Measure and Integration

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Module No. # 05

Lecture No. # 16

Measurable Functions on Measure Spaces

Welcome to lecture number 16 on measure and integration. If you recall, in the previous lecture we had started looking at the notion of measurable functions on measure spaces. We will continue that study of measurable functions and their properties and then we will specialize on measurable functions on measure spaces. Then, we will look at the space of measurable functions when x is real line and the sigma algebras ((.)) Borel sigma algebra or Lebesgue sigma algebra. If there is time, we will start looking at the integration of nonnegative simple measurable functions. Let us recall what we had been doing.

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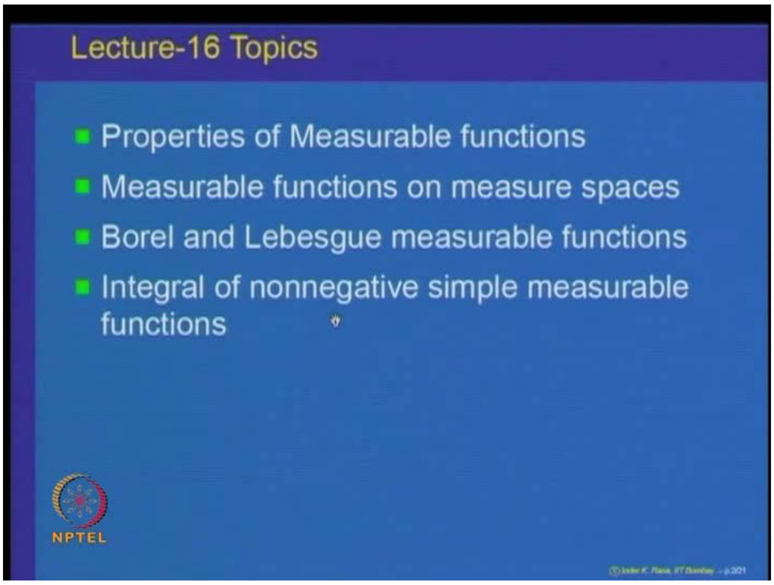

We had been looking at properties of measurable functions; that is what we will be continuing doing; then, we will look at measurable functions on measure spaces and look at Borel and Lebesgue measurable functions and then look at integral of nonnegative simple measurable functions.

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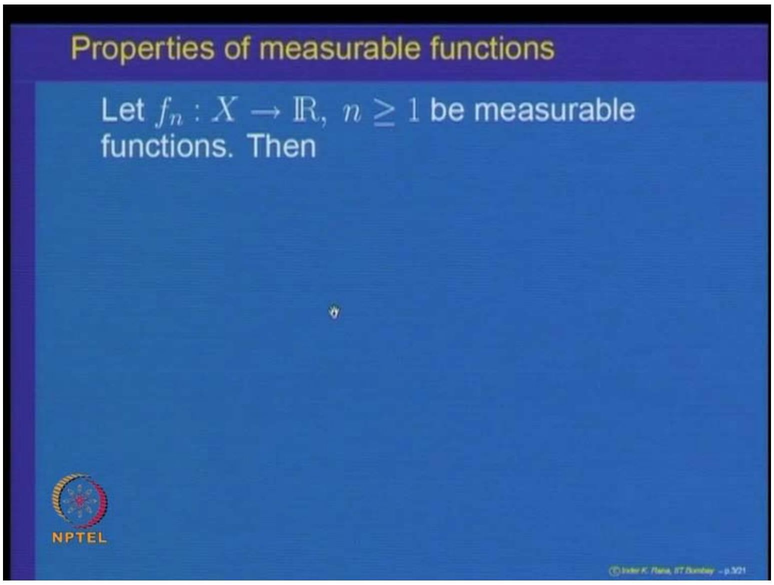

Let us just recall what we had done regarding measurable functions.

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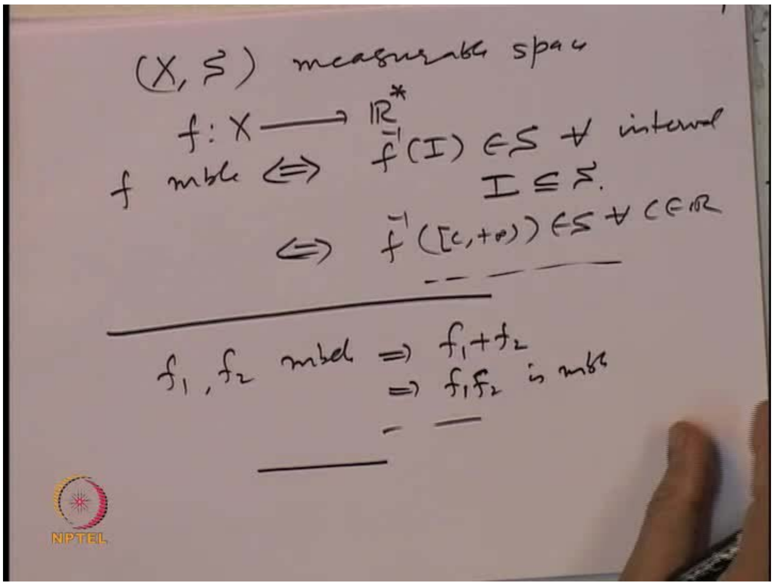

We had said that if X, S is a measurable space and f is a function from X to R star, then saying that f is measurable is the same as saying the inverse image of every interval I belongs to S for every interval I contained in S. There were equivalent ways of defining measurability in terms of special intervals like this. This is same as if and only if f inverse of intervals of the type C to infinity belong to S for every c belonging to R and so on. Then, we looked at what is called the algebra of measurable functions. We proved that if f_1 and f_2 are measurable, then it implies f_1 plus f_2 is measurable; it implies f_1 into f_2 is measurable and so on.

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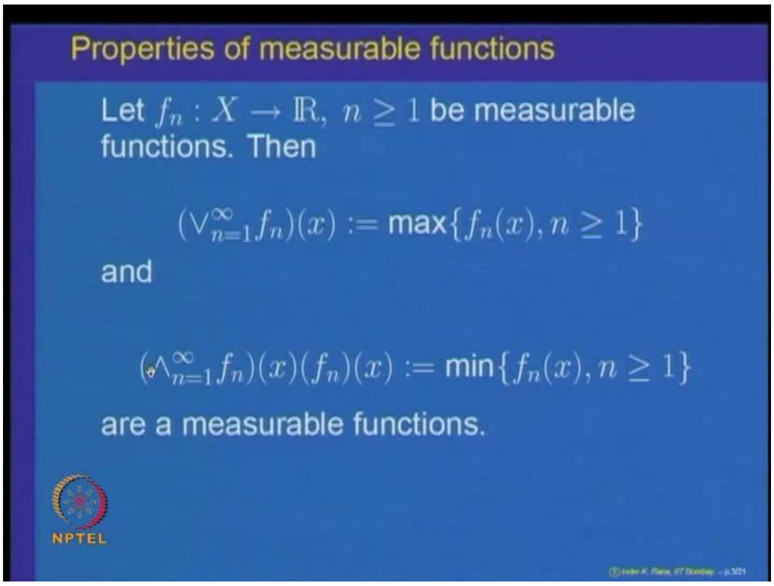

Today, we look at the properties of sequences of measurable functions. We want to prove the following. Look at a sequence f_n of measurable functions; then, look at the function what is called the maximum of f_ns. This is a function denoted by V n equal to 1 to infinity f_n of x is defined as the maximum of f_n of x, n bigger than or equal to 1; this is called the maximum of the sequence f_n. Similarly, we have the notion of minimum of f_ns which is denoted by wedge 1 to infinity f_n of x equal to minimum; this is extra (Refer Slide Time: 03:31); that is the definition. The claim is that if f_n is a sequence of measurable functions, then the maximum and the minimum are also measurable functions. Let us prove this.

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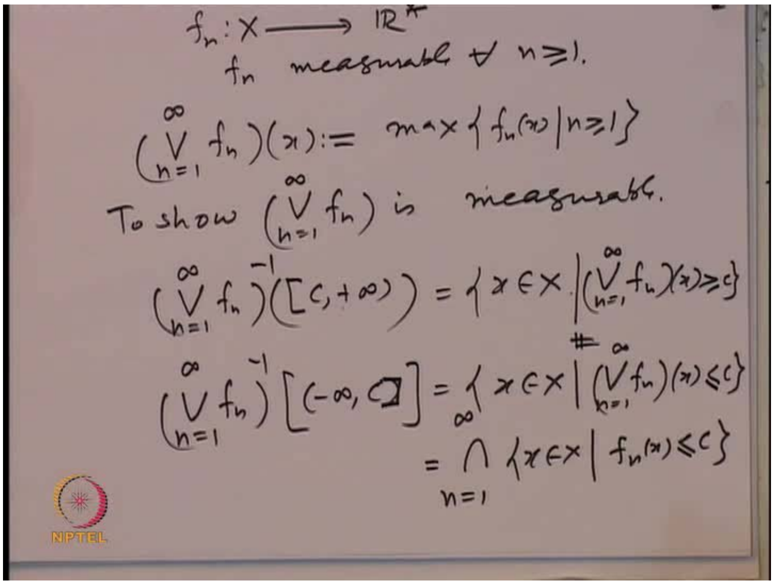

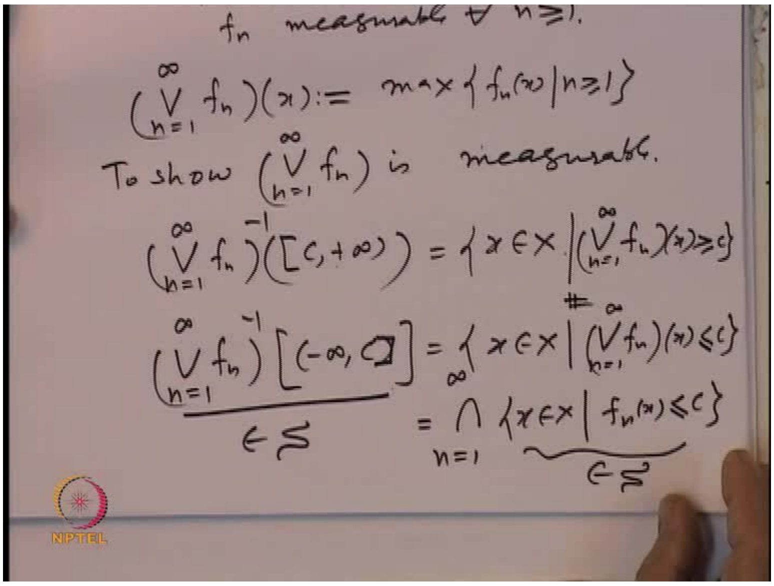

f_n is defined on X to R star and f_n is measurable for every n bigger than or equal to 1. We defined maximum n equal to 1 to infinity of f_n of x to be equal to maximum of f_n of x, n bigger than or equal to 1; this is the definition of this maximum. We want to prove or to show that this function V n equal to 1 to infinity f_n is measurable. To prove that, we can use any one of those conditions which we had defined earlier for measurability.

Let us look at maximum n equal to 1 to infinity f_n of inverse of the interval; let us look at say c to infinity. That means what? That means this is all x belonging to X such that n equal to 1 to infinity f_n of x is bigger than or equal to c. To prove this, we have to convert some of this relation into individual f_ns because each individual f_n is measurable.

((.)) saying that maximum is bigger than or equal to c; that means at least one of them crosses over c; that is way of doing it, but let us look at the equivalent criteria; let us look at the sets f_n inverse the maximum is less than or equal to c; that is equal to minus infinity to c; let us look at that.

This is same as x belonging to X such that the maximum value f_n of x is less than c. Now, if maximum of something is less than c, then each one of them has to be less than c; that is the reason instead of using this kind of intervals (Refer Slide Time: 06:20), it is more convenient for the maximum to use this kind of intervals because then this set can be written as intersection n equal to 1 to infinity of x belonging to X such that f_n of x is less than c. This may not be exactly true; so, let us take this is closed (Refer Slide Time: 06:47) because maximum is less than or equal to c; then every one of them will be less than or equal to c; that is okay.

We look at the intervals of the type minus infinity to c closed and that is intersection of these sets and each one of this sets belongs to the sigma algebra S because each f_n is given to be measurable. So, it is an intersection of elements in the sigma algebra; this set also belongs to the sigma algebra S (Refer Slide Time: 07:19). That means this proves the fact that the maximum of f_ns is a measurable function. A similar proof will work for the minimum.

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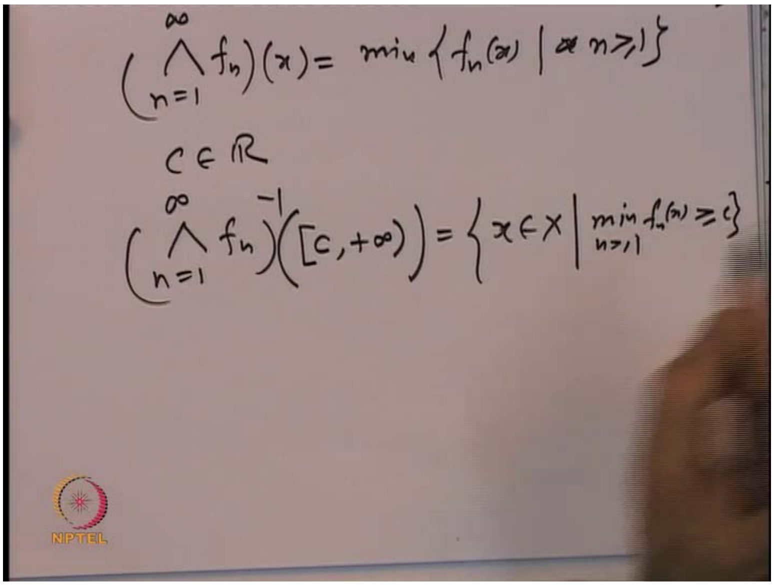

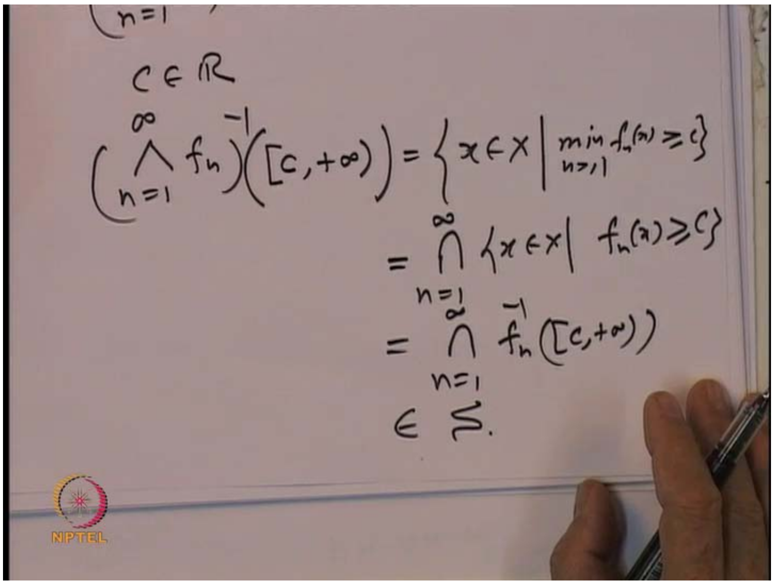

Let us look at the wedge n equal to 1 to infinity f_n. That is defined as the minimum of f_n of x for n bigger than or equal to 1. The claim is that this is a measurable function. Once again for any c belonging to R, let us look at the minimum of f_n, n equal to 1 to infinity inverse of some type of interval we want to show it belongs to S. Let us try looking at the minimum; minimum of this is bigger than c; let us try this. This is all x belonging to X such that the minimum of f_n of x, n bigger than or equal to 1 is bigger than or equal to c.

If the minimum of some certain numbers is bigger than or equal to c, then each one of them has to be bigger than or equal to c because even if one is smaller, then the minimum will become smaller. This is equal to intersection of n equal to 1 to infinity of all x belonging to X such that f_n of x is bigger than or equal to c. That implies this is nothing but intersection n equal to 1 to infinity of f_n inverse of C to plus infinity and each f_n being measurable, this is the measurable set; it implies that this is a set in S.

We have proved that if f_n is a sequence of measurable functions, then we look at the maximum or the minimum of this sequence of measurable functions and both of them are again measurable functions. As a consequence of this, we prove that the limit of a measurable function is also a measurable function; that is our next aim – to prove that the limit of measurable functions is also a measurable function.

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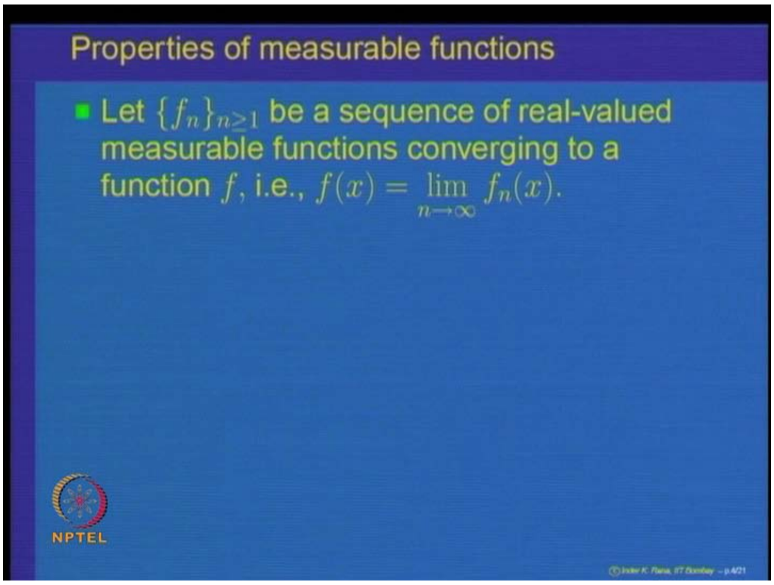

For that, let us understand what is the limit of a function.

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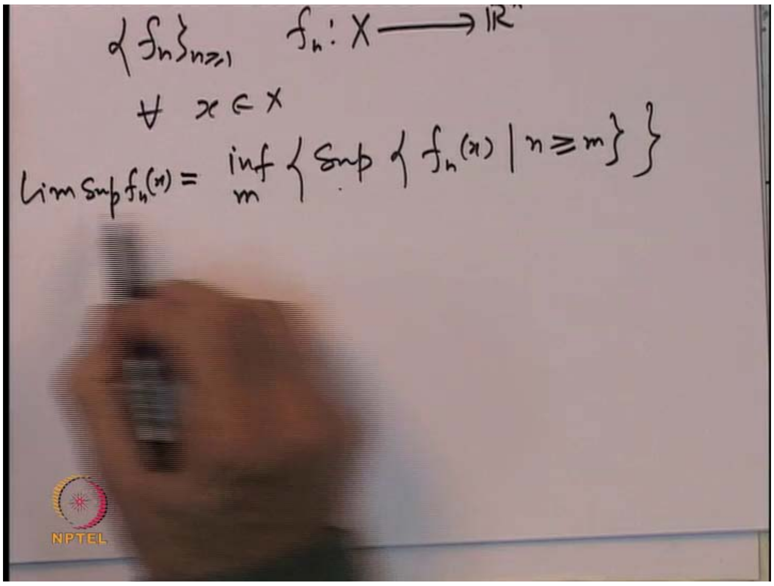

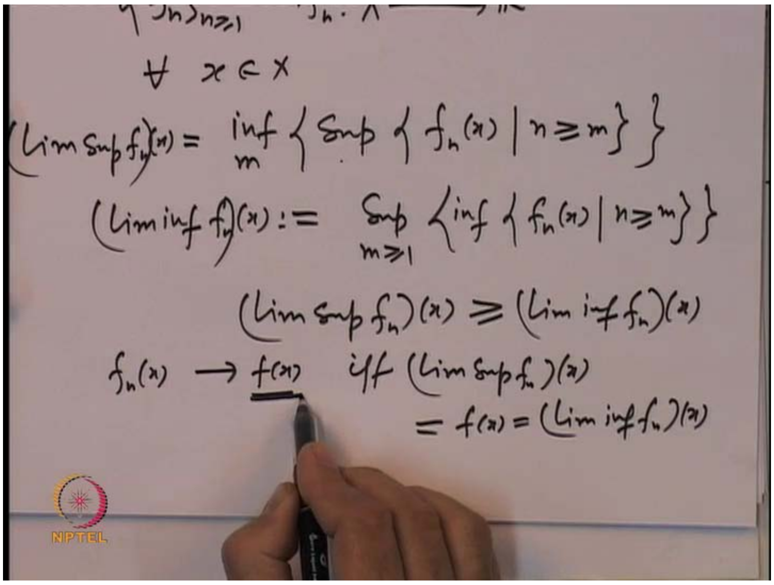

Let us look at a sequence f_n of functions; each f_n is defined from x to R star. To define the notion of the limit of f_n at a point x, what we do is for every x belonging to x, let us look at the maximum or the supremum of f_n of x for n greater than or equal to some stage m. This number, the supremum, depends on m and then we take the infimum over all ms; this is the supremum.

First take the supremums and then take the infimums; this gives you a function; this is called limit superior of f_n at the point x; this is called the limit superior of the sequence of functions f_n of x at the point x. Similarly, limit inferior of f_n of x is defined as follows: you first take the infimum of f_n of x for n greater than or equal to some stage m and then look at the supremum for all m bigger than or equal to 1; this is called the limit superior (Refer Slide Time: 11:35).

You must have seen in your elementary analysis classes that limit superior of f_n of x is always bigger than or equal to limit inferior of f_n of x; this inequality always holds. The sequence f_n of x converges to f of x if and only if limit superior f_n of x is equal to f of x is equal to limit inferior of f_n of x. These are elementary facts from basic analysis about when is a sequence of real numbers convergent; it says that for any sequence of real number or extended real numbers, you can define the concept of limit superior and also you can define the concept of limit inferior; limit superior is defined by looking at the supremum of the sequence A_n from some stage m onwards and then this supremum depends on m; so, look at the infimum of all these supremums; that is called the limit superior.

Similarly, limit inferior is defined as first taking the infimums of the sequence f_n of x from some stage m onwards and then looking at the supremums of these numbers which depend on m. One proves that the limit superior of a sequence is always bigger than or equal to limit inferior and the sequence is convergent if and only if the limit superior is equal to limit inferior. In case you have not come across these concepts, I strongly suggest that you pick up a book on elementary analysis and revise the concepts of limit superior and limit inferior; we are going to use that fact now here to prove that f is measurable. What is f? f of x is nothing but limit superior and limit inferior. The only thing to show is that the limit superior and limit inferior are both measurable functions.

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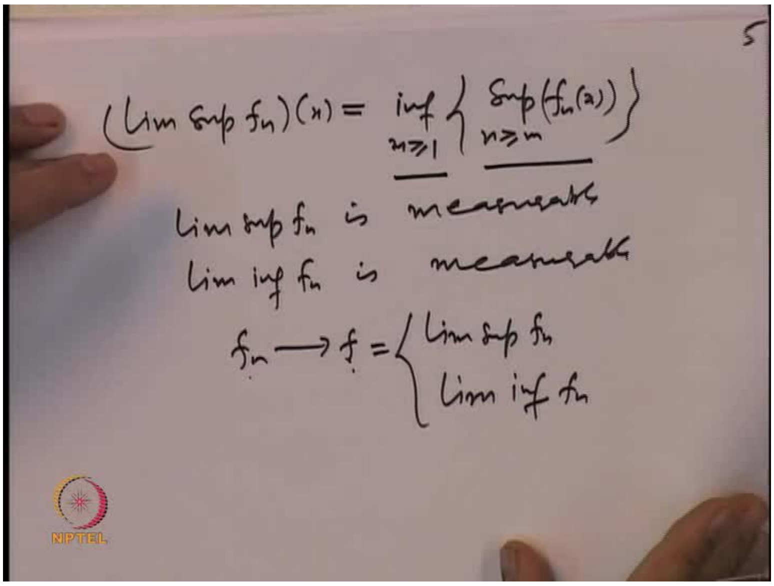

Limit superior of f_n is nothing but first taking the supremums of f_n of x from n bigger than or equal to n and then taking infimums m bigger than or equal to 1. Just now we have shown that if f_n is a sequence of functions, then the supremums, the maximums, are also measurable functions. This is a measurable function and the infimum of measurable functions is a measurable function (Refer Slide Time: 14:30). This implies that limit superior f_n is measurable. Similarly, limit inferior f_n is measurable and saying that f_n converges to f is the same as saying this f is equal to limit superior f_n or also equal to limit inferior of f_ns. That proves the fact that f_n converges to f for every point x and f_ns measurable implies f is a measurable function. So, limits of measurable functions are also measurable functions.

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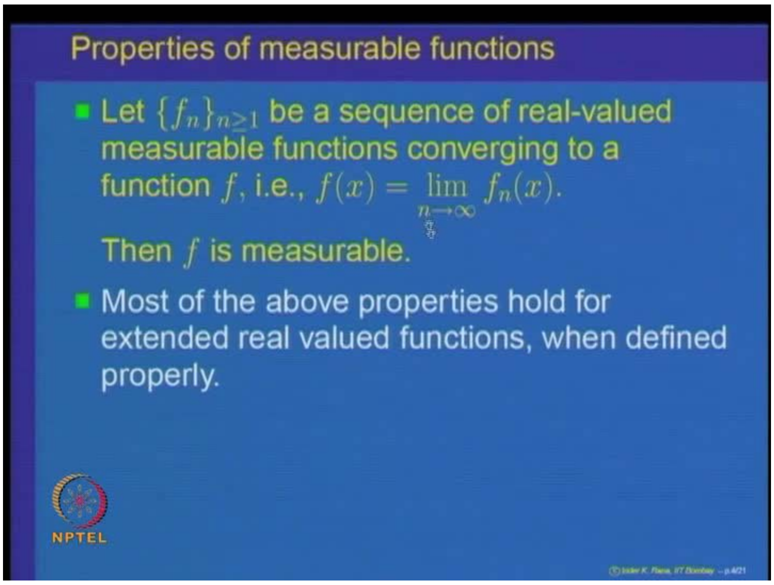

This proves the theorem that the class of all measurable functions is nice; it is closed under taking point-wise limits. Now let us observe that most of these properties hold for extended real valued functions also when properly defined.

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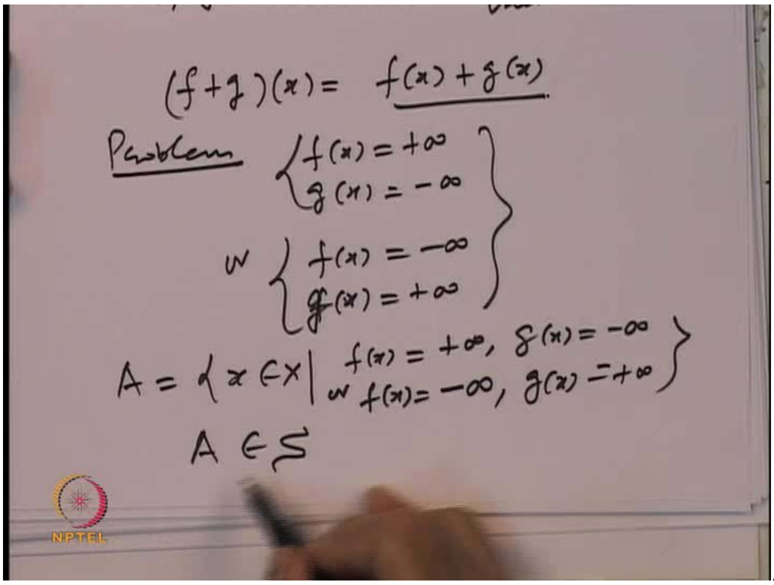

The only thing to observe is the following: if f and g are extended real-valued functions and you want to define f plus g, care has to be taken because you will like to define it as f of x plus g of x, but the problem comes if f of x is equal to plus infinity and g of x is equal to minus infinity. Then what will be this number (Refer Slide Time: 16:19)? That is not defined or f of x is equal to minus infinity and g of x is equal to plus infinity; even then, the problem comes; this number is not defined.

What one does is to ((.)) suitably define means look at all the points, call the set as A where all x belonging to X where either of these things happen – where f x equal to plus infinity, g x is equal to minus infinity or f of x equal to minus infinity and g of x equal to plus infinity. Now, one observes that this set A is in the sigma algebra because f of x is equal to plus infinity belongs to sigma algebra and intersection with that belongs to sigma algebra; so, this set A is in the sigma algebra. A is the set on which the problem can come.

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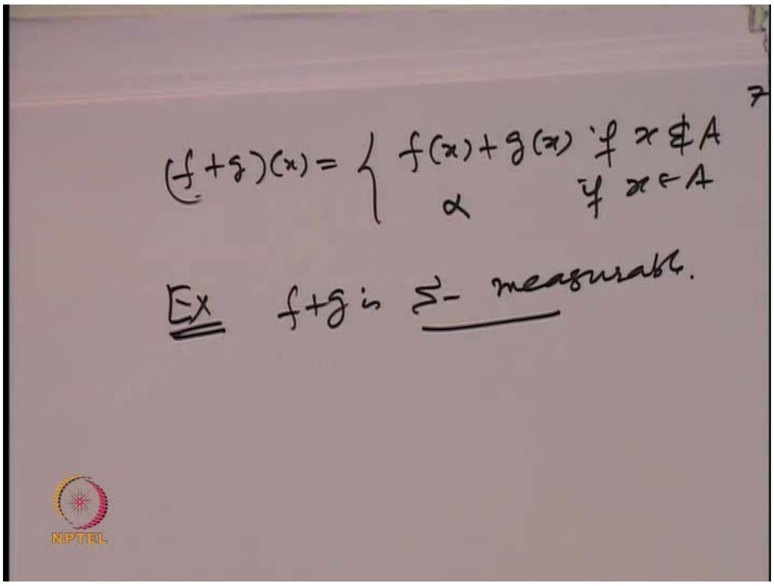

What one does is one defines f plus g of x to be equal to f of x plus g of x if x does not belong to A; you can define it as any number alpha if x belongs to A. With this definition, it easy to observe; let me leave it as exercise for you to show that if I define it this way with any value alpha on the set A, then f plus g is S measurable. That is what I mean by saying that the above results, most of these properties, hold for extended real-valued functions also when those functions are appropriately defined (Refer Slide Time: 18:03). We will not go much into the details of this; one can easily verify these things.

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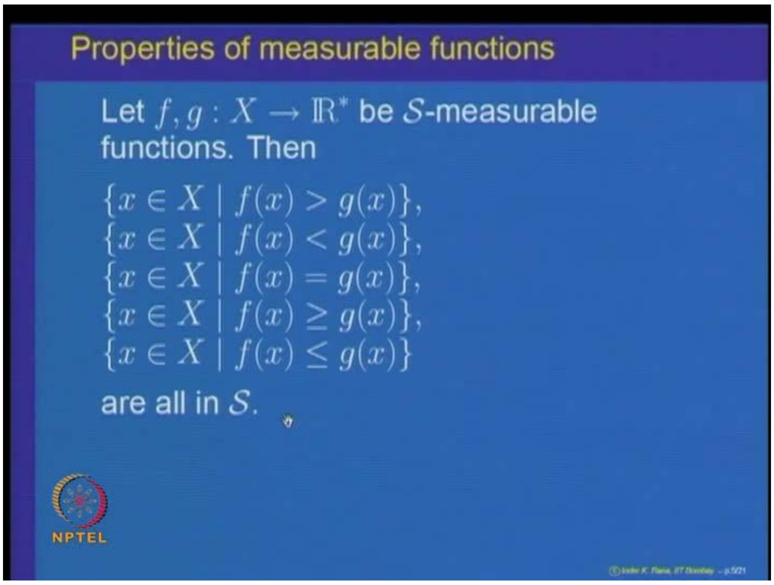

Another property of measurable functions is the following. Let us look at two functions f and g which are measurable. Then the following holds. Look at the sets x belonging to X such that where f of x is bigger than g of x or the set x belonging to X where f of x is strictly less than g of x or x belonging to X where f of x is equal to g of x and similarly, where f of x is bigger than or equal to or f of x is less than or equal to. Our claim is all these sets are in the sigma algebra S. Let us look at to proof of one of them and others will follow similarly.

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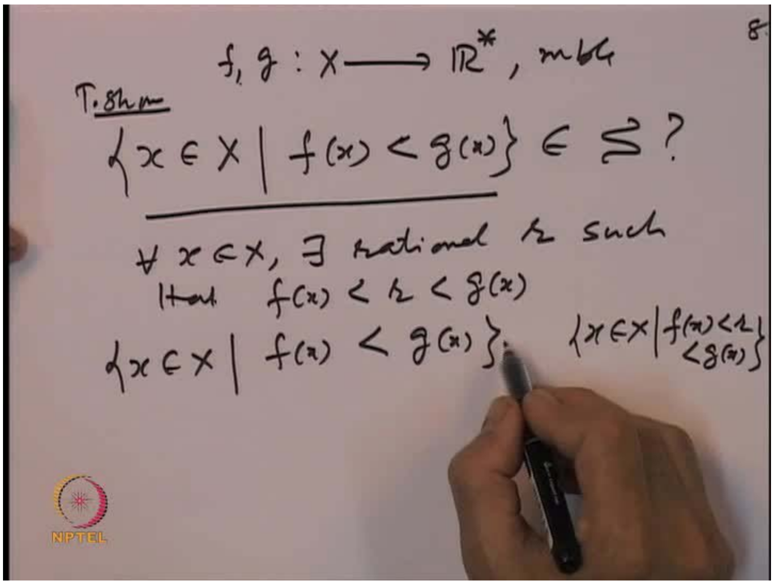

f and g are functions X to R star measurable. Let us look at the set x belonging to X such that f of x is strictly less than g of x. Our aim is to show that this belongs to the sigma algebra S. Since we are given f and g are both measurable, we are given the property that f of x less than or equal to some real number belongs to the sigma algebra and similarly g of x less than or equal to a real number belongs to the sigma algebra.

The objective is try to interpret this sets in terms of union intersections of something of sets of the type where f of x is less than something and g of x is less than something. For that, we observe that for any x if f of x is less than g of x, then there must be a rational number in between them; for every x belonging to X, there exists a rational r such that f of x is less than or is less than g of x. Here, we are using the fact that rationals are dense on the real line.

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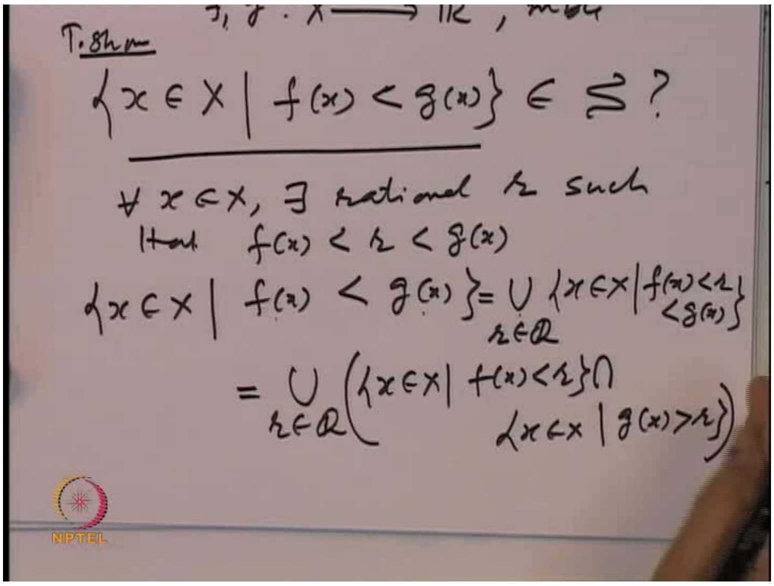

With this property, we can write x belonging to X such that f of x is less than g of x; this implies that x belonging to X such that for some rational, f of x is less than r is less than g x; conversely if for some r this is true, then obviously f of x is equal to g of x; so, the claim is this is equal to union over r belonging to rational numbers (Refer Slide Time: 21:11).

This is the only crucial point in this that the set f of x less than g of x can be written as a union over all rationals such that f of x strictly less than r strictly less than g of x. Now, observe this set is an intersection; so, I can write it as r belonging to Q. This set is where f x is less than r and g x is bigger than r (Refer Slide Time: 21:42). So, it is x belonging to X such that f of x less than r intersection with the set x belonging to X such that g of x is bigger than r.

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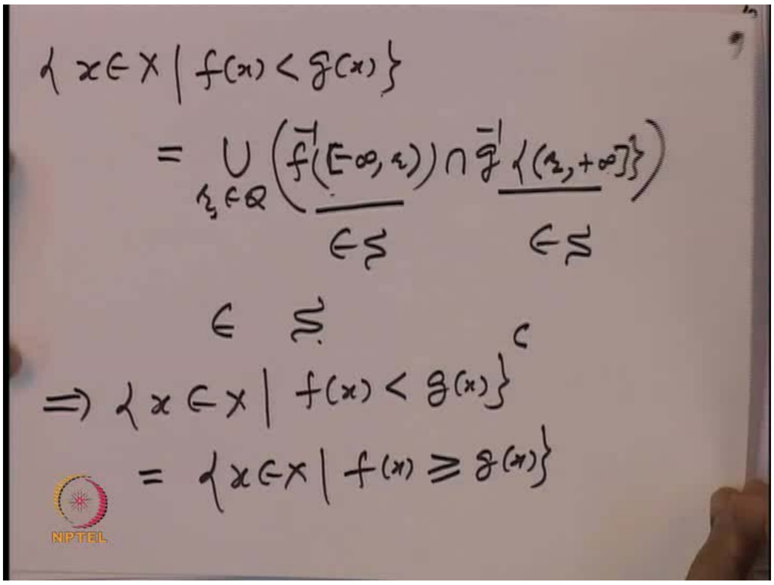

(Refer Slide Time: 22:03) What we have done is we have interpreted the set x belonging to X such that f of x less than g of x as… Let us just look at the set again. This is union over r (Refer Slide Time: 22:14). That is equal to union over r belonging to Q. What is this set? This is f inverse of f of x less than r; that is, less than r means it is minus infinity to r. The second set is nothing but g inverse of g of x bigger than r; so, it is r to plus infinity.

The set f of x less than g x is written as union over rationals intersections of these two sets (Refer Slide Time: 23:01). Now f and g being measurable, this set belongs to the sigma algebra; g being measurable, this set belongs to the sigma algebra; it is intersection and so the whole set belongs to the sigma algebra; intersection belongs to sigma algebra; rationals are countable and so this is the countable union of elements in the sigma algebra; so, this belongs to the sigma algebra.

What we have shown is the set x belonging to X such that f of x strictly less than g of x belongs to the sigma algebra S. That proves the first property of the theorem. Now if you take just the complement of this set, this also implies that x belonging to X such that f of x less than g of x the complement of this set what will be that? That is all x belonging to

X such that f of x is bigger than or equal to g of x; that set also belongs to the sigma algebra. (Refer Slide Time: 24:05)

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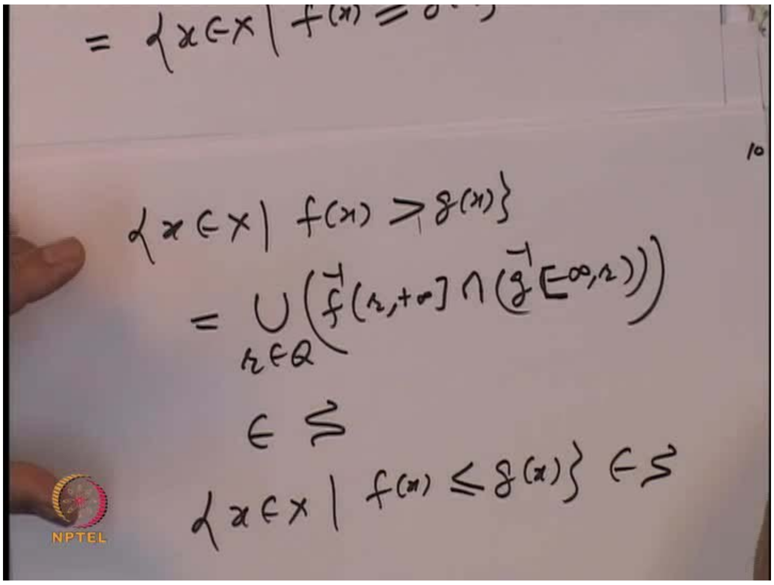

Similarly, we said that f of x less than g of x belongs to it. Let us write x belonging to X such that f x is strictly bigger than g x is also in the sigma algebra, because by similar arguments I can write this as the union over all rationals of f inverse of… f x bigger than and so that will be r to plus infinity intersection with g inverse of minus infinity to r.

By similar argument where we had f of x less than, again we can interpret f x is bigger than g of x and so there must be a rational in between; so, that must be true; that will imply that this belongs to the sigma algebra. Saying that f of x bigger than g of x belongs to the sigma algebra is okay. If you take the complement of this, that is nothing but x belonging to X such that f of x less than or equal to g x also belongs to the sigma algebra because this is the complement of the set in the sigma algebra. (Refer Slide Time: 25:28) Measurable sets have nice properties, namely if f and g are measureable, then operations involving measurable functions give you again sets in the sigma algebra. These are nice properties and we will see use of these properties soon.

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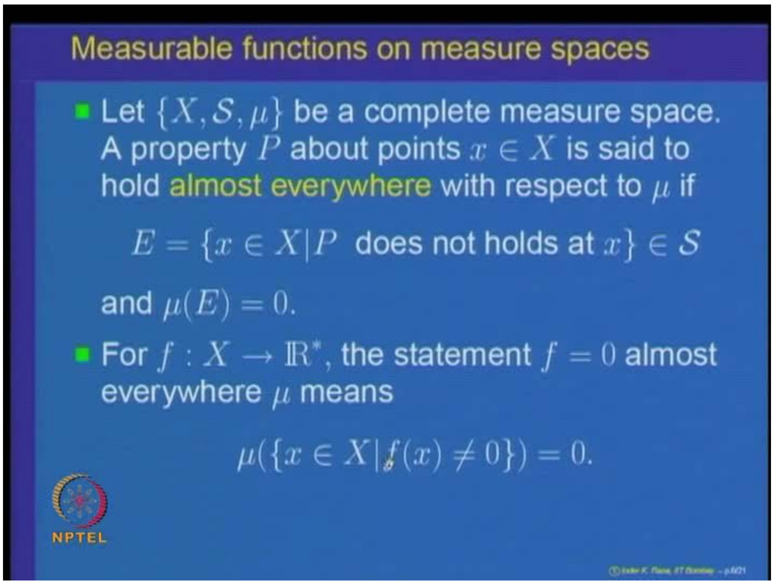

With this, we complete the study of measurable functions on measurable spaces. Next, we are going to look at measurable functions which are defined on measure spaces; they also a play a role later on. We want to look at functions f which are defined on the set X taking extended real values and on X there is a sigma algebra S and a measure mu given on S.

Let us first define what is the meaning of the notion of almost everywhere. We say that a property P about the points x is set to hold almost everywhere with respect to the measure mu if… Look at the set of points X for which the property P does not hold at x; look at all those points of x such that the property P does not hold at the point x. This is a subset; this subset belongs to the sigma algebra and mu of E is equal to 0. What we are saying is except for a set of measure 0, the property holds. That is why we give it a name that the property P holds almost everywhere with respect to the measure mu. Let me illustrate this with some examples.

Let us take a function f and we look at the statement that f is 0 almost everywhere. f is a function which is an extended real-valued function defined on the set X. We want to say that this function is 0 almost everywhere. Look at the set of points where f is not 0. What will that statement mean? That set where f is non-zero should be an element in the sigma algebra and its measure should be 0; so, x belonging to X such that f of x is not 0 should be an element in the sigma algebra and its measure… Measure can be defined only when the set is in the sigma algebra so mu of that set is equal to 0. The statement that f is equal to 0 almost everywhere will mean mu of, measure of, the set of points where f x is not 0 is 0.

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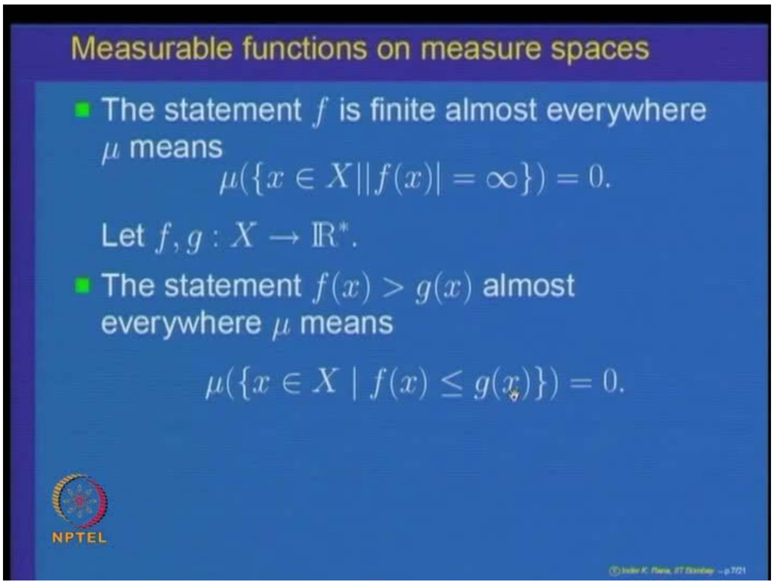

Let us look at another illustration of this. The statement that f is finite almost everywhere – what will that mean? That will mean look at the set of points where f is not finite; that means what? f is an extended real-valued function and so it can take the value plus infinity or minus infinity. The set of points x belonging to X such that mod of x is equal to plus infinity (that is same as either f of x is plus infinity or f of x is equal to minus infinity) is in the sigma algebra and mu of that set is equal to 0.

Saying a function f is finite almost everywhere means the set of points where it can take the value plus infinity or minus infinity is the set of measure 0. Similarly, let us look at two functions f and g and let us look at the statement that f is strictly bigger than g almost everywhere. f is strictly bigger than g almost everywhere – what will that statement mean?

That means the set of points where f x is not strictly bigger than g of x and that is the same as the set of points where f of x is less than or equal to g of x – the complement of that statement is f of x less than or equal to g of x and these set of points have got measure 0. Saying that f of x is strictly bigger than g almost everywhere with respect to mu means mu of the set where this statement is not true and that is f of x less than or equal to g of x is 0.

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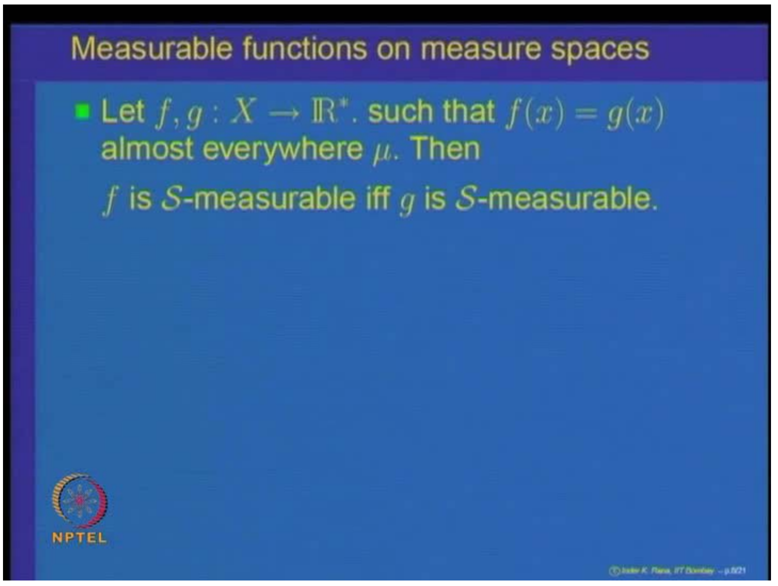

This concept of almost everywhere is quite useful when looking at measurable functions. Let us prove the property that if f and g are two extended real functions such that f of x is equal to g of x almost everywhere mu, then measurability of one of the functions f implies the measurability of the other function g; so f is S-measurable if and only if g is S-measurable. Let us prove this property that if two functions are equal almost everywhere, then the measurability is not changed – measurability of one.

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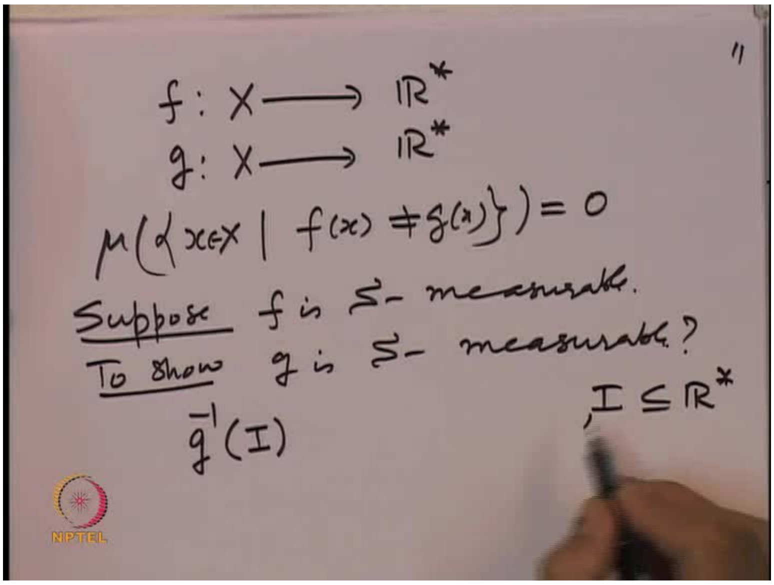

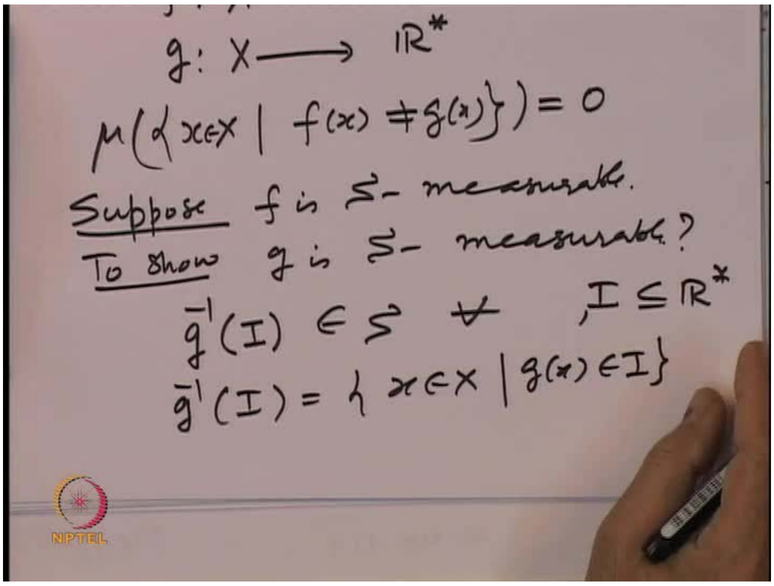

Let us look at f from X to R star; g is also from X to R star. We know that the set of points x belonging to X such that f of x not equal to g of x has mu measure equal to 0. Let us suppose f is S-measurable; we need to show g is S-measurable. To show that g is S-measurable, let us look at g inverse of any interval I and g inverse for every interval I, I an interval in R star; then we want to show that g inverse of I belongs to S for every interval I (Refer Slide Time: 31:55). Now, we have to transform this property, this set, into something regarding f. Let us look at g inverse of I is same as all x belonging to X such that g of x belongs to I. This is a subset of the set X.

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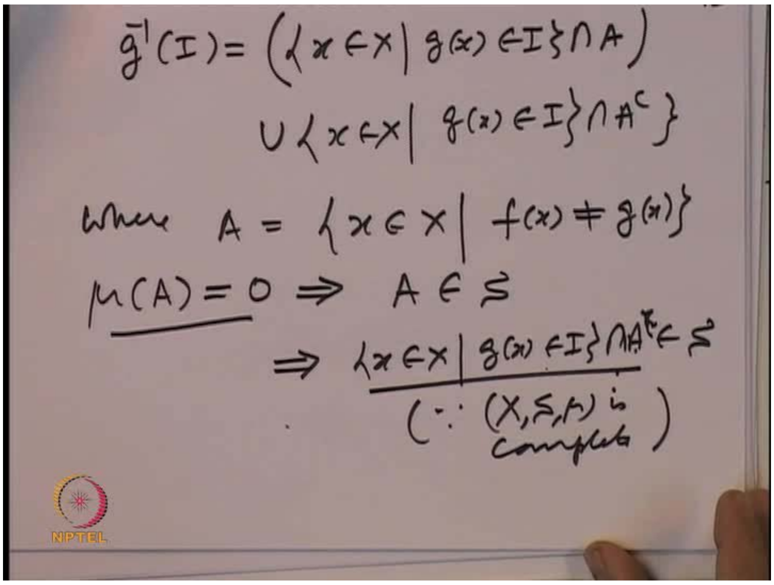

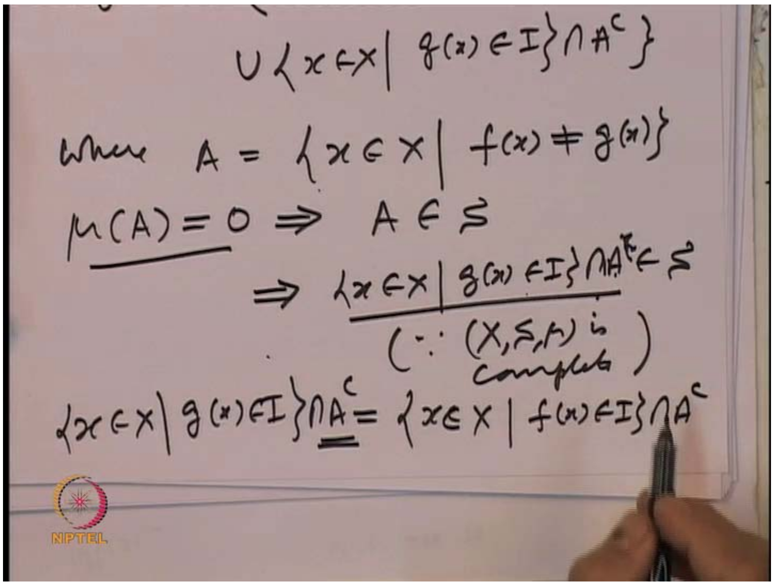

What I can do is I can write g inverse of I as intersection of x belonging to x such that g of x belongs to I intersected with the set A and also union; so, intersect with A complement; so, g inverse of I I have intersected with A and A complement. It is a union of these two sets; it is x belonging to X such that g of x belongs to I intersection A complement.

Now, let us look at the first set. This is g x belonging to I intersection A (Refer Slide Time: 33:11). What was the set A? What is the set A? A is the set x belonging to X where f of x is not equal to g of x. We are given that mu of A equal to 0. That automatically implies that A belongs to the sigma algebra S; that automatically implies that the set x belonging to X such that g of x belongs to I intersection A also belongs to the sigma algebra. So, this set also belongs to the sigma algebra. Why? It is because this is a subset of A and A is a set of measure 0; so, this is a set of measure 0 and we have already assumed our measure spaces are complete.

This is so because the measure space X, S, mu is complete; we have made the assumption that we are working on complete measure spaces; that shows the importance of complete measure spaces. This set belongs to A (Refer Slide Time: 34:36). In the other part A complement, on A complement f is equal to g; so, I can replace it by A complement; so, the set x belonging to X such that g of x belongs to I intersection A complement is the same as the set x belonging to X where f of x belongs to I intersection A complement, because on A complement f is equal to g. That means what? g inverse of I is written as g inverse of I intersection A.

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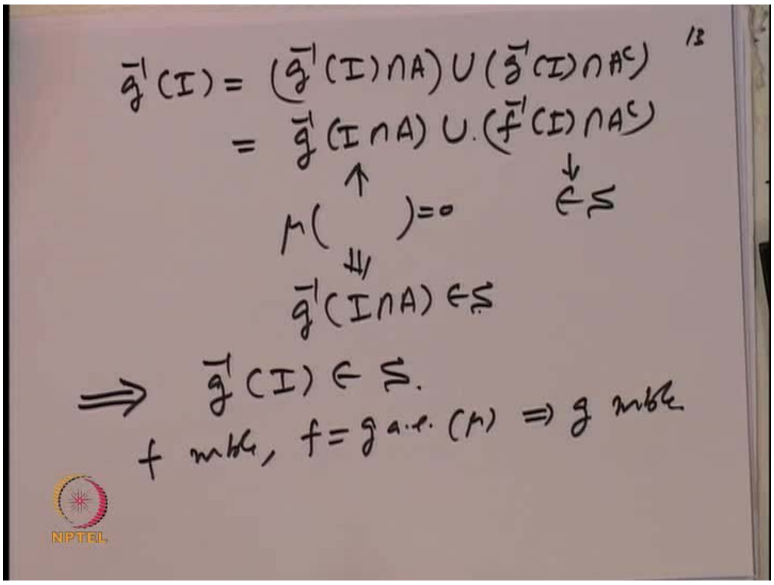

Let us just rewrite this statement again. What we are saying is g inverse of I can be written as g inverse of I intersection A union g inverse of I intersection A complement. That is same as g inverse of I intersection A union f inverse of I intersection A complement, because on A complement f is same as g. This is a set of measure 0; mu of this set is equal to 0 (Refer Slide Time: 35:52). It implies that this set g inverse of I intersection A belongs to the sigma algebra.

This set f is measurable; it implies this set is in the sigma algebra; A is in the sigma algebra; so, A complement in the sigma algebra; intersection is in the sigma algebra; so, this element belongs to the sigma algebra (Refer Slide Time: 36:11). This is a union of two elements in the sigma algebra; this implies that g inverse of I also belongs to the sigma algebra S. We have shown f measurable f equal to g almost everywhere mu implies g measurable; that is the importance of measurable functions equal almost everywhere, but keep in mind we have used the fact that underlying measure space is a complete measure space.

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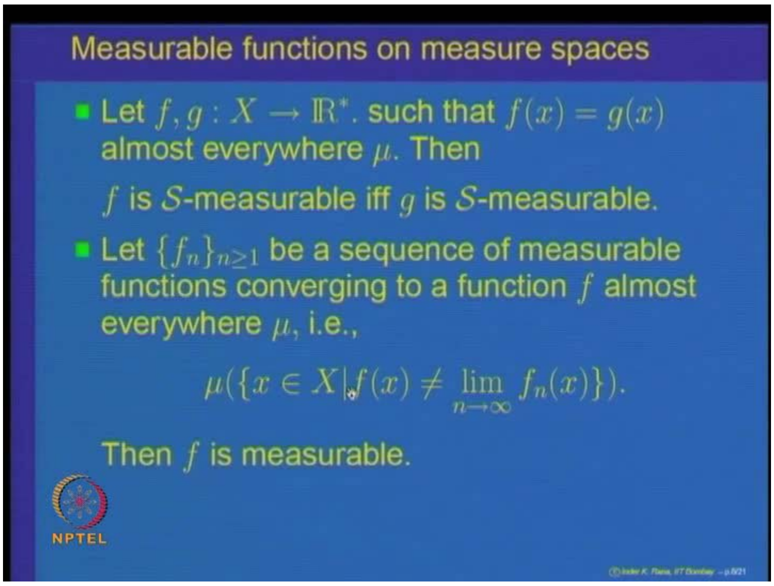

This says that if f is measurable, you can change its values on a set of mu measure 0 and still the function will remain measurable. Another impetration of this result is if f is measurable and you change its values on a set of measure 0 and call that function as g, that is measurable; that is quite an important fact. Another application of this concept of almost everywhere is the following; look at the sequence f_n – a sequence of measurable functions converging to a function f almost everywhere; that is, the set of points where f x is not equal to the limit has got (this set has got) measure 0. Then, the claim is the f is also measurable. Just now we proved that if a sequence f_n of measurable functions converges to f, then f is measurable; now we are saying that if f_ns are defined on a complete measure space and f_n converges to f almost everywhere, even then this property remains true. Basically the idea is same as before; so let us just look at how does one write the proof of this statement.

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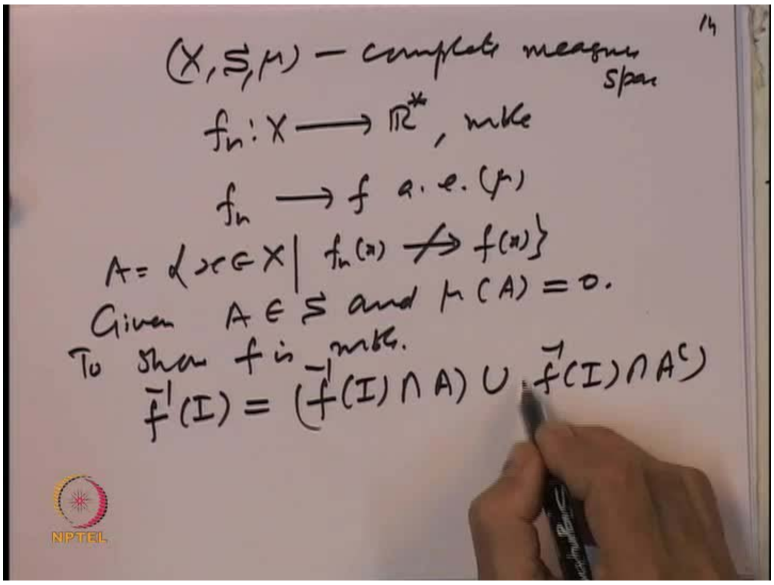

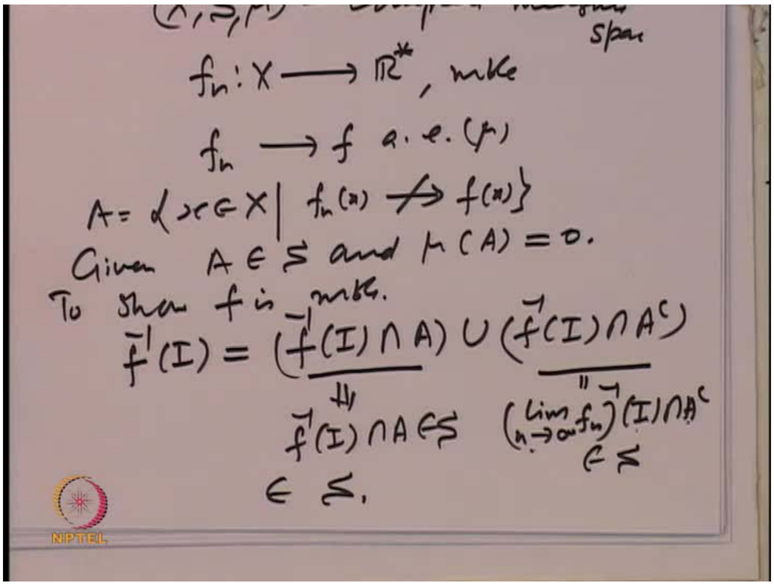

We have got a complete measure space X, S, mu; we have got a sequence f_n of functions; f_ns are measurable and f_ns converge to f almost everywhere mu. That means look at the set A equal to x belonging to X such that f_n of x does not converge to f of x. Then, what is given to us is that this set A belongs to the sigma algebra and mu of A is equal to 0. We want to show that f is measurable. Once again, for any interval I look at f inverse of I. I can write it as f inverse of I intersection A union f inverse of I intersection A complement.

We are given A is a set of measure 0. This is a subset of A (Refer Slide Time: 39:33); so, that is a set of measure 0. This implies that f inverse of I intersection A belongs to the sigma algebra S because this is a set of measure 0 and our underlying measure space is complete. On this portion A complement, f_n is converging to f; so, this f I can write it as limit n going to infinity of f_n inverse of I intersection A complement. This set is same as this (Refer Slide Time: 40:05).

We know f_ns converge to f on A complement; that is a measurable set; this is an element in the sigma algebra, because on A complement f_n is converging; if we restrict ourselves to A complement, then that must be an element in the sigma algebra; so, both belong to the sigma algebra; this belongs to the sigma algebra S (Refer Slide Time: 40:36). We can exploit the concept of almost everywhere when dealing with complete measure spaces. This implies that if f_n is a sequence of measurable functions converging to a function f almost everywhere, then the limit also is a measurable function. This emphasizes the property of something holding almost everywhere.

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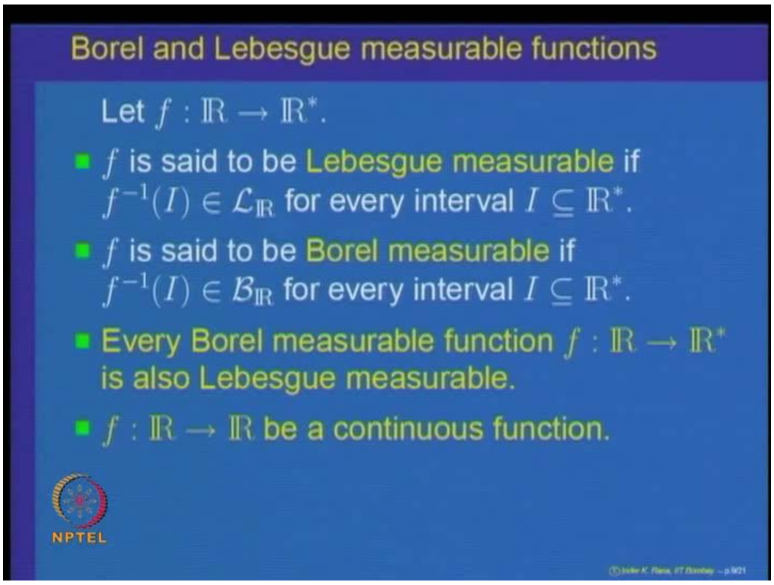

Now, let us specialize the case when our underlying set is the real line. Then we have got two sigma algebras; when x is equal to real line, then we have got two sigma algebras. One is the Borel sigma algebra and the other is the sigma algebra of Lebesgue measurable sets. We have shown that the sigma algebra of Borel subsets is a subclass of Lebesgue measurable sets.

When we are looking at functions defined on real line taking values as extended real numbers, there are two possibilities to analyze whether the function is measurable with respect to the Borel sigma algebra or measurable with respect to the Lebesgue sigma algebra. There are two notions of measurability as far as the real line is concerned and we will separate them out. We will say a function is Lebesgue measurable if the inverse image of every interval in R star is a Lebesgue measurable set.

If the inverse image of every interval in R star is a Lebesgue measurable set, then we will say that the function is Lebesgue measurable; we will say a function is Borel measurable if for every interval in R star will pull back its ((.)) image in R is a Borel set in R. Here is the difference: Lebesgue measurable requires that the inverse image is in the Lebesgue sigma algebra – sigma algebra of Lebesgue measurable sets and f inverse of I in B_R says its inverse image is always a Borel set in R.

It is obvious because Borel subsets form a subset of R; it is obviously clear that every Borel measurable function is also a Lebesgue measurable function, because inverse image of every interval is in B_R and B_R is the subset of L_R; so, every Borel measurable function is also a Lebesgue measurable function. For example, let us look at a function which is continuous. If f: R to R is continuous function, then it is going to be a Borel function. Let us prove that every continuous function is a Borel measurable function and hence also Lebesgue measurable.

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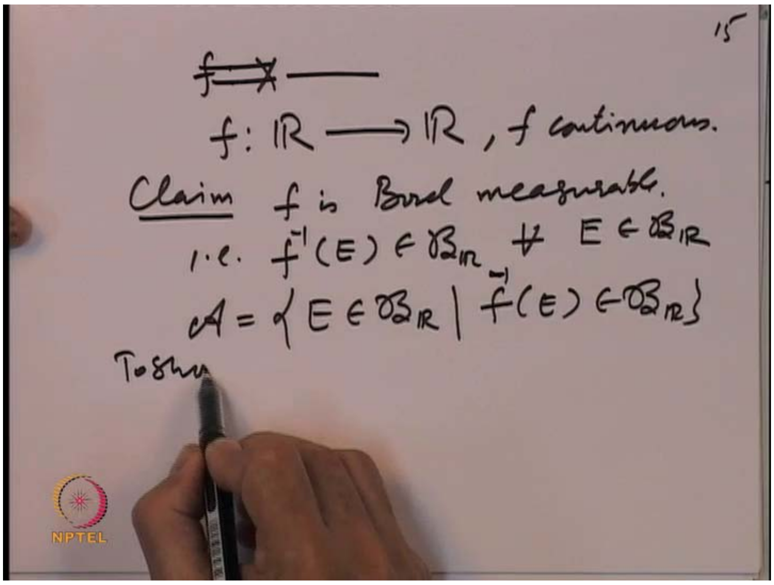

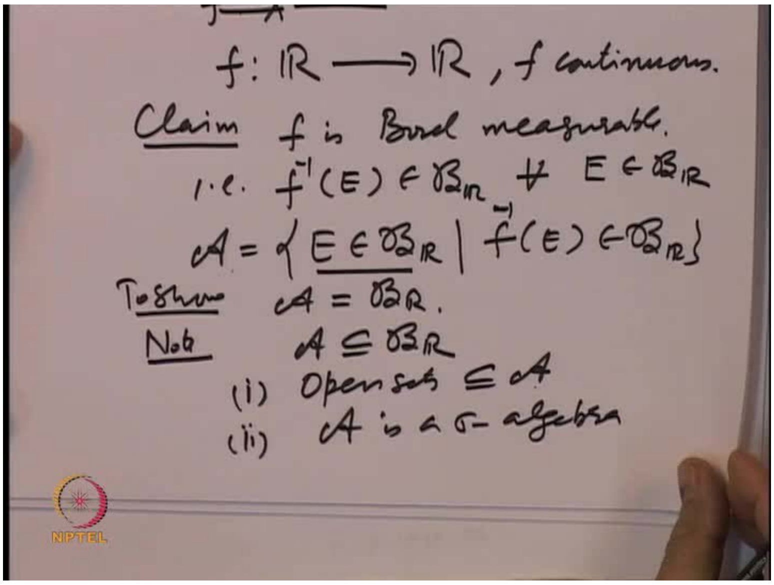

f is a function which is defined from X. Sorry, X is real line. f is the function defined from R to R and f is continuous. The claim is that f is Borel measurable; that is, f inverse of any set E belongs to B_R for every set, say, E belonging to B_R. The continuity of a function can be expressed in terms of open sets. Let us look at the class A of all subsets E belonging to B_R such that this property is true: f inverse of E belongs to B_R.

What do we have to show? Saying that f inverse of E belongs to B_R for every E in B_R is equivalent to saying to show that this A is equal to B_R; that is where we are going to use our sigma algebra technique. To show that A is equal to B_R, note that A is already a subclass of B_R because we are picking up sets in B_R; to show that A is equal to B_R, we have to show that B_R is inside A. For that, we will show two steps: (i) – open sets are contained in A; second, we will show that A is a sigma algebra because once A is a sigma algebra and includes open sets, it must include the smallest sigma algebra generated by the open sets, that is, B_R; so B_R will be inside A and we will be through. To prove these two facts it is quite obvious because of the given condition (Refer Slide Time: 45:45).

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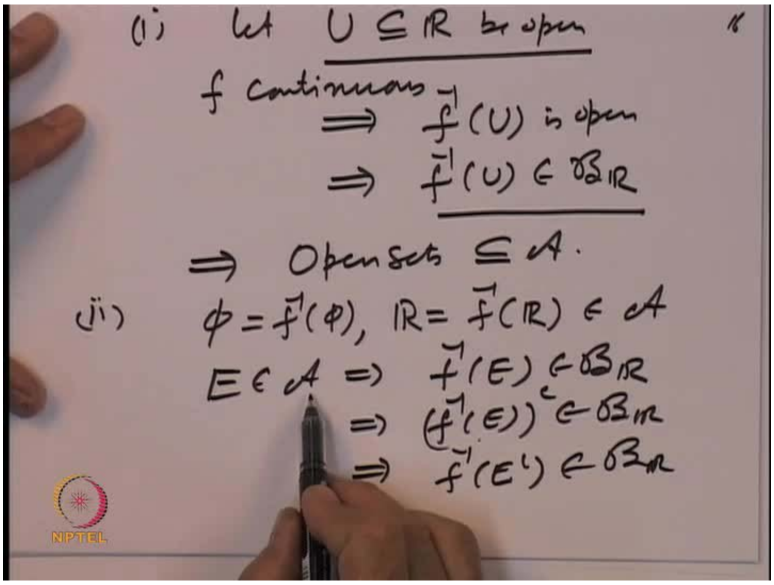

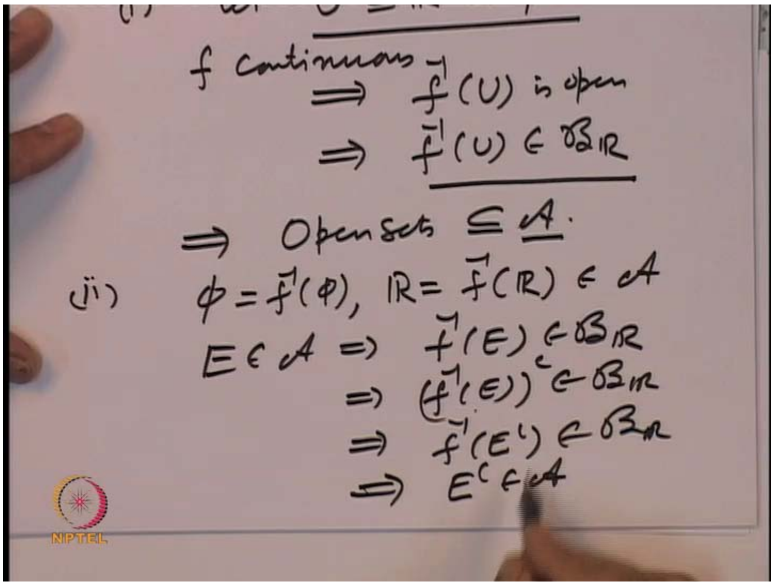

Open sets belong to A and so let U contained in R be open; F continuous implies that f inverse of U is open; hence, this means f inverse of U belongs to B_R. What we have shown is if U is open, then f inverse of U is in B_R; that implies that the open sets are inside A. A is a sigma algebra; that is more straightforward. Let us observe; the empty set is equal to f inverse of empty set and R is equal to f inverse of R; both belong to A because empty set and the whole space are equal to this; this is obvious – empty set and the whole space belong.

The second property if E belongs to A implies f inverse of E belongs to B_R; that implies f inverse of E complement belongs to B_R; that implies this set is same as f inverse of E complement belongs to B_R. What we have shown is if E belongs to A then f inverse of E complement belongs to B_R and that implies that E complement belongs to A; so that means E ((.)). So, the class A includes the empty set, includes the whole space and it is closed under complements.

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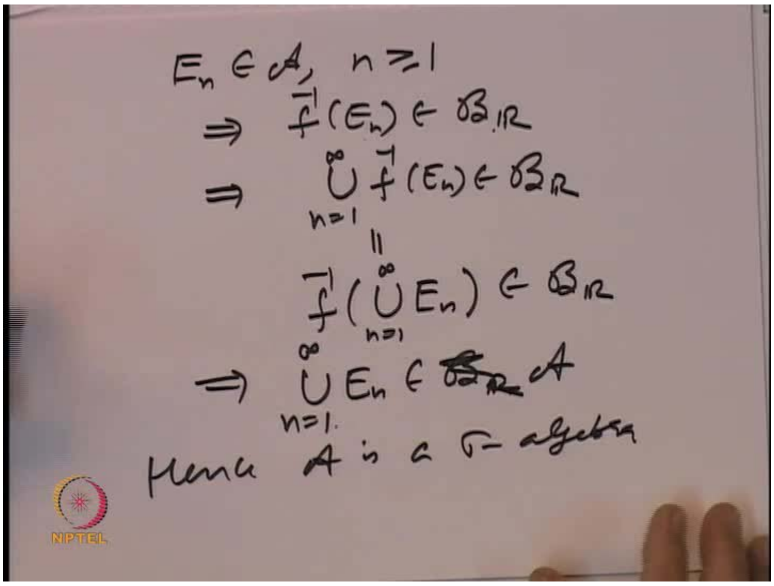

Finally, let us show that it is also closed under countable unions. Let us take sets E_n belong to A, n bigger than or equal to 1. That means what we are given is that f inverse of E n belongs to the sigma algebra B_R because the property E_n belongs to A means the inverse image is in B_R. That implies B_R is the sigma algebra; that implies union 1 to infinity F inverse of E_n belongs to B_R.

Now, a simple observation: this set is same as f inverse of union E_n, n equal to 1 to infinity and that belongs to B_R. If E_ns belong to A, then f inverse of the union belong to B_R; that means union of n equal to 1 to infinity E_ns belong to A. If E_ns belong to A, then F inverse of the union belongs to B_R and that means the union belongs to A. Hence, we have shown that A is a sigma algebra of subsets of A and it includes open sets; so, it must include B_R and hence this is equal. That proves that every continuous function from R to R is Borel measurable and hence it is also Lebesgue measurable (Refer Slide Time: 49:19). So, all topologically nice functions continuous functions become Lebesgue measurable on the real line.

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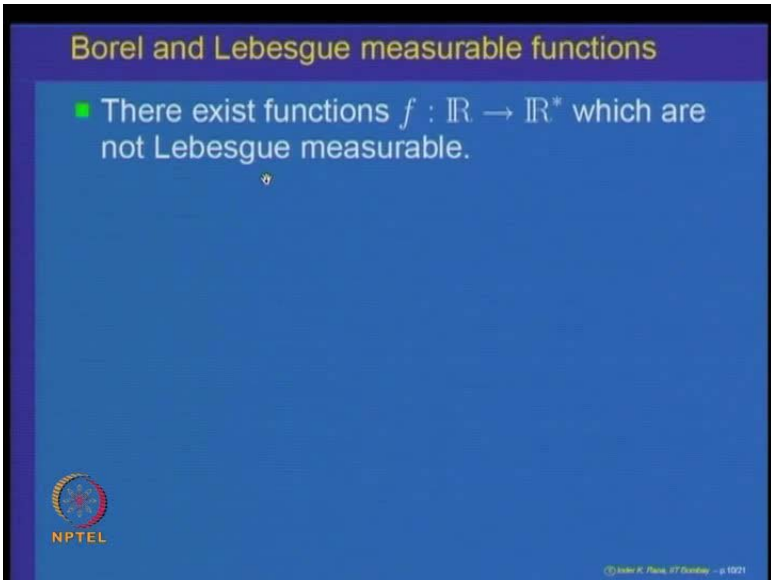

Let us look at some more properties. We showed that every Borel function is Lebesgue measurable. There exists functions first of all R to R star which are not Lebesgue measurable. To prove that, we have to simply observe that there are sets.

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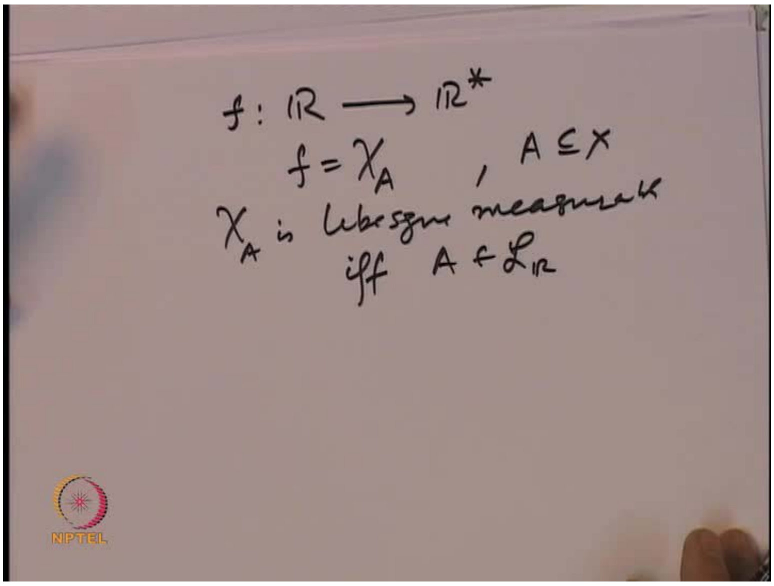

Let us go back and recall that for a function f from R to, say, R star, let us look at f equal to indicator function of a set A where A is a subset of X; recall chi of A is Lebesgue measurable if and only if A belongs to L of R. If you can produce a set which is not Lebesgue measurable, then the indicator function will not be Lebesgue measurable. The answer to this question do there exist non-Lebesgue measurable functions depends upon whether there are non-Lebesgue measurable sets. If you recall, we had proved the fact that the non-Lebesgue measurable sets exist; that question is related to basic set theory. If you assume Axiom of Choice, then we constructed non-Lebesgue measurable sets.

So assuming Axiom of Choice, one can claim that there exist functions which are not Lebesgue measurable. By the same reasoning, one can ask the question: do there exist functions which are Lebesgue measurable but not Borel measurable, because every Borel measurable is Lebesgue measurable? For that, by the same logic again if you pick up a set A which is Lebesgue measurable but not a Borel set, then the indicator function of that set is going to be a function which is going to be B_R measurable but not Borel measurable.

These two questions – whether there exist non-Lebesgue measurable functions and whether there exist functions which are Lebesgue measurable but not Borel – get tied up with the fact that the Lebesgue-measurable subset is a proper subset of power set of R and B_R is a proper subset of the Lebesgue-measurable sets. With that, we conclude the study of properties of measurable functions.

Let me just recall; the measurable functions are functions defined on the underlying set X with properties that the inverse image of every set E in the Borel sigma algebra of extended real numbers is again in the sigma algebra – on the domain space; that is, S. This is a property about the inverse images of sets being in the sigma algebra S. We will see how this property plays a role in our further study of study of integration; we will do it in the next lecture; we will start the notion of integration for measurable functions. Thank you.