Measure and Integration

Prof. Inder K. Rana

Department of Mathematics

Indian Institute of Technology, Bombay

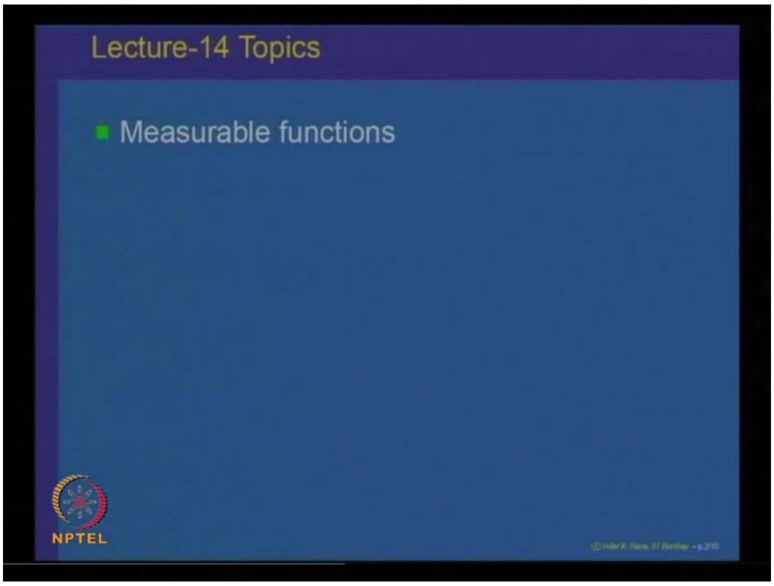
Module No. # 05

Lecture No. # 14

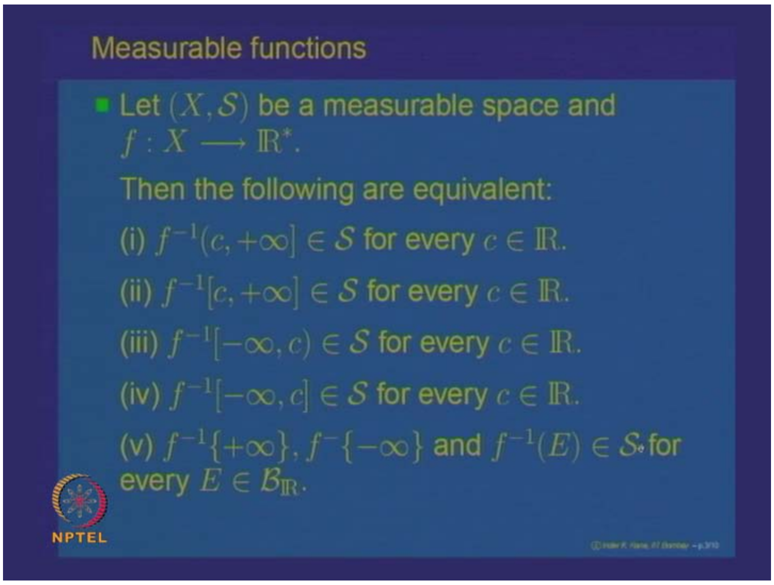
Measurable Functions

Welcome to lecture number 14 on measure and integration. Today, we will start looking at functions on measurable spaces; these are called measurable functions.

(Refer Slide Time: 00:33)



(Refer Slide Time: 00:36)



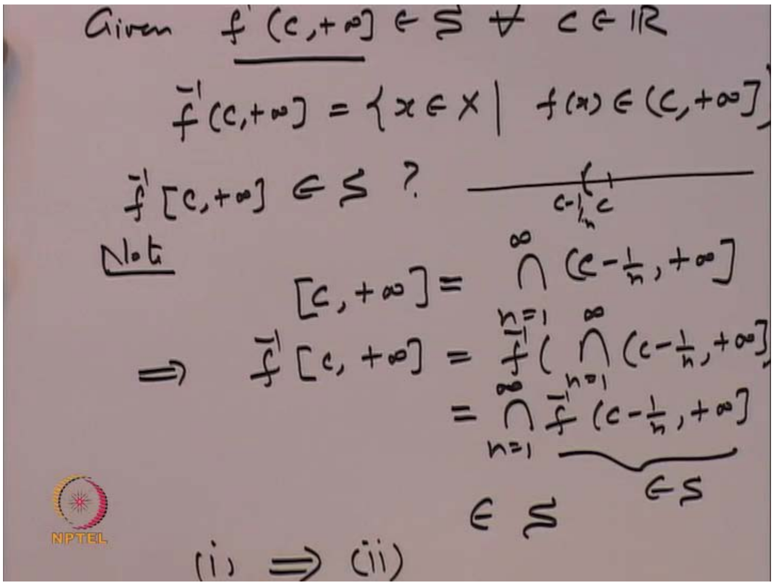
To start with, we assume that we have a measurable space X, S. X is a set and S is a sigma algebra of subsets of the set X. We have a function f defined on X taking extended real values; R star denotes the set extended real line; that is, the set of all real numbers together with plus infinity and minus infinity and with ((.)) possible operations that we had defined earlier. We will be looking at functions which are extended real values defined on the set X.

To start with, we want to prove the following; namely, for this function f, the following statements are equivalent: inverse image of the open interval c, closed at infinity – if you take the inverse image of any such interval, then that belongs to the sigma algebra S. We will show that this is equivalent to saying that the inverse image of the closed interval c to infinity belongs to S for every c, the real number. This is equivalent to saying that the inverse image of the interval minus infinity to c, minus infinity closed and c open, also belongs to the sigma algebra S.

Then, we will show that this is also equivalent to saying that the inverse images of all the intervals of the type minus infinity to c, c closed, belongs to S for every c belonging to R. We will show that these four are equivalent to each other and also these are all equivalent to the following; namely, the points f inverse of plus infinity and the set f inverse of minus infinity along with f inverse of every set E, E a Borel set in R, belong to S.

We will show that for a function f defined on X taking extended real numbers as the values, these five conditions are equivalent. The methodology is going to be this: we will prove (i) is equivalent to (ii); (ii) is equivalent to (iii); (iii) is equivalent to (iv); and any one of them is equivalent to (v).

(Refer Slide Time: 02:57)



Let us start proving these properties. First property: we are given that f inverse of c to plus infinity belongs to S for every c belonging to R. We want to prove the same property for f inverse of …. Keep in mind what is f inverse; f inverse of c to plus infinity is the set of all points x belonging to X such that f of x belongs to c to plus infinity. This is a set of all points x in the domain which are mapped into the interval c to infinity. f inverse does not mean that the function is invertible or anything; this is a symbol used (Refer Slide Time: 03:42); it is a pull-back of the points which go into this c to plus infinity.

We want to look at f inverse of closed interval c to plus infinity; we want to show that this belongs to S. To show that, let us observe the simple set-theoretic equality; namely, the closed interval c to plus infinity can be written as intersection of look at the open interval c to c minus 1 by n to plus infinity and look at the intersection of all these intervals. Keep in mind here is c and here is c minus 1 over n (Refer Slide Time: 04:30).

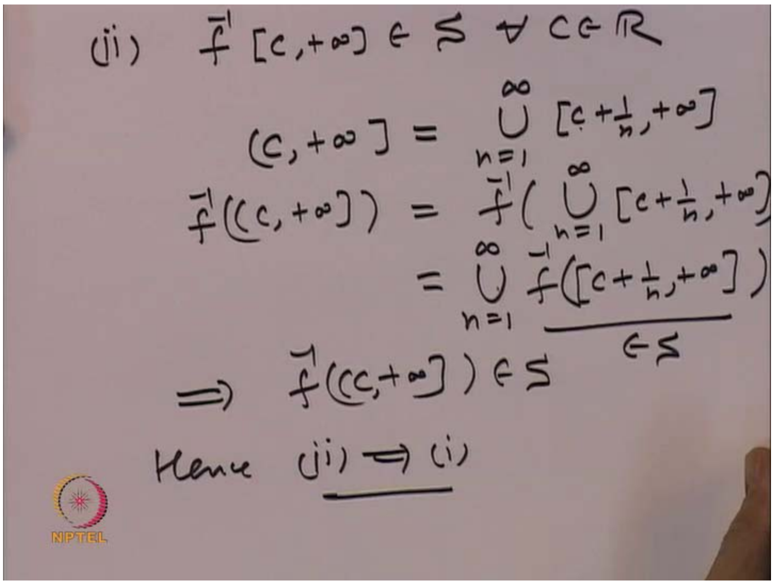
If you take this open interval, this open interval c minus 1 by n to plus infinity includes this closed interval c to infinity for every n; the intersection also we had included.

Actually, this is equal because any point which is slightly bigger than c can be excluded by taking n as sufficiently large. So, c minus 1 over n converges to c; that is the basic idea. This is a simple identity about intervals which should be easy to prove.

This implies that the f inverse of c to plus infinity is equal to f inverse of intersection n equal to 1 to infinity of c minus 1 over n to plus infinity. Here is another simple observation that the inverse images of intersections are same as intersection of the inverse images. This is equal to f inverse of c minus 1 over n to plus infinity closed. We are given that whenever the interval is of the type c to plus infinity, c open, the inverse image is in S; so, each one of these sets belongs to S; S is a sigma algebra; so, intersection belongs to S; so, this belongs to S (Refer Slide Time: 05:49).

Basically, what we have done is the closed interval c to plus infinity is written as an intersection of open intervals c minus 1 over n to plus infinity. Observing that the inverse images of intersections are intersections of inverse images, we get that f inverse of the closed interval c to plus infinity belongs S. We have proved (i) implies (ii).

(Refer Slide Time: 06:20)

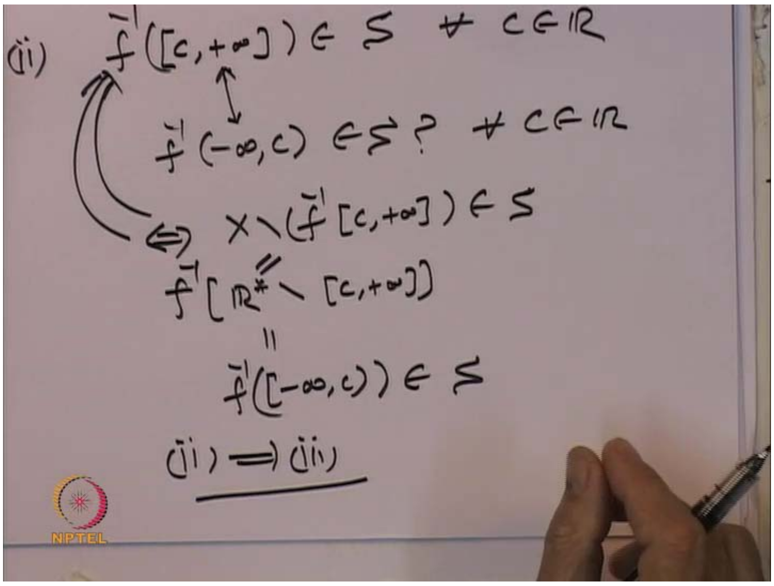


Let us show that (ii) also implies (i). What is the statement (ii)? Statement (ii) says f inverse of the closed interval c to plus infinity belongs to S for every c belonging to R; that is the statement (ii). We want to now prove the same thing for open intervals. The idea is the open interval c to plus infinity can be expressed as union of the closed intervals c plus 1 by n to plus infinity, n equal to 1 to infinity. That is quite easy to verify; the interval c plus 1 by n to infinity is inside the interval c to plus infinity; so, this union is inside it; converse is easy to check, because c plus 1 over n goes to c; this is actually equal to c to infinity.

Once again, observe that the inverse image of c to plus infinity is equal to f inverse of the union n equal to 1 to infinity c plus 1 over n to plus infinity. Once again, a simple observation is that the inverse images of the union is union of the inverse images; that gives us that this is f inverse of c plus 1 over n to plus infinity. We are given that each one of them belongs to S. This is a union of sets in S; S is a sigma algebra; it implies that this set also, f inverse of c to plus infinity, belongs to S (Refer Slide Time: 08:10).

Hence, we have shown that (ii) implies (i). So, (i) implies (ii) and (ii) implies (i). Thus, we have shown that the statement (i) implies statement (ii) and the statement (ii) implies (i). If you see the proofs carefully, in both of them we have just tried to represent a closed interval as an intersection of open intervals; also, in the (ii) implies (i) we have tried to use the fact that you can represent an open interval as a union of closed intervals. Similar facts are used in proving the remaining statements; let us just prove the remaining statement, namely, (ii) implies (iii).

(Refer Slide Time: 09:01)

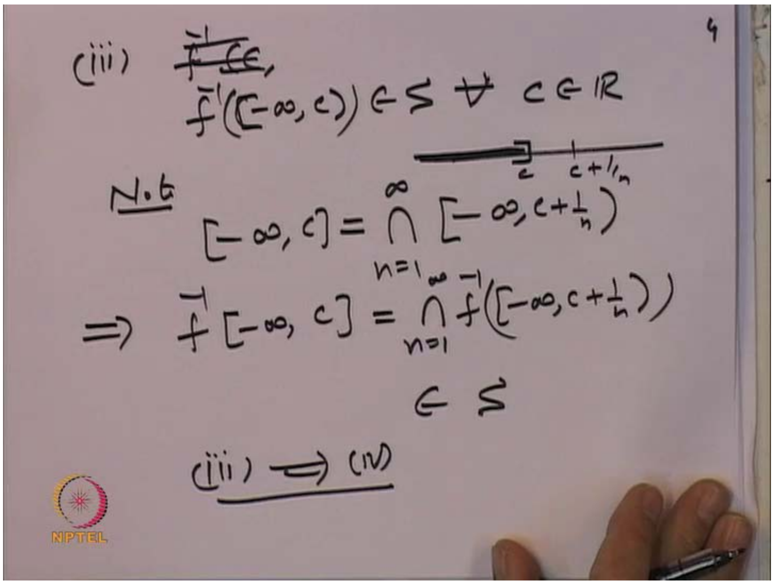


Let us prove (ii) implies (iii). The statement (ii) is regarding closed intervals. We are given that f inverse of this belongs to S for every c belonging to R; that is the statement (ii) which is given. We want to show that f inverse of minus infinity to c open belongs to

S for every c belonging to R. If you look carefully, this interval and this interval are related with each other (Refer Slide Time: 09:33). They are complements of each other.

The given statement implies that because this belongs to S, X minus f inverse of c to plus infinity also belongs to S. Here is a small observation: the complement of the inverse image is nothing but the inverse image of the complement. This set is equal to f inverse of R star minus c to plus infinity. That is equal to f inverse of minus infinity to c open, because here c is closed; so, this also belongs to S, because S is a sigma algebra; if a set belongs, its complement belongs; so, this belongs to S. All these statements are reversible; if this belongs, then its complement belongs; these are all if and only if statements. So, (ii) implies (iii) is obvious by taking complements.

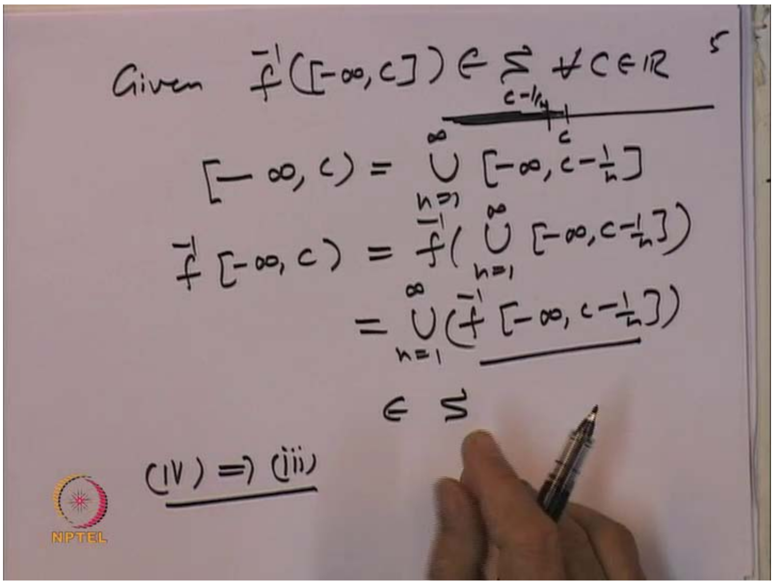
(Refer Slide Time: 11:01)



Let us prove (iii) implies (iv). The statement (iii) implies what is given to us is f inverse of minus infinity to c open belongs to S for every c belonging to R. Here, we want to conclude this closed interval…. Note that the closed interval to c is equal to… We want to include the point c inside; it is nothing but look at minus infinity to c plus 1 over n – the open interval. Here is c and here is c plus 1 over n (Refer Slide Time: 11:49). This interval – the closed interval – is already inside c plus 1 over n for every n.

If I take the intersection of all this, that will give us the closed interval minus infinity to c; it is similar to the earlier argument. This implies that f inverse of minus infinity to c is equal to f inverse of the intersection; that is, intersection of the inverse images – f inverse of minus infinity to c plus 1 over n open; each one of them belongs to n and so this belongs to S. So (iii) implies (iv). If f inverse of minus infinity to c open belongs, then f inverse of minus infinity to c closed also belongs. So, (iii) implies (iv).

(Refer Slide Time: 12:49)



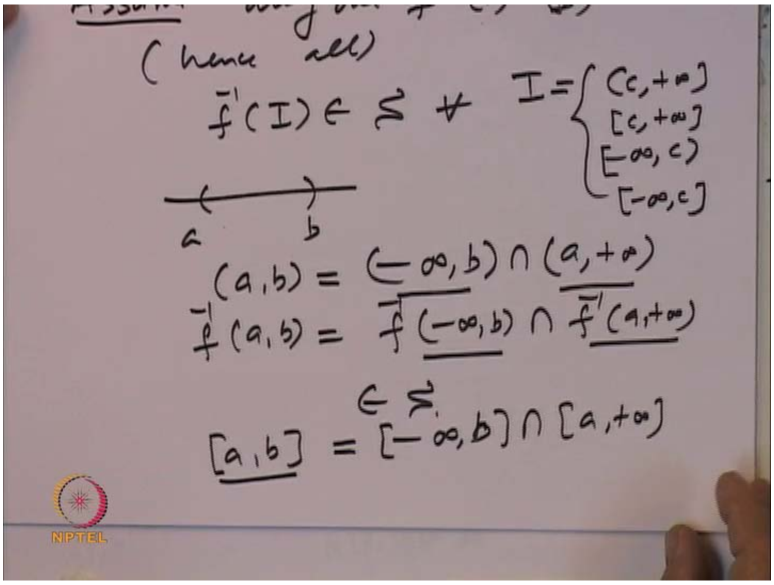
Let us prove the converse statement namely (iv) implies (i) (Refer Slide Time: 12:45). We are given f inverse of minus infinity to c closed belongs to S for every c belonging to R. We want to look at f inverse of the open interval c. Once again, it is a similar situation; that is, minus infinity to c; it is here (Refer Slide Time: 13:14). We want to look at the open interval. Let us look at the union of intervals minus infinity to c minus 1 over n, n equal to 1 to infinity.

Here is c minus 1 over n (Refer Slide Time: 13:31). These are all inside it – closed interval; the unions will give us this open interval. Once again, taking the inverse images minus infinity to c is equal to f inverse of the union n equal to 1 to infinity; f inverse of the union is union of the inverse images; that gives us minus infinity to c minus 1 over n. Each one of them is given to be inside S; that implies that this belongs to S; so, (iv) implies (iii) is also true.

What we have shown till now is that the first four statements are equivalent to each other (Refer Slide Time: 14:27). The first statement was about intervals of the type c to infinity, open; the next one was c closed; next was minus infinity to c. Inverse images of all these types of intervals are inside S; all these statements are equivalent to each other.

Now, let us prove that this implies that f inverse of plus infinity and f inverse of minus infinity and f inverse of every Borel set is inside S.

(Refer Slide Time: 15:05)

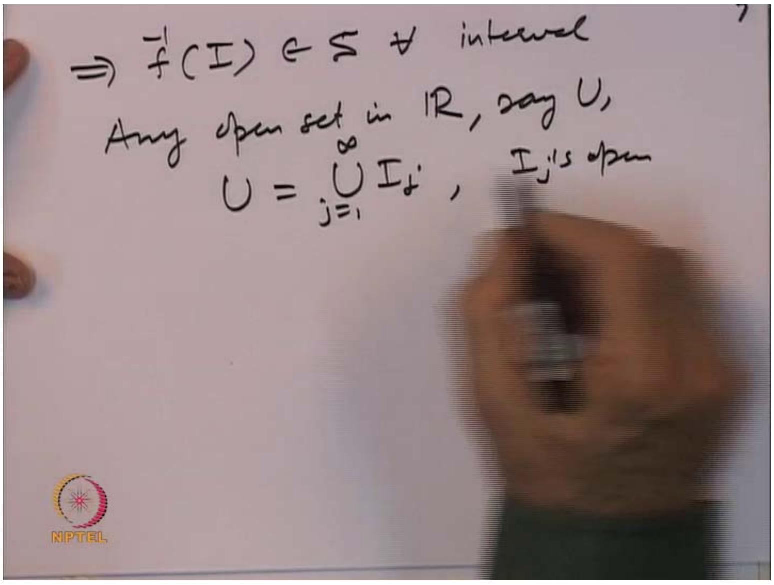


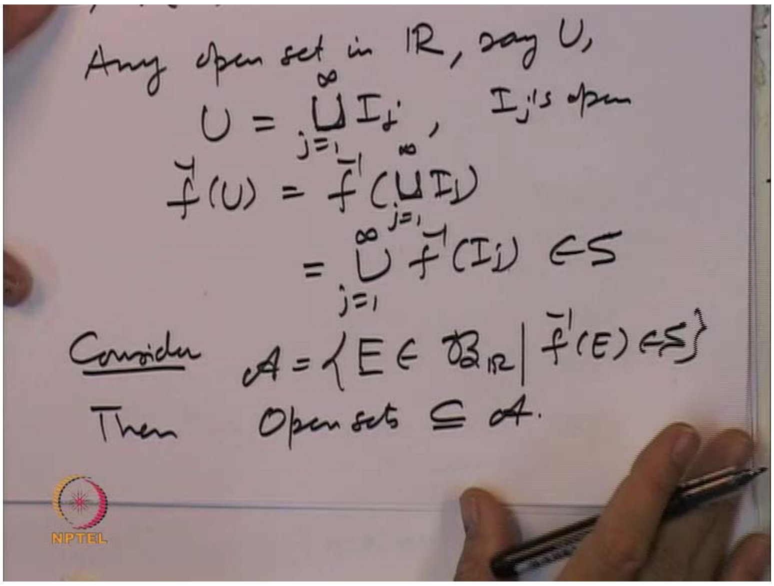
Let us assume any one of (i) to (iv) and hence all because they are equivalent. So, we know that f inverse of an interval belongs to S for every interval I which looks like c to plus infinity or looks like closed c to plus infinity or it looks like minus infinity to c open or minus infinity to c closed. For all these four types of intervals, any one of the first four statements implies they belong to S.

Now, look at any other interval. Supposing I is an open interval a to b. We can write this open interval a to b as minus infinity to b open interval intersection with the open interval a to plus infinity. We know that inverse image of this interval belongs to the sigma algebra and inverse image of this belongs to the sigma algebra (Refer Slide Time: 16:21). That will give us that the inverse image of a, b is equal to f inverse of minus infinity to b intersection f inverse of a to plus infinity. Both belong to the sigma algebra and so this will belong to the sigma algebra S.

What I am trying to say is that any one of the statements (i) to (iv) implies that inverse image of every open interval also belongs. Similarly, we can take actually a closed interval also; for example, a closed interval a, b can be written as minus infinity to b intersection a to plus infinity. A similar argument will imply that inverse of this interval also belongs to S.

(Refer Slide Time: 17:15)

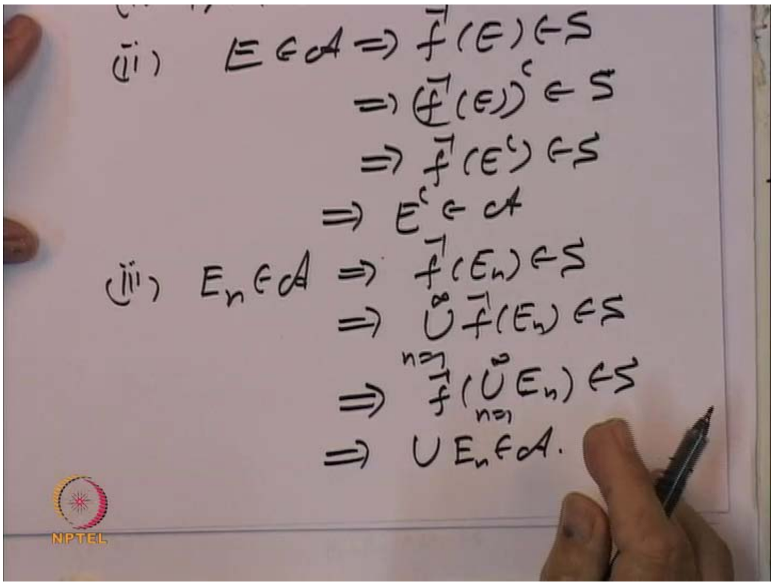




If we assume any one of the statements (i) to (iv), that implies that f inverse of every interval belongs to S for every interval. Recall that any open set in R is a countable union of open intervals; say, the open set is U; then, U can be written as union of I_js, j equal to 1 to infinity, I_js open. Actually, you can write it as a disjoint union of open intervals also – countable disjoint union of open intervals.

So, f inverse of U will be equal to f inverse of disjoint union of I_js which is the same as union of f inverse of I_js, j equal to 1 to infinity; actually, disjoint is not needed, but anyway, that is okay. f inverse of every open interval belongs to S; so, this belongs to S (Refer Slide Time: 18:24). If we assume any one of the four conditions, then that implies that f inverse of every open set is in the sigma algebra. Here is the sigma algebra technique: consider the class A of all subsets in B_R such that f inverse of E belongs to S. Just now we showed that the open sets are inside A; it is easy to check that A is a sigma algebra. Let us check that A is a sigma algebra.

(Refer Slide Time: 19:30)

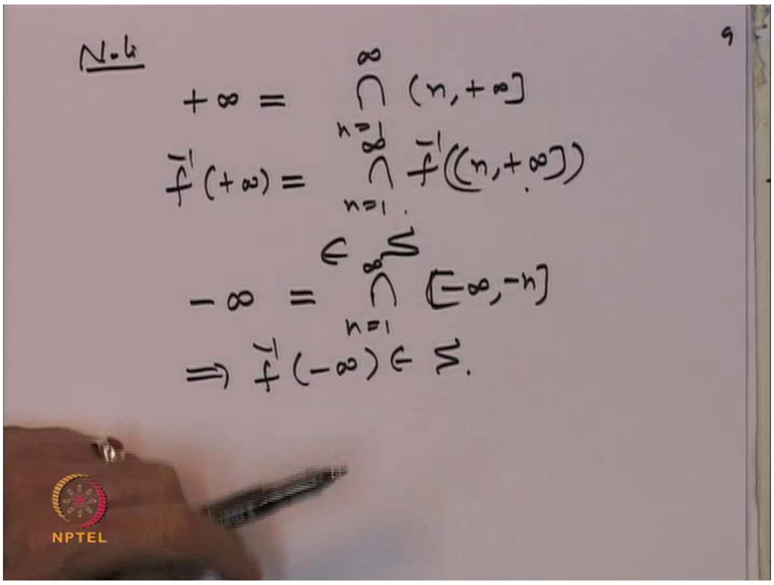


Why is A a sigma algebra? Clearly, the empty set and the whole space R belong to A, because X and the empty set belong to S. Secondly, let us observe that if a set E belongs to A, that means that f inverse of E belongs to S; that implies that f inverse of E complement belongs to S, because S is a sigma algebra; that is same as f inverse of E complement belongs to S.

It implies E complement belongs to A; so, A is closed under complements. Finally, if E_ns belong to A, it implies that f inverse of E_n belongs to S; it implies union of f inverse of E_ns also belongs to S, because S is a sigma algebra. Hence, that implies that union of inverse images is inverse image of the union; so, union E_n also belongs to S; it implies union E_ns belong to A. We have verified that A is a sigma algebra (Refer Slide Time: 20:53)

This is a sigma algebra including open sets. So, it must include the Borel sigma algebra inside, but it is already a subclass of Borel sets; so, A is equal to the class of Borel sets. That means if we assume any one of those first four conditions in the statements that we just now stated, then that implies the statement that f inverse image of every Borel set is in the sigma algebra S (Refer Slide Time: 21:24). Let us verify that the inverse images of the points plus infinity and minus infinity are also….

(Refer Slide Time: 21:33)

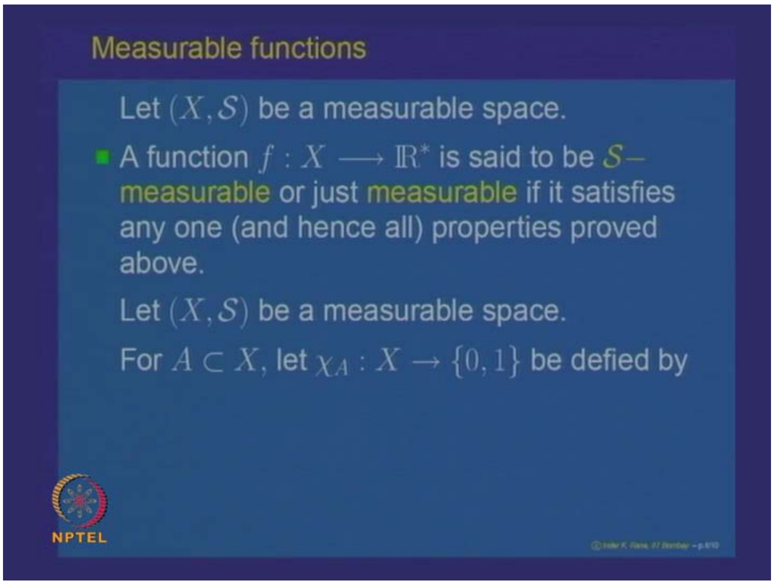


Note that plus infinity can be written as intersection of n to plus infinity, n equal to 1 to infinity. f inverse of plus infinity is equal to intersection n equal to 1 to infinity f inverse of n to plus infinity. f inverse of this is equal to f inverse of the right-hand side (Refer Slide Time: 21:59). The right-hand side is the intersection; so, it is the intersection of the inverse images. Each one is an interval; inverse image of each one of the intervals belongs to S; so, intersection belongs to S; so, this belongs to S.

A similar argument for minus infinity will imply, because minus infinity can be written as intersection of n equal to 1 to infinity of minus infinity to minus n. The inverse image of this will be intersection of inverse images and will imply that f inverse of minus infinity belongs to S. We have shown that if you assume any one of those four conditions stated above, then that implies that the inverse image of the point plus infinity and inverse image of every Borel set belong to the sigma algebra S.

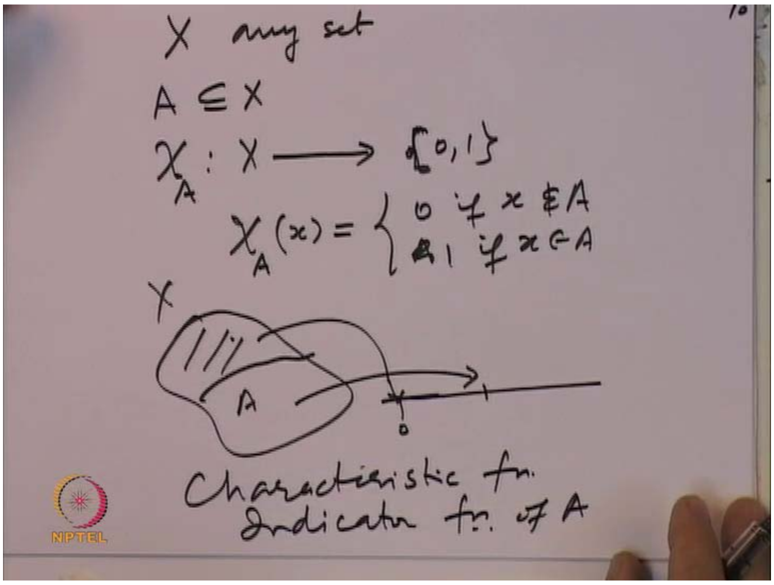
The converse statement is obvious, because every interval is a Borel set. Saying that statement (v) implies any one of the four statements above is obvious, because every interval is a Borel set; that is a special case. We have proved this theorem; namely, for a function f defined on a set X taking extended real-valued functions, all these five conditions are equivalent to each other (Refer Slide Time: 23:19). If you assume any one of them, then other ((.)) will also hold.

(Refer Slide Time: 23:40)



A function which satisfies any one of these conditions is called a measurable function. A measurable function on X taking extended real values is a function which satisfies any one of those five conditions as stated here (Refer Slide Time: 23:54). These are going to be an important class of functions for us to deal with. Let us look at some examples. The first example is that of what is called the indicator function of a set. Let us look at what is called the indicator function of a set X.

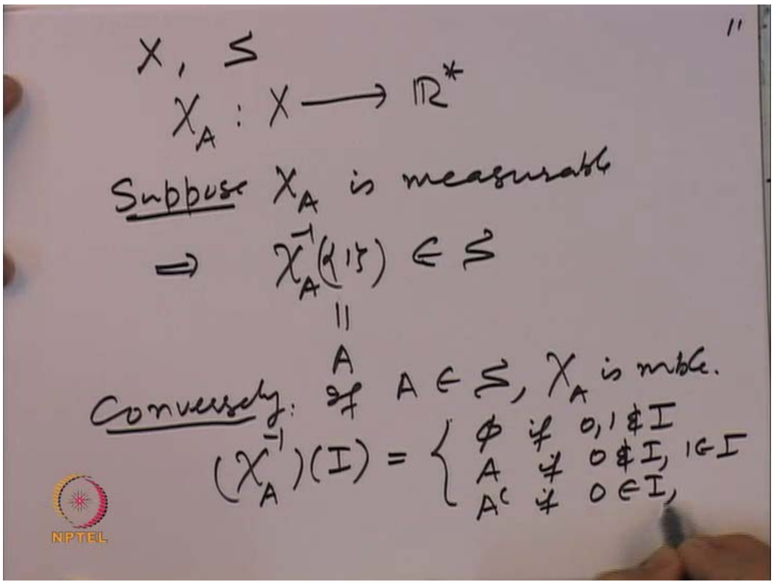
(Refer Slide Time: 24:15)

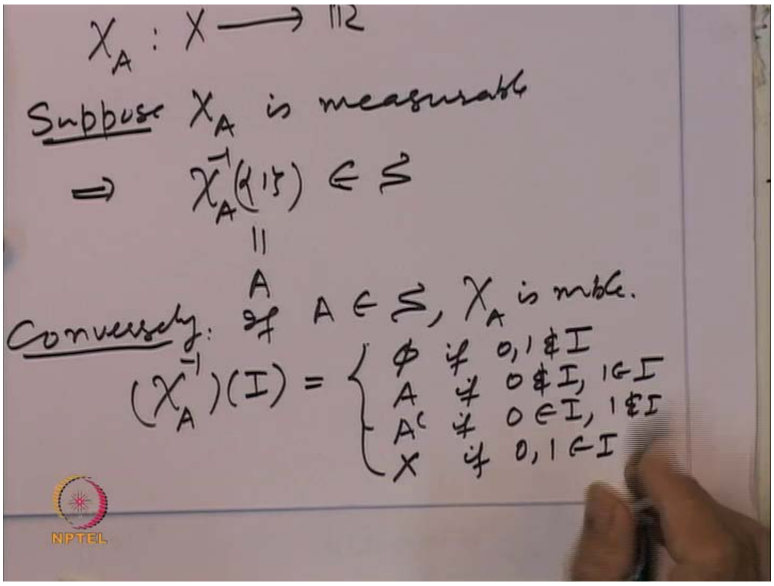


Let us take any set X and A is a subset of X. We define a function called the chi of A; this is the Greek letter chi (Refer Slide Time: 24:27) and lower suffix A. It is a function on X taking two values 0 or 1. This is called the characteristic function or the indicator function. This function takes a value; at a point x, the value is 0 if x does not belong to A; at the point A, the value is 1 if x belongs to A.

Here is the set X; here is the set A. On A, it gives the value 1; outside A, it gives the value 0. It is a two-valued function; the points where it takes the value 1 is exactly the points in the set A; so, this is called the characteristic function or the indicator function of the set A. This is called the indicator function of the set A.

(Refer Slide Time: 25:42)



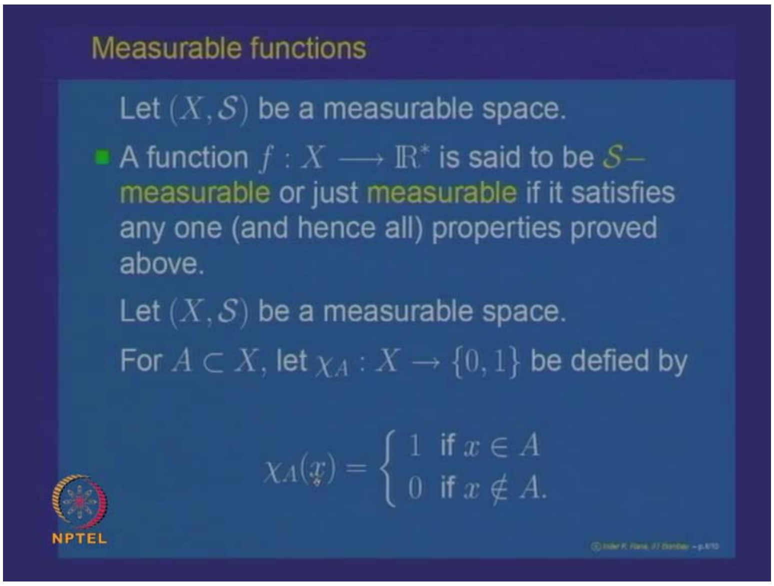


X is a set; S is a sigma algebra; we have got the indicator function A of the set A on X taking, of course, only two values. We can consider it as a function taking extended real values. We want to know if it is measurable. Suppose the indicator function of A is measurable. That implies that if I look at chi_A inverse of the singleton point 1, that belongs to S, but what is that value? What are the points where it takes the value 1? That is precisely A; that is the set A; so, A belongs to S.

If the indicator function is measurable, then we get A belongs to S. Conversely, if A belongs to S, we claim that chi of A is measurable. For that, look at chi_A inverse of any interval I. What is that going to be? The inverse image of an interval is going to be equal to the empty set if 0 or 1 does not belong to the interval I, because then there is no point which goes to the interval; it is equal to A if 0 does not belong to I and 1 belongs to I; similarly, it is A complement if 0 belongs to I and 1 does not belong to I; it is equal to X if both 0 and 1 belong to I.

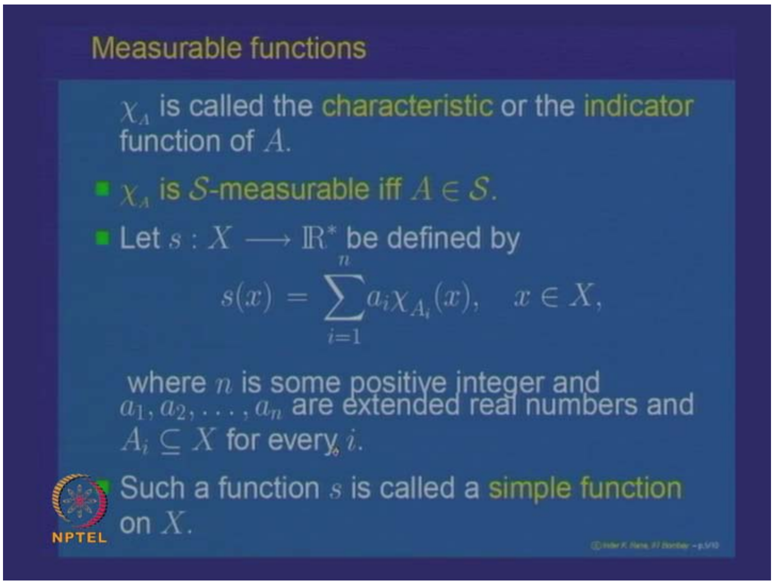
It is an empty set or it is a set A or A complement or X and all of these are elements of the sigma algebra S. The inverse image of every interval is in S; hence, the indicator function is a measurable function. What we have shown is that the indicator function is measurable (Refer Slide Time: 28:00).

(Refer Slide Time: 28:03)



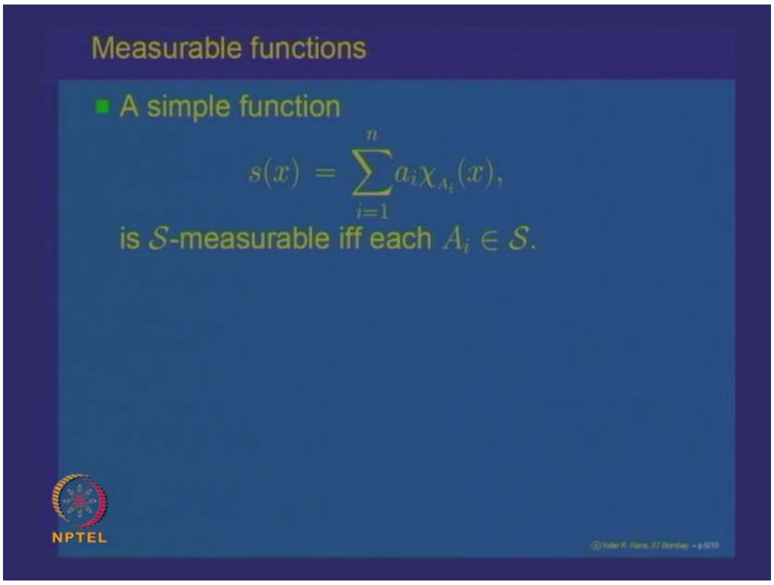
This indicator function is defined as 1 if x belongs to A and 0 if x does not belong to A.

(Refer Slide Time: 28:09)



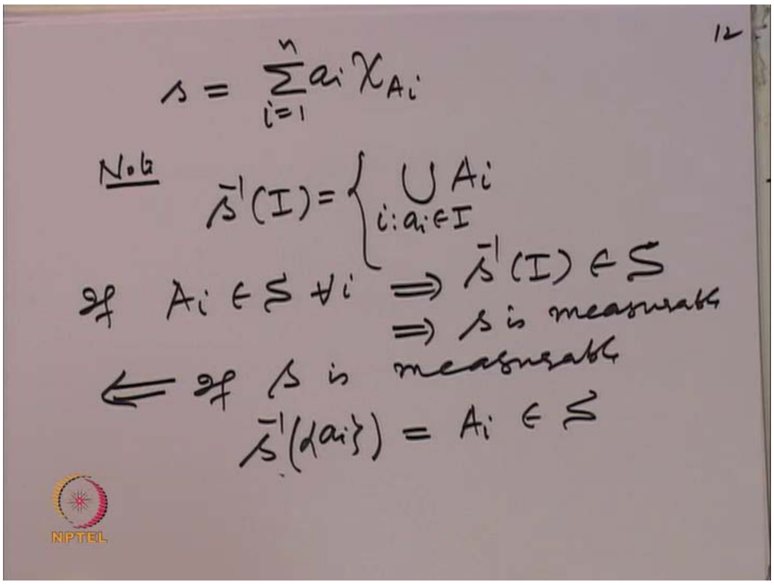
The characteristic function is measurable if and only if the set A belongs to S; this is the simplest example of a measurable function. Let us consider a linear combination of the indicator functions. Suppose s is a function defined on X such that s of X is equal to a_i times the indicator function of a set A_i at (( )) at x, i equal to 1 to n. Look at sets A_1, A_2 up to A_n – subsets of X; look at their indicator functions and take a linear combination of them – a_i times the indicator function of A_i; such a function is called a simple function on X. Such a function is called a simple function on X.

(Refer Slide Time: 29:09)



Our claim is that a simple function is measurable if and only if each one of the A_is belongs to S.

(Refer Slide Time: 29:20)

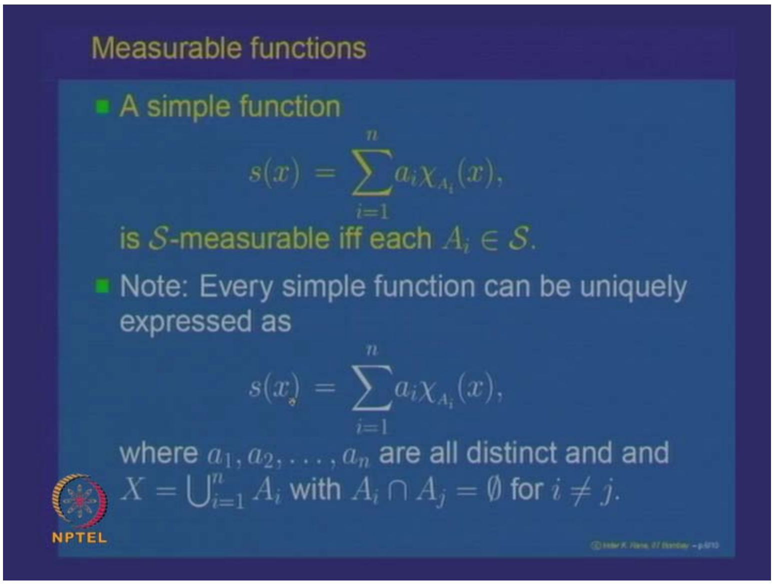


We want to prove the simple function S which is sigma a_i indicator function of A_i, i equal to 1 to n is measurable if and only if…. Note: to check measurability, we have to look at s inverse of an interval I. What is that going to be? The function s takes values small a_is on the set A_i; this is the main thing to be observed – a finite linear combination of the indicator functions is a function which takes only finite member of values, namely, a_1, a_2 up to a_n and the value small a_i is taken on the set capital A_i.

What will be s inverse of I? That will be union of those sets A_i union over i such that a_i belongs to the interval I. Clear? Let us once again observe that s takes values A_1, A_2 up to A_n. Look at the inverse image of an interval I; look at those is such that a_i belongs to the interval I;. the pull-back of this will be the set A_i. Look at the unions of these A_is; so, s inverse of I is union of A_is.

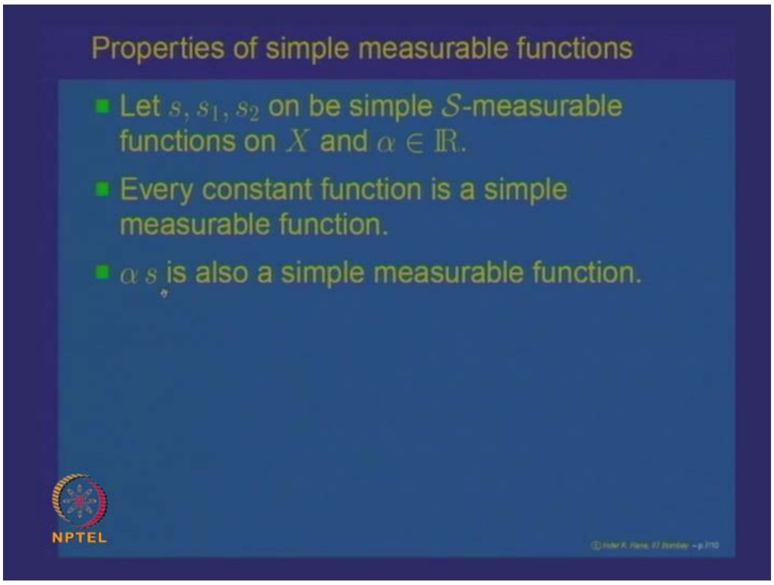
If each A_i belongs to S for every i, this will imply that s inverse of intervals belongs to S. It implies that s is measurable because the inverse image of every interval belongs to I. This interval is in extended real numbers (Refer Slide Time: 31:11); plus infinity and minus infinity are included in this; so, it is measurable. Conversely, if s is measurable, then look at s inverse of singleton a_i; that will be equal to A_i. Hence, measurability implies this belongs to S. Of course, here, one has to take slight care; we can assume that all the A_is are distinct, because if they are not distinct we can put together those A_is into one box. That says that a simple function is measurable if and only if all the sets involved in the representation ((.)) a_i times chi of A_i are all measurable.

(Refer Slide Time: 32:16)



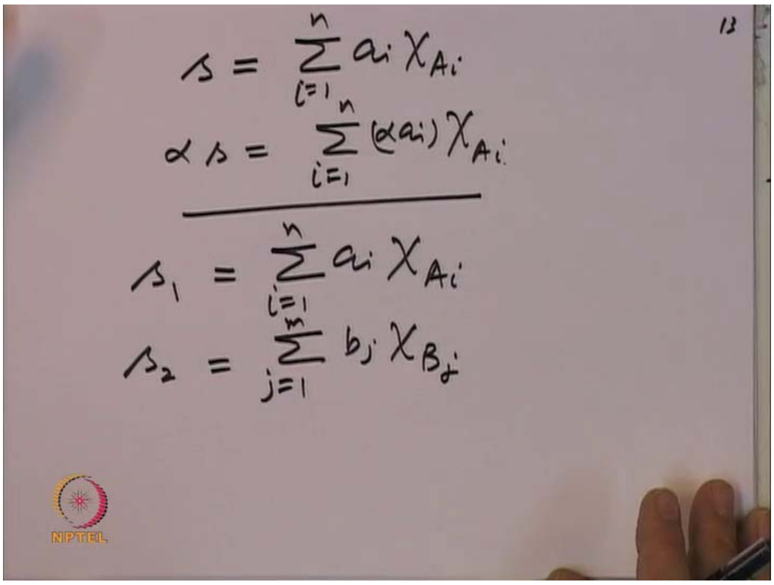
As observed just now, we have given a simple function s; we can also write it in the form, we can represent as, summation a_i indicator function of A_i where all the a_is are distinct and these capital A_is are disjoint, because if sets are not disjoint we can put them together; if the same value is taken on two distinct sets, then we can put them together in one box and call that set as a new A_i. This is sometimes called a standard representation of a simple function where the a_is are distinct and these capital A_is form a partition of the whole space X. A simple function is nothing but a finite linear combination of indicator functions; that is an example of a measurable function.

(Refer Slide Time: 33:12)



We will study some more properties of this class of simple measurable functions. Let us start; let s, s_1 and s_2 be simple measurable functions and alpha be a real number. First of all, we want to observe that every constant function is a simple measurable function. What is a constant function? A constant function is nothing but a function which takes a single value everywhere on the set. We can think of it as this: if the constant value taken is c, then it is c times the indicator function of the whole space X. Every constant function is simple measurable, because it is a constant multiple of the indicator function. Alpha times a simple function is also a simple measurable function because of the fact that if…

(Refer Slide Time: 34:08)

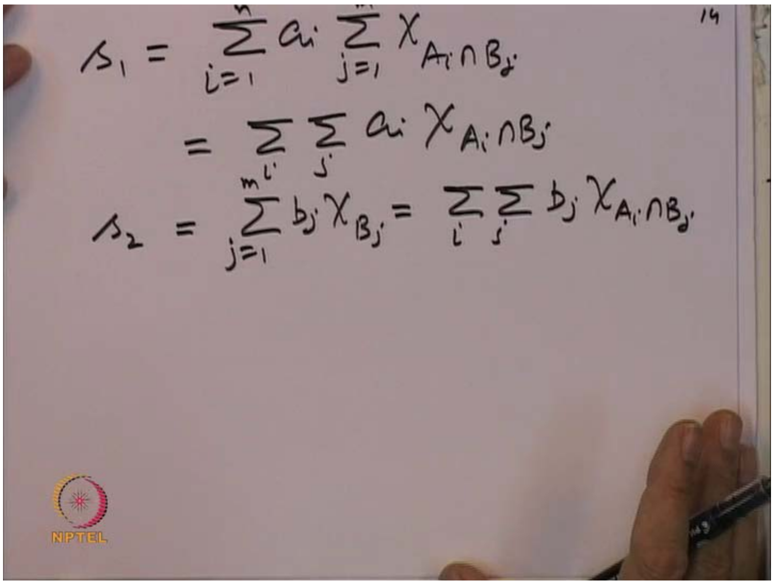


If a simple function s is equal to summation a_i chi A_i, i equal to 1 to n, then alpha times s is equal to sigma alpha a_i times chi of A_i, i equal to 1 to n. alpha s is again a simple function and only its values have changed, but the sets on which these values are taken remain the same. Clearly, it indicates that if s is measurable then alpha s is also a measurable set.

The next property we want to check is that if s_1 and s_2 are two simple measurable functions, then s_1 plus s_2 is also a simple measurable function. Let us take a function s_1 which is sigma a_i chi A_i, i equal to 1 to n. Let us say s_1 has the representation sigma a_i chi A_i and s_2 has the representation j equal to 1 to m b_j chi of B_j. Whenever one is dealing with more than one simple function, the idea is to try to bring the sets involved in the representation to be the same.

We have a standard representation that union A_i is equal to X; here, union B_js is also equal to X. Then you can write s_1 as…. Each A_i can be decomposed into a union of the B_js. You can write i equal to 1 to n a_i chi of A_i intersection B_j and union over js. Each A_i can be intersected with union of B_js. Here is an observation: if you have two sets A and B and they are disjoint, then A union B is equal to chi of A plus chi of B; this we leave for you to verify: the indicator function of the union of two sets is equal to sum of the indicator functions whenever the sets are disjoint.

(Refer Slide Time: 36:49)

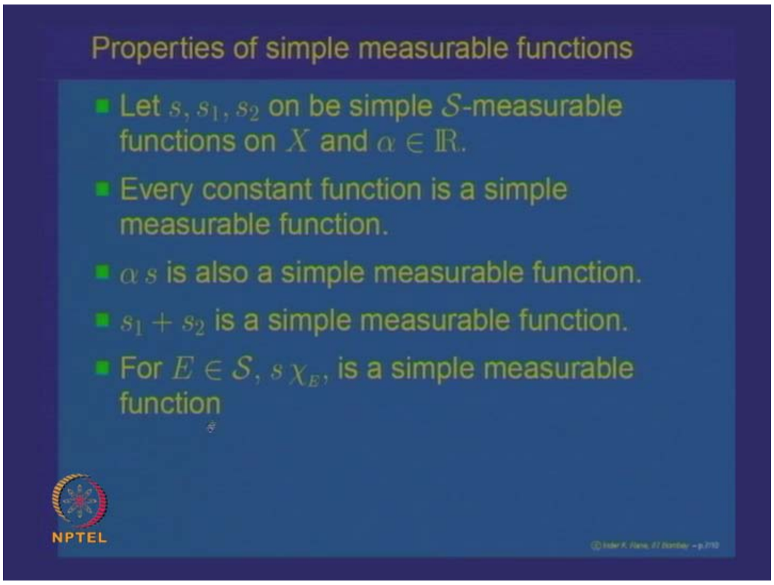


Using that, we can write s_1 as summation i equal to 1 to n a_i summation over j equal to 1 to m chi of A_i intersection B_j. This is the same as summation over i summation over j a_i chi of A_i intersection B_j. Similarly, for the second simple function s_2 which had the representation b_j chi of B_j, j equal to 1 to m, we can write this as summation over i summation over j b_j chi of A_i intersection B_j.

What we are saying is that whenever we are given two or a finite number of simple functions, we can assume without loss of generality that the indicator functions involved are of same sets. s_1 is equal to summation over i summation over j a_i times indicator function of A_i intersection B_j. Similarly, s_2 can be written as summation over i summation over j b_j indicator function of A_i intersection B_j. Then, what is s_1 plus s_2? s_1 plus s_2 is nothing but summation over i summation over j of a_i plus b_j indicator function of A_i intersection B_j.

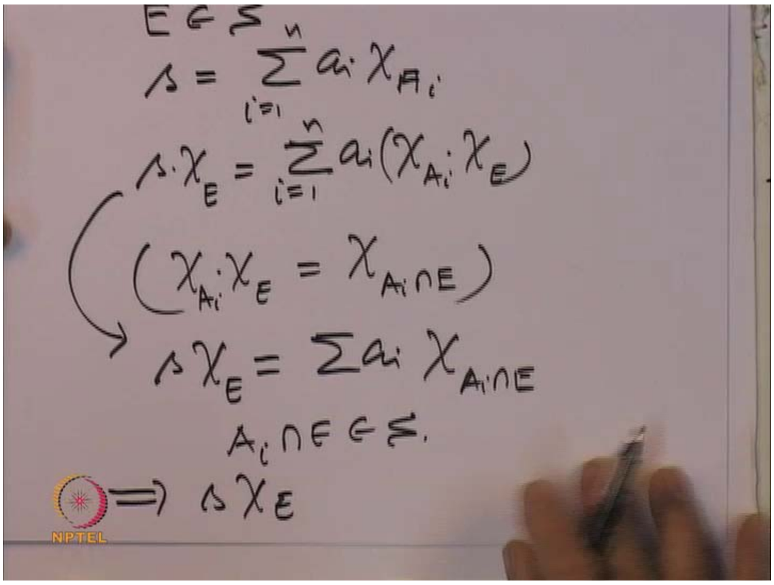
That should be clear, because if I take a point x, then if x belongs to A_i intersection B_j, then s_1 gives the value a_i and s_2 gives the value b_j; sum will give the value a_i plus b_j; outside, the value is 0; so, one does not have to bother. s_1 plus s_2 can be given the representation summation over i summation over j a_i plus b_j of this (Refer Slide Time: 38:55). Since A_i belongs to the sigma algebra and B_js belong to the sigma algebra, that implies A_is intersection B_js also belong to the sigma algebra. So, s_1 plus s_2 is written as a linear combination of indicator function of sets which are in the sigma algebra; that implies s_1 plus s_2 is measurable.

(Refer Slide Time: 39:30)



This proves the property that the class of simple measurable functions is closed under addition. The first property said it is closed under scalar multiplication; this says it is closed under addition. Next, let us take any fixed set – any set E in the sigma algebra – and multiply S with the indicator function of E; then, the claim is this is also a simple measurable function; that comes from a very simple observation.

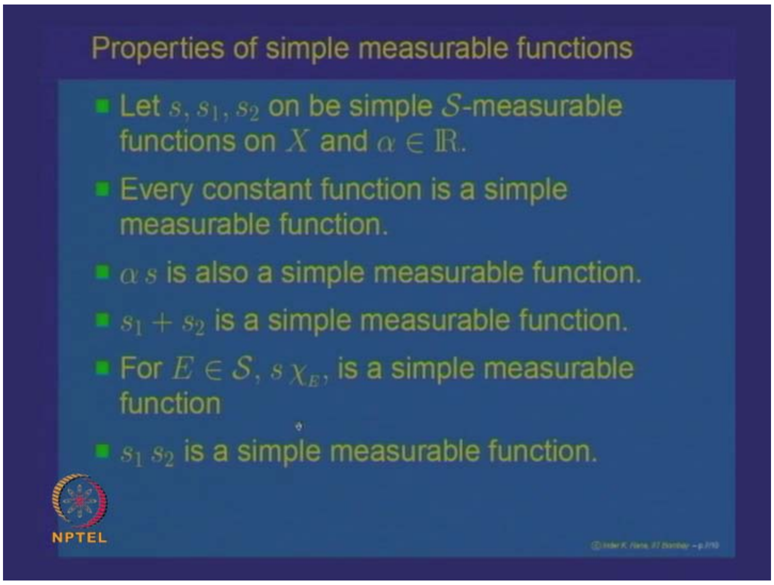
(Refer Slide Time: 39:59)



Let us take a set E belonging to S and s is a simple function which is sigma a_i indicator function of A_i. Then, s times the indicator function of E is nothing but summation i equal to 1 to n a_i indicator function of A_i times indicator function of E. Here is an observation: the product of indicator functions is nothing but the indicator function of the intersection; the product of indicator functions is equal to indicator function of the intersected set.

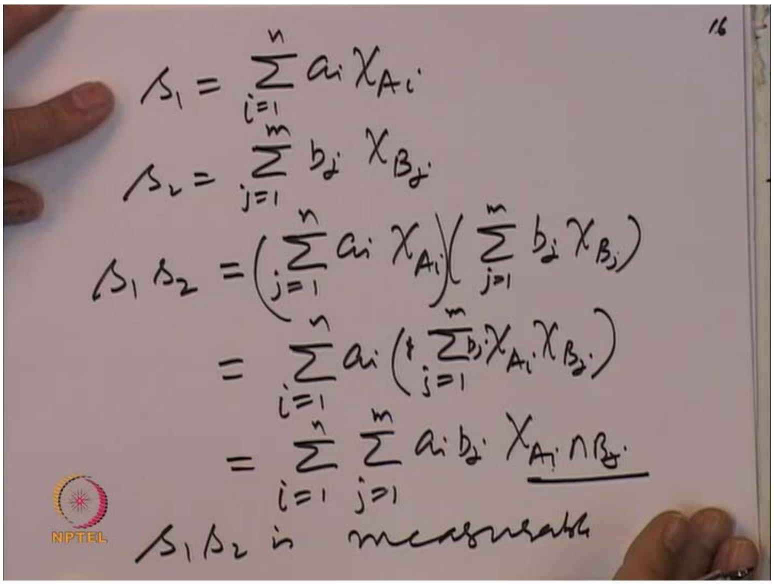
If we use this, then the function s times indicator function of E can be written as sigma a_i indicator function of A_i intersection E. It is again a linear combination of indicator functions of sets A_is intersection E. Since A_is belong to the sigma algebra and E belongs to the sigma algebra, this belongs to the sigma algebra (Refer Slide Time: 41:07). The function s multiplied by the indicator function is a linear combination of characteristic functions of sets which are in the sigma algebra; this implies s into indicator function of E is measurable; that proves our next property.

(Refer Slide Time: 41:29)



Using this, it is easy to check that a product of simple measurable functions is also a simple measurable function.

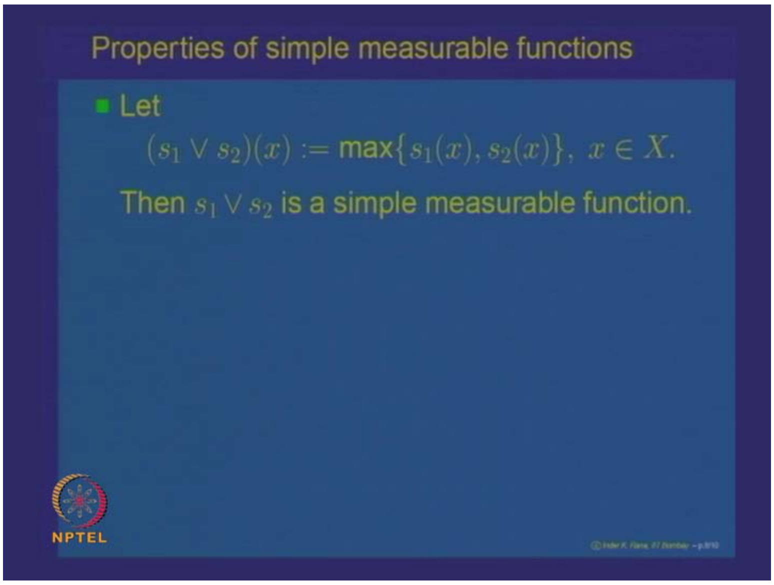
(Refer Slide Time: 41:43)



For that, let us take s_1 is sigma a_i indicator function of A_i and s_2 is sigma j equal to 1 to m b_j chi of B_j. Then, s_1 multiplied with s_2 is nothing but this; we can do distributive law; 1 to n a_i of chi A_i summation b_j 1 to m chi of B_j. We can write this as summation over i 1 to n a_i summation over j 1 to m chi of A_i chi of B_j into that constant b_j; let us write that b_j here (Refer Slide Time: 42:20)

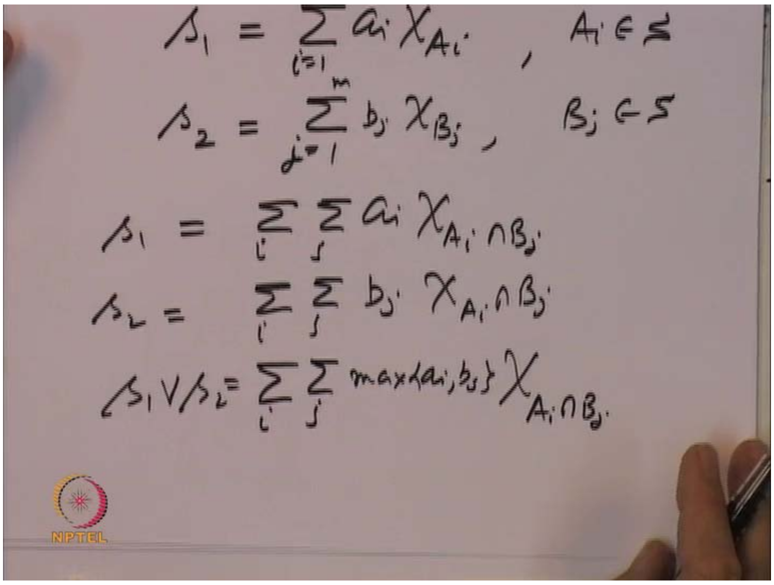
Anyway, we need not have done that much; we could have just said that is chi, indicator function, of A_i times s_2; each one of them is a simple. Anyway, this can be written as summation over i 1 to n summation over j equal to 1 to m a_i b_j chi of A_i intersection B_j. Once again, s_1 into s_2 is a linear combination of indicator function of sets where A_i belongs to S because s_1 is measurable, B_js belongs to the sigma algebra S because s_2 is measurable; the intersection is measurable; so, s_1 into s_2 is measurable and product of simple measurable functions is again measurable (Refer Slide Time: 43:40).

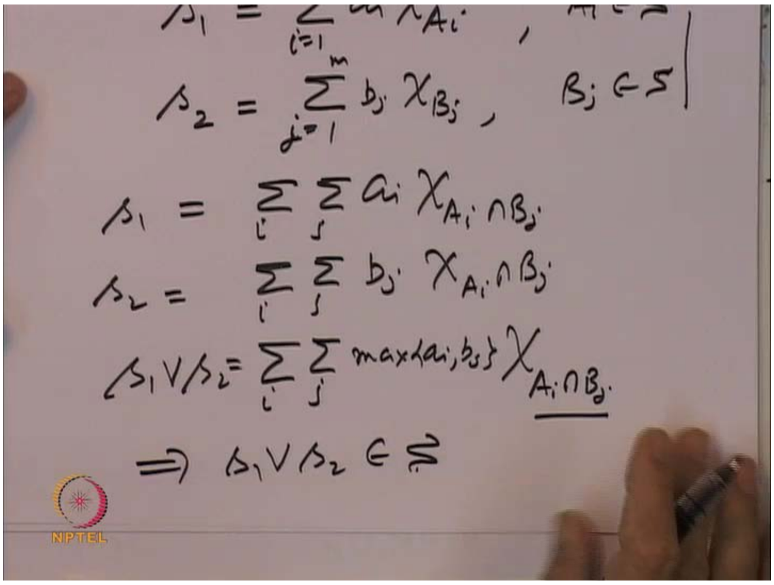
(Refer Slide Time: 43:41)



Let us go a step further. Given two simple functions s_1 and s_2, let us define what is called maximum of these two function s_1 V s_2. What is s_1 V s_2? This is the function whose value at a point x is defined as the maximum of the numbers s_1 of x and s_2 of x. At every point x, compare the values of s_1 and s_2; whichever is higher, define the value to be that number. The claim is that s_1 V s_2 is also a simple measurable function. Once again, the technique is same as for the sum.

(Refer Slide Time: 44:26)



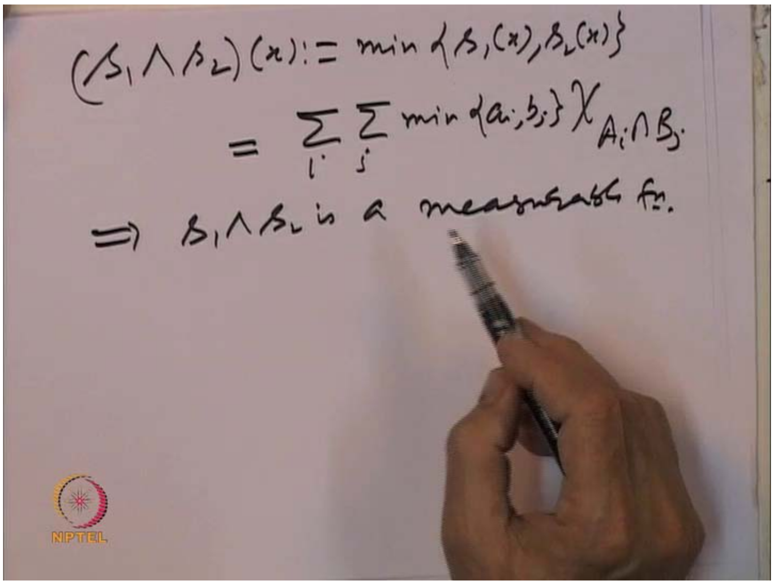


Let us write s_1 is equal to sigma a_i chi of A_i, i equal to 1 to n and let us assume s_1 is simple. That means all the A_is are in the sigma algebra S. Similarly, s_2 is measurable. Let us write s_2 as j equal to 1 to m and b_j chi of B_j where B_js belong to S. Now, we bring them to the common sets as before. Let us write s_1 as sigma over i sigma over j a_i chi of A_i intersection B_j and s_2 equal to sigma over i sigma over j b_j chi of A_i intersection B_j.

Then, s_1 maximum s_2 at any point x…. We want to define what will be at any point x the value of this (Refer Slide Time: 45:26). Look at the point x; it will be in either one of the sets A_i intersection B_j. Then, s_1 will give the value a_i and s_2 will give the value b_j and the maximum of that has to be put. It is maximum of a_i, b_j if x belongs to A_i intersection B_j. So, s_1 V s_2 is nothing but summation over i summation over j of this (Refer Slide Time: 45:57).

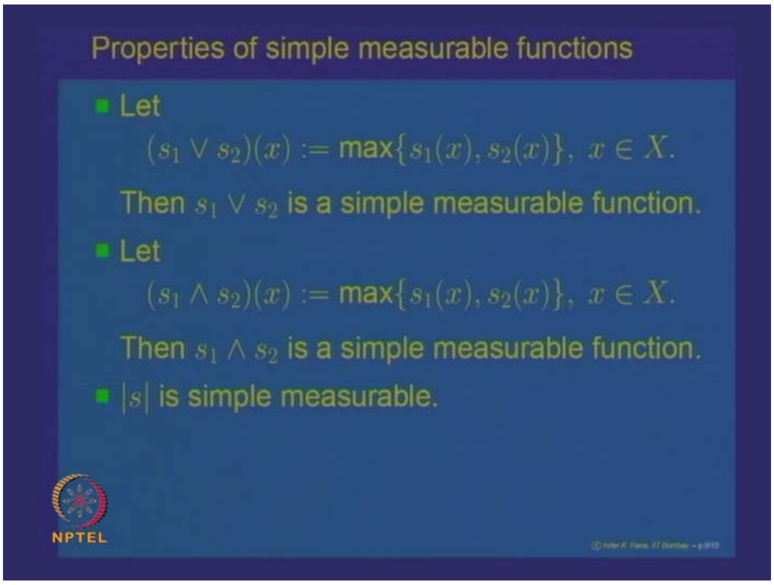
This is once again a finite linear combination of characteristic function where the sets involved are in the sigma algebra. This will imply s_1 wedge s_2 belong to S (Refer Slide Time: 46:18). A similar argument will imply that the corresponding minimum of the two simple measurable functions is also a measurable function. What is the minimum function?

(Refer Slide Time: 46:28)



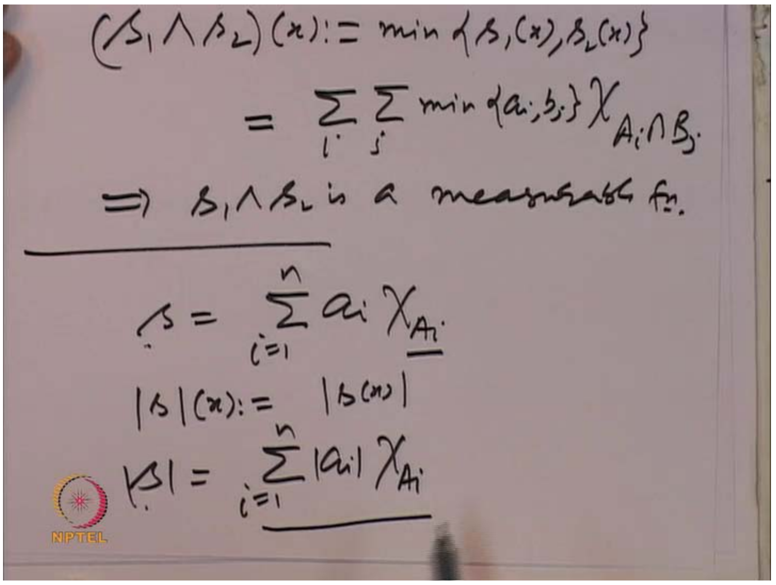
s_1 wedge s_2 at a point x is defined as the minimum of s_1 of x, s_2 of x; that is called the minimum of the two functions. We want to show that also is a simple measurable function. Once again, if s_1 is defined as this and s_2 is defined as this (Refer Slide Time: 46:48), then what is s_1 wedge s_2? This can be written as simply sigma over i sigma over j minimum of a_i, b_j into indicator function of A_i intersection B_j. Once we write it that way, it becomes clear that the minimum also is a… This implies that s_1 wedge s_2 is a measurable function whenever s_1 and s_2 are measurable functions.

(Refer Slide Time: 47:32)



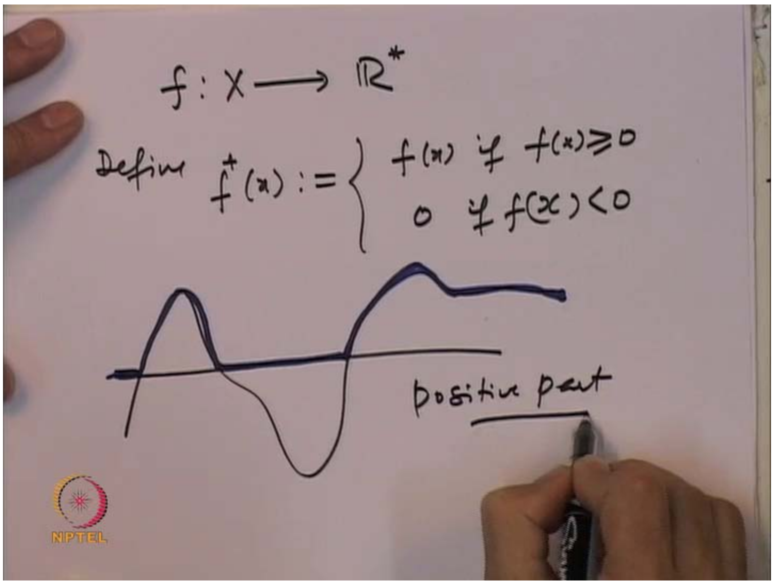
Not only the maximum but the minimum also is a simple measurable function whenever s_1 and s_2 are measurable functions. Let us finally prove that if s is simple measurable, then mod s is also a simple measurable function. There are many ways of looking at this.

(Refer Slide Time: 47:57)



If s is equal to sigma a_i chi of A_i 1 to n, then what is mod s? Mod s is a function defined at x to be equal to mod of s of x. Mod s is nothing but sigma mod of a_i, i equal to 1 to n into indicator function of A_i; this also is measurable, because if s is measurable, each A_i is a measurable set and mod s is a linear combination of indicator function of sets which are measurable.

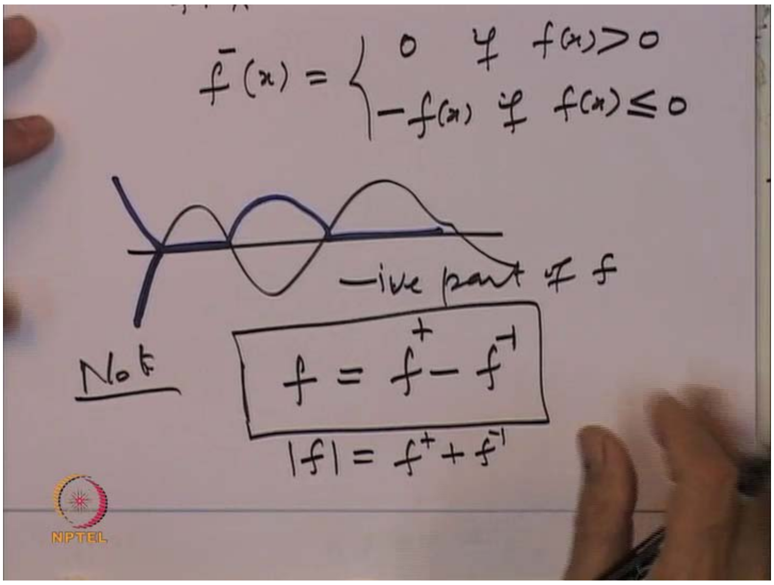
(Refer Slide Time: 48:49)



At this point, it is worth noting a few things about mod of a function (Refer Slide Time: 48:48). Let us take any function f from X to R or R star. Let us define f plus of x to be a function on x as follows. It is equal to f of x if f of x is greater than or equal to 0; it is 0 if f of x is less than 0. What we are saying is look at the value of the function f of x; if it is bigger than or equal to 0, then you keep the value of function as it is; as soon as it goes below, you cut it off by the value 0.

If this is the function f of x, then what is f plus? When it goes below, you keep the value to be 0 because is going below; because it is up, you keep it as it is (Refer Slide Time: 49:53). Now, it is going below and you keep the value to be 0; now, it is going up. This is the function f plus; this is called the positive part of the function; this is called the positive part of the function.

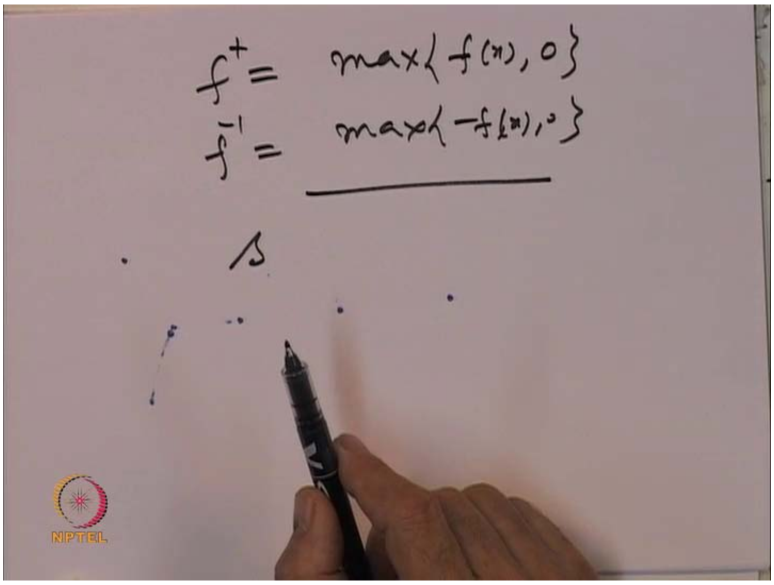
(Refer Slide Time: 50:24)



Similarly, we can define what is called the negative part of the function to be as follows. Given a function f from X to R star, the negative part of the function f of x is defined as 0 if f of x is bigger than 0. As soon as it becomes positive, we make it 0; we make it equal to minus of f of x if f of x is less than or equal to 0; keep in mind the negative. If this is the graph of the function, then what do we do?

We look at the graph; as soon as it is below, we keep it as it is; it is 0 if f of x is positive; so, on the positive part we keep it here; when it is below, we reflect it up. So, it is this, this, this, this and so on (Refer Slide Time: 51:16). This is called the negative part of f. Let us observe that the function f is written as the positive part minus the negative part. Every function can be represented as the positive part and the negative part; both these functions are nonnegative functions; mod of f can be written as f plus plus f minus; that is the mod f.

(Refer Slide Time: 52:04)



You can also think of the positive part f plus as the maximum of f and the constant function 0; f minus can be thought of as maximum of minus f and 0; this is another way of looking at it. For a simple function, saying that mod f is measurable can also be looked at because if s is measurable, simple function is measurable, the maximum of simple function and 0 is measurable; the positive part is measurable; the negative part is measurable; hence, mod f will be also measurable.

We will continue properties of measurable functions in our next lecture. In the next lecture, we will prove an important theorem; namely, we will look at how sequences of measurable functions behave whether the limits of sequences of measurable functions are measurable or not. Thank you.