Measure and Integration
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Lecture No. # 17
Integral of Nonnegative Simple Measurable Functions

Welcome to lecture 17 on measure and integration. Today, we will start the topic of integration. First, I will explain the building blocks for the integration and how the process will be done.

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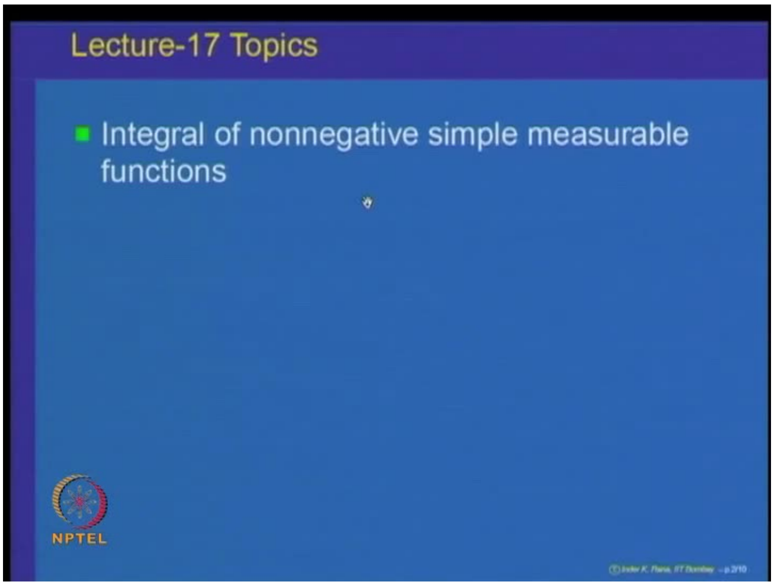

So, topic for today’s discussion is integral of nonnegative simple measurable functions. See, the basic idea is, we want to define the notion of integral for a function f defined on a set X, taking values in R star.

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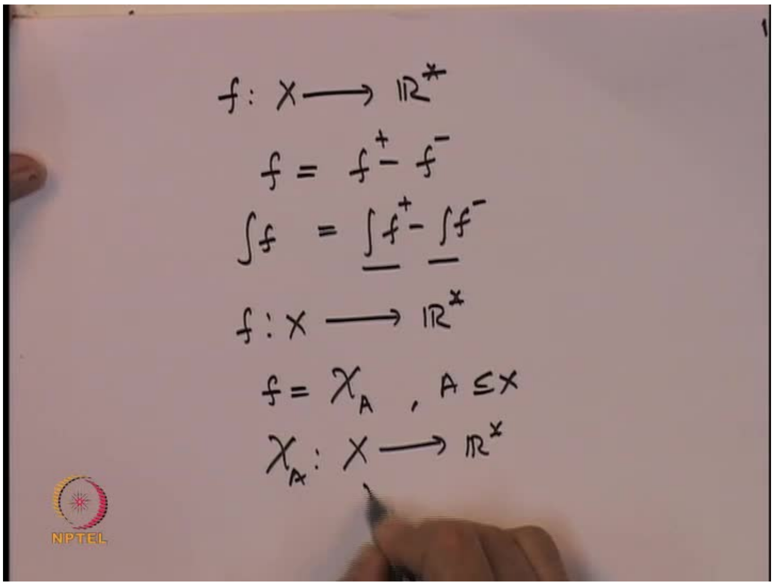

So, now for a function f, it is, we can represent a function f as the positive part minus the negative part. The advantage of doing this is that f plus and f minus, both are nonnegative functions, and so it is, and integration being a linear process, so integral of f is going to be equal to integral of f plus minus integral f minus. So, it is enough to define the notion of integral for nonnegative functions and for nonnegative functions f on X to R star, we recall, that we can take it as we look at functions, which are first of all very simple functions.

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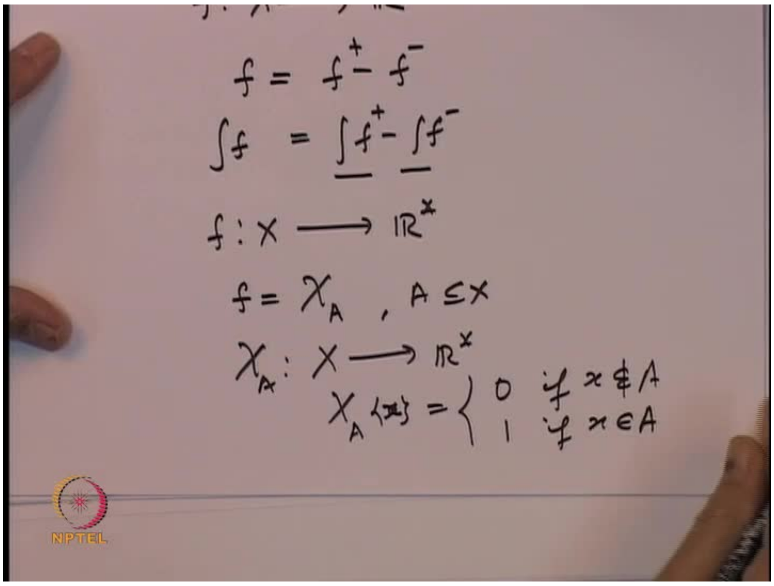

For example, let us look at a function f, which is the indicator function of a set X, of a set A, A contained in x. This is a function, which takes only 2 values. So, indicator function of A is a function on X taking values in R star. So, chi of A at 0 at a point x is equal to 0, if x does not belong to A and is 1, if x belongs to A.

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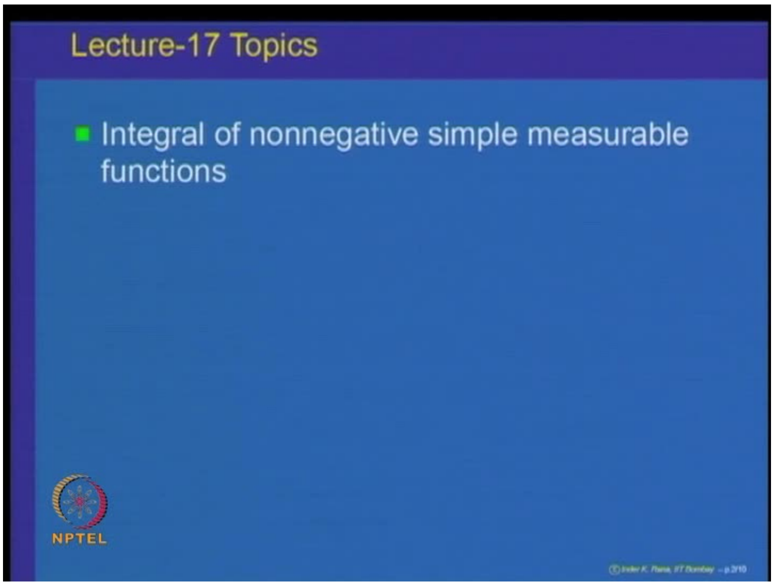

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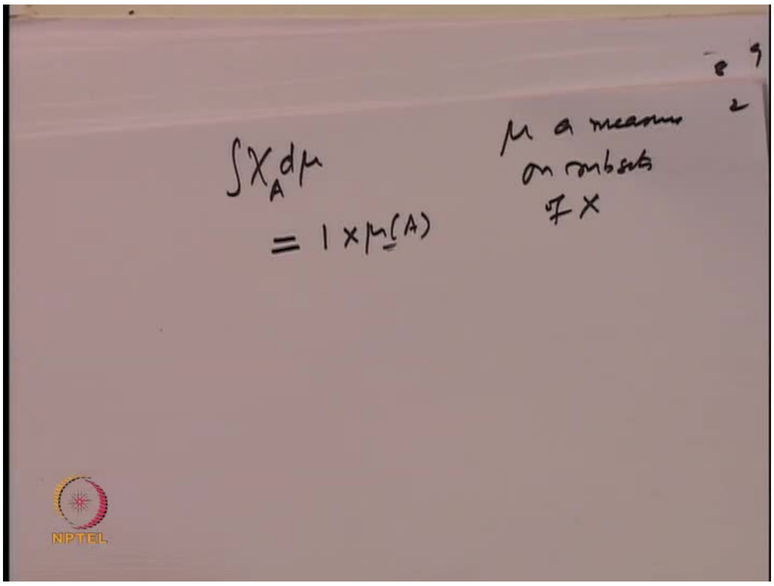

So, you can, one can think of this function taking only 2 values. Now, the value where it is 0, the integral, so we want to define the notion of integral and this is going to be with respect to a measure mu on X. So, mu, a measure on subsets of X, so we are going to write it as X A d mu. So, what it should be on A? The value is 1, so we would like to put it as 1 times mu of A; in some sense, mu of A is the size of a set and 1 is the height. So, this is, in the sense, the area of the, we know, the graph of the function.

So, let us look at functions, which are going to be linear combinations of indicator functions.

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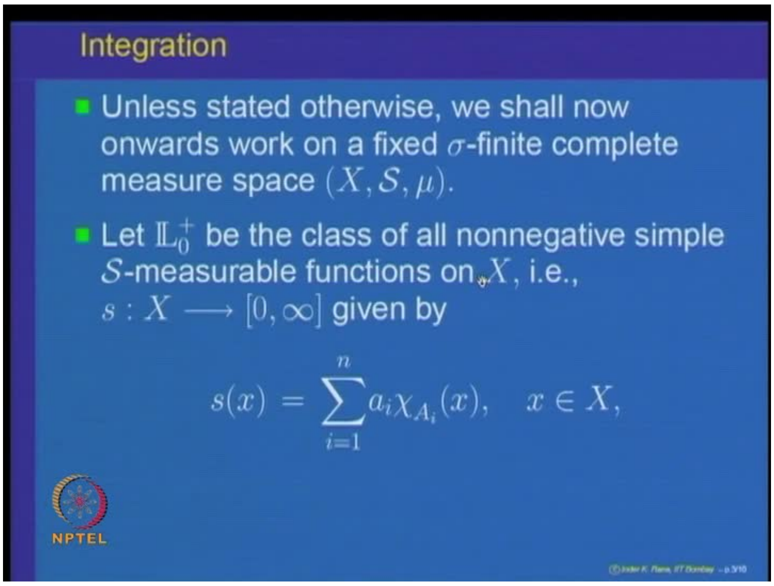

So, we start looking at the integral of nonnegative simple measurable functions. So, let us recall, so we will fix our notation, that from now onwards, we are going to work on a measure space X, S, mu, where X is a set, S is a sigma - algebra of subsets of X and mu is a measure on defined on S, and this is a complete measure space. That means, that all sets A, such that mu of A is 0 implies, that A and all its subsets are inside S.

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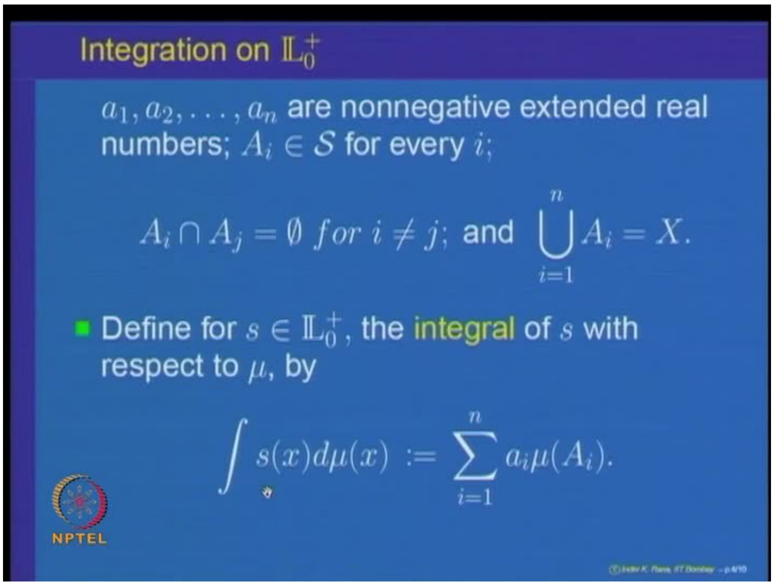

So, let us denote by L lower 0 upper plus to be the class of all nonnegative simple S-measurable functions on X. So, now, let us recall, what was a nonnegative simple measurable function s. It is a function defined on X in nonnegative values and it is, it has a representation s of x is equal to sigma i equal to 1 to n a i times the indicator function of the set A i evaluated at x; x belonging to X, where a i's a 1, a 2, a 3, a n are extended real numbers and the sets A i's are in the sigma algebra S, so they are in the sigma algebra, A i's are in the sigma algebra S and they are pairwise disjoint, that means, A i intersection A j is empty for i not equal to j and the union of these sets is equal to X.

So, this is going to be the class of nonnegative simple measurable functions. For such class of, for functions in this class, we are going to define the notion of integral. So, for a function s in this class, if its representation is as given before, so if s of x is equal to sigma a i, indicator function of capital a i, then its integral is defined as s d mu. So, integral is denoted by integral sign s of x d mu x to be a i, that is, the value of the function on the set a i times the measure of a i mu of A i. So, the integral of s with respect to mu, as written here, is defined as sigma a i times mu of a i. a i is the value taken on the set A i, so a i times the size of the set a i, so mu of A i.

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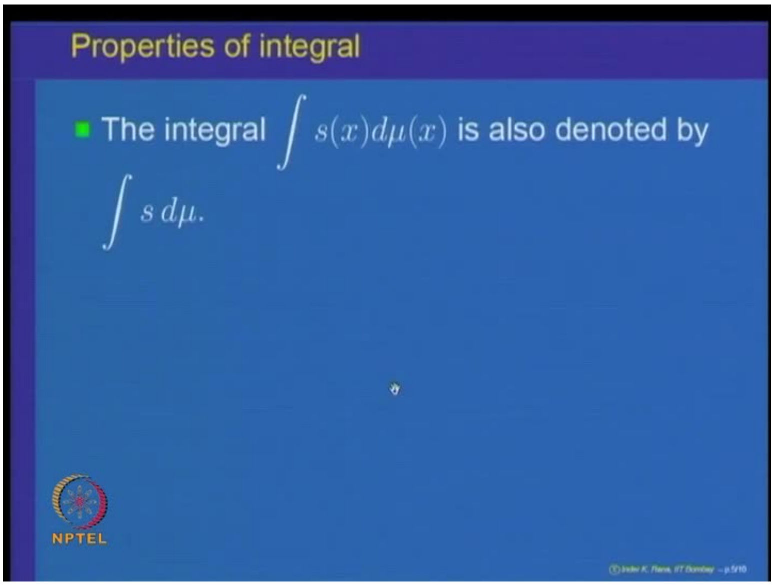

Sometimes we do not indicate the variable x, we just write as s d mu to be the integral of the simple function s, nonnegative simple measurable function s, with respect to mu.

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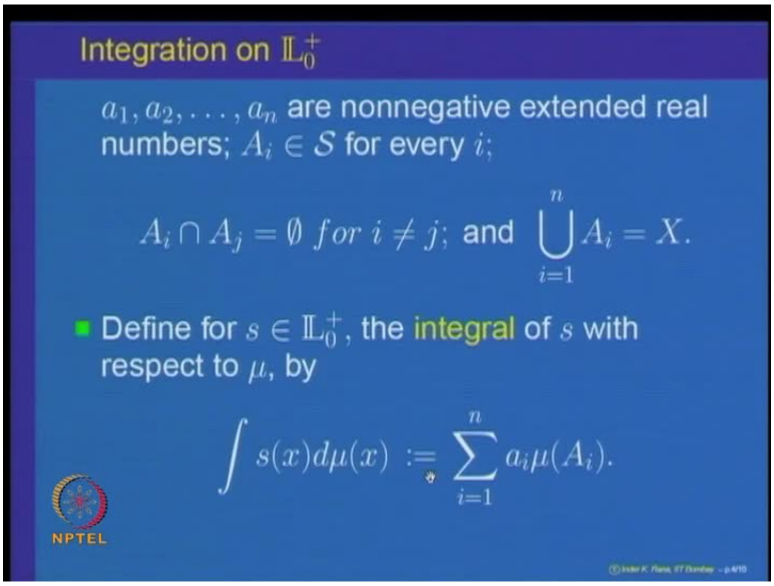

And let us note here, that our representation, the integral is with respect to a representation of the function.

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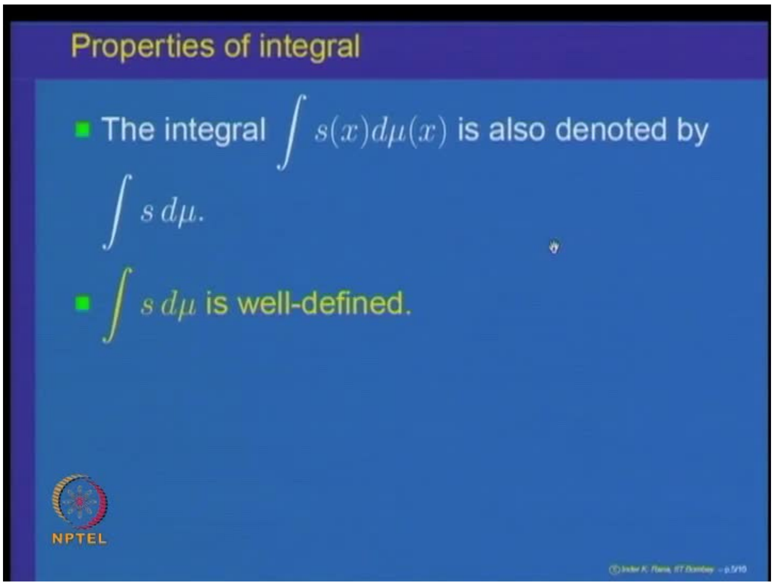

So, first of all we would like to show, that integral s d mu is well defined; so, let us prove that the integral is well defined.

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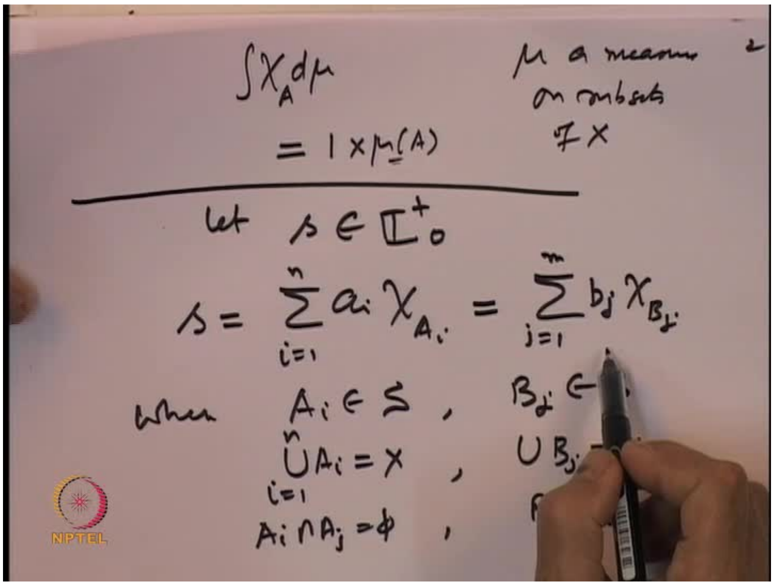

So, let us take a function s belonging to L plus 0, so it is a nonnegative simple measurable function. So, let us say, s is written as sigma a i indicator function of A i, i equal to 1 to n, also representable as sigma j equal to 1 to m of some b j chi of B j, where these sets A i's belong to the sigma algebra S, B j's belong to the sigma algebra S and union of A i's is equal to x and union of B j's is also equal to x, and these sets are disjoint. So, A i intersection A j is empty and B i intersection B j is empty for i not equal to j.

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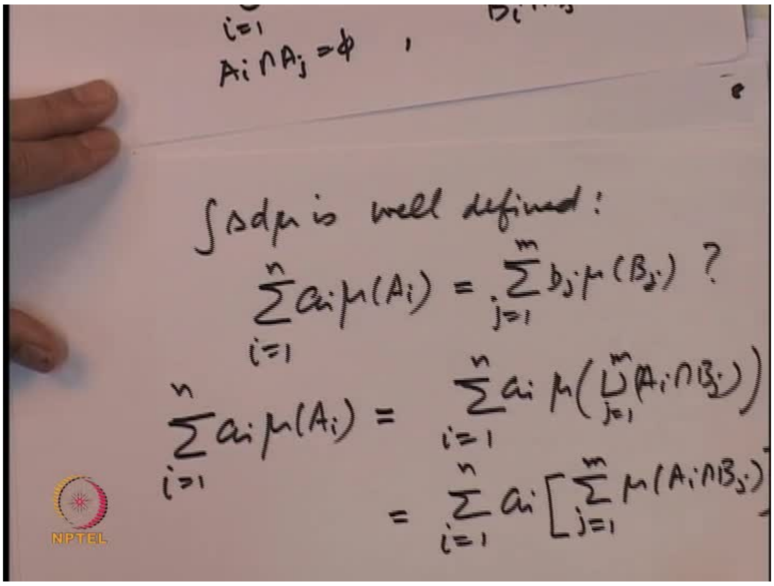

So, let us say, that set s, a simple function s has got 2 representations possible, so what we want to show? We want to show, that the integral of s, so integral s d mu is well defined and that means what. So, mathematically, that means, we have to show, that sigma a i mu of A i 1 to n is equal to sigma j equal to 1 to m b j mu of B j. So, this is what we have to show.

So, let us start. So, sigma a i mu of A i, i equal to 1 to n, I can write it as sigma i equal to 1 to n a i and then, mu of this A i can be written as union of A i intersection B j, j equal to 1 to m because union of B j's equal to x, so A i intersection x and that is same as this. Now, this is a, B j's are disjoint. So, these sets are A i intersection B j's, for i fix are disjoint. So, by using finite additive property of the measure, we have this is equal to i equal to 1 to n a i and this is nothing but sigma j equal to 1 to m mu of A i intersection B j.

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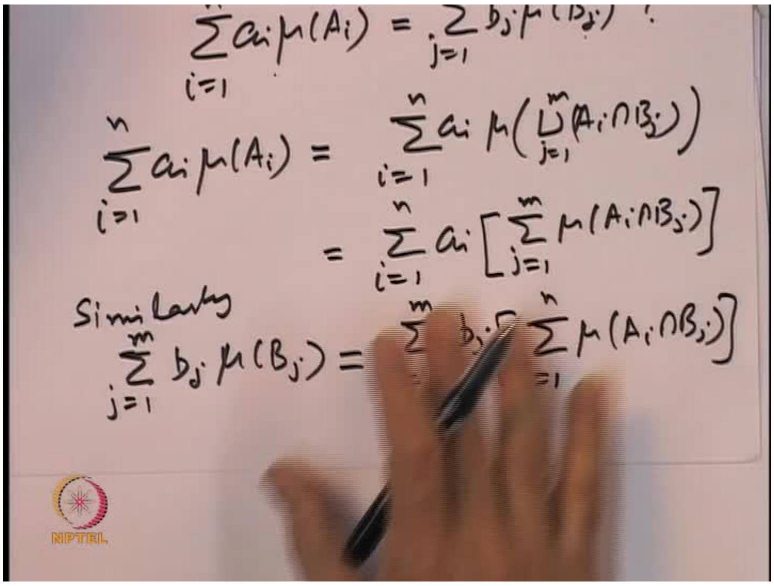

Similarly, we can write the other side, that is, sigma j equal to 1 to m of b j mu of B j to be equal to sigma j equal to 1 to m b j sigma i equal to 1 to n mu of A i intersection B j. So, the left hand side here is written as this sum; the right hand side is written as this sum.

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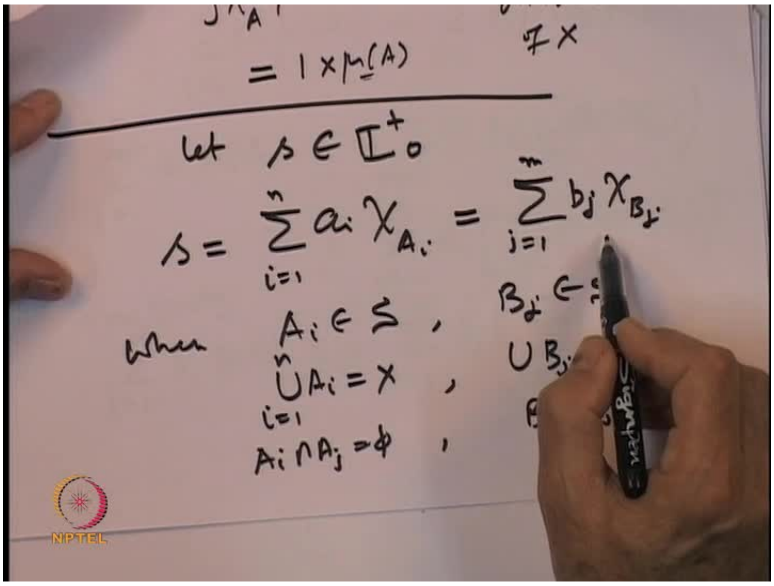

Now, we want to show, that these 2 sums are equal. Now, let us observe, that given that the function s has got 2 representations, this equal to this, so how is this function calculated?

At a point x, if X belongs to A i, the value is a i and on the other hand, it may belong to some B j, the value will be b j. So, that force is one to say, that if X belongs to A i intersection B j, then a i must be equal to b j.

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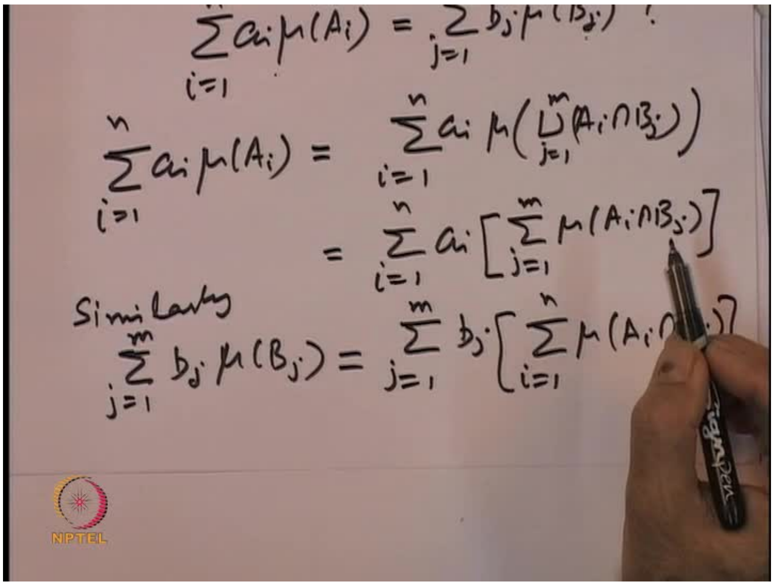

So, this is the crucial thing to note here, that if s, a simple nonnegative simple measurable function is given 2 representations, one is sigma a i, capital A i, indicator function of A i and sigma b j, indicator function of B j, then for x belonging to A i intersection B j, the value of s of x on one hand is a i, other hand is b j. So, A i must be equal to B j. So, this is the crucial thing to note.

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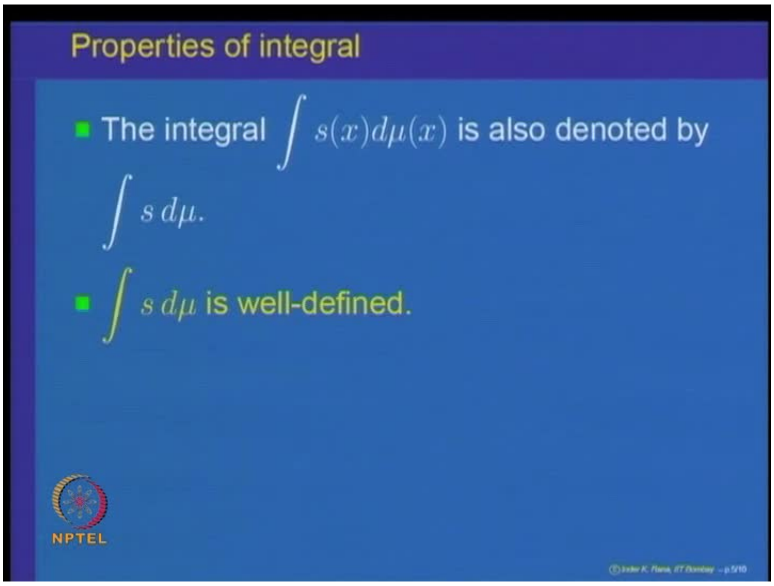

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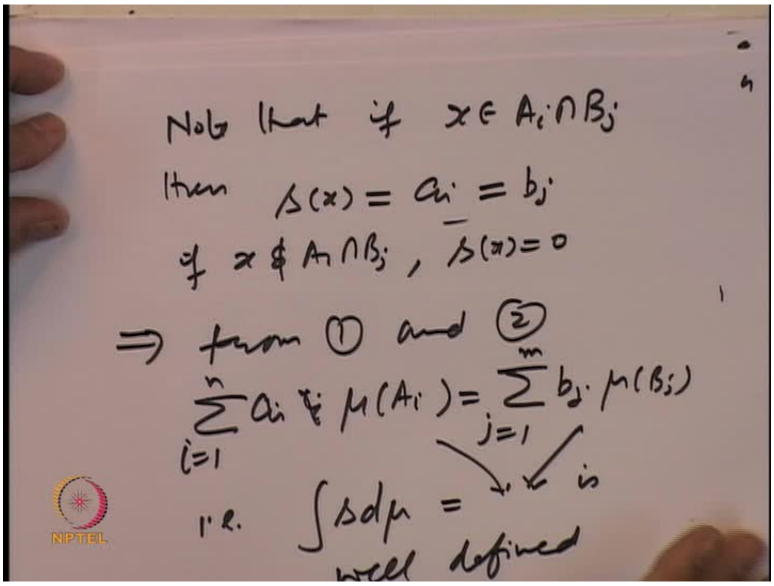

So, let us make this observation and write it out. So, note that, if x belongs to A i, if x belongs to A i intersection B j, then s of x is equal to a i; it is also equal to b j. So, a i is equal to b j and if x does not belong to A i intersection B j, then s of x is equal to 0.

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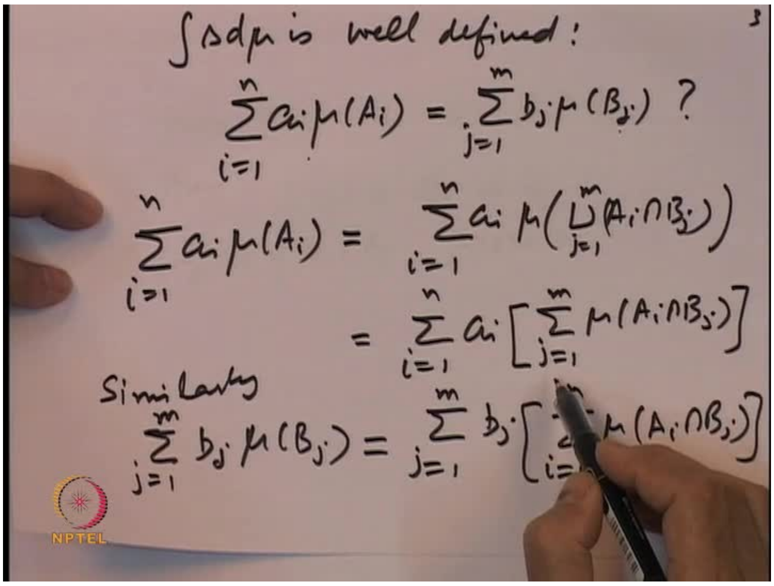

So, that means, in this summation, whenever x belongs to A i intersection B j, this a i is going to be equal to b j, otherwise, in this sum, the term does not matter. So, that proves the fact, so that will imply from these 2 equations, from equation 1 and equation 2, so this implies from equation 1 and 2, that sigma a i i equal to 1 to n of mu A i is equal to sigma j equal to 1 to m b j mu of B j. So, that is, integral s d mu can be defined as either of these sums, so is equal to either this or this, is well defined; so, the integral of a nonnegative simple measurable function. So, we can choose any representation of, we can choose any representation of the nonnegative simple function and define its integral in terms of that.

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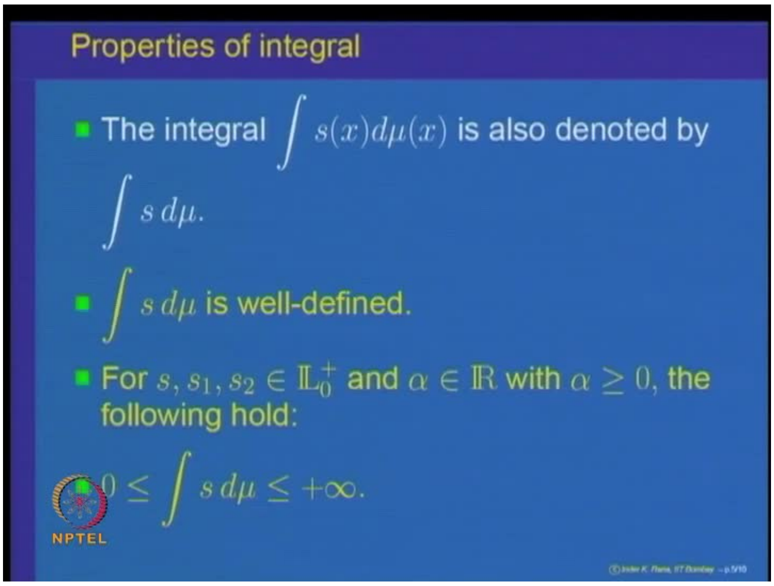

Next, let us look at properties of this integral. So, we are going to look at functions s, s 1, s 2, which are nonnegative simple measurable functions; alpha will be a real number; alpha bigger than or equal to 0. Then, we are going to look at what happens to various properties.

So, 1st observation is that integral s d mu is a nonnegative number, it could be equal to plus infinity. So, integral s d mu is an extended nonnegative real number, that is obvious, because what is s d mu? Integral of s d mu is summation of a i's times mu of a i's, all the terms are nonnegative. So, this is a nonnegative number, so this is an obvious property.

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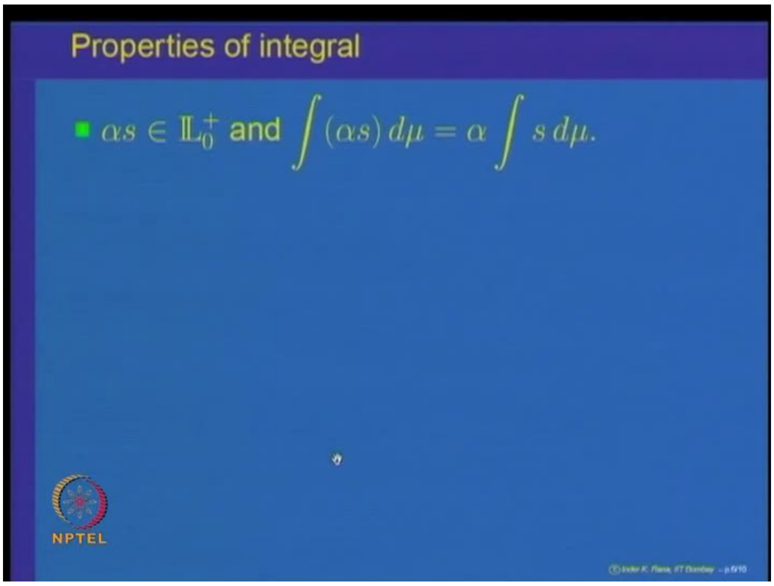

The 2nd property we want to check, that for a nonnegative simple function s, alpha s belongs to L plus 0 plus and the integral of alpha s d mu is same as alpha times integral of s d mu.

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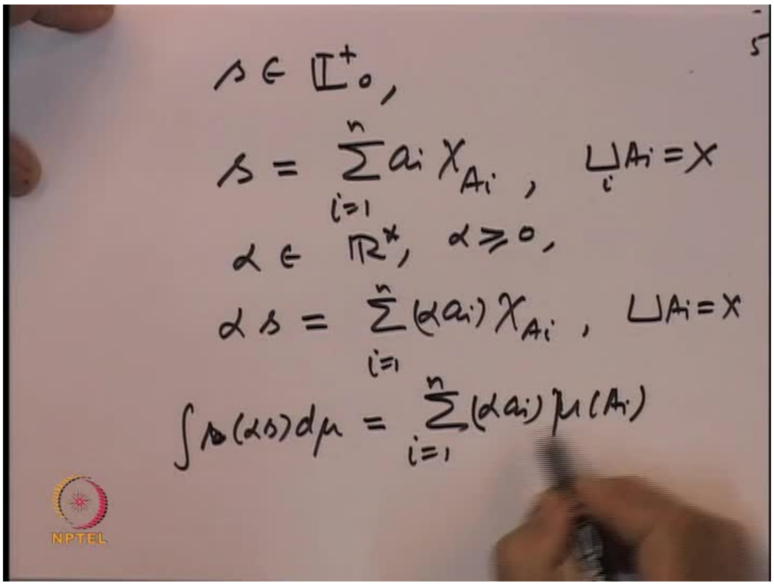

So, let us check, that, so s belongs to L plus 0, is a nonnegative simple measurable function. So, let us write, let us write s is equal to sigma a i indicator function of A i, where union A i is equal to X.

So, whenever it is a partition, we will write as this - square bracket union over i equal to X and alpha is a nonnegative alpha belonging to R star, alpha bigger than or equal to 0. Then, alpha of s, so as the representation is alpha a i chi of A i and A i's are still a partition of X, but that means if this the representation, so integral alpha s d mu integral of alpha s with respect to mu is going to be equal to, by our definition, i equal to 1 to n alpha a i times mu of A i.

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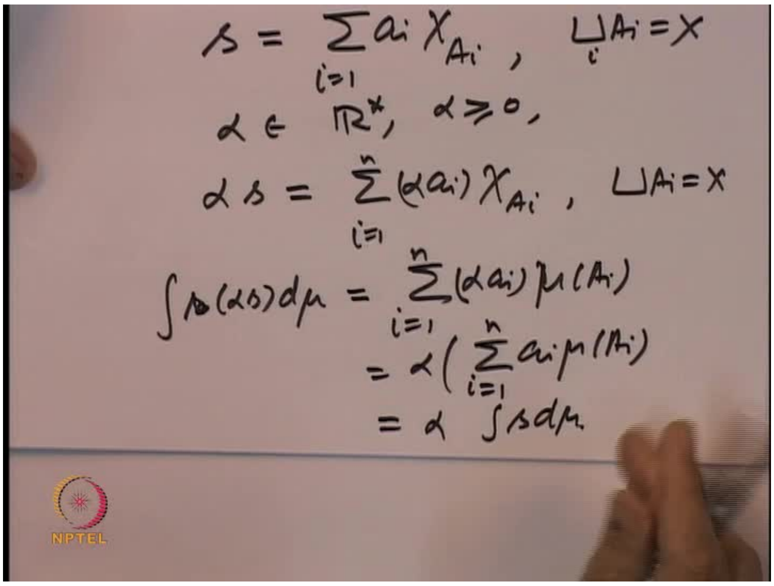

And this is the finite sum, nonnegative, everything. So, alpha comes out, alpha times the summation of i equal to 1 to n of a i mu of A i and that is nothing but alpha times integral of s d mu.

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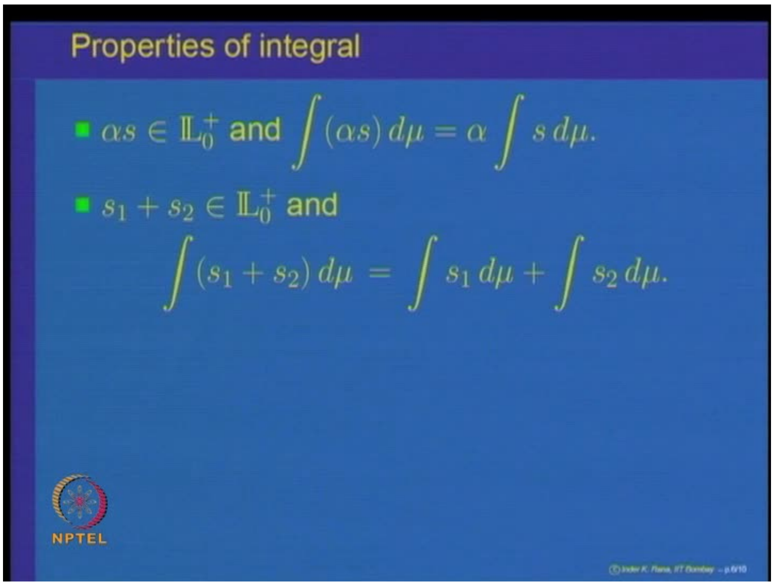

So, that proves the property, that the integral of nonnegative simple functions is, if you multiply it by a constant alpha, then the alpha comes out, so the integral of alpha s d mu is equal to alpha times s d mu.

Next, we want to show, that it is a linear operation. So, we want to check, that if s 1 and s 2 belong to L 0 plus, then s 1 plus s 2 belong to L 0 plus, that we have already checked, but we will check it again today also, and the integral of s 1 plus s 2 d mu is integral of s 1 plus integral of s 2.

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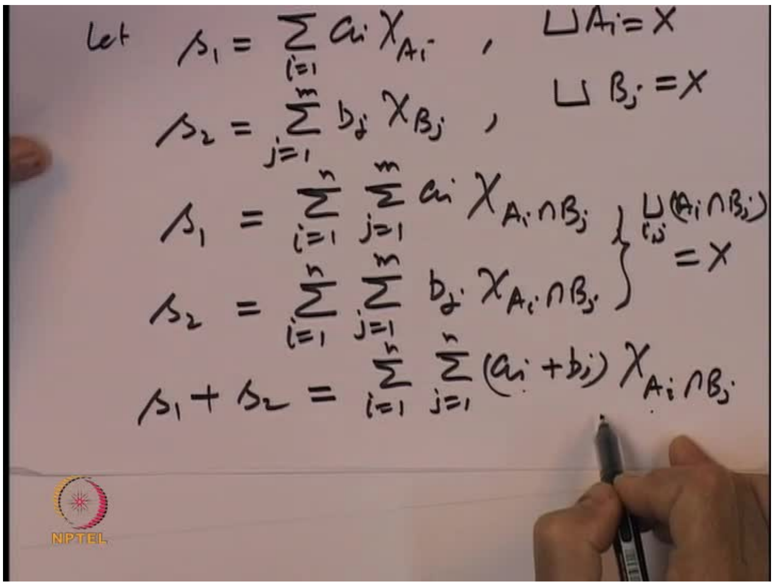

So, for such things, we have, so let us take a function s 1, s 2 belonging to L plus 0, so nonnegative simple measurable function. So, let us write, let s 1 be equal to sigma a i chi of A I, where A i's form a partition of X. And let us write s 2 sigma j equal to 1 to m b j chi of B j union B j's partition of X.

So, if you recall, we had said, that we can bring both s 1 and s 2 a common partition and what is that common partition? A i intersection B j. So, what we are saying is, we can write s 1 as sigma i equal to 1 to n sigma j equal to 1 to m a i chi of A i intersection B j. And also, similarly, s 2 can be written as i equal to 1 to n sigma j equal to 1 to m of b j chi of A i intersection B j. Now, here note, that union over i and j A i intersection B j, that is the partition of the whole space, so that is equal to X.

So, this is the point to be, sort of, noted, that whenever we are given 2 functions, s 1 and s 2, with 2 representations, which involves some partitions A i and partition B j, then we can bring them to a common partition, namely A i intersection B j. And now we can define, what is s 1 plus s 2. So, s 1 plus s 2 is going to be equal to sigma over i 1 to n sigma over j equal to 1 to n a i plus b j chi of A i intersection B j.

That is clear because on A i intersection B j s 1 is a i and on A i intersection B j s 2 is b j. So, s 1 plus s 2 will be equal to a i plus b j on A i intersection B j.

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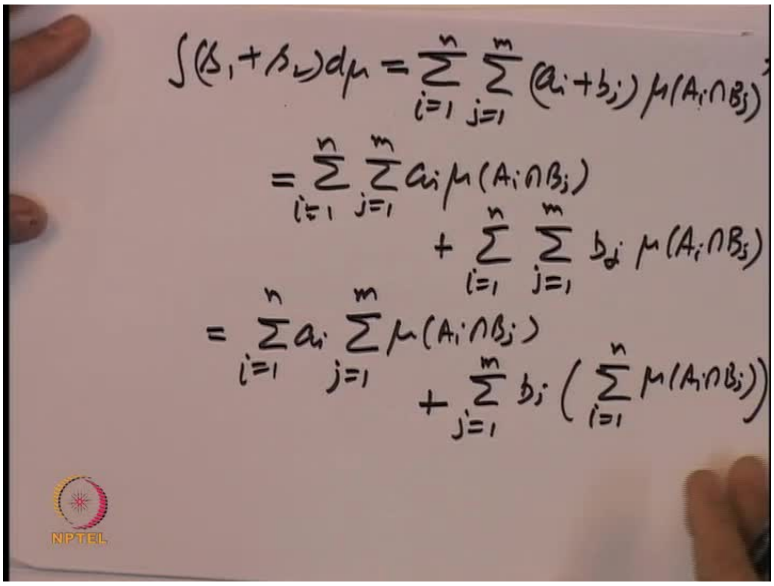

So, once we have got a representation of s 1 plus s 2, we can define, what is the integral of s 1 plus s 2. So, this representation gives us, that integral of s 1 plus s 2 d mu is equal to summation over i 1 to n summation over j 1 to m of a i plus b j into mu of A i intersection B j. So, because this is the representation, so a i plus b j is a value on the set A i intersection B j. So, integral is going to be equal to summation over i summation over j of a i plus b j, the value on the set A i intersection B j.

Now, the right hand side, we can write that is equal to 2 terms, one is summation over i summation over j of a i times mu of A i intersection B j plus the 2nd term, summation i equal to 1 to n summation j equal to 1 to m of b j. So, a i and second term is b j mu of A i intersection B j and now these are all finite sums. So, we can write, the first term as sigma i equal to 1 to n, take a i outside and this is summation of mu of A i intersection B j because this is summation over i only, so you can take it out, j equal to 1 to m of A i intersection B j plus. Here, summation over j and summation over i, so we will write it as summation over j first b j and inside is summation over i equal to, I have interchanged the order of summation, they are finite terms only, finite sums only, so that is allowed. So, that is, 1 of mu of A i intersection B j.

And now, we observe, that the 1st sum by the finite additivity property of the measure is nothing but mu of A i and this summation over i, this sum is nothing but mu of B j because A i's form a partition of x and here B j's form.

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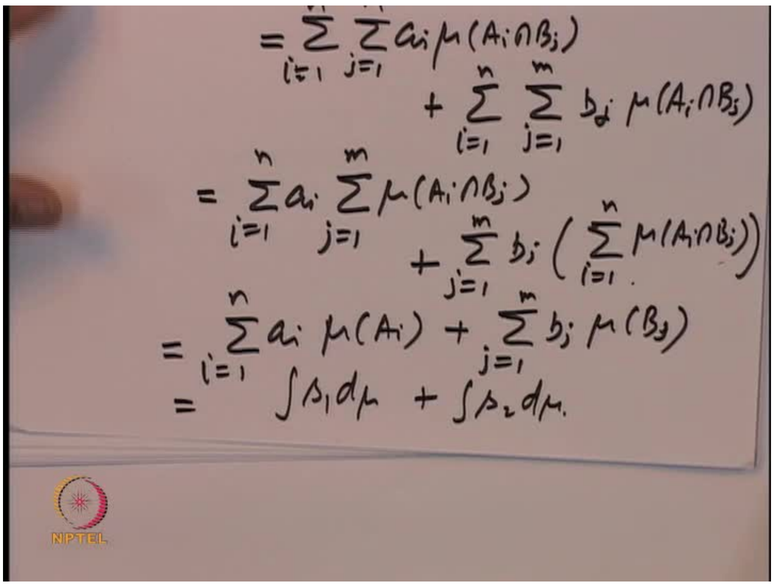

So, 1st term is equal to summation i equal to 1 to n a i mu of A i plus summation j equal to 1 to m b j of mu of B j and now clearly, this is integral of s 1 d mu plus the 2nd term is integral s 2 d mu.

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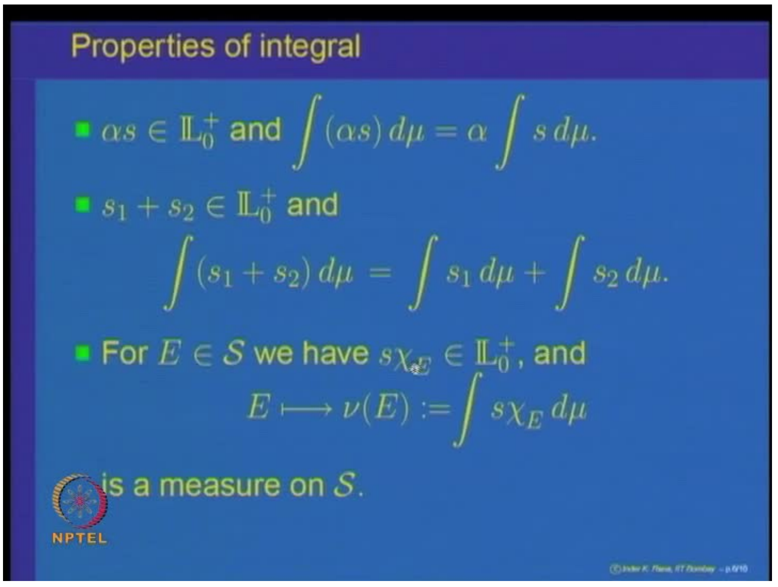

So, that proves the fact, that integration is a linear process, namely integral. If s 1 and s 2 are in L 0 plus, s 1 plus s 2 also is in L 0 plus and the integral is of s 1 plus s 2 is equal to integral of s 1 plus integral of s 2.

Next property we want to check is the following, that for a set if E is a set in the sigma algebra S and we multiply s nonnegative simple measurable function by the indicator function of E, then that function also belongs to L 0 plus; that again, we had checked it earlier when we defined simple nonnegative functions. So its integral is defined and we want to check, that E going to nu of E, which is integral of s indicator function of E d mu is actually a measure on S. So, this gives a method of generating more measures on this sigma algebra E, so let us prove this property.

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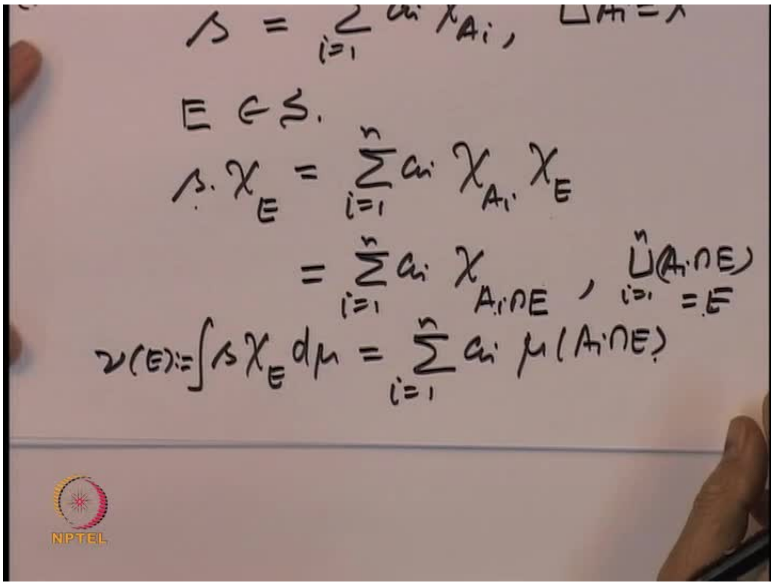

So, let us take a function, let us take a nonnegative simple measurable function L plus 0 s of given by sigma i equal to 1 to n a i indicator function of A i, where union of A i is equal to X and E is a fix set in the sigma algebra S. Then, s times the indicator function of E, so multiply this equation on both sides by indicator function, that is, i equal to 1 to n a i chi A i multiplied by chi of E.

And now, here is the observation, that the product of indicator function of 2 sets is nothing but the indicator function of the intersection. So, this can be written as i equal to 1 to n a i. This product indicator function of A i into indicator function of E can be written as the indicator function of A i intersection E.

So, that is only observation one has to make and now, so s times indicator function of E is given by this, so where union of A i intersection E, what will be that? That is the disjoint union giving you the set E and on E compliment this function is 0. So, if you like you can add 1 more term here, 0 times the indicator function of E compliment, but that is not so normally, whenever the, that kind of a set, that term will not mention it here. So, automatically on the compliment it is 0 and that gives you the partition of the set.

So, this means, s of indicator function of E is a i times indicator function of A i intersection E, where these things form a partition. So, that implies, s times the indicator function of E is a nonnegative simple measurable function, and what is the integral of that? So, integral of s chi of E d mu is equal to sigma i equal to 1 to n a i mu of A i intersection E. So, that is the integral of this function. So, we want to prove, that if we call this as nu of E that is a measure.

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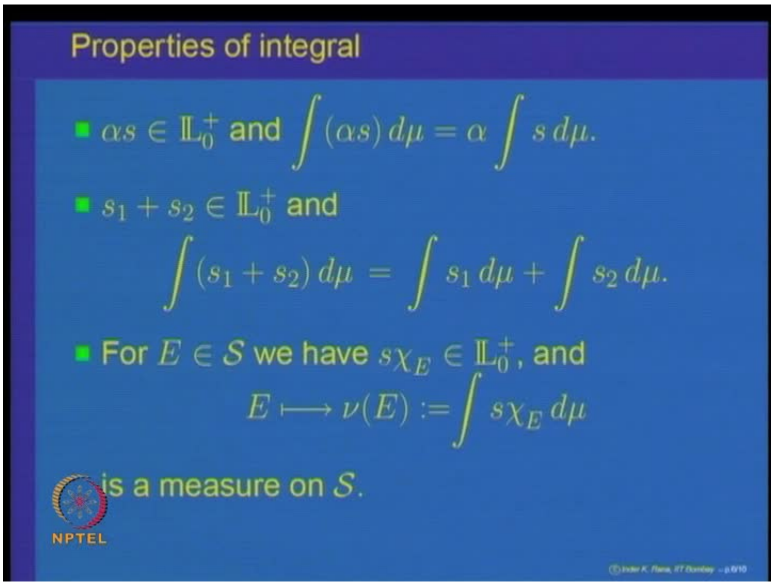

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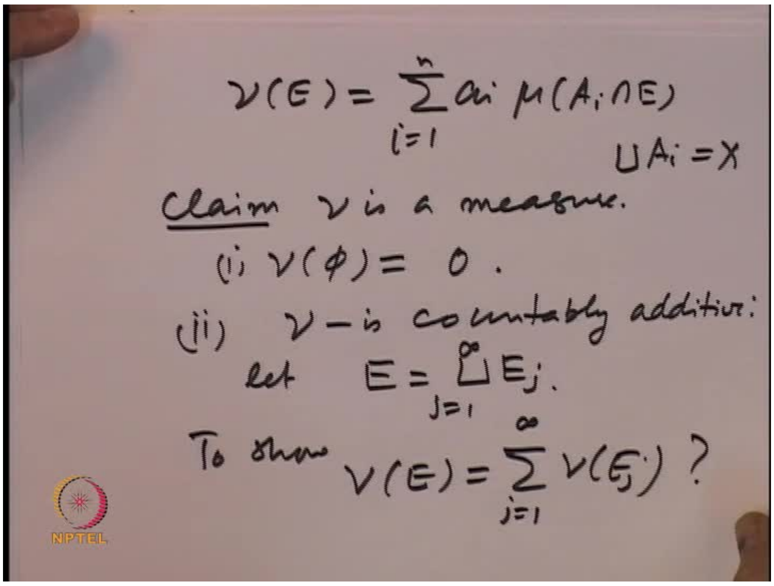

So, let us check that property, to check it is a measure, what we have to check. So, nu of a set E is defined as sigma, by our previous calculations, a i times mu of A i intersection E i equal to 1 to n, where A i's are partition of X and A i's they are in the sigma algebra always.

So, claim, nu is a measure. So, what is to be checked? nu of empty set equal to E is empty set. So, mu of A i intersection E, that is empty set, so that is 0, so it is equal to 0.

What is the 2nd property we want to check? nu is countably additive. So, for that, so let us write, let E be equal to union of E j j equal to 1 to infinity, where all the sets are in the sigma algebra. So, we want to show that. To show, nu of E is equal to sigma nu of E j j equal to 1 to infinity. So, that is what we have to show; so, let us compute both sides and show the required property.

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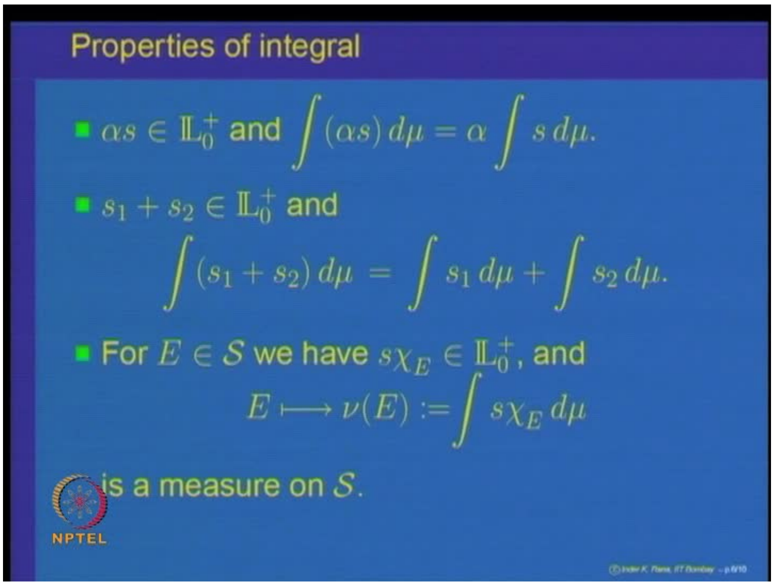

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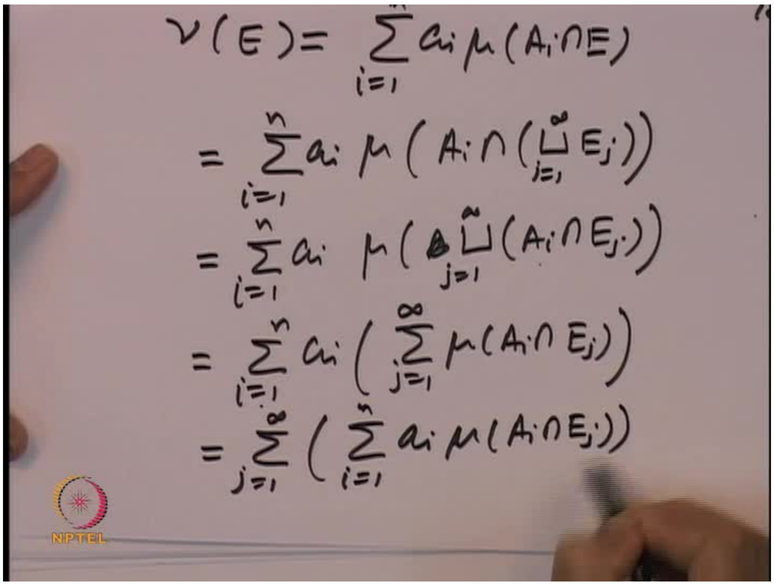

So, let us look at nu of E. So, nu of E is equal to sigma i equal to 1 to n a i mu of A i intersection E; by definition mu of E is defined as this thing. What is E? Let us put the value of E, so it is i equal to 1 to n a i mu of A i intersection, union, disjoint union E j j equal to 1 to infinity. That is by the definition of, by the fact, that E is a disjoint union of E j's, but that we can write it as summation i equal to 1 to n a i mu of… So, this is nothing but, so we can write it as disjoint union over j 1 to infinity of A i intersection E j, by the distributive property of intersection over union.

So, this is a countable disjoint union of sets in the sigma algebra, so by the countable additive property of the measure mu, this term is equal to summation i equal to 1 to n a i summation j equal to 1 to infinity of mu A i intersection E j. And now note, that we have got 2 sums here, one is summation a i and other is summation j equal to 1 to infinity, and all are nonnegative extended real numbers, so we can interchange the order of integration without any problem.

So, we can write this as summation over j first, then summation over i 1 to n a i mu of A i intersection E j, so we write this as this. So, now note, that this term, summation over i a i mu of A i intersection E j is nothing but the nu of e j.

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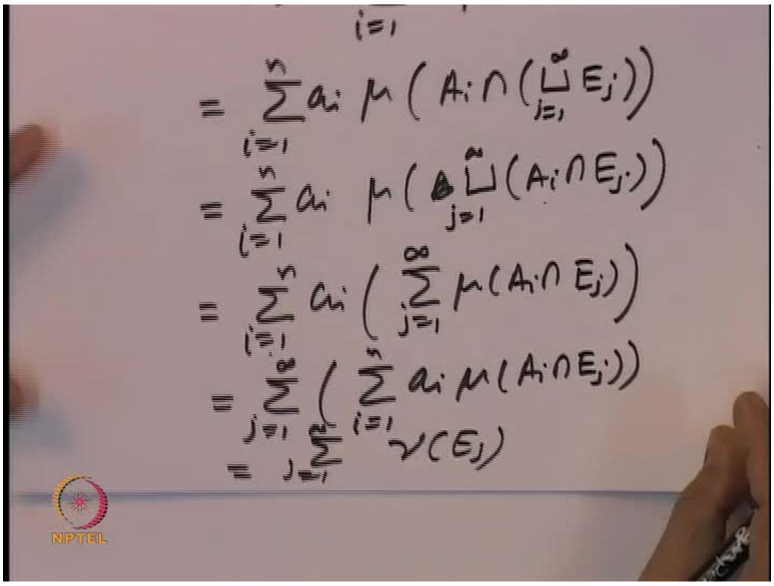

So, by definition, this is summation over j equal to 1 to infinity, so this is nu of E j.

So, we have shown, that nu of E is summation nu of E j's, whenever E is equal to union of disjoints, pairwise disjoint sets E j. So, that proves, that mu is a measure.

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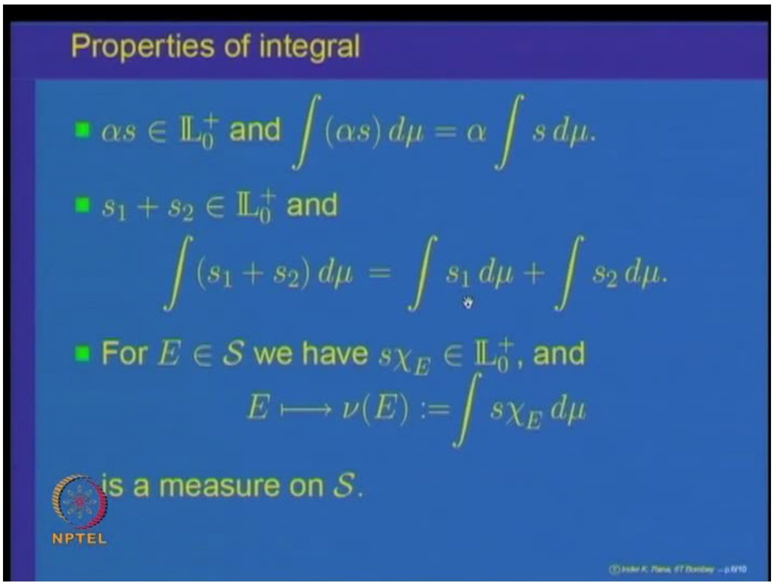

So, we have proved this property. Also, that for a set E in S, the integral s times indicator function of E is a nonnegative simple measurable function and if its integral is denoted by nu of E, then nu of E is a measure as E varies over measurable sets.

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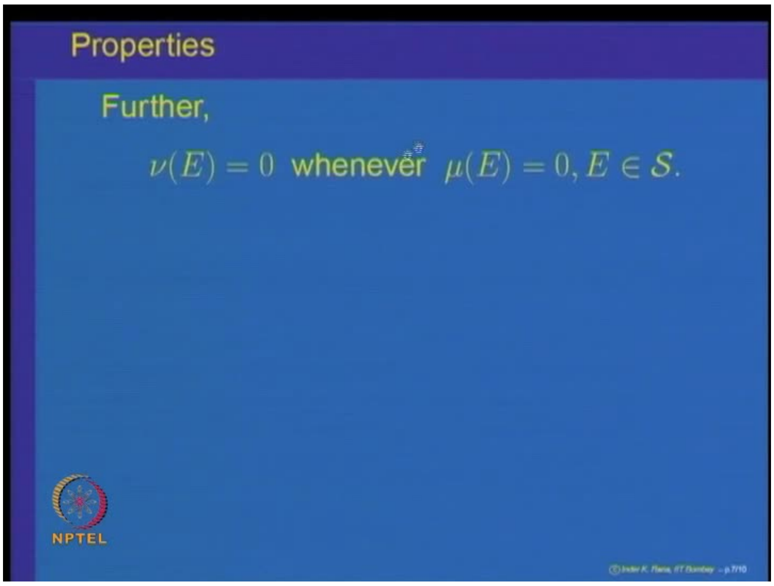

And this measure has a very nice property. So, this nu, measure nu of E has a very nice property, that nu of E is 0, whenever mu of E is 0.

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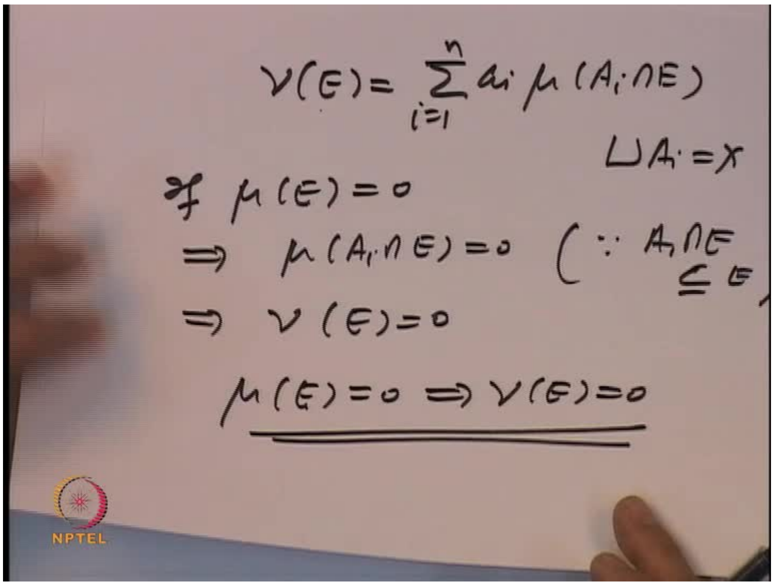

So, let us just check that property again, check that property, the nu of E is defined as summation i equal to 1 to n a i mu of A i intersection E, where union of A i's is equal to x. So, if mu of E is equal to 0, that will imply, that mu of each A i intersection E is also 0 because A i intersection E is a subset of E and mu is a measure; so, mu is also monotone; so, mu being monotone, mu of A i intersection E is less than or equal to mu of E, which is equal to 0, so that means, this is equal to 0.

So, implies, mu of each term in the definition of mu of E is 0, that means, nu of E is 0. So, this nu measure, which is defined via integration of nonnegative simple functions has the property, that mu of E equal to 0, implies nu of E equal to 0.

This is a very special property, so it relates 2 measures - mu and nu. That means, it says, whenever E is a set of measure 0 for mu, it is also a set of measure 0 for nu. And later on, almost in the end of the course, we will characterize such measures. Whenever 2 measures are related by this, there is a theorem, which says, that mu must be representable as integral with respect to mu. So, we will come to that theorem a bit late in our course when we are finished integration and some more properties of it.

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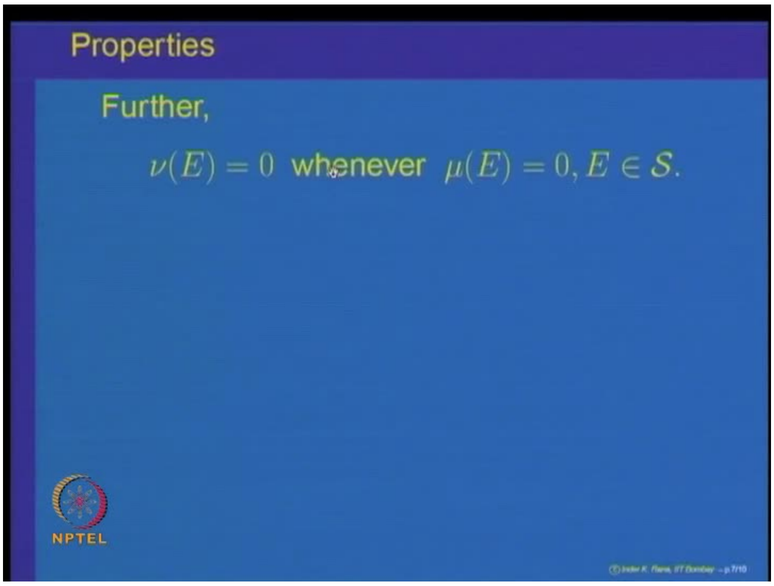

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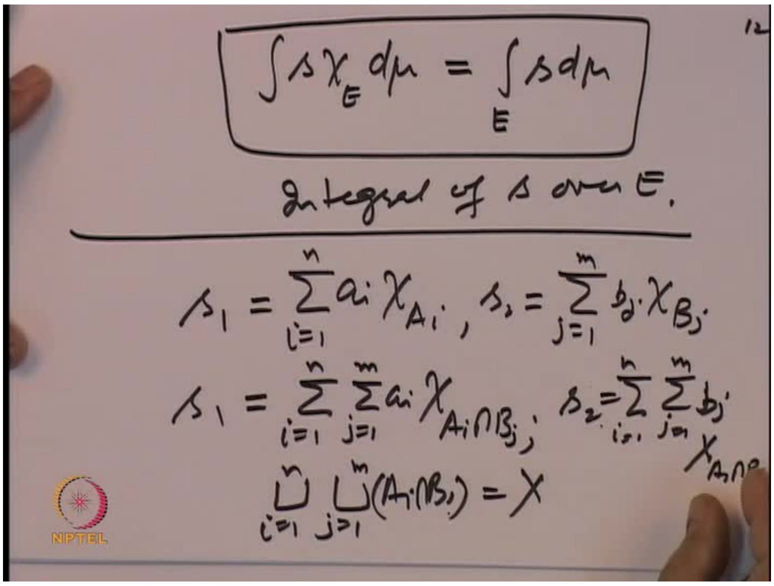

So, this nu of E, which is written as, which is the integral, is having a special property, and let us also mention, that integral of s indicator function of E d mu is also written as integral E of s d mu. So, this is another way of writing, so this is called integral of s over E; so, this, so we say, this is integral of s over the set E. So, that is the notation we will follow because outside E, s is 0 in this representation.

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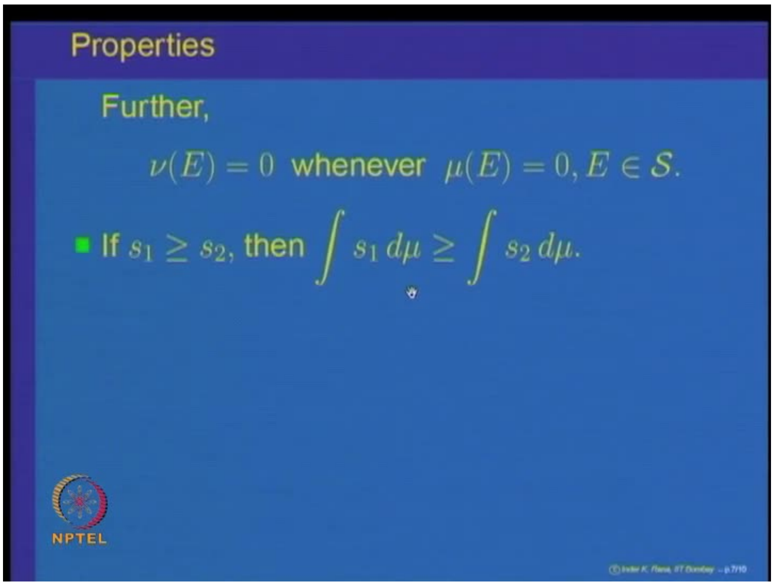

So, next property we want to check is that if s 1 is bigger than s 2, then integral s 1 is bigger than integral s 2. So, let us check that property. So, let us write s 1, which is nonnegative simple measurable function as sigma a i indicator function of A i and s 2 as sigma b j chi of B j j equal to 1 to m.

So, as we had mentioned, whenever you want to do some analysis regarding 2 simple functions - s 1 and s 2, bring them to a common partition. So, we will write, this is also equal to sigma over i 1 to n s 1 can be written as sigma over i sigma over j 1 to m a i times indicator function of A i intersection B j and s 1 can be written as this, and we can write s 2 as sigma over i sigma over j 1 to m of b j times the indicator function of A i intersection B j.

So, now and union of A i B j intersection B j j equal to 1 to m union i equal to 1 to n is a partition of X. So, now, they have common partitions and when you say s 1 is bigger than s 2, means what?

So, let us take a point x. So, if x belongs to X, then it belongs to 1 of A i intersection B j, and s 1 has the value a i and s 2 has the value b j. That means, s 1 of x, which is a i must be bigger than or equal to s 2 of x, which is b j on A i intersection B j.

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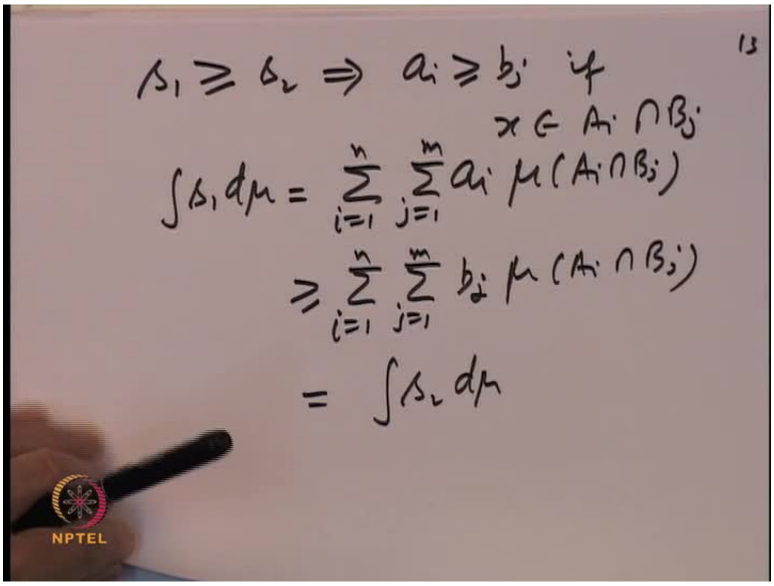

So, that means, if this is the representation, so then, s 1 bigger than s 2 implies, that a i is bigger than or equal to b j, if x belongs to A i intersection B j. Once you observe that, now problem is solved, so what is integral of s 1 d mu? That by definition is sigma over i 1 to n sigma over j 1 to m of a i mu of A i intersection B j and a i is bigger than b j if x belongs to this. So, this is bigger than or equal to sigma i equal to 1 to n sigma j equal to 1 to m of b j mu of A i intersection B j and which is equal to integral of s 2 d mu.

So, integral of s 1 is bigger than integral of s 2 if s 1 is bigger than or equal to s 2. So, that proves the next property.

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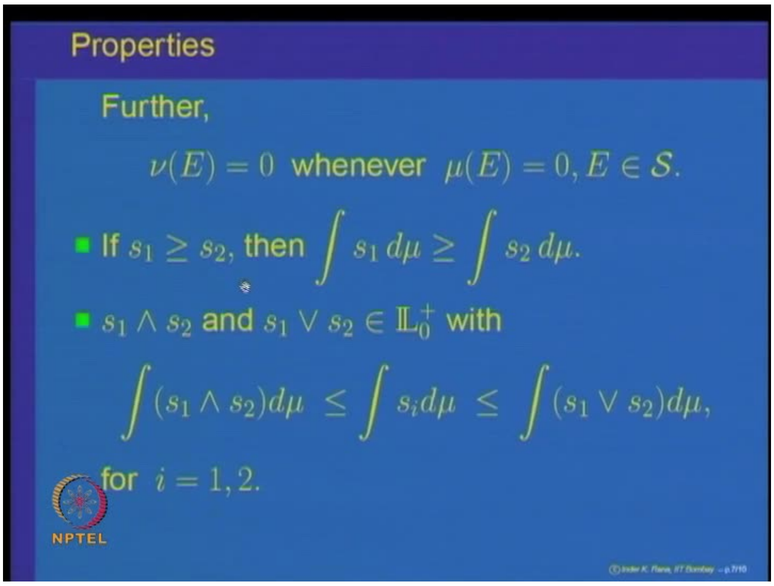

Now, we are going to look at special functions. If s 1 and s 2 are nonnegative simple measurable functions, then we want to look at s 1 v s 2 and s 1 wedge s 2

Recall how as s 1 v s 2 defined. s 1 v s 2 was defined as the maximum of s 1 and s 2 and similarly, s 1 wedge s 2 was defined as the minimum of s 1 and s 2. So, and we had shown, that if s 1 and s 2 are nonnegative simple measurable functions, then the maximum of s 1 and s 2 and the minimum of s 1 and s 2 are also nonnegative simple measurable functions.

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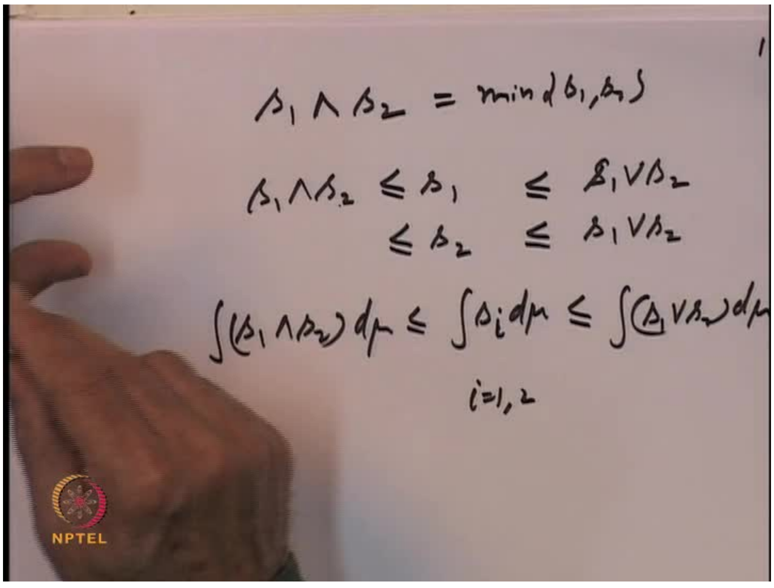

So, we want to check this property. Now, here, that integral of s 1 wedge s 2 is less than s i integral less than equal to integral of the next one, but that is obvious, because if s 1 and s 2 are nonnegative simple measurable functions and you look at s 1 wedge s 2, that is the minimum of s 1 and s 2. Then, clearly, s 1 wedge s 2 is a minimum, so it is going to be less than or equal to s 1 and also going to be less than or equal to s 2, and s 1 v s 2, s 1 v s 2, the maximum is going to be bigger than s 1 and s 2 both; so, it is going to be less than or equal to s 1 maximum s 2.

So, what we are saying is, s 1 wedge s 2 is less than or equal to both s 1 and s 2 and both s 1 and s 2 are less than or equal to maximum of s 1 and s 2, and just now and all are simple functions, so what we have proved just now? So, that we will say, that the integral of s 1 s 2 the minimum of s 1 and s 2 d mu is less than integral of s 1, also less than integral of s 2. So, less than or equal to integral s i d mu i equal to 1 and 2 and both these integrals are less than or equal to integral of s 1 wedge s 2 d mu.

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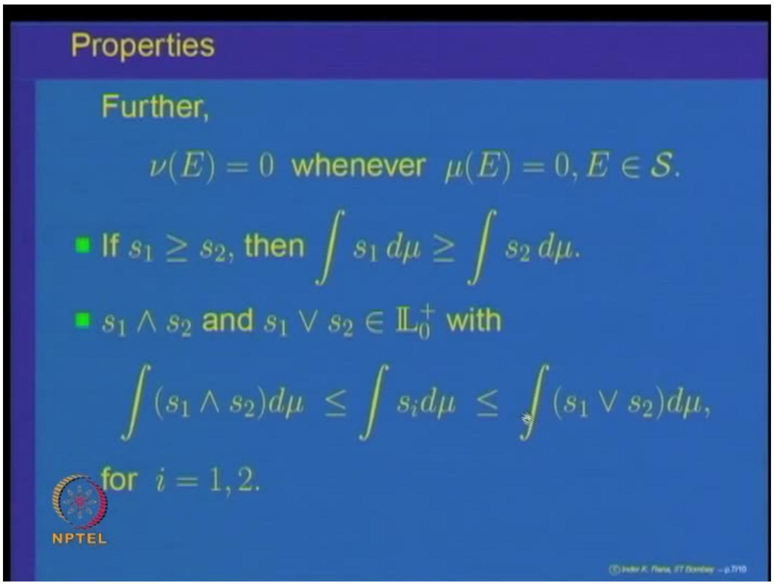

So, that proves the required property and that follows from the earlier property on, that if s 1 is less than or equal to bigger than or equal to s 2, then integral s 1 is bigger than or equal to integral s 2.

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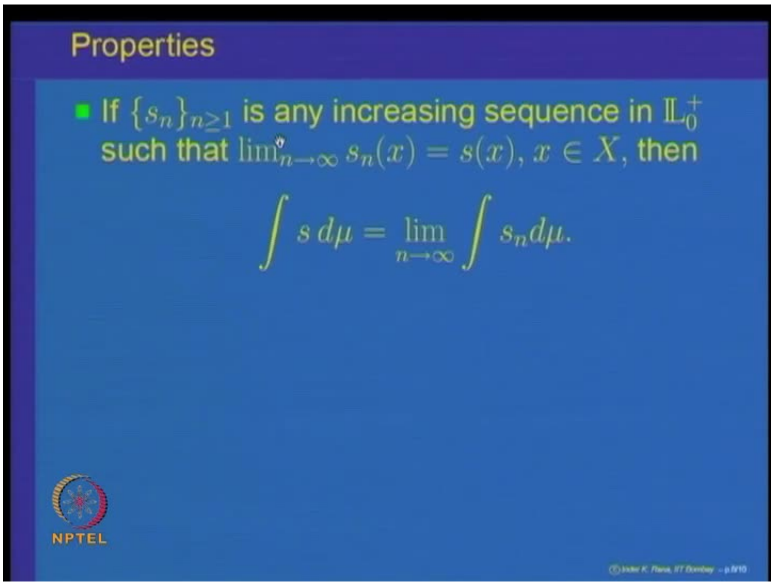

And now, let us look at a property how does this integral behave with respect to limiting operations. So, we want to claim, that if S n is a sequence, increasing sequence in L 0 plus, so it is an increasing sequence of nonnegative measurable functions increasing to a simple function s of x, then integral s d mu is limit n going to infinity integral s n d mu.

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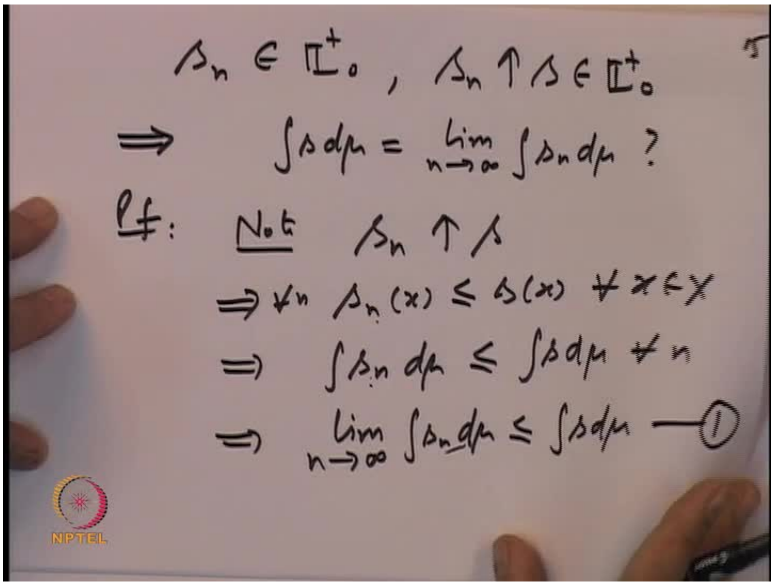

So, this is the 1st, in the sense, non-trivial argument required. So, s 1, s n are functions in L 0 plus nonnegative simple measurable functions. s n is increasing to s, s belonging to L plus 0 nonnegative simple measurable. We want to show, this implies, that integral s d mu is equal to limit n going to infinity of integral s n d mu; so, this is what we want to show.

So, now, let us start observing. So, first, what is the proof of this? So, note, what we have given is s n is increasing to s, so that means what? If s n is increasing to s, that means, that s n of x is going to be less than s of x for every x belonging to X, so that is obvious from this, if this, then this implies s n is increasing to s, implies each s n x is less than or equal to s of x.

Now, s n is a simple function, s is a simple nonnegative simple measurable function, s n is less than or equal to this for every n, so that implies, that integral of s n d mu is less than or equal to integral s d mu for every n. So, integral s n d mu is less than or equal to integral s d mu and integral s n d mu is an increasing sequence of extended nonnegative extended real numbers, so implies, that the limit of that, which are this, may be equal to plus infinity, s n d mu is also less than or equal to integral s d mu.

So, here is, that n is a sequence of nonnegative extended real numbers, n less than or equal to a implies a n's are increasing, so limit of a n will be less than or equal to a, wherein extended real numbers, keep in mind.

So, we have proved, so call it as 1, so we have proved, in the required equality we have proved, that right hand side limit n going to infinity integral s n d mu is bigger than or equal to integral s d mu.

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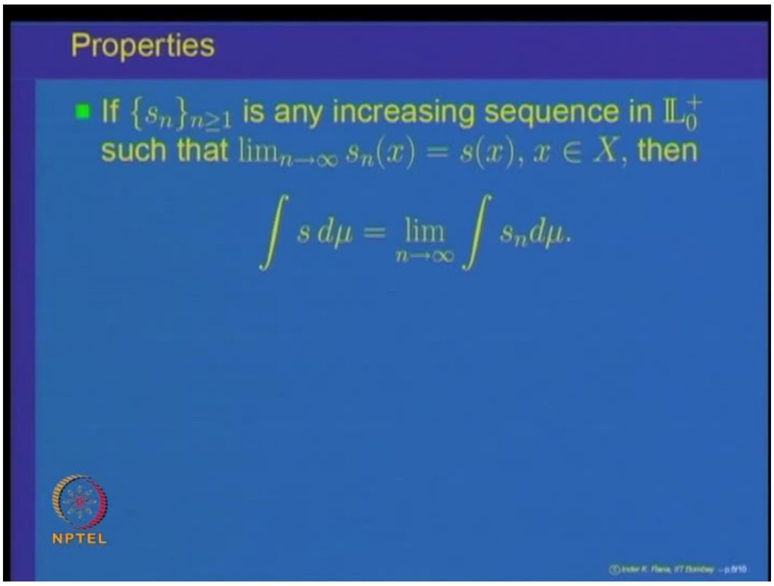

We want to prove the other way round, the inequality also. So, to do that, here is a, here is a small manipulation, that we have to do. So, for that, what we do is the following.

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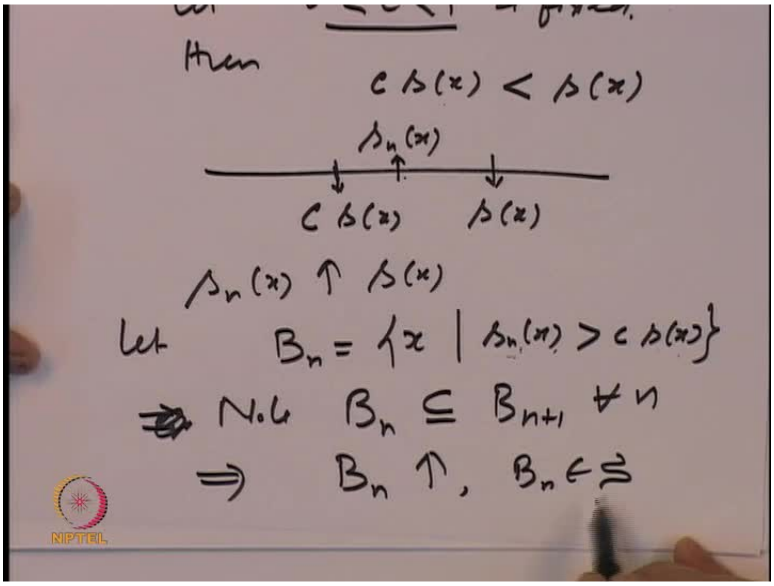

So, let us fix, let a number c between 0 and 1 be fixed, then c times s of x for any point x, c times s of x is going to be strictly less than s of x. so, here is c times s of x and here is s of x, so let us fix c between 0 and 1 and look at the end, for any point x.

Let us look at c times s of x, then the 1st observation, because c is between 0 and 1, c is strictly less than 1, so c times s of x will be less than s of x, so it will be somewhere here.

And now, s n x is increasing to s of x, s n x, so after some stage s n x must be on the right side of c times s of x, so after some stage it must be on the right side, so this is the picture that happen. So, let us write, let us define B n to be the set of all x, such that s n of x is bigger than c times s of x.

So, collect all those points where this is going to happen, where s n of x is bigger than, see, this stage will depend upon n, so now, so that means, implies, that first of all, so let us note, that B n plus, if s n x is bigger than c of s n, then s n plus 1 is anyway bigger than s n of x.

So, because s n is increasing, so s n plus 1 x is going to be bigger than B n, so if, so that means, this B n is inside B n plus 1 for every n, that means, that is, B n is an increasing sequence. So, implies, B n is an increasing sequence, so that is the first observation because all B n's, s n is increasing. So, if x belongs to B n, then s n x is bigger than c times s of x, but s n, s n is increasing, so s n plus 1 x is going to be bigger than s n of x. So, if s n x is bigger than c times s of x, then s n plus 1 also is going to be bigger.

So, x belonging to B n implies x belongs to B n plus 1, that means, B n is a subset of B n plus 1, that means B n is an increasing sequence of sets. And also observe, that each B n is an element in the sigma algebra s, each B n is an element in the sigma algebra s because B n is, where s n is bigger than c times s. All are simple measurable functions and we observe that such sets are in the sigma algebra.

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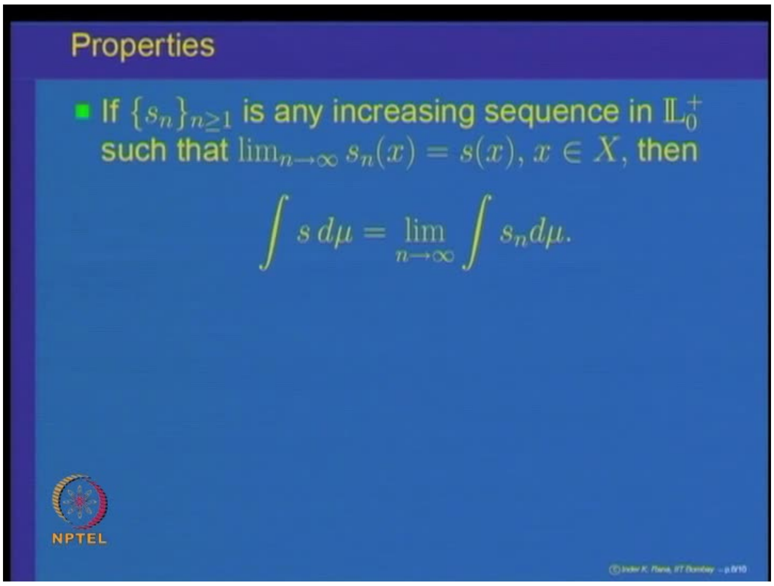

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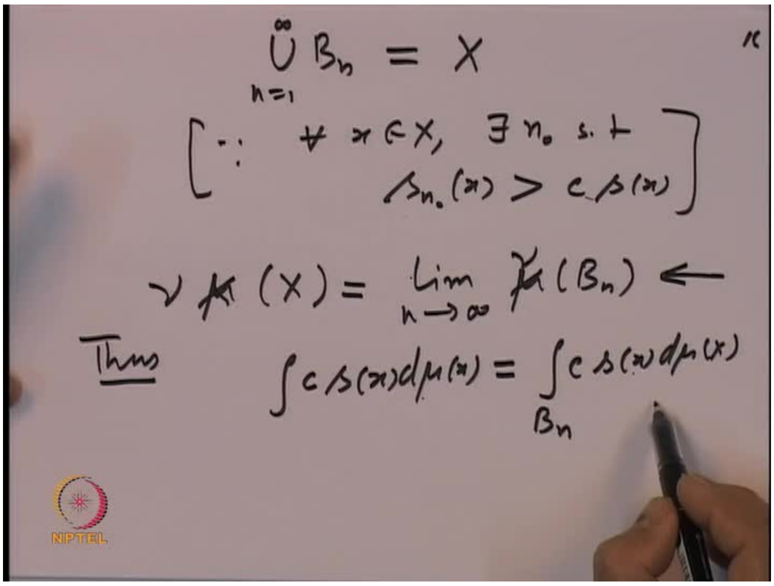

So, B n is an increasing sequence of sets in the sigma algebra s. And let us observe, what is the union of these B n's? So, union of B n's n equal to 1 to infinity, obviously it is contained in x because all are subsets of x, but by the fact, that for every x, this picture that we observed here, for every x there is going to be some stage after which s n is going to be bigger than x because c times s x is strictly less than this.

So, that fact implies, that this union is equal to x because, so observation here is, because for every x belonging to x, there is a stage n naught, such that s n naught x is bigger than c times s of x. That is because s n x is converging to s of x, because s n x is going to increase to s of x, so it has to crossover the, this point c times s of x, otherwise it cannot reach that point.

So, B n is an increasing sequence of sets in the sigma algebra and their union is equal to x, and mu is a measure, countable additive, and we have proved equivalent way of saying, that mu countably additive, is same as saying whenever a sequence of sets a n is increasing, then mu of a n's must increase to mu of a. So, by that fact, mu of x must be equal to limit n going to infinity mu of B n's, so that must be true.

So, now, let us use all these facts and look at, now, so thus, if we look at integral of c times, integral of c times s of x d mu x, you look at this integral. So, first of all we claim, that this is equal to integral of c times s of x d mu x over B n's.

So, 1st observation we want to make and that is, because if we look at this as a measure, if we look at this as a measure, mu of B n, just now we proved integral over sets of simple function over sets is s measure. So, look at that measure nu, nu and B n is increasing into x, so nu B n must go to nu of x.

So, this fact, we are using for this is the fact, we are using for not mu, but we are using for nu and where, what is nu? nu is integral of c s over B n.

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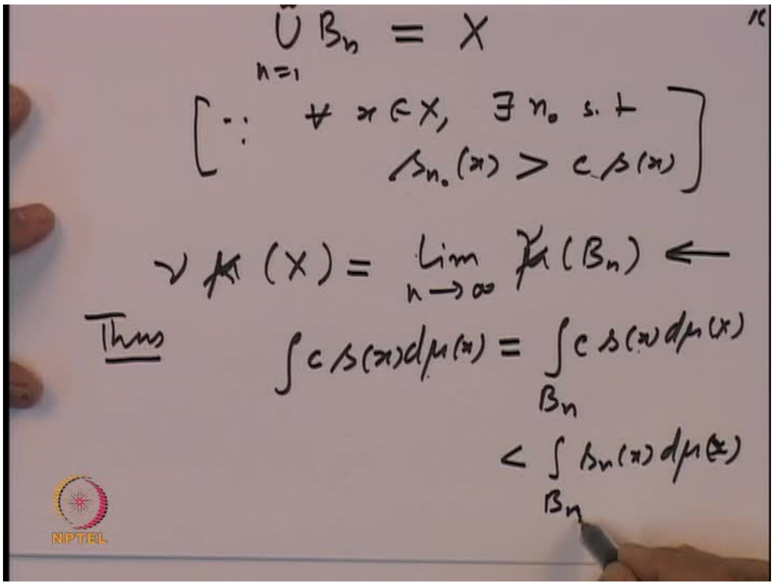

So, that is the fact we are using here. So, that means, this is equal to, so, now on B n, what is happening on the set B n? On B n, s n of x is bigger than c times, so that means, c times s of x is less than, so it is less than integral over B n of s n of x d mu x because that is a definition of the set B n.

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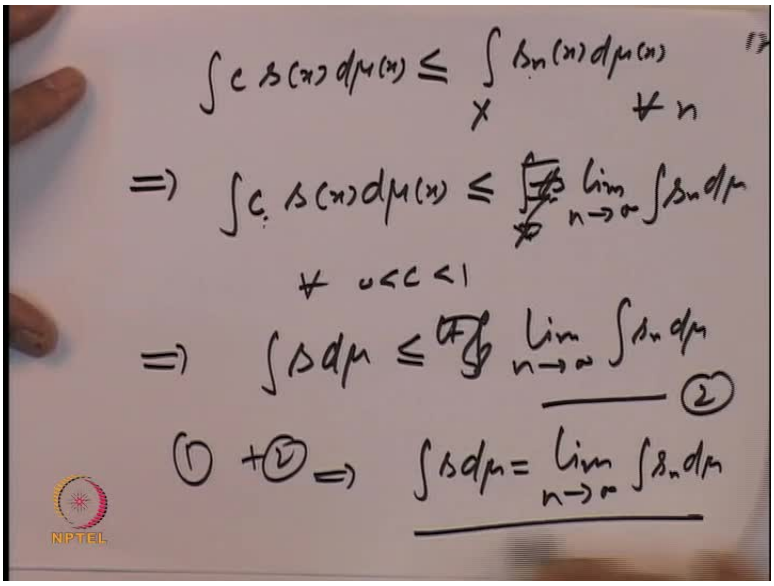

So, we are replace c s x, we are using the fact integral of s 1 is less than integral of s 2, whenever s 1 is less than or equal to s 2, so this is less than or equal to this. Now, B n is a set, subset of x, so this integral I can replace and say, that this is less than, so this is less than or equal to integral over the whole space x s n x d mu x.

So, what we are saying is, by this analysis what we have shown is that the integral c times s of x d mu x is less than or equal to this for every n, and because this happens for every n and s n, this integrals are increasing sequence of numbers. So, this implies, that integral c times s of x d mu x is also less than or equal to integral over x of s limit, so is less than or equal to limit n going to infinity of integral s n d mu.

Now, this holds for every c between 0 and 1, so I can take the limit c goes to 1. So, implies, that integral of s d mu is also less than or equal to integral limit, less than or equal to limit n going to infinity of integral s n d mu. So, that is my other way round inequality 2. So, we approved both ways, inequalities, 1 and 2.

So, 1 if we recall, we are already shown 1, that integral s n d mu less than integral s d mu, that was 1 we proved, and now, we proved integral s d mu. So, 1 plus 2 imply, that integral s d mu is equal to limit n going to infinity integral s n d mu.

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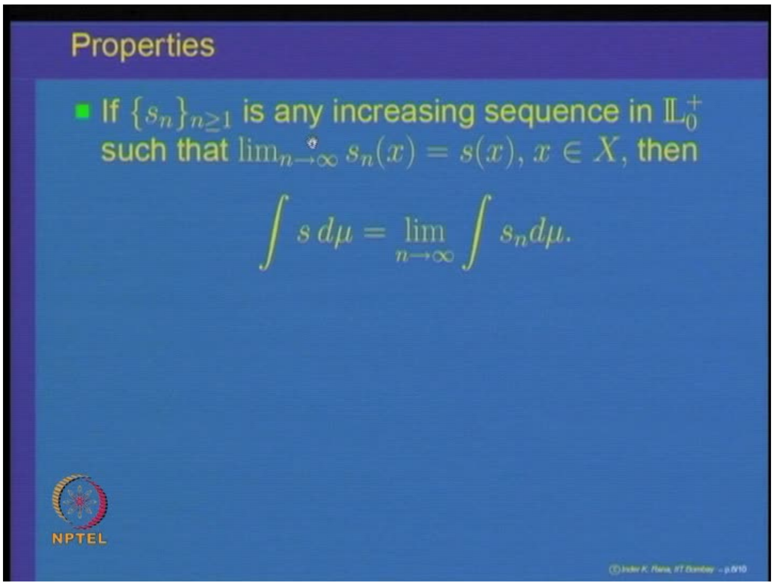

So, that proves the result, the required result namely, that integral of s n, if s n is increasing sequence in L plus, then you can interchange, so what is s? That is a limit. So, integral of the limit is equal to limit of the integrals, whenever s n is increasing nonnegative simple functions.

So, is a nice property for increasing sequences, so at this stage, one can ask the question that we have proved, that if s n is an increasing sequence nonnegative simple measurable functions increasing to s, then integral of s n's converge to integral of s. Will this property hold for decreasing sequences, namely if s n is decreasing nonnegative simple functions, decreasing to s? Can we say, that integral of s n's will decrease to integral s? We do not know that fact, at present we cannot prove, at present, this fact. In fact, many more properties of such things we will explore as we extend the notion of integral.

So, we will stop here today and analyze next time another way of representing integral of nonnegative simple measurable functions and then, go out to define integral of nonnegative measurable functions. We will extend the notion of integral form nonnegative simple measurable functions to nonnegative measurable functions, we will do it next lecture.

Thank you