Measure and Integration

Prof. Inder K. Rana

Indian Institute of Technology, Bombay

Module No. # 05

Lecture No. # 15

Properties of Measurable Functions.

Welcome to lecture 15 on Measure and Integration. In the previous lecture, we had defined what is called a measurable function on a measurable space X and then we had looked at some equivalent ways of looking at measurable functions.

We looked at examples of what are called simple measurable functions; they are nothing but finite linear combination of indicator functions of subsets of the set x.

The simple measurable functions are sort of the core of the class of all measurable functions.

We showed that some of simple measurable functions, product of simple measurable functions and, maximum and minimum of simple measurable functions, are all simple measurable functions.

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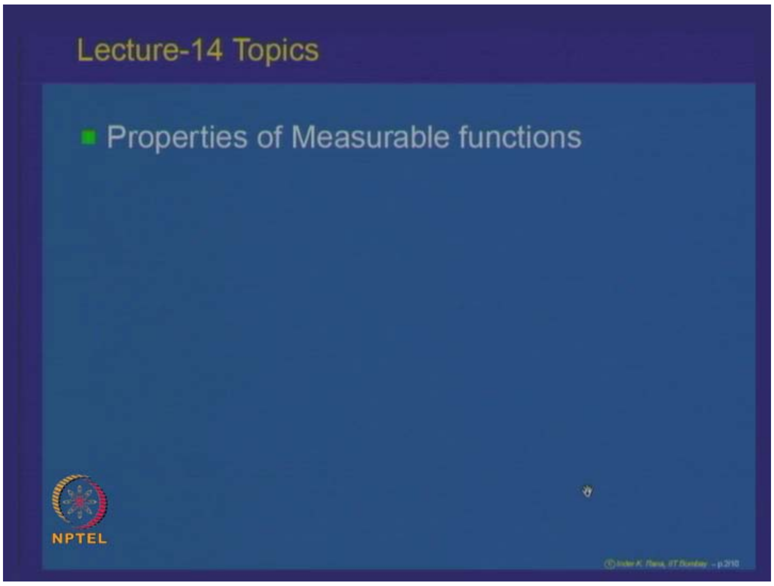

Today, we will start looking at some general properties of measurable functions f; then we will characterize measurable functions in terms of simple measurable functions. So, todays talk is going to be mainly on properties of measurable functions.

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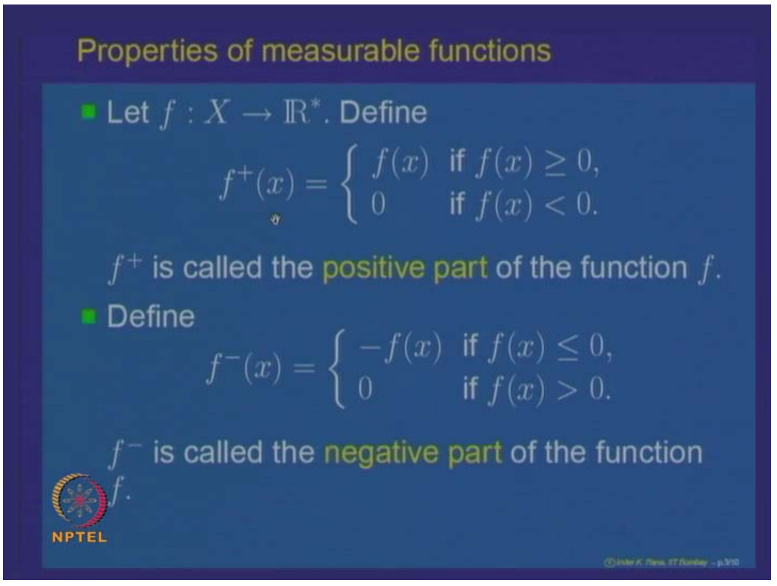

So, let us recall, for a simple function we define what is called the positive part and the negative part of a function. This can be defined for any function, let f be a function defined on X taking extended real valued functions- extended real values R star.

Then we define what is called the positive part of the function. That is denoted by f plus of x. This is again a function on the space X and it is defined as, f plus of x is equal to f of x, if f of x is bigger than or equal to 0 and it is defined as 0 if f of x is less than zero.

Essentially, what we are saying is, look at the graph of the function f of x. As long as the graph remains above the x axis, keep the function as it is; so, f of x is kept as it is, if f of x is bigger than or equal to 0.

As soon as the graph goes below the x axis, we define its value to be equal to 0. So, this is called the positive part of the function. Note that for any function f, the positive part of the function is again a function on the space X, but it takes only non negative values.

Similarly, we can define the negative part of the function to be a function on X, again such that it is denoted by f minus of x. So, f minus of x is equal to minus of f of x; keep in mind, we are putting a negative sign here if f of x is less than or equal to 0. That means, as soon as the function is on x axis or below the x axis, we reflect it against x axis and put its value as minus of f of x. So, if f of x is negative, this will always be a non negative quantity and it is function is defined as 0 if f of x is bigger than 0.

Essentially, once again it is looking at the graph of the function; as long as the graph of the function is above the x axis, we put it as the value equal to 0 and it is minus of f of x, if f of x is less than or equal to 0.

So, these are called the positive part and the negative parts of the function and it is quite obvious from the definition that f can be written as, f plus minus of f minus.

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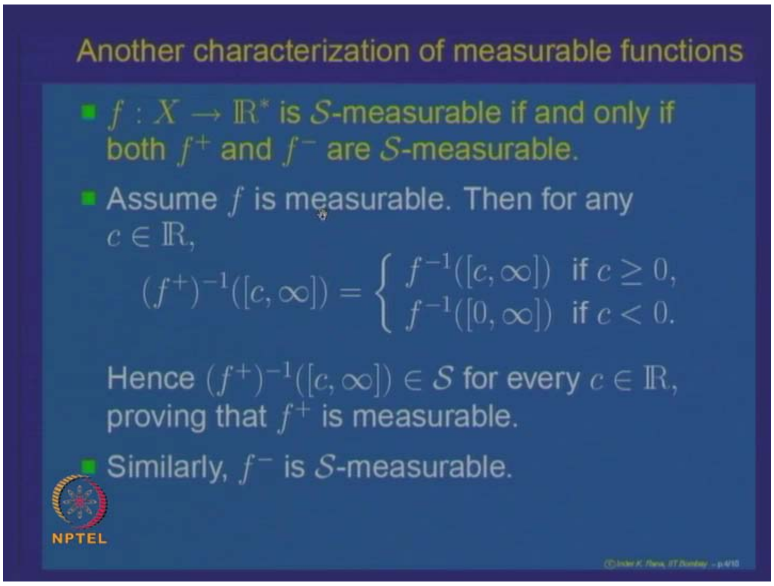

We want to prove that, if f extended real valued is a measurable function- we want to show that, in that case the positive part and negative part are both measurable functions; conversely if positive part and negative part are measurable, then the function f is measurable.

So namely saying, a function f on X taking an extended real valued is measurable, if and only if, both f plus and f minus are measurable functions. The proof is where the simple. Let us assume, first f is measurable then for any point c in R, look at the inverse image of the enclosed intervals c to plus infinity- f plus inverse image closed interval c to infinity.

So that we know, is because f of x, f plus of x is equal to f of x, if f of x is bigger than 0. If this value c is bigger than 0, then f plus of x will be equal to f of x; this inverse image of the closed interval c to infinity under f plus is nothing but the inverse image of f, of the interval c to infinity, if c is bigger than or equal to 0 because in that case, f of x is always going to be positive. And it is equal to the inverse image of the interval 0 to infinity. I, if c is negative, because then we do not want to look at the remaining part.

So in either case, because f is measurable both these sets are in the sigma algebra. So, f inverse f plus inverse of c to infinity- belongs to the sigma algebra, if f is measurable; that proves that f plus is measurable. A similar argument will prove that f minus is also measurable.

Essentially, what we are saying is, the f plus inverse- that is a inverse image of the interval c to infinity under f plus can be represented as inverse image of an interval of f, under f of some interval and both are f measurable, implies that both whenever that f inverse of an interval is a set in the sigma algebra.

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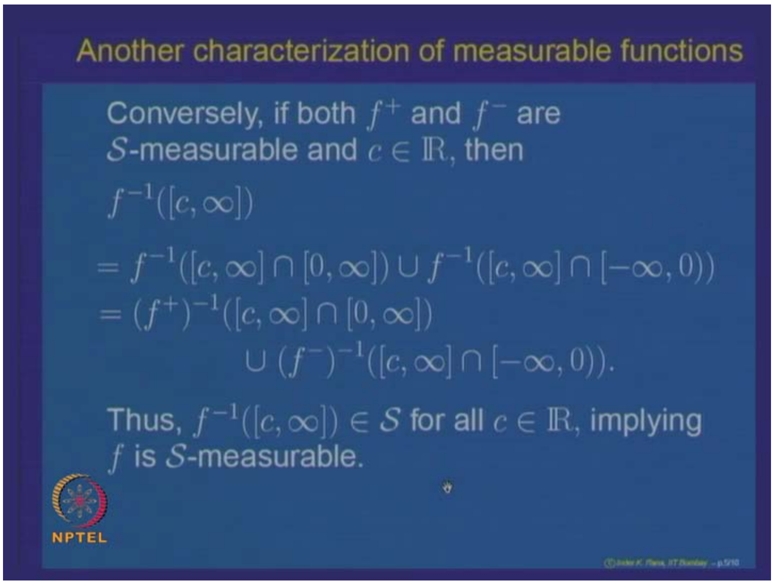

So, we have shown that if f is measurable, then f plus and f inverse or f minus are both measurable. Let us prove the converse part, suppose both f plus and f minus are measurable; let us take a point c belonging to R, then we should show that the inverse image, f inverse of c to infinity belongs to the sigma algebra S for every c in R.

So, let us fix the c and look at this. We have to interpret this in terms of inverse images of some intervals in terms of f plus and f minus. Now, let us observe that f, the inverse image of the interval c to infinity can be decomposed into two parts: namely, f inverse of c to infinity and intersection 0 to infinity.

Look at the intersection of c to infinity with 0 to infinity, the intersection of this interval c to infinity with minus infinity to 0. Thus, interval c to infinity is decomposed into two parts: it has intersection with minus infinity to open interval 0 and it has intersection with close interval from zero to infinity.

So, f inverse of c to infinity is nothing but f inverse, the inverse image of interval c to infinity intersection with 0 to the part of the interval, which lies in the positive part; inverse image of the part of the interval, which lies in the negative part, but that means in the first part, we are looking at whenever the function is in 0 to infinity, that means function is non negative.

The first inverse image is nothing, but the inverse image of this- in same interval under f plus, similarly; the second one is an inverse image of the interval c infinity into intersection with minus infinity to 0 with respect to f inverse.

So, the part of the function which lies in 0 to infinity is written as inverse image under f plus of an interval and the other part is written as inverse image under f minus.

Since, both f and f minus- f plus and f minus are measurable functions, so these two sets f plus inverse of that interval and f minus inverse image under that interval, these both are sets in the sigma algebra S. So, their union is also in the sigma algebra S, that means f inverse of c to infinity is in the sigma algebra for every c- belonging to R.

And that proves that this is a, f is a measurable function. We have shown that if function f is measurable, if and only if, X positive part and negative part both are measurable functions.

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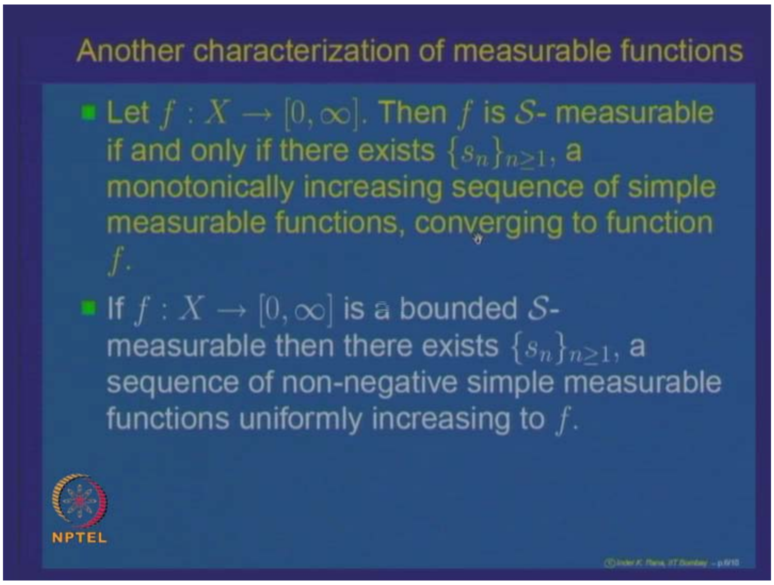

Next, let us look at some more properties of measurable functions; we want to give a characterization of measurable functions in terms of simple functions. So, we start with a non negative function, let f be a non negative function on defined on X, taking values in 0 to infinity.

We want to show that this function f is measurable, if and only if, there exists a sequence s_n of simple monotonically increasing functions ~~and~~again in fact non negative. We can also say: they are non negative simple measurable and non negative sequence of functions, which are monotonically increasing to the function f.

That means if a function f is non negative and measurable, this can happen if and only if S can be written as a limit of simple functions, which are non negative and the sequence s_n is increasing.

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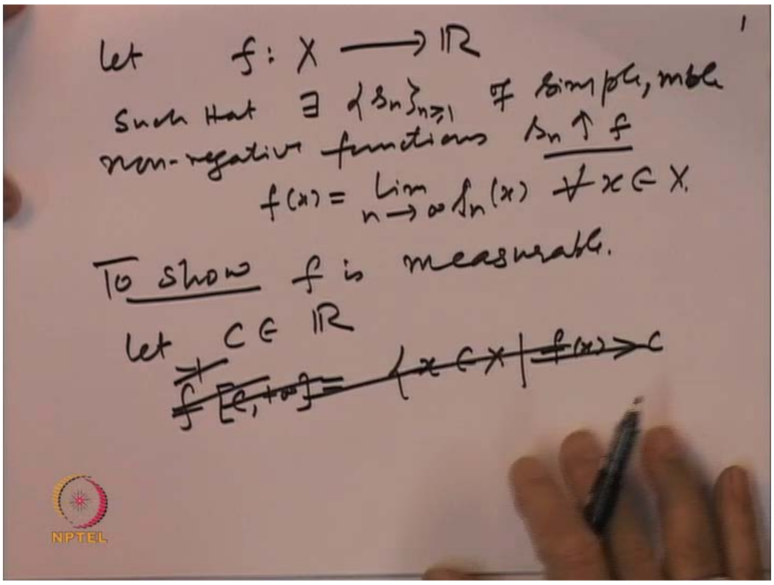

So, let us prove this fact- let us start with- let f, is from X to R and we are given a sequence, such that there exists a sequence s_n, of simple non negative functions; s_n increasing to f, that means what? That means f of x is limit of n going to infinity of s_n of x for every x- belonging to X and this s_n are a monotonically increasing. so, this is one this is written as this.

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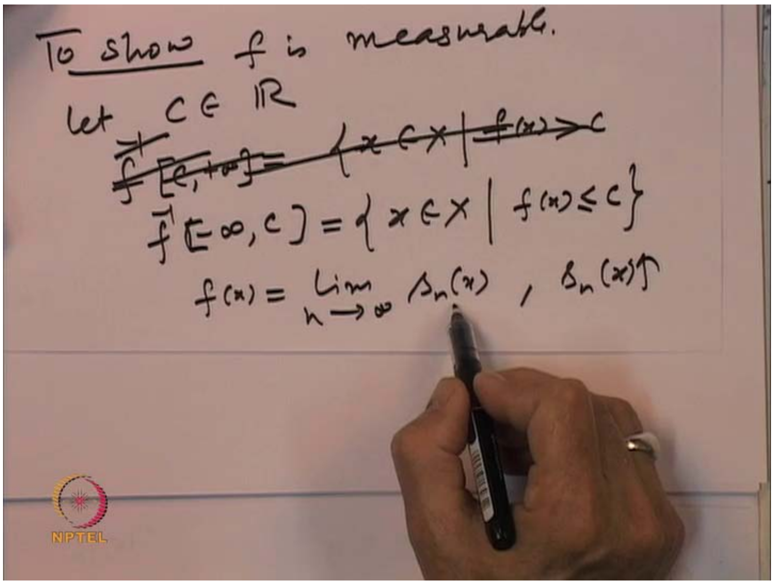

To show non negative simple functions, each the simple, of course measurable; so to show that f is measurable, now let c belong to R and let us look at the inverse image of the interval c to plus infinity, what is that? That is all x belonging to X, say that f of x is bigger than c or it will be easier, if we will look at the other sets.

So let me instead of this- let me look at the set, which is f inverse of minus infinity to c. We will soon see why I am taking this instead of the earlier set. Why I am taking this, because this proof becomes slightly simpler. So, what is this? This is all x belonging to X such that f of x is less than or equal to c.

What is f of x? Recall, just now we said f of x is limit n going to infinity of s_n of x and s_n x is increasing. So, if the limit of s_n x, which is f of x is less than or equal to c- that means each s_n has to be less than or equal to c, because even if one goes above c then the limit has to be bigger than c.

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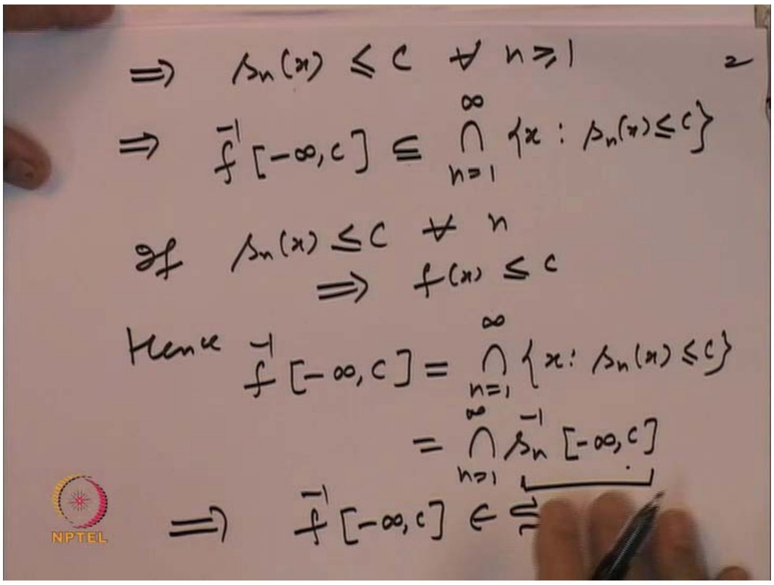

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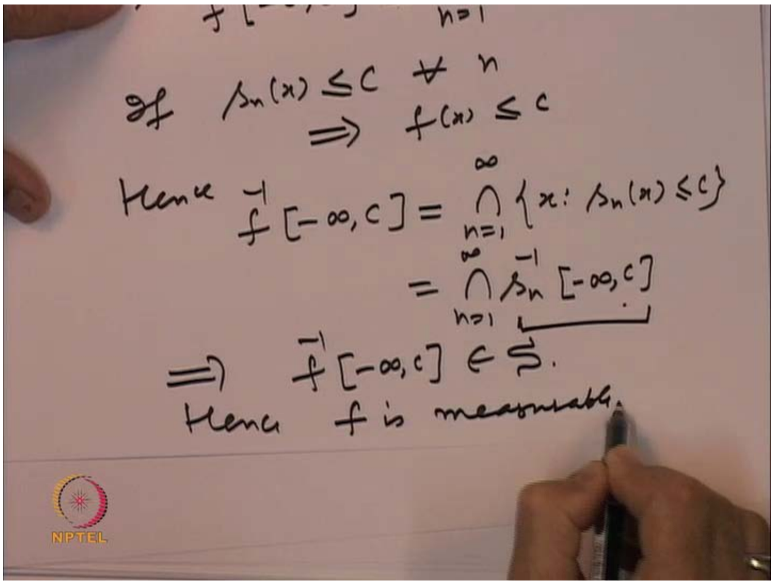

So, this condition implies that s_n x is less than or equal to c for every n bigger than or equal to 1.

So implies, that the set f inverse minus infinity to c- if x is such that f of x is less than or equal to c, that implies s_n of x is less than or equal to c.

So, that means this is contained in this is for every n, this contained intersection n equal to 1 to infinity of all x; such that s_n of x is less than or equal to c and if conversely- if, so note, if s_n of x is less than or equal to c for every n, then at this- automatically, implies that f of x, because f of x is the limit that is also less than or equal to c.

Hence, what we ~~have~~ are saying is, this is actually an equality- so, n is f inverse of minus infinity to c can be written as intersection n equal to 1 to infinity of x.

Such that s_n of x is less than or equal to c and that is same as n equal to 1 to infinity. So, this is s_n inverse of the interval minus infinity to c and s_n is being a simple measurable functions, each one of them is an element- each one of these sets is an element in the sigma algebra S. So, implies that f inverse of minus infinity to c belongs to the sigma algebra S, that means hence f is measurable.

So, what we have shown is, if there exists a- what we have shown is, if f can be written as a limit of increasing sequence of simple measurable and non negative simple measurable functions, then f is measurable.

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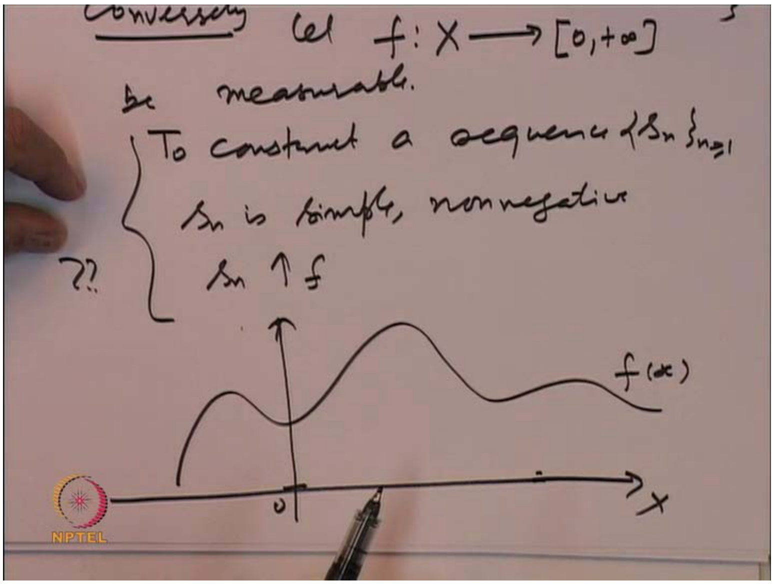

Let us look at the converse part of it which is going to be slightly not so obvious. So, conversely let f from X be a non negative function 0 to plus infinity, is measurable. We want to show, to construct a sequence s_n of functions; say that each s_n is simple, ~~non negative~~non-negative and s_n is increasing to f.

So, this is all we want to do. This construction is intuitively very obvious, but needs to be explained so, let us look at in the picture and let us draw a picture of the function. So, let us draw- this is x axis and this is the values, that takes the real number and their all values are non-negative. So, the graph is going to be above the x axis and this is going to be the graph of the function of f of x.

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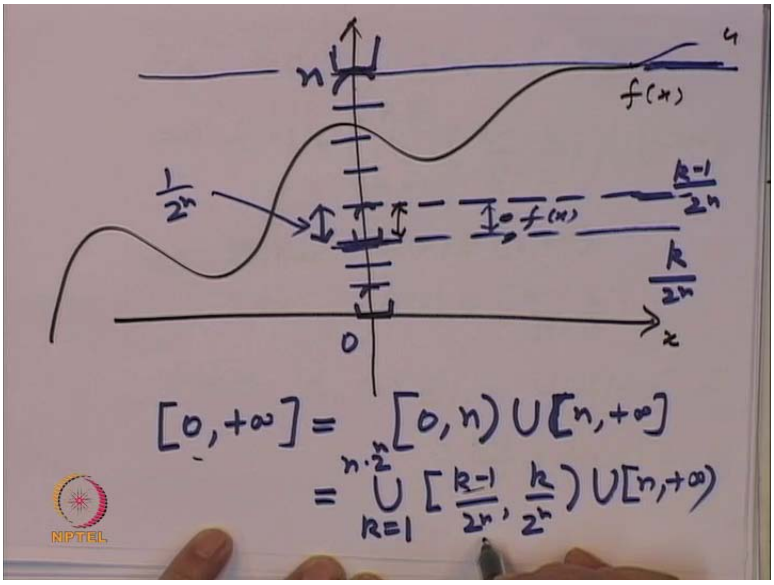

And the range of the function is a subset of 0 to infinity. So, what we are going to do is? We are going to partition the range first, into smaller intervals; to do that let me draw a picture on a slightly bigger piece so that, we are able to look at it. so this is the graph of the function f of x and this is x.

So let us mark of, let us put a point n here and this is the point 0. So, from 0 to n and then from n to onwards we have dividing the range of the function. The range is a subset of 0 to infinity. So, we had divided the range and so I am writing 0 to plus infinity as equal to 0 to n open and union to the closed n to plus infinity.

So, I have divided the range into two parts and now next, what do we do is the portion 0 to n, this portion from 0 to n where I am going to divide for every n into smaller pieces of lengths 2 to the power n. Cut it into pieces such that, the length of each piece is nothing but 1 over 2 to the power n so this is a length of each piece.

So, let us call the interval- this is my general interval so the upper point here will denoted by say, k minus 1 by 2 to the power n and the lower part is k by 2 to the power n. So, I am going to write this is equal to and we are going to look at: open at the bottom, closed at the bottom and open at the top.

I am going to write as union of intervals of the type k minus 1 by 2 to the power n, k by 2 to the power n union that other part we leave it as it is n to plus infinity.

This union, how many such small pieces will be there? Total length from 0 to n each has got: sub interval and has got length. So, 2 to the power n- this starts with k equal to 1 and goes up to n times 2 to the power n. so this is what we do

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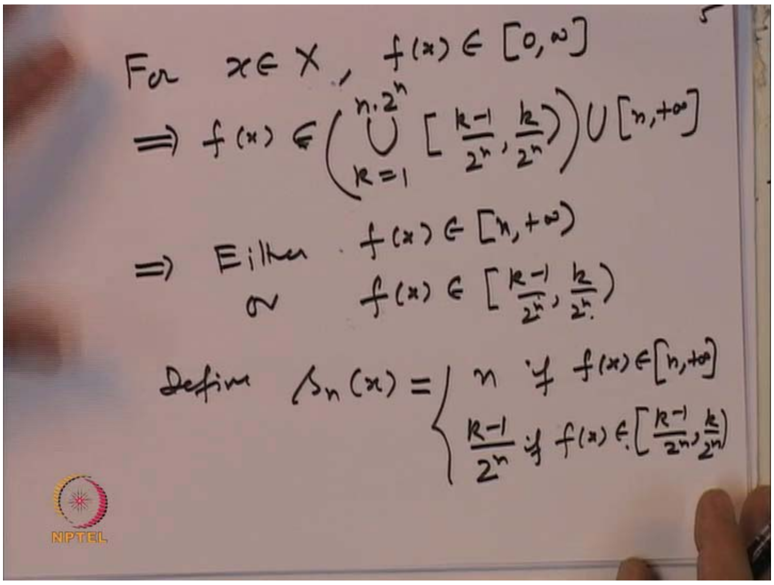

Now for every n, I want a function s_n of x and for any x, look at the value of f of x. so, let us look at those values. For every x belonging to X f of x belongs to 0 to infinity.

So, that means this union is a disjoint union so, which are let us write f of x belongs to that partition- that we have design. So, k equal to 1 to n times 2 to the power n of the interval k minus 1 by 2 to the power n to k by 2 to the power n open and this union, n to plus infinity.

Now, if f of x belongs to this and this is a disjoint union. So f of x will belong to only 1 of them, implies either f of x belongs to n to plus infinity or f of x will belong to 1 of the sub intervals; let us call it as k minus 1 by 2 to the power n over k by 2 to the power n.

So, these are the two possibilities and now we want to define a function $s_{n.}$ So, to define s_n of x we want to define what should be the value of it. See if the value is bigger than n: if f of x is bigger than n, then let us keep the value to be equal to n. This happens if f of x belongs to n to plus infinity.

And let us define it: if f of x is inside this k minus 1by 2 to the power n to k by 2 to the power n, then let us take the lower value. So let us define, s_n of x to be equal to the power n, if f of x belongs to k minus 1 by 2 to the power n to k by 2 to the power n.

This is how we are going to define the function s_n, so let us look at in from the picture point of view. What I am trying to do is? So, because this (Refer Slide Time: 22:20) is the range of the function, f of x is going to be in somewhere else.

So if f of x is above n, above this line n say for example that is happening here and this is the function, then for all these points my s_n is going to be to be this constant. As soon as the function crosses n, the value of s_n is going to be that point and all the other possibility that f of x lies in 1 of this interval.

So let us say f of x is here, so this is my f of x and then what do I want? My value of s_n should be such that the difference between s_n and f should be small and this is the smallness we have created. So, I will define my s_n to be the lower value and so, this is going to be my s_n.

So, as soon as the function is inside this strip and wherever the function is inside the strip the value is this lower value. If it crosses n, then that is the constant value n. So, the function s_n x is defined to be equal to this, so this s_n of x is n if f of x is bigger than or equal to n and if it is strictly less than n, then it belong to 1 of those small pieces and that is defined as the lower value of that interval, as the value or the function s_n of x.

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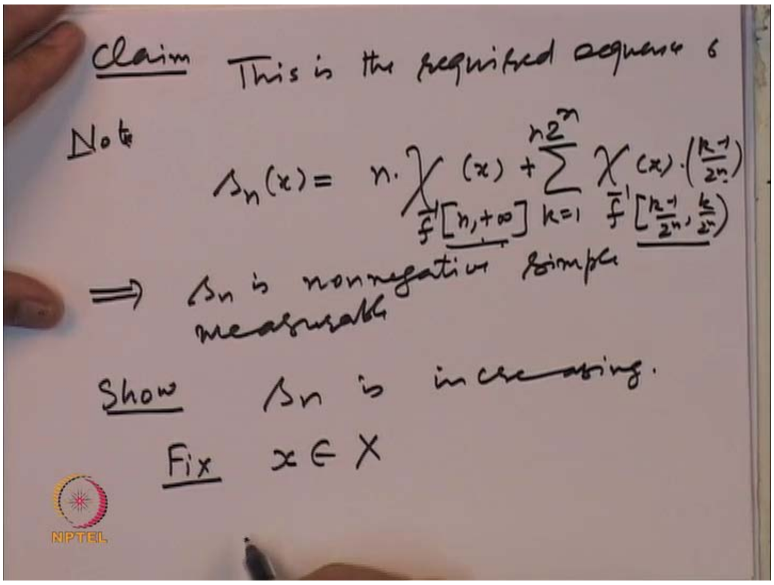

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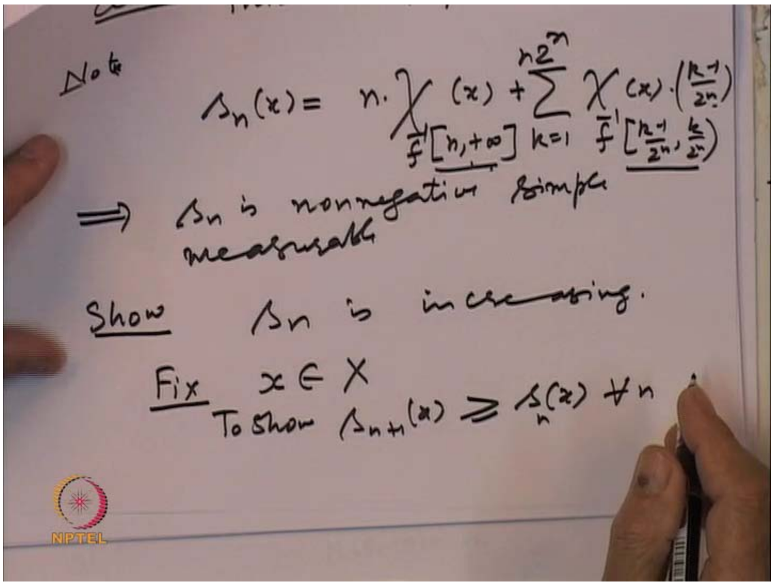

So, our claim is that this is the required sequence. So, claim this is the required sequence and so, let us first observe what is s_n? s_n is defined as this, so I can write s_n of x to be equal to n times on this interval.

So, it is n times the indicator function of the interval n to plus infinity of x. If the point belongs here then this number will be 1 and the value will be n plus; if it lies in that interval k minus 1 by 2 to the power n to k by 2 to the power n.

The value is this so, it is this value that times the indicator function of that set. It is summation k equal to 1 to 2 to the power n times 2 to the power n of the indicator function. So, what is that set? That is nothing but the f inverse image of k minus 1 by 2 to the power n and k by 2 to the power n open of x.

So, from the picture and from the earlier formula we get that my function is this function and it is clear from this, that it implies s_n is non negative, simple measureable. Why it is non negative? Because n is value, taken are either n or the sorry in multiplied this by k minus 1 by 2 to the power n that value we forgot to multiply.

So, either you take the value k minus 1 by 2 to the power n or n and all are non negative number. So, this is a non negative function and it is a finite linear combination of characteristic functions. This is not n to infinity and this is f inverse.

So, that we just- we should be careful. So it is n, if f of x belongs to this that means x belongs to f inverse of n to plus infinity and f of x belongs to this interval means if x belongs to f inverse of this interval.

So, that proves that s_n is measurable. Now, why are these sets measureable? It is linear combinations of indicator functions of sets and this set is measurable because f is measurable. This set is measurable, because f is measurable. So f is measurability of f, implies that inverse images of intervals are elements in the sigma algebra. So these are the elements of the sigma algebra and hence, s_n of x is a linear combination of indicator functions of sets in the sigma algebra S, so it is a simple measurable function.

And now let us prove that, this is S. So claim we want to show is that s_n is increasing.

So let us fix x belonging to X to show s_n plus 1 of x is bigger than or equal to s_n of x for every n.

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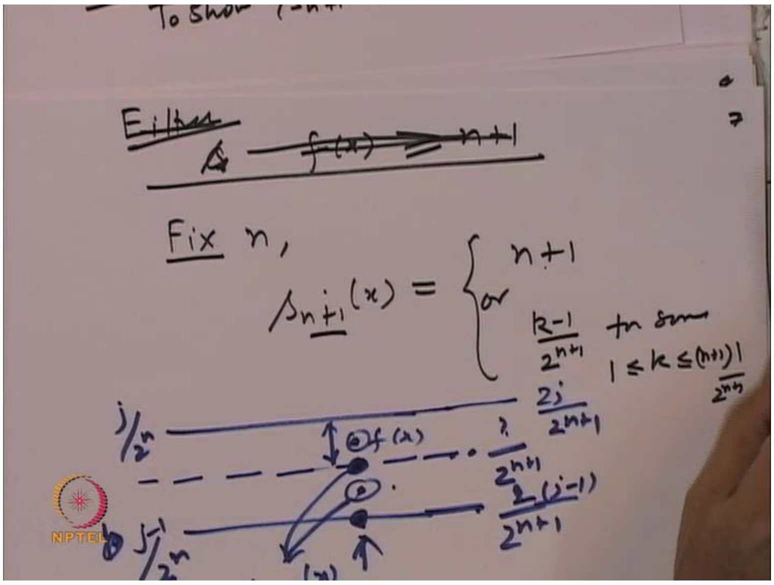

So, let us look at that why is that true? Now look, what is the value of so, either f of x is bigger than or equal to n plus 1, which is know. Let us look at slightly differently so, I want to prove that s_n is increasing.

So to prove the increasing part, let us fix n so we want to look at what is s_n plus 1 of x, say that is going to be depended upon whether- so, either it is n plus 1 or it is going to be some k minus 1 over 2 to the power n plus 1; for some k between 1 and n plus 1 times, 1 over 2 to the power n plus 1.

So what we are saying is, s_n plus 1of x either it will be bigger or it will be equal to n plus 1. That is the case if f of x is bigger than or equal to n plus 1 or it will be equal to 1 of lower values of 1 of the sub intervals at the n plus 1th stage.

In the n plus 1th stage we will be dividing the interval into 2 n plus 1 part. So, let me write this and draw this pictures slightly here. To understand what is happening So, here is- so, this is- here is, let us say j by j minus 1 by 2 to the power n and that is j to the power 2 to the power n at the nth stage where, f of x is somewhere in-between.

Now, at the next stage what we are doing? We are going to divide this into two equal intervals: so that, this part is something divided by 2 to the power n plus 1 and this part is something divided by 2 to the power. So, what will be this? This will be 2 times j minus

1 divided by 2 to the power n plus 1 and this part would be 2 j divided by 2 to the power n plus 1 and that is the middle line in between.

So, now my s_n plus 1 x depending on f x, if f of x is here. If this is f of x, then s_n plus 1 is this is the value of s_n plus 1. So, this is the value of s_n plus 1 and if f is here, then this is the value of s_n plus 1; so, either f of x will be here or it will be here.

So, if f of x is here it lies in that interval of length 2 to the power n plus 1, then the values are lower n point.

So value of s_n is here, but in that case what is the value of this- the value of s_n plus 1. So this is the value of s_n plus 1 x and what is the value of s_n? That is always going to be equal to this value.

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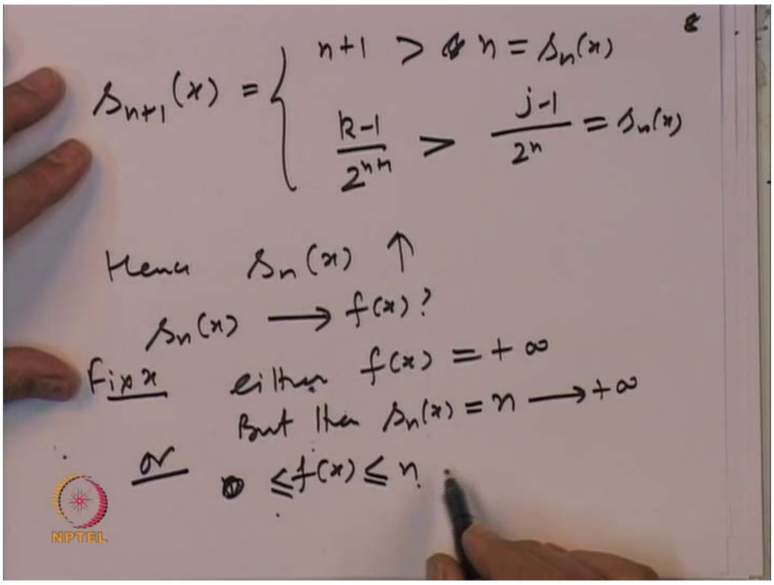

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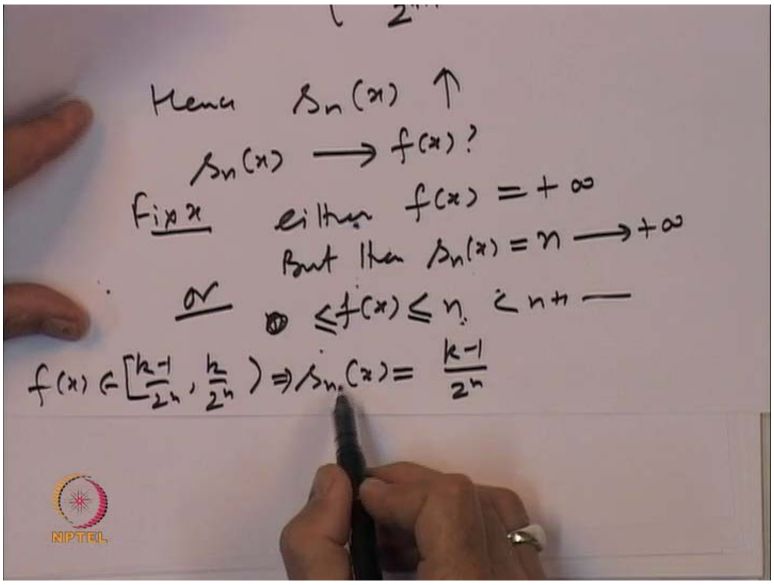

And if f of x is this, then this is the value of s_n plus 1 x, either will be here or it will be here and s_n plus 1 of x will always be here. So, this value is less than or equal to this value. So, that analysis let us just to write it is equal to this. In either case, s_n plus 1 of x, either it will be n plus 1 or it will be which is bigger than n, which is equal to s_n of x.

If not if it is below, then the value is if it is in one of those intervals, then the value is going to be some k minus 1 over 2 to the power n plus 1, which is always going to be equal to 2 to the power n. The lower value here let us it is difficult to write those symbols so that lower value this k minus 1. So, this is k minus 1 that is less than or equal to j minus for some j and that will be equal to j minus 1 over 2 to the power n which will be equal to s_n of x.

Geometrically, it is quite clear that what is happening. So, either if f of x is here in between then either the value of s_n plus 1 is this value or if f of x is here, then s_n plus 1 is this value which is the value of s_n also.

So in either cases, this implies that hence, s_n x is increasing and let us prove that s_n x converges to f of x, that the limit is equal to f of x. So if we fix x, if x is fixed so either f of x is equal to plus infinity that is 1 possibility, but then if this is plus infinity, what is s_n of x? That is always, f of x is always bigger than n and for any n s_n x is going to be equal to n, which goes to plus infinity and s_n goes to infinity.

So, if f of x is plus infinity, then s_n x is equal to n for every n and hence it goes to plus infinity or what is second possibility? f of x is not infinity, that means it is a real number so it will lie between some n and it will be less than or equal to some n.

So it will be some 0 less than or equal to n, then in that case, there exists a n, say that this is happening.

So, it also will be less than n plus 1 and so on. So, what is s_n of x in that case? In that case s_n of x is going to be some k minus 1 over 2 to the power n.

If this is less than this, then f of x will belong to one of the intervals k minus 1 over 2 to the power n and k by 2 to the power n. So, implying that is the same.

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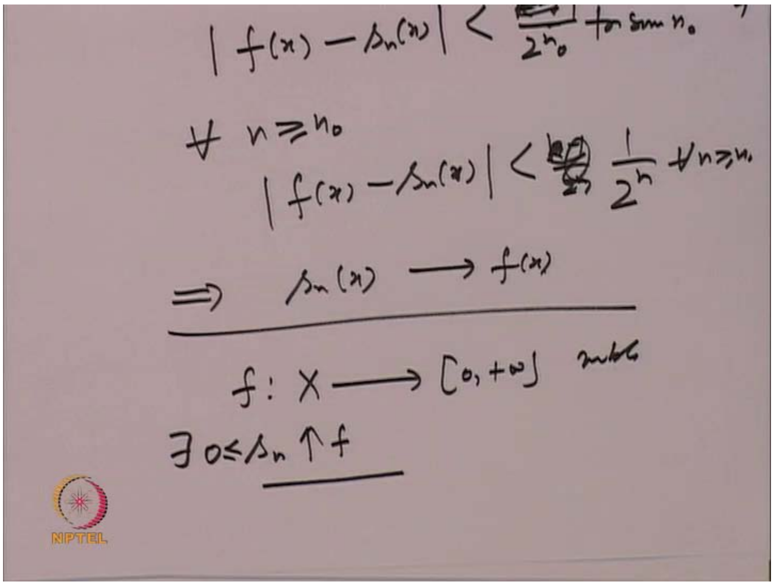

What is the difference between s_n and this? S_n is the lower value and f is something in between and that implies, that the absolute value of f of x are, actually f of x is bigger so minus s_n of x will be less than k minus 1 by 2 to the power n, for some n and that means if this is happening for some n let us say for some n_o, then for every n bigger than or equal to n_o f of x minus s_n of x is less than those k minus 1 by 2 to the power n, for some k and some n. So, it will be less than one over two to the power and the difference will be at the most.

Both lies in the interval of length 2 to the power n_o, so it will be less than 1 over 2 to the power n, for every n bigger than n_o and that implies that s_n x converges to f of x.

So let me just go over to the construction once again, to understand because this is an important construction. It says that, I want to given a function f, which is non-negative measurable and we want to construct a sequence of simple functions, which are non negative, which are increasing and they converge to f of x. So what we do is, we divide the range and this is the range of the function, which is a subset of it.

So divide into a partition the range, so partition into 0 to n and this is 0 and this is n, union n to infinity upwards so, this is the portion and the portion 0 to n is divided into sub intervals, each of length 1 over 2 to the power n.

So this will look like k minus 1 by 2 to the power n, k by 2 to the power n and k equal to 1. From 1 to how many such intervals will be there? Each of length 2 to the power n, total length is n so, n times 2 to the power n.

So this is, we have partitioned range and now given a point x, f of x either it will be beyond n or given an n, either it will be beyond n or it will be between 0 to n.

If it is beyond, then we define s_n of x to be equal to n. So, if the value of f of x is bigger than n, then we define it to be equal to n and if it is not, then it will be between the interval 0 to n, so it will (Refer Slide Time: 37:39) fall into 1 of the sub intervals, see somewhere here in some k minus 1 by 2 the power n to k by 2 to the power n.

So, we define thus the lower value of that interval, that is k minus so this is k and this is k minus 1, so the lower value to be equal to the value of the function s_n x.

So this sequence is increasing and see for any point x, f of x and s_n will be at the most difference of 1 over 2 to the power n. For n large enough or if not, then s_n will go to infinity.

So, that is the idea that it converges and increasing once again comes from the fact that we are taking the lower value at every stage. So at any stage, either (Refer Slide Time: 38:29) s_n plus 1 is bigger than n plus 1and in that case it will be bigger than n, also s_n plus 1 will be bigger than s_n. If not it will be in 1 of those sub intervals are length 2 to the power n plus 1, but how did you get those? So that this total length is 1 over 2 to the power n so when we want to divide the next stage from s_n to s_n plus 1, we divide it into two equal parts. If f of x is here, then s_n plus 1 is the lower value here or if s_n plus 1 is here, it is a lower value. So in either case, s_n is always going to be the lower value. So that says it is increasing and convergent. So that proves the theorem.

That shows what we have shown is the following: given a function f: X to 0 to plus infinity measurable then there exists a sequence s_n or non negative functions, which are simple and measurable increasing to f.

So that is what we have to prove. Let us come back to the theorem (Refer Slide Time: 39:37) which said that, this is one of the key theorems in the motion and in for the concept of measurable functions, that every non negative measurable function can be approximated and can be obtained as a limit of non negative simple measurable functions.

This non negative simple function can be slightly to be an increasing sequence, so you can approximate a non negative function as a limit of increasing sequence of non negative simple measurable functions.

So this immediately gives us a corollary, for functions which are not non negative but let us before that, let me just observe that this in the proof. If the function f is bounded, then the sequence can be chosen to be s_n, to be uniformly increasing to f and not only it converges point wise to f, you can actually claim that it converges to f uniformly.

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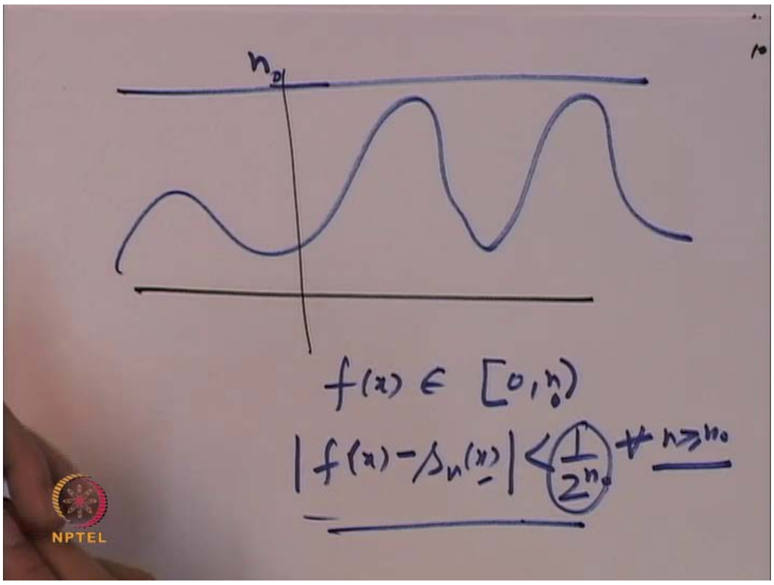

So to prove that, it converges to f uniformly if we just observe because the function is bounded and so let us just observe that if the function is bounded, then that means f is a bounded function.

So it is graph and there is going to be n, so that the graph of the function always stays below this. Once n is fix, that means f of x is always going to belong to 0 to n, for some n and for that f of x let us say n_o and n_o is the bound for the function then f n x minus s_n of x is going to be less than 1 over 2 to the power n_o for every n bigger than n_o.

So this works for all, given that bigger than 0. I can select n_o, say that this is true and that means the same application will works for every x; that means the sequence s_n converge is uniformly to f of x.

So this is an observation, which we may not be need~~ing~~ it, but it is good to observe that, if f is a bounded measurable function that this theorem says that, if f is a bounded measurable function which is non negative, then there is a increasing sequence of non negative simple functions uniformly converging and uniformly increasing to f.

So this is the case for when the function is non negative in the general case~~;~~. ~~f~~For a general measurable function~~.~~, ~~W~~we can look at the positive part and the negative part of the function. Approximate the positive part by a sequence and approximate the negative part by a sequence of simple functions.

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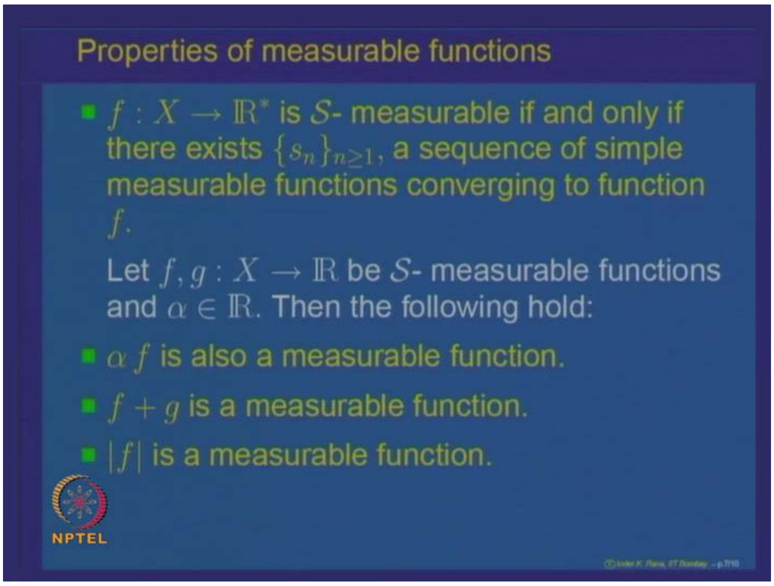

And then look at the difference of the ~~2~~two and that will give us a sequence of simple measurable functions, converging to f and they will not to any longer be monotonic. So as a consequence we ~~have,~~have that if f is not necessarily non negative function and if f is measurable function, then there exists a sequence of simple functions converging to it.

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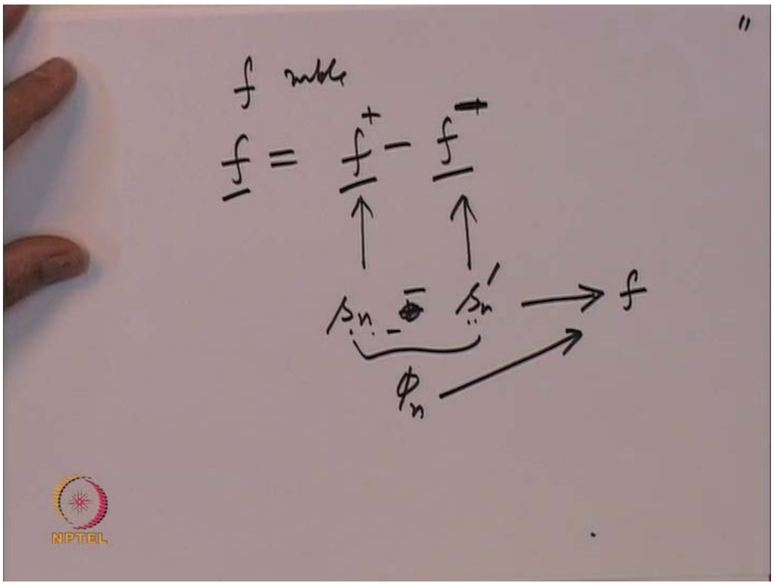

So what we have saying is if f is measurable, then I can write f is equal to f plus minus f minus and we just now observed that f measurable, then implies both of them are measurable and for this there is a sequence s_n, which is increasing to simple measurable functions non negative increasing to this.

There is another sequence call it as, say s_n` which is again non negative simple measurable functions, increasing to f`. So if I look at this plus this, then that sorry this minus this minus, then this will converge to f. So, call this as your new sequence so this is called as phi$_n$ and phi$_n$ is a sequence of the difference of simple measurable functions is measurable. So this is a s_n, which is a simple measurable function and s_n` is a simple measurable function. So, phi$_n$ is a simple measurable function.

s_n converges to f plus and s_n` converges to f minus, so the difference will converge to the difference which is f.

Only thing is this phi$_n$ converge to f, but we cannot say phi$_n$ are increasing any more. Each one of them is increasing, about the difference may not be increasing. So that proves that for a general measurable function is measurable, if and only if the sequence of simple measurable functions converging to f.

So now, let us look at some more general properties of measurable functions. let us take so, we are going to look at various properties: given two functions f and g, which are measurable, given a scalar, whether sum of the measurable functions is measurable or not, whether the part of measurable functions is measurable or not and whether scalar multiple of a measurable function is measurable or not.

So let us list all the properties which are true: first says if f is measurable and alpha is a scalar, then alpha times f is also a measurable function. So for that, this alpha could actually be any extended real number also depending upon because we are taking only keep in mind I am taking only real valued functions for the time being.

f and g are both real valued functions which are measurable and alpha is a real number. So the claim is alpha f, which is again a real valued function is measurable. Now, we can for this- we can apply our sequential criteria because f is measurable. So there is a sequence of simple measurable functions converging to it.

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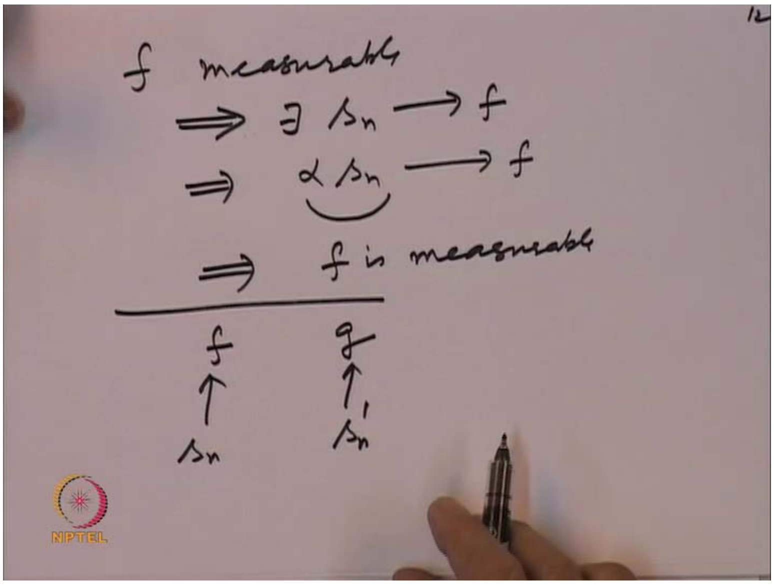

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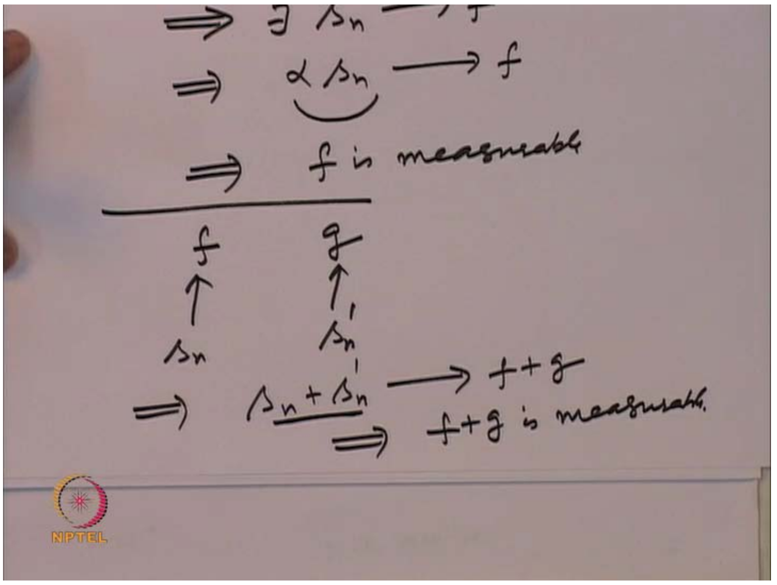

And so look at the sequence and let us look at the proof of this, that says f measurable and implies there exists a sequence s_n of simple measurable functions converging to f. But that implies, by the properties of sequence it is alpha s_n converges to f, because s_n is simple measurable. For a constant time, a simple measurable function is again a measurable. So, this is a sequence of simple measurable functions converging to f and implies by the previous theorem f is measurable.

The same proof works for sum of two functions. Let us say f and g are two measurable functions, we want to prove that f plus g is measurable. So, f measurable implies there is a sequence s_n of simple measurable functions converging to it and g is measurable so there is a sequence s_n` of simple measurable functions converging to it. So, that implies that $s_n$ plus $s_n$` converges to f plus g.

And this is once again for every n, this is a sum of simple measurable functions. So, this is again a simple measurable function. We got a sequence of simple measurable functions which converges to an f plus g and implies that f plus g is measurable.

So that implies f plus and we have proved the next step namely if f and g are measurable, then f plus g is also measurable.

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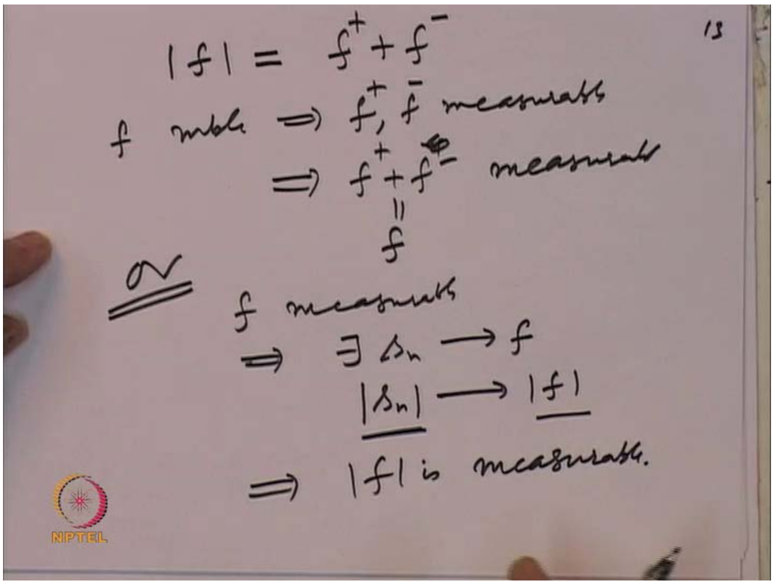

Let us look at next property if f is measurable, then mod f is also measurable. Why is mod f measurable? You can look at ~~2~~two different ways and now we have got enough techniques to conclude this, see either we can write mod f is equal to f plus plus f minus.

This is observation which will play a role later on and also this is the positive part of the function and this is the negative part of the function, f measurable implies both f plus and f minus are measurable. Implies f plus plus f minus is measurable and this is precisely my f.

So that is 1 way of looking at it or you can also look at some sequence point of view. f measurable implies, there is a sequence s_n of simple measurable functions converging to f, but then a simple argument which work for sequence is what I have already seen that mod of s_n converges to mod f and observation if s_n is simple, then mod s_n is also simple for every n. So, this is a sequence of simple measurable function converging to mod f that means mod f is measurable.

So, either you can look at sequences or you can look at the positive part and negative part. Either one will help you to conclude that if f is measurable, then mod f is also measurable. Let me look at (Refer Slide Time: 49:32) another property of measurable functions namely, that if we are seeing this property for simple functions. That if E is a set in the sigma algebra S and if f is measurable, then product of f times indicator function of E is also a measurable function.

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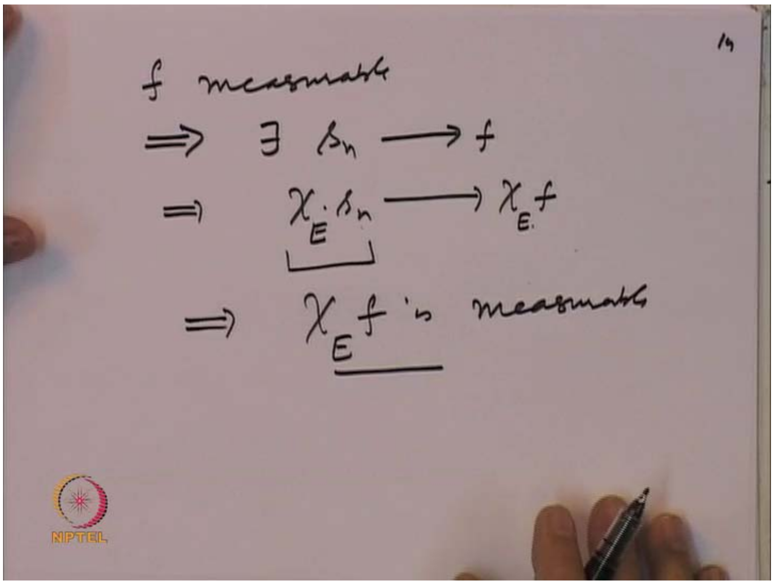

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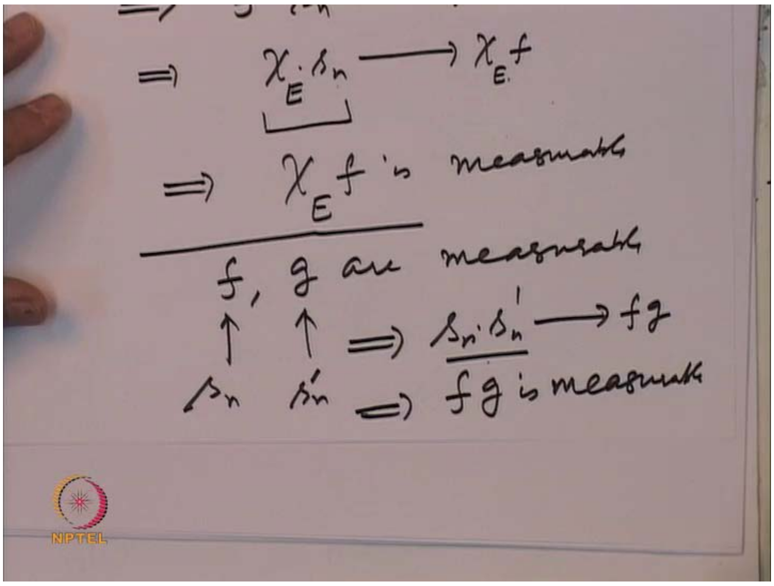

So once again, we can take the help of the criteria just now proved. f measurable implies that there exists a sequence of simple measurable functions converging to f at every point, but once that is true if s_n converges to f, then that implies look at chi_E times s_n that will converge to chi_E times f.

Because this remains multiplying by a function so, this converges to simple properties of sequences and now observe that this is, if s_n is simple measurable function, then the indicator function of E times s_n is also a simple measurable function that converges to indicator function of E times. So that implies, that indicator function of E times f is measurable.

In fact, we can go a step further and prove that you can multiply by the same argument. Suppose, f and g are measurable then for f we have got a sequence s_n of simple measurable functions converging to f and we have got a sequence s_n of simple measurable functions converging to s_n` converging to g.

So that implies, if I multiply s_n` that converges to f times g and product of simple measurable functions that we have already seen, is again a simple measurable function. So, a sequence of simple measurable function converging to f of g, that means implies that f and g are measurable.

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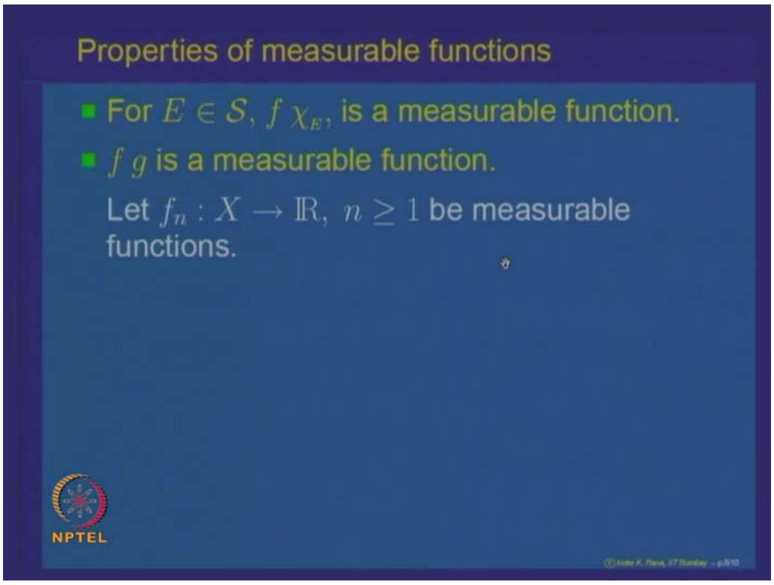

So product of measurable functions is also measurable. I think we will close here today and look at the sequences of measurable functions next time. So, today what we have proved?

We have looked at the important criteria, characterization of measurable functions: a function f defined on a set X taking extended real valued functions is measurable, if and only if keep in mind it is a characterization so f measurable f function defined on x is measurable, if and only if we can find a sequence of simple functions converging to it.

If f is non negative we can find this sequence of simple functions s_n, which is increasing and converging to f. If an addition we know that f is a bounded measurable function, then you can have the sequence of simple functions s_n which converges uniformly to f. so that are the important criteria.

We have seen some applications today and we will see more applications later on also and then we looked at the algebra of measurable functions. We proved that if f is measurable, then scalar times f is also a measurable function and if f and g are measurable, then f plus g is also measurable, f into g is measurable and the mod f is also measurable.

This is for the real valued functions and in case the functions are extended real valued then, while defining f plus g and f into g you have to be slightly careful because f may take the value plus infinity at a point and g may take the value minus infinity. Then how will you define f plus g?

So for such kind of problematic sets, we can separate out a set A on which f of x is plus infinity and all g of x is equal to minus infinity or f of x is minus infinity and g of x is the plus infinity.

So on this set A, we may not be able to define what is f plus g. But, outside that we can define f plus g and this set where f is plus infinity and g is equal to minus infinity or other way around is a measurable set; is in the sigma algebra. We can change the values, we can define f plus g to be equal to anything we like and still that f plus g will be a measurable function.

Modifications of the algebra of measurable functions properties still remain true when the functions are extended real valued. We will continue the study of sequences of measurable functions in next lecture. Thank you.