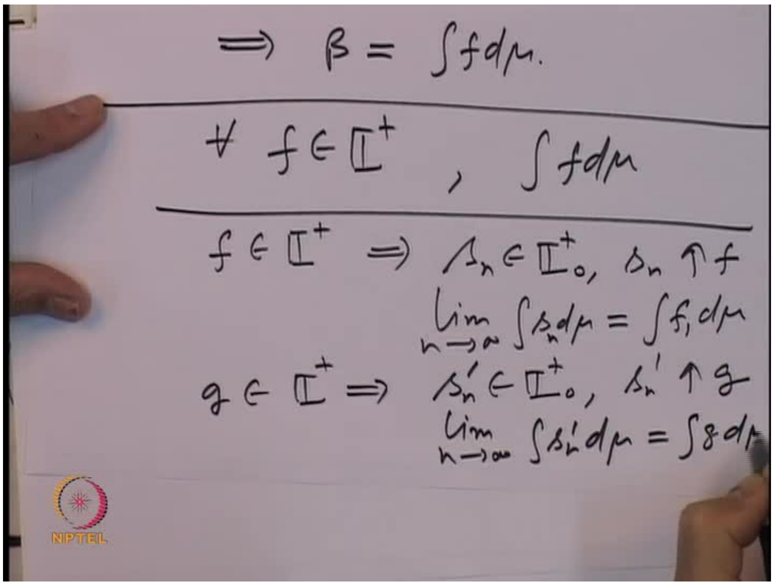
(Refer Slide Time: 23:39)



In this case integral f d mu is equal to plus infinity that implies beta is also equal to plus infinity that is beta is equal to plus infinity is equal to integral f d mu. Now, let us look at the case when this integral is finite, in case integral f d mu is finite that means we know that integral s n d mu converges to integral f d mu.

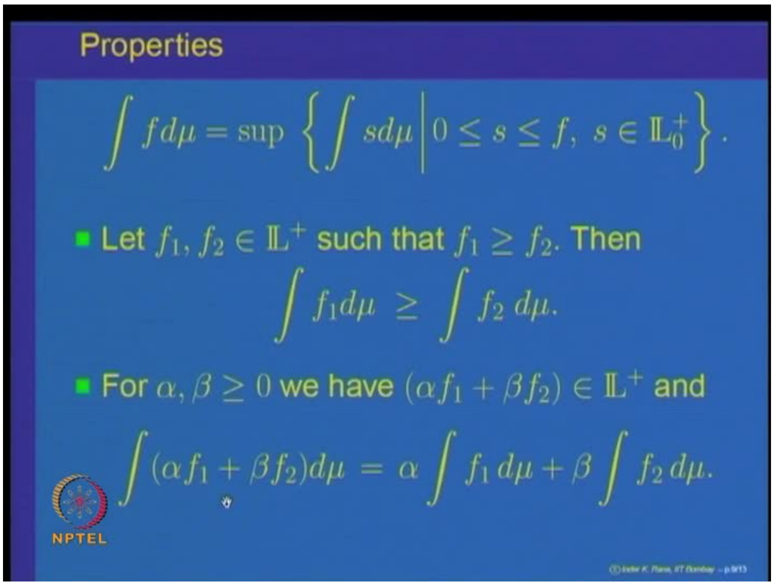
For every epsilon bigger than 0, there is some n naught such that integral f d mu minus integral s n naught d mu is less than epsilon that means, integral f d mu is less than or equal to integral s n naught d mu plus epsilon and s n naught is one function, which is less than or equal to f. So, this is less than or equal to beta plus epsilon. Integral f d mu is less than or equal to beta plus epsilon and this holds for every epsilon implying integral f d mu is less than or equal to beta.

(Refer Slide Time: 25:04)



Once again we have proved that integral f d mu is less than or equal to beta and clearly beta is less than or equal to integral f d mu that is obvious, because beta is the supremum; overall nonnegative simple functions s d mu less than or equal to integral of beta is the supremum. So, definition beta is the supremum of integral s d mu, where s is less than or equal to f, integral s d mu is less than or equal to integral f, so beta is always less than or equal to integral of f d mu. Hence, this implies beta is equal to integral of f d mu.

(Refer Slide Time: 25:47)



So that proves another way of defining the integral of a nonnegative simple measurable function that if f is a nonnegative simple measurable function, then its integral can also be defined as the supremum over all integral s d mu, where s is a nonnegative simple measurable function.

Using these two definitions, let us prove that various properties of the integral for every function f in L plus, we have defined integral f d mu. Now, we are going to look at the properties; the first property that we are saying is if f 1 is bigger than f 2, then integral f 1 is bigger than integral f 2. That follows from the above definition itself because integral f 1 d mu is going to be the supremum over integrals of all nonnegative simple measurable functions s such that s is less than or equal to f 1.

Similarly, for f 2 it is going to be a supremum over all nonnegative simple measurable functions, s less than or equal to f 2, but if s is less than or equal to f 2 and f 2 is less than or equal to f 1, so s is going to be less than or equal to f 1. This supremum for f 1 is taken over a larger class and then that of f 2 that supremum is going to be for integral f 1, the supremum is going to be bigger than or equal to integral over f 2, so that follows directly from here.

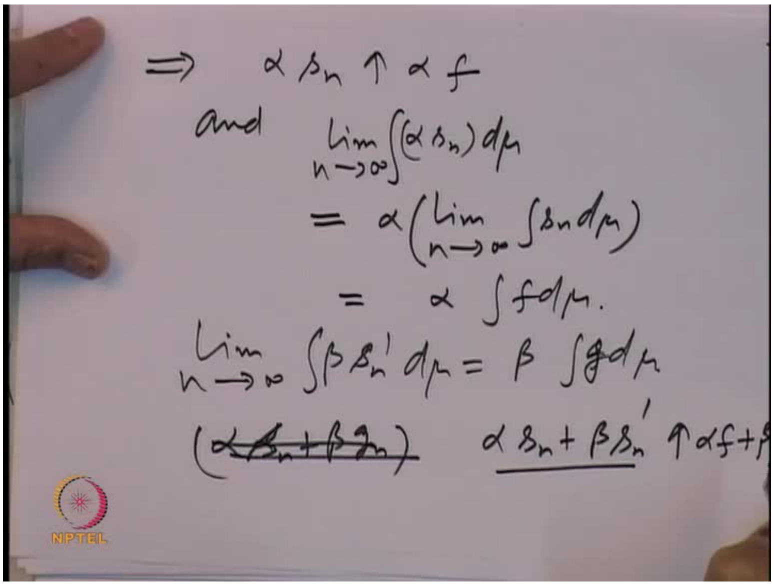
From the above definition that this integral is supremum over integral of nonnegative simple measurable functions below f. As a consequence of this we immediately have this theorem that if f 1 is bigger than f 2, then integral f 1 d mu is bigger than or equal to integral f 2.

Next, let us look at the linearity property of this integral namely if alpha and beta are nonnegative real numbers, extended real numbers then and f 1 and f 2 are in L plus. Then alpha times f 1 plus beta times f 2 belongs to L plus and integral of alpha f 1 plus beta f 2 is equal to alpha times integral f 1 plus beta times integral f 2.

To prove that let us so f belongs to L 1, L plus implies there is a sequence s n of nonnegative simple measurable functions, s n increasing to f and limit n going to infinity integral s n d mu giving us the integral of f 1 d mu. Similarly, g belongs to L plus implies that there is a sequence let us call it as an prime of nonnegative simple measurable functions s n prime increasing to g and its limit n going to infinity integrals of s n prime d mu giving us the integral of gd mu.

From these two let us just simply observe that if s n increases to f then alpha s n will increase to alpha f. Similarly, beta s n prime will increase to beta of g and integral of alpha s n for nonnegative simple measurable functions is equal to alpha times integral s n.

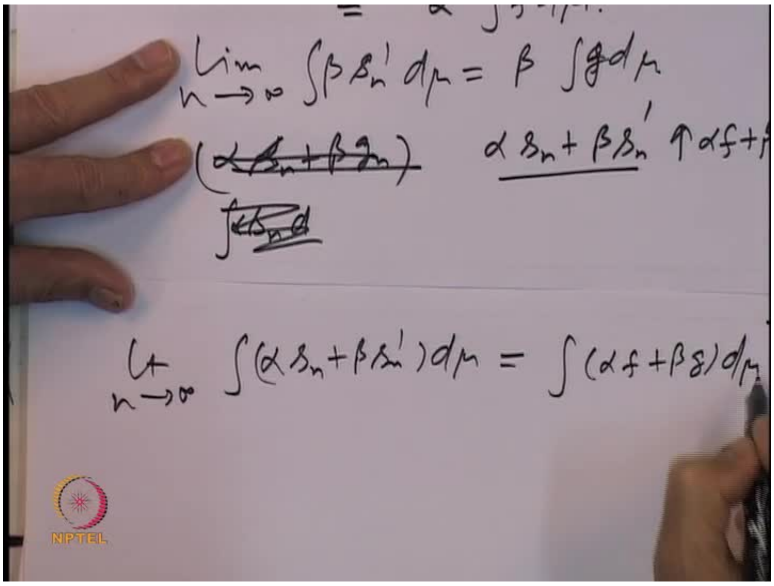
(Refer Slide Time: 29:44)



So, combining all these properties we will have the required result. Let us just write it out implies that alpha s n increases to alpha of f and limit alpha s n, n going to infinity integral d mu will be equal to alpha times limit n going to infinity of integral s n d mu, because for nonnegative simple measurable functions alpha times s n is same as alpha times the integral and that is equal to alpha integral f d mu.

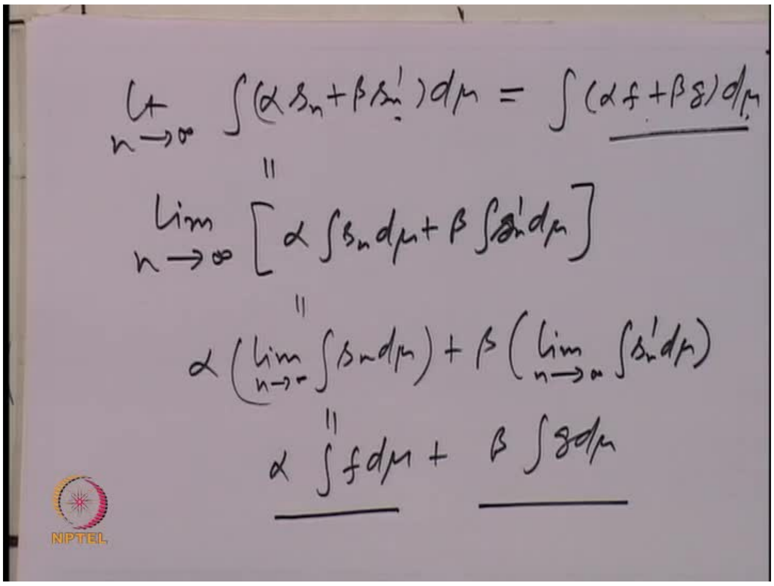
Similarly, limit of n going to infinity of beta s n prime integral d mu will be equal to beta times integral of gd mu. On the other hand, if you look at the sequence alpha s n plus beta s n dash, then this is a sequence of nonnegative simple measurable functions and that increases to alpha f plus beta g by the properties of sequences.

(Refer Slide Time: 31:11)



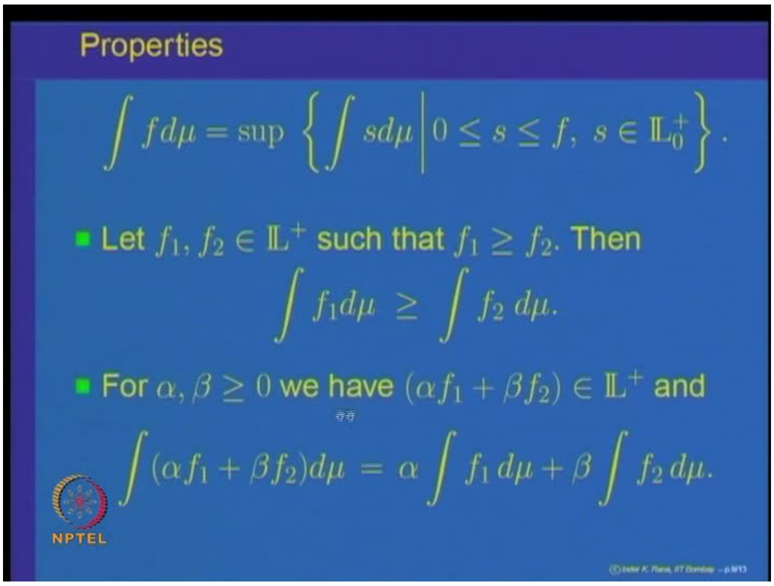
As a result, we will have that the integral s alpha s n d mu, let us write because of this we have the property that the integral alpha s n plus beta s n dash d mu limit n going to infinity will be equal to integral of alpha f plus beta gd mu.

(Refer Slide Time: 31:48)



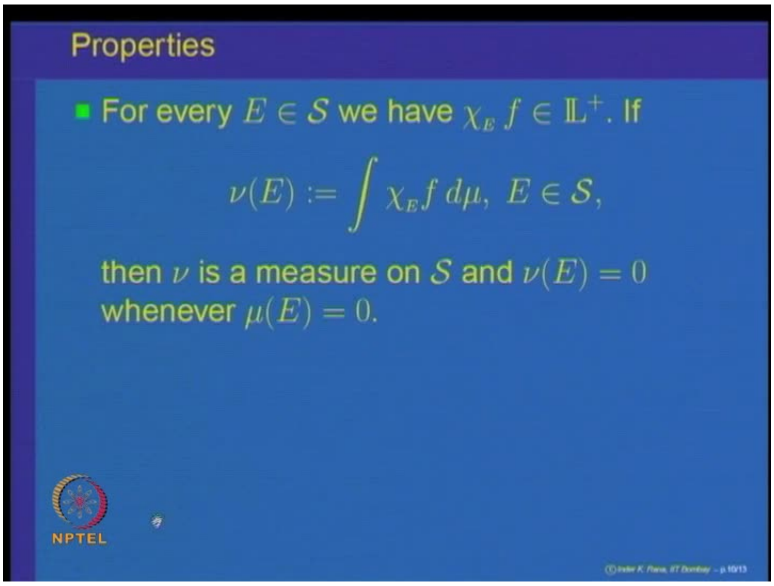
But, integration is linear for nonnegative simple measurable functions; this side is nothing but, limit n going to infinity of alpha integral s n d mu plus beta integral g n d mu. By the properties of limits of sequences, this is equal to alpha times limit integral of s n d mu plus beta times sorry this not this is s n dash beta times limit n going to infinity of integral s n dash d mu. We know this is alpha times integral f d mu plus beta times integral gd mu.

(Refer Slide Time: 32:53)



So, integral of alpha f plus beta g is equal to alpha times integral f plus beta times integral of g. That proves the property that then alpha and beta are nonnegative extended real numbers, then alpha f 1 plus beta f 2 belongs to L plus that we are already shown also. Now, we are claiming that the integral of alpha f 1 plus beta f 2 is equal to alpha times integral of f 1 plus beta times integral of f 2, we have written for g so that is same as for this.

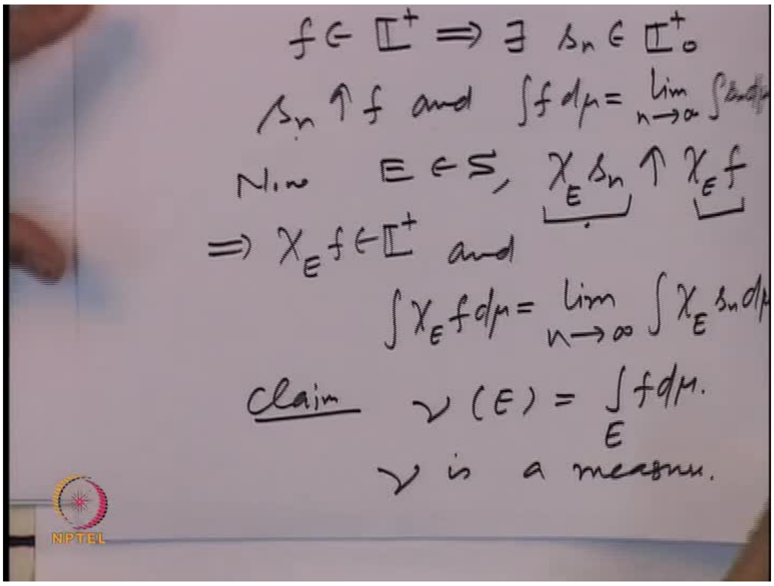
(Refer Slide Time: 33:18)



Next, we prove an important property and an extension of the earlier version for nonnegative simple functions namely, for every measurable set E if we look at the function the indicator function of E times f, then that is also a nonnegative measurable function. If its integral is denoted as nu of E, nu of E is the integral chi E f d mu where E belongs to S then this is a measure, this nu is a measure on the sigma algebra S and has the property that nu of a set E is 0 whenever mu of the set E is equal to 0.

Let us prove this property also. This property again we are going to use the fact that the integral of a nonnegative simple measurable function is given by as a limit of the integrals of an increasing sequence of nonnegative simple measurable functions.

(Refer Slide Time: 34:25)

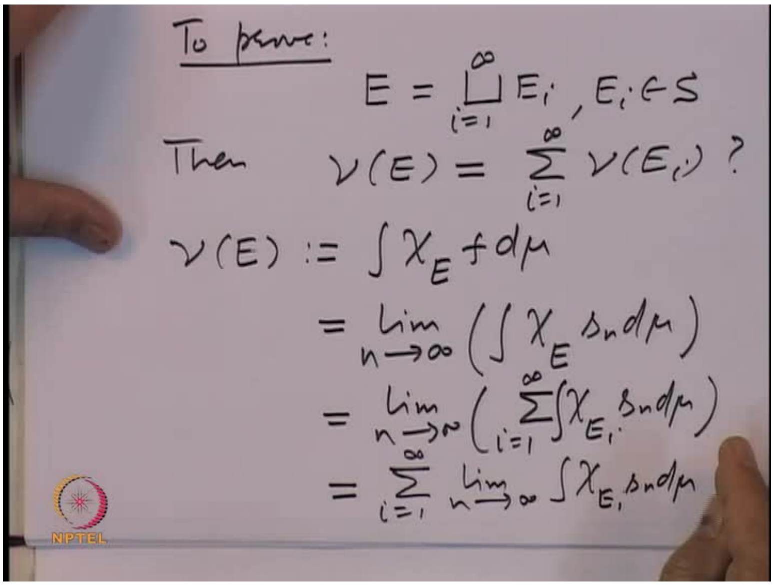


So, f belonging to L plus implies we have a sequence s n belonging to L plus 0 such that s n increases to f and integral f d mu is written as limit n going to infinity of integral s n d mu. That is by the fact that f belongs to L plus and integral of f is defined as limit of integral s n d mu, for any sequence s n which increases to f.

For E a set in the sigma algebra S because s n is increasing to f, so clearly indicator function of E times s n will increase to indicator function of E times f. Observe we have done it earlier also, then this is a nonnegative simple measurable function, it is increasing to this function that implies the indicator function of E times f is a nonnegative measurable function.

Because we have got this sequence increasing to this nonnegative measurable function, so integral of the indicator function of E times f d mu is nothing but, limit n going to infinity of integral indicator function of E times s n d mu times integral of s n d mu. This is how the integral is defined.

(Refer Slide Time: 36:30)



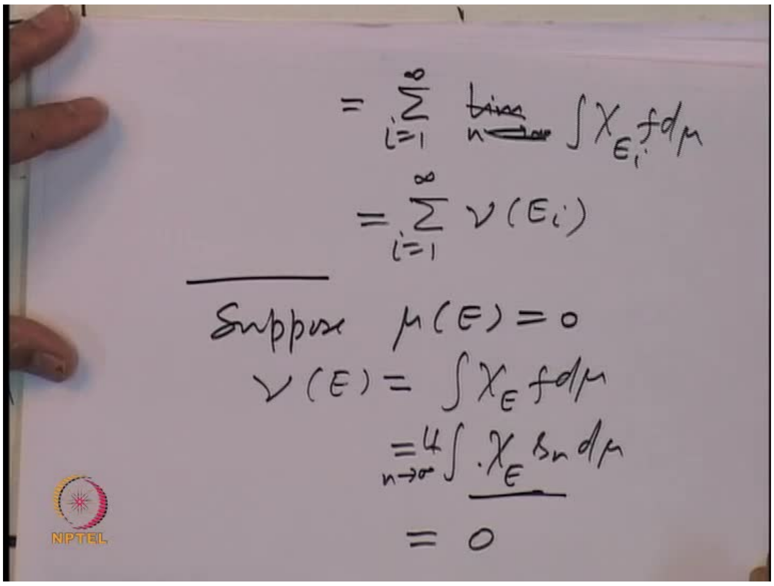
Now, we want to claim that if you call this number nu of E as integral over E f d mu, then the claim is nu is a measure. So, what we have to prove is the following, to prove this what we have to show that if a set E is a countable disjoint union of sets E i, E i's in the sigma algebra S. Then we want to show that nu of E is equal to sigma nu of E i, i equal to 1 to infinity, this is what we have to show.

Let us start looking at nu of E. By definition nu of E just now we saw that nu of E is nothing but, integral of the indicator function of E times f d mu. So, integral of E f d mu is nothing but, limit of integral indicator function of E s n d mu by the fact that s n is increasing to f, just now observed that.

This can be written as limit n going to infinity of integral indicator function of E s n d mu. This is just from the fact that s n is increasing to f, so indicator function of E times s n will increase to indicator function E times f. Hence, integral of indicator function of E times f is nothing but, the limit of the integrals of the indicator function E s n d mu.

Now, this E is a disjoint union of sets E i that implies let us write this limit n going to infinity of this, I can write as summation chi Ei of s n d mu, i equal to 1 to infinity. Here we are using the fact that E is a disjoint union of sets E i and for a nonnegative measurable function s n, if you integrated over a set E then that is a measure. That is the property, so the corresponding property for nonnegative simple measurable functions which we had already proved is true. We are using that fact to bring it here. This is limit of a series i equal to 1 to infinity of sorry this is an integral here integral of chi E E is union, so this is integral of the union.

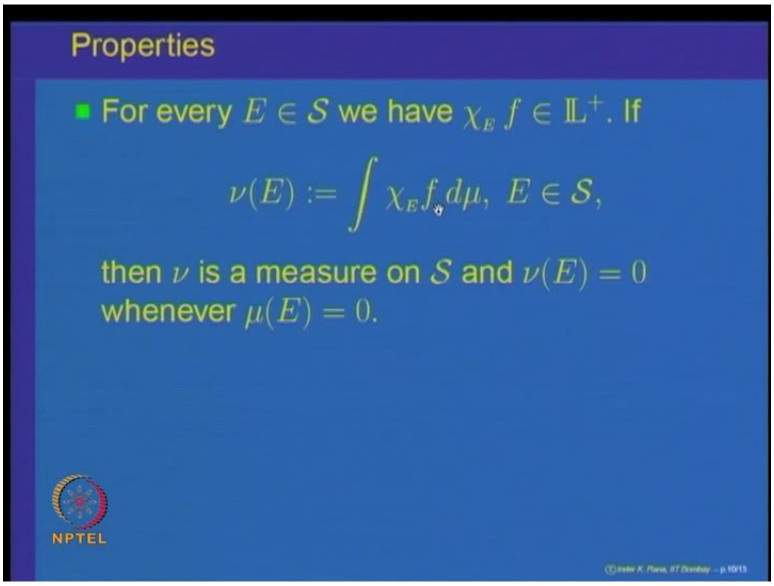
(Refer Slide Time: 39:57)



Now, I am going to interchange this summation and limit; summation i equal 1 to infinity that is allowed because all the quantities involved are nonnegative, this interchange is possible. So, I can write it as summation i equal to 1 to infinity limit n going to infinity of integral chi E i s n d mu.

Now, simply we observe that this last quantity is nothing but, summation i equal to 1 to infinity limit n going to infinity. The last quantity is nothing but, nu limit of n going to infinity that is, that limit is nothing but, integral of chi E i f d mu, because s n is increasing to f. So, chi E i times s n increases to chi E i times f this limit of n going to infinity integral of chi E i s n d mu is nothing but, integral of chi E i times f that value you put. That is nothing but, our definition of nu of E i.

(Refer Slide Time: 40:57)



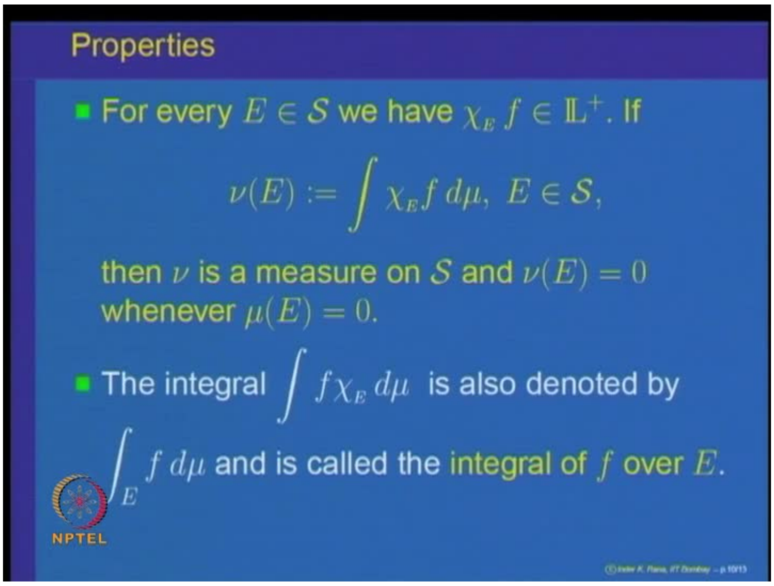
That proves that nu is a measure on E i. Once again, observe that here we have use basically what we have done; we have use the fact that f is a limit of incase in sequence of nonnegative simple measurable functions, so integrals of nonnegative simple measurable function that sequence gives you integral of f. Then go to that sequence, use the property for nonnegative simple measurable functions that property is true so and come back.

Finally, to prove that nu of E equal to 0 implies mu of E equal to 0 that is suppose mu of E equal to 0, then what is nu of E? Nu of E which was defined as integral chi E of f d mu, which was nothing but, integral of chi E times s n d mu limit of that, let us write limit n going to infinity of this but, this mu of E being 0, this integral is 0 so this is equal to 0.

Once again, for a nonnegative simple measurable function its integral over E is 0 if mu of E is 0, so that property is being used once again here. This proves the fact that this measure nu of E, which is constructed as integral of chi of E f d mu is a special measure which has the property that its null set nu of E is 0 whenever mu of E is 0.

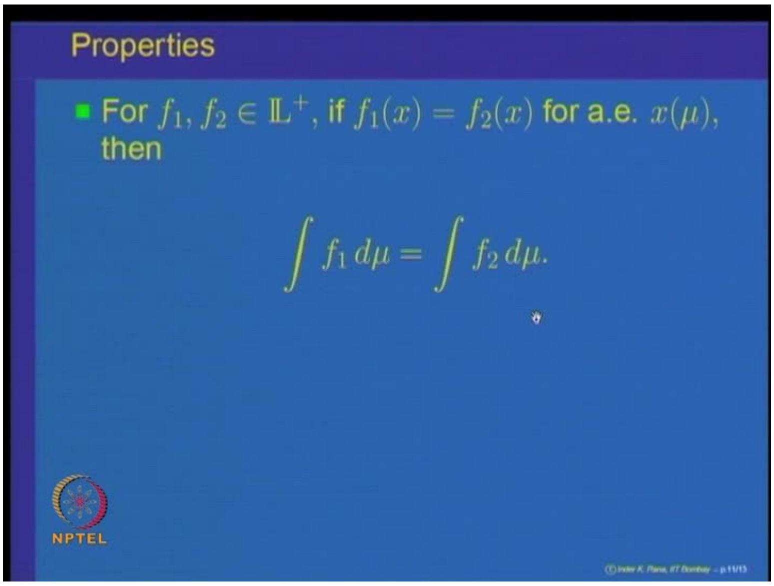
So, still now what we have done? We have defined the integral of a nonnegative simple measurable function as a limit of integral of nonnegative simple measurable functions. Because if f is nonnegative measurable it is a limit of nonnegative simple measurable functions which increase to this function f. So, integrals of those nonnegative simple measurable functions are defined. Take their limit and define integral of f to be limit of the integrals of nonnegative simple measurable functions. Using this, we have proved that this integration is linear.

(Refer Slide Time: 43:42)



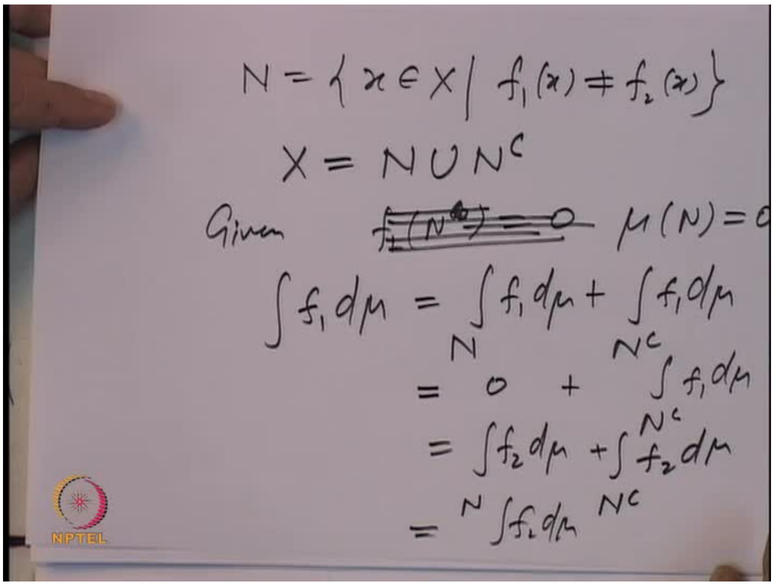
The next property we want to analyze is, how does this class of nonnegative measurable functions and the operation of integral behave for sequences in the class L plus.

(Refer Slide Time: 43:45)



Here is the first important theorem that we are going to prove; before that let us just prove just a simple observation that if f 1 and f 2 are nonnegative simple; f 1 is equal to f 2 almost everywhere, then integral of f 1 is equal to integral of f 2.

(Refer Slide Time: 44:08)

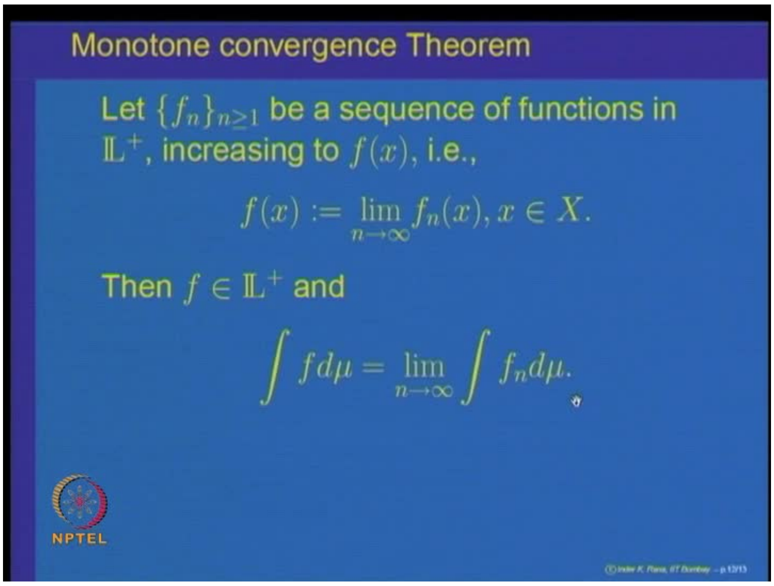


That property is quite obvious, because let us write the set N to be the set all x belonging to X where f 1 x is not equal to f 2 of x. Then the whole space can be written as N union N complement. So, f 1 of N complement we are given f 1 is equal to f 2 almost everywhere; where they are not equal this set has got f 1 sorry we are given that sorry we are given that this set N f 1 is equal to f 2 almost everywhere, where they are not equal that is a set of measure 0, so mu of n is equal to 0.

Now, integral of f 1 d mu can be written as integral over N f 1 d mu plus integral over N complement of f 1 d mu. This is by the fact just now we proved that integral over a set is a measure, so this is integral over N, N integral over N complement that gives you integral over the whole space. So, mu of N being equal to 0, this is the first term is 0 plus integral over N complement f 1 d mu.

But this is also same as integral over N of f 2 d mu, because measure of N is 0 and on N complement f 1 is equal to f 2, I can write as N complement f 2 d mu that once again is equal to integral f 2 d mu, so integral f 1 d mu is equal to integral of f 2 d mu. That essentially says that the integral of a function does not change if it changes its values on a set of measure 0.

(Refer Slide Time: 46:04)



Let us now come back to the property that I was trying to state earlier namely, if we have a sequence f n of nonnegative simple measurable functions and f n's increase to f that is f of x is equal to limit of f 1 x, then the claim is f belongs to L plus and integral f d mu is equal to limit n going to infinity integral of f n d mu. So, this is one of the important theorems in our subject; it is called monotone convergence theorem.

Monotone, because we are looking at sequence f n which is an increasing sequence; it is a sequence of function which is increasing. So, it is monotonically increasing sequence, increasing sequence of nonnegative measurable functions increasing to a function f; we already seen that f will be a measurable function and it is nonnegative.

But, the important thing is integral of f; f is the limit integral of the limit is equal to limit of the integrals. That is the important property we want to prove for integral for nonnegative measurable function. The proof of this theorem requires some construction and we do not have time today to complete the proof, so we will do the proof of this theorem next time. So, we stop here today by having stated the monotone convergence theorem and look at the proof of this in the next lecture. Thank you.