Measure and Integration

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Module No. # 04

Lecture No. # 13

Characterization of Lebesgue measurable sets

Welcome to lecture number 13 on Measure and Integration. If you recall, in the previous lecture, we had been looking at the Lebesgue measurable sets and its properties. We will continue that study of Lebesgue measurable sets and its properties today itself.

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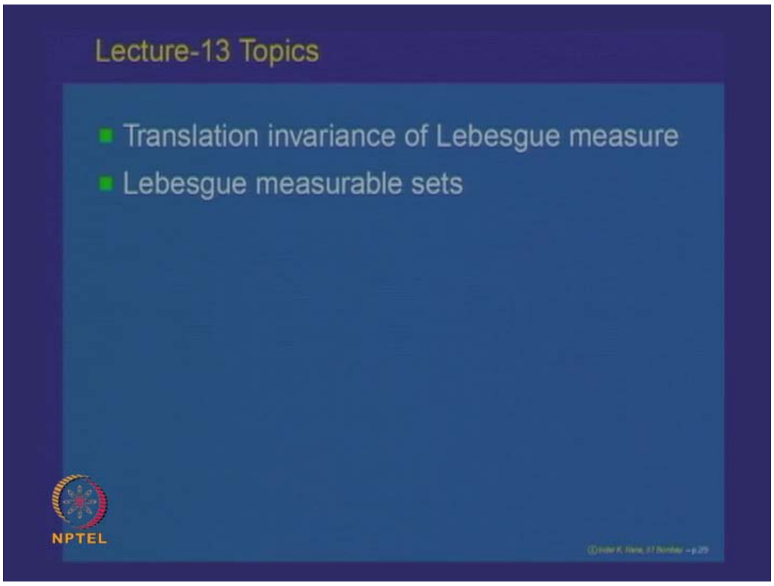

We will be specially looking at the translation invariance property of the Lebesgue measure, and then Lebesgue measurable sets. These are the topologically nice subsets of the real line.

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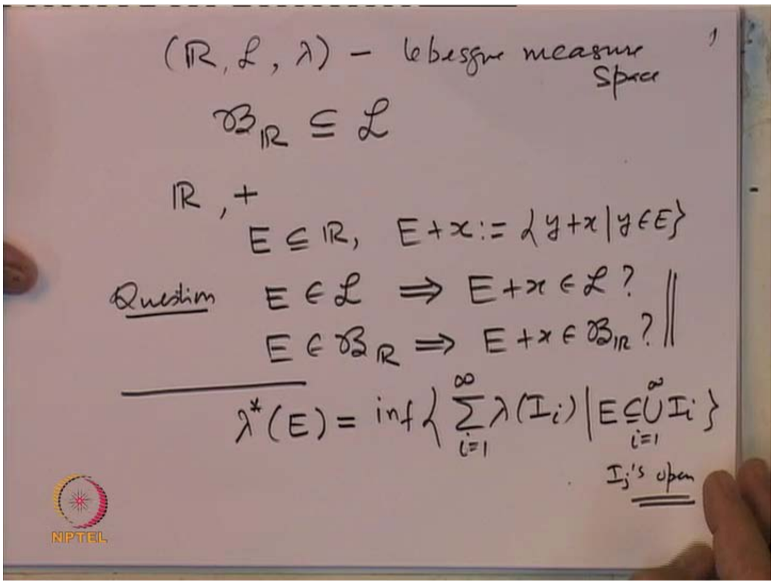

Let us recall that we had defined what is called the Lebesgue measurable sets and that gave us the space – the real line, the Lebesgue measurable sets, and the Lebesgue measure. So, this was called the Lebesgue measure space. The Borel sigma algebra of real line; the Borel subsets of the real line form a sub sigma algebra of the Lebesgue measurable sets.

These properties we had seen and now, today, what we are going to look at is the following: Recall that on the real line, there is a binary operation of addition. You can add real numbers. So, this operation can be used to transform subsets of the real line.

Let us take a set E contained in real line and define what is called E plus x. So, E plus x is defined as all elements Y plus x, such that y belongs to E. So, it is the set E, which is translated by an element x.

The question is E belongs to L; if E is Lebesgue measurable, does this imply that E plus x is Lebesgue measurable? Similarly, we will also look at the second question namely, if E belongs to B_R; if E is a Borel subset of real line, does it imply E plus x belongs to B_R? So, these are the two questions that we will start analyzing.

The importance of these two questions is, is the class of Lebesgue measurable sets invariant under translations and is the class of Borel subsets invariant under the group operation of translation on the real line? So, to start with, we will answer these two questions.

To answer the first question, let us recall that Lebesgue measure is nothing, but the restriction of the outer Lebesgue measure. The Lebesgue outer measure for real line is defined as the infimum of sigma lambda of intervals I_I, where these intervals form a covering of E is a subset of union I_i's; i equal to 1 to infinity and I_i is all intervals. Keep in mind the remark you said; (Refer Slide Time: 3:40) you can choose these intervals I_i's and I_j's to be open, if necessary.

Whether you take all possible coverings of E by intervals or all possible coverings of E by open intervals, both will give you the same value namely, the Lebesgue outer measure of the set E.

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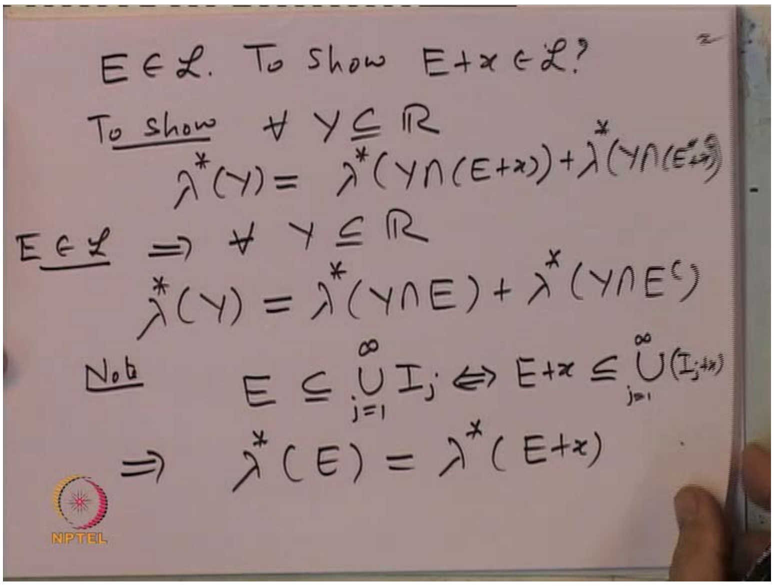

Let us start with the set E, which is Lebesgue measurable to show that E plus x is also Lebesgue measurable. So, this is the question.

To show that, recall what is the definition of a measurable set. To show that for every subset Y of real line, we should have Lebesgue outer measure of Y is equal to Lebesgue outer measure of Y intersection E plus x plus Lebesgue outer measure of Y intersection E plus x complement. So, that is what we have to show.

Now, let us start observing that we are given that E is Lebesgue measurable. So, E, Lebesgue measurable implies that for every subset Y of real line, the Lebesgue measure of Y is equal to outer Lebesgue measure of Y intersection E plus outer Lebesgue measure of Y intersection E complement.

Now, our aim is to transform this E to E plus x, which means we should be looking at the properties of the outer Lebesgue measure of a set in terms of translation. So, let us note that if a set E is covered by a union of intervals I_j; j equal to1 to infinity, that is, if and only if E plus x is covered by union of the translated intervals, that is, I_j plus x; j is equal to 1 to infinity. That means every covering of the set E gives a corresponding covering of the set E plus x by the intervals I_j plus x. Note that if I_j is an interval, I_j plus x also is an interval.

If E is covered by intervals I_j, you get a corresponding covering of E plus x by the intervals I_j plus x. Conversely, given a covering of E plus x, we can construct back a covering of E by translating by minus x. So, obviously, this property implies that Lebesgue measure of a set E is same as the Lebesgue measure of the set E plus x. So, this observation implies that the Lebesgue measure of a set remains invariant under translations.

This is an important property of the Lebesgue outer measure that we are going to use. To conclude that, if E is Lebesgue measurable, then E plus x is also Lebesgue measurable.

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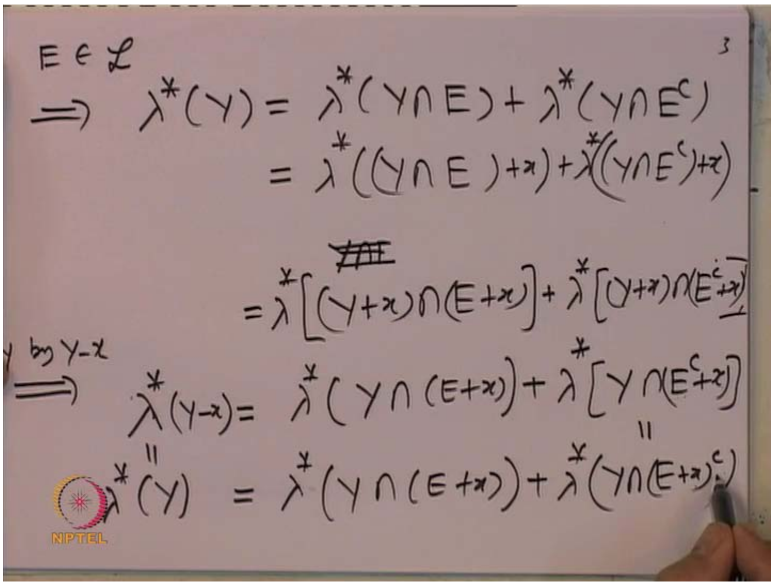

As we said E, Lebesgue measurable implies that for every subset Y, we have got lambda star of Y is equal to lambda star of Y intersection E plus lambda star of Y intersection E complement. Now, observing that Lebesgue measure is invariant under translations, we can write this as lambda star of Y intersection E plus x plus lambda star of Y intersection E complement plus x. So, we are using the fact that lambda star is translation invariance.

Now, I will tell you a simple observation that Y intersection E plus x is same as Y plus x intersection E plus x. This means that if you take intersection and translate, that is same as translating and intersections. So, that is a simple set theoretic property.

The first term is equal to lambda star of Y plus x intersection E plus x. Similarly, the second one will give you, that is, lambda star of Y plus x intersection E complement plus x. Using the fact that lambda star is translation invariant, intersection translation is same as translation intersection. They commute with each other (Refer Slide Time: 09:03). Now, this property is true for every subset of Y. So, I can replace Y by Y minus x. So, implies replace Y by Y minus x, we get… this is also equal to… so, lambda. Let us replace lambda star of Y minus x is equal to lambda star of…; (Refer Slide Time: 09:37) Y plus x minus x, so, that is, Y intersection E plus x plus lambda star of… Y plus x minus x, so, that is, Y intersection E complement plus x. In the above equation, we have replaced the set Y by Y minus x. So, this is true.

Now, observe lambda star of Y minus x is same as lambda star of Y. We are translating Y by x or minus x; that does not affect. So, for every subset Y, we get lambda star of Y is equal to lambda star of Y intersection E plus x plus lambda star of Y intersection E complement plus x.

Now, a simple observation tells you that (Refer Slide Time: 10:26) this set E complement plus x is same as E plus x complement. So, first take the complement, and then translate, that is, same as saying first translate, and then complement. It is purely a simple set theoretic exercise, which usually will be able to verify easily.

We get lambda star of Y is equal to lambda star of the first Y intersection E plus x plus lambda star of Y intersection E plus x complement (Refer Slide Time: 11:02). Hence, this implies that E plus x is Lebesgue measurable.

We have proved the first property namely, if you take a Lebesgue measurable set E and translate, then the translated set also is Lebesgue measurable. Another way of saying the same thing is that the Lebesgue measurable sets are translation invariant. They remain (( )) class of Lebesgue measurable sets is translation invariant.

We have already seen that the Lebesgue outer measure is translation invariant. So, that means that the length function is translation invariant on the class of all Lebesgue measurable sets. So, this proves the first property. We have answered the question that if E is Lebesgue measurable, then E plus x is also measurable.

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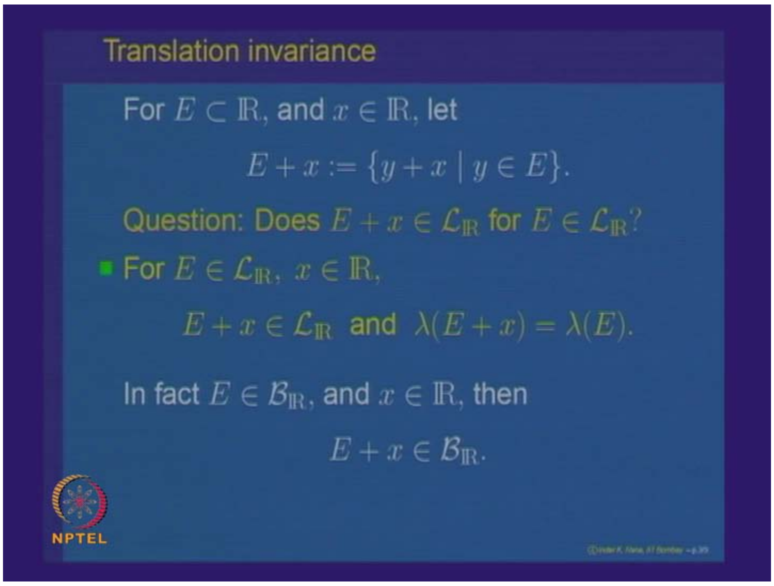

Let us look at the next question that if E is a Borel set, can we say that E plus x also is a Borel set. To answer this question, we need some topological properties of the real line. So, let us look at the topological properties. What we are going to look at…

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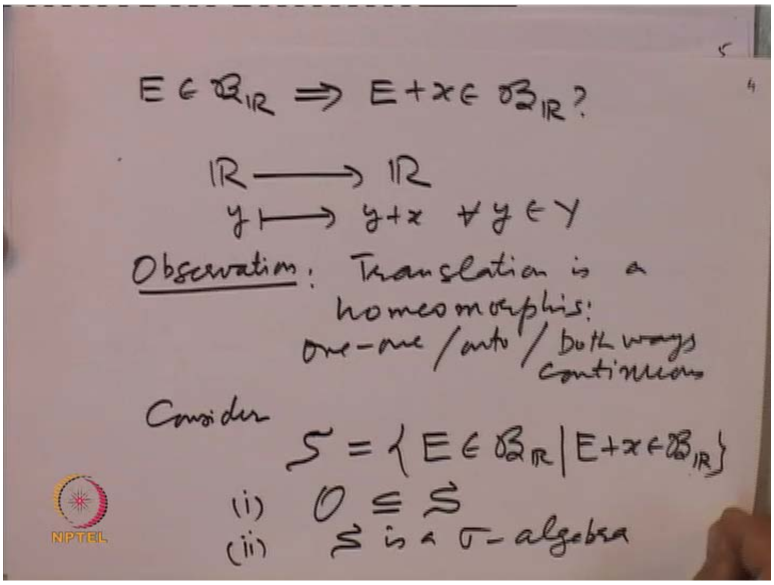

The question is, if E is a Borel subset of real line, does that imply E plus x is also a Borel subset of real line? For that, let us observe; consider the map from real line to real line, where y goes to y plus x for every y that belongs to Y. x is fixed, so this is translation.

This is a translation map from real line to real line and the observation is that this map translation is a homeomorphism. What does homeomorphism means? That is, it is one-one, onto, and both ways continuous. That means (Refer Slide Time: 13:23) this is continuous and the inverse map because this one-one; onto; that is also continuous. So, this is an important property; very basic, but yet important property that we are going to use to prove that if E is a Borel set, then E plus x is a Borel set.

To prove our requirement, let us consider the collection say S. S is the collection of all subsets, which are Borel and which have that required property namely, E plus x belongs to B_R. So, look at all Borel subsets of the real line, such that their translated set is also a Borel set.

We are going to prove two things about this. (Refer Slide Time: 14:14) (i) that the class of all open sets is subsets of S and (ii) we will prove that S is a sigma algebra. Once these two properties are proved, all open sets are inside S and S is a sigma algebra. So, these two properties will imply that the sigma algebra generated by the class of all open sets is inside S and that is nothing, but the Borel sigma algebra.

The Borel sigma algebra will come inside S and that will prove the required property. So, we have to only prove these two facts namely, that if you take an open set, then it belongs to S. That means, if I take an open set and translate, that should be a Borel set, but that is again obvious by the fact that translation is a homeomorphism.

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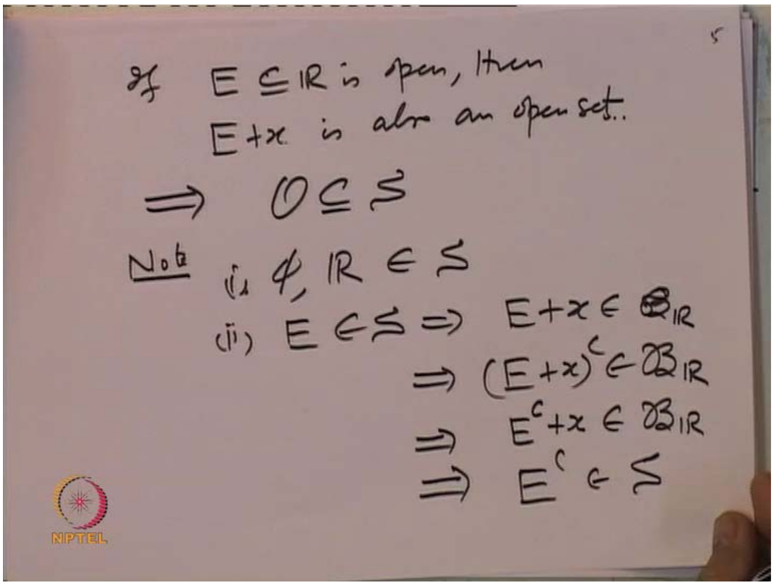

If E contained in R is open; is an open set, then the set E plus x is also an open set. So, that is also an open set. In fact that is very simple property to prove because if E is open, then in every point there is an open interval inside E and the translated one will be inside E plus x.

That is easy to verify or one can simply verify by saying that E plus x is an open map or the translation is a homeomorphism. So, this is an open set and this implies the first property that all open sets are subsets of S.

The second thing that it is a sigma algebra. So, for that note that empty set the whole space are both open, so they belong to S. Second, if a set E belongs to S that implies E plus x belongs to S is a Borel set, but that implies (Refer Slide Time: 16:35) that this is a sigma algebra. So, E plus x complement is also a Borel set because this is a sigma algebra, so must be closed under complements.

Now, a simple observation that this set E plus x is nothing, but E plus x complement is same as E complement plus x. (Refer Slide Time: 16:55) This is same as this. So, that belongs to B_R and that implies that E complement belongs to S, which is same as saying E complement.

(Refer Slide Time: 17:08) The class S of Borel subsets of real line say that translates, or Borel sets include empty set the whole space; it is closed under complements. Let us prove that this is also closed under countable unions.

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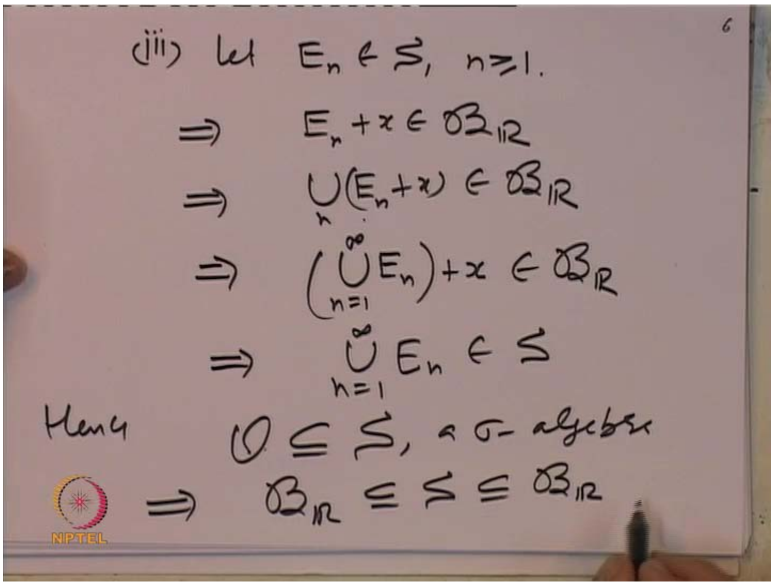

Third property namely, let E_n be a sequence of sets in S implies that E_n plus x is a Borel set in R. Borel sets being a sigma algebra implies that union of E_n plus x also belongs to B_R. Now, here is a simple observation that this set is same as… you first take the union, and then take the translation. It is same as first translating, and then taking the union. So, that belongs to B_R.

Basically, translation commutes with all set theoretic operations. That is the observation we have been using again and again. So, this belongs to B_R (Refer Slide Time: 18:23). That implies union n equal to 1 to infinity E_n is a set in S. So, we have proved. Hence, open sets are inside S, a sigma algebra. So, implies that the Borel sigma algebra generated by open sets, which is the Borel sigma algebra; is inside the smallest sigma algebra generated by open sets namely, the Borel sigma algebra must also come inside S and that is a subset of B_R. Hence, all are equal. This proves the second fact namely, if E is a Borel set, then its translation is also a Borel set.

Once again it emphasizes the use of the technique that we had called as the sigma algebra technique namely… We wanted to prove that for every Borel set E, the translation is a Borel set.

Note that we have collected all the sets, which have this property, and we proved two facts namely, the open sets are inside this collection S and S is a sigma algebra. So, that implied that the smallest sigma algebra generated by open sets, which is nothing, but the Borel sigma algebra; also comes inside S. The technique we have been using and we will be using quite often is called the sigma algebra technique.

We want to prove some properties about subsets of a set x. Collect them together and try to show that collection forms a sigma algebra and includes the generators of the required sigma algebra. So, we have proved that under translation, the measurable sets, the collection of Borel sets are very well behaved. We get a translation invariant measure on real line and that is the length function.

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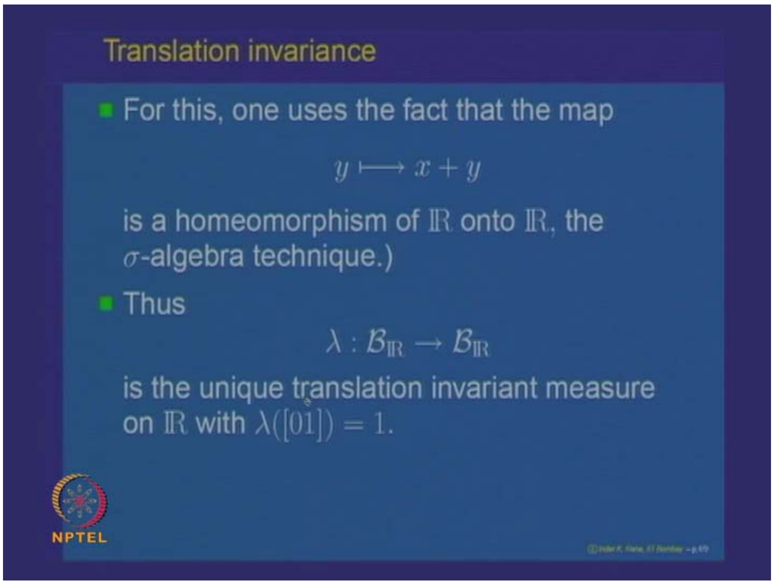

This gives us the fact that lambda, the length function on B_R, the Borel sigma algebra of subsets of real line is the unique translation invariant measure such that the length of the interval 0, 1 is equal to 1. This is a very important property of the length function.

If you observe on the real line, there is a notion of addition and we just now pointed out that the group operation x, y goes to x plus y is a continuous map. One can also easily check that x going to minus x, that is, the inverse and the group operation is also a continuous map. That is summarized by saying that the real line is a topological group.

On the real line, there is a topological structure, a metric, a topology, and a group structure. That two behave very well with respect to each other saying that the group operations x, y goes to x plus y and x goes to x inverse. Both are continuous maps. So, one says such a thing is called a topological group.

The real line with addition and the usual metric; the normal metric forms what is called a topological group. We have shown that on this topological group, there exists a translation invariant measure. On the sigma algebra of subsets of it, there exists a translation invariant measure.

This is a very important fact and that can be generalized to what are called a locally compact topological groups. On every locally compact topological group, there exists an invariant measure because the group may not be abelian. So, one has to make a specific thing. There exist a left invariant or a right invariant measure on every locally compact abelian group. That plays an important role in doing analysis on such groups.

Just a pointer that you may come across in higher or other courses in the studies of higher mathematics that on every locally compact topological group, there exist an invariant measure. That actually is called Haar measure on the topological group. So, Lebesgue measure on the real line is an example of Haar measure on the locally compact topological group; real line under addition and the usual multiplication, usual metric space topology. So, these were the properties of Lebesgue measurable sets; these are the group structure.

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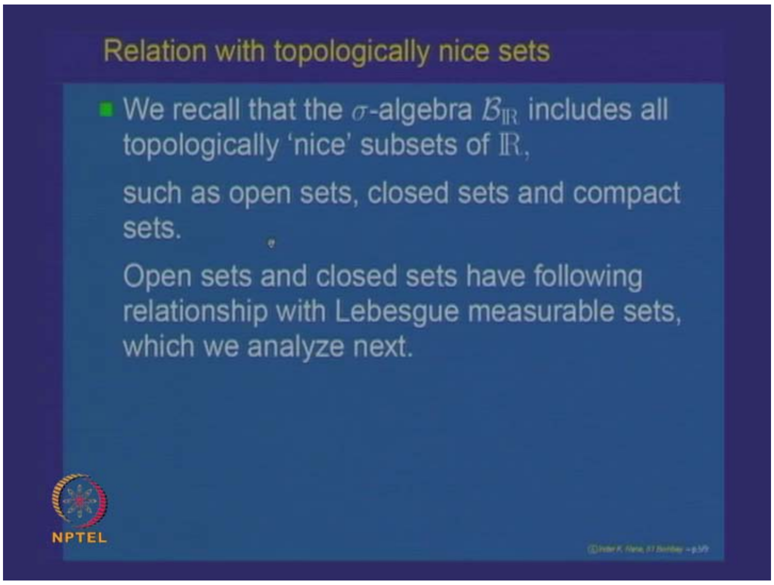

Now, let us look at the properties of the Lebesgue measurable sets with respect to topologically nice subsets of the real line. Namely, topologically nice sets are open sets and closed sets. So, we will prove; we will actually analyze and characterize measurability of sets in terms of open sets and closed sets.

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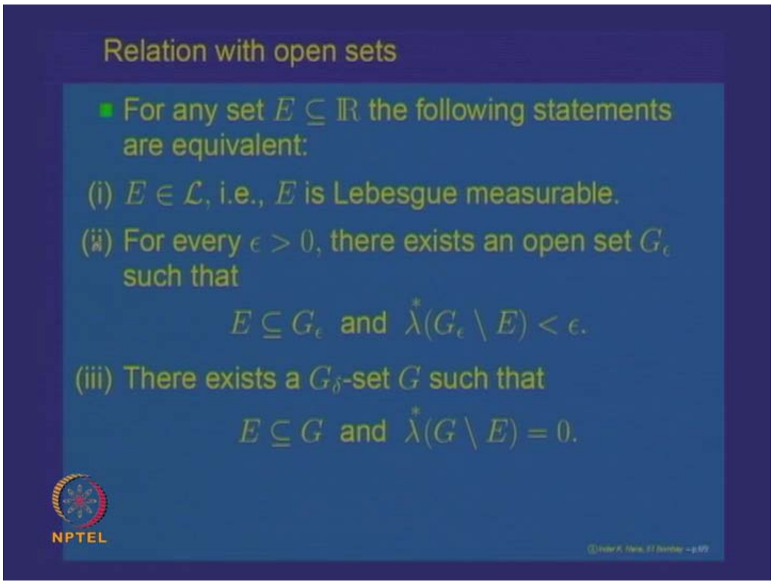

This is the precise theorem that we are going to prove. Namely, if E is any subset of the real line, then the following statements are equivalent. Saying that a set E is Lebesgue measurable is equivalent to saying that for every epsilon bigger than 0, there exist an open set G lower epsilon such that E is a subset of G epsilon. That means the open set includes E and the difference between the two sets has got outer measure small. So, that means there is a little difference between a Lebesgue measurable set and open set, which covers it.

We will show that for every set, E is Lebesgue measurable. If and only if for every epsilon bigger than 0, there exists an open cover of it such that the difference between the cover and the set, that is, lambda star of G epsilon minus E is small. We will show that this is equivalent to saying that there exist a G delta set G such that the set includes E and the difference has got measure zero.

A set G delta set; What is a G delta set? G delta set is nothing, but a countable intersection of open sets. So, subsets of the real line or in any metric space, which are countable intersections of open sets are called G delta sets.

Let us prove this theorem (Refer Slide Time: 25:37). Saying that these three statements are equivalent is saying that if one of them holds, then the other one also holds. So, what we are going to do is, we will assume one and show that (i) implies (ii). Then, we will show that if you assume statement (ii), then that implies statement (iii) and if you assume statement (iii), then that implies (i). That will imply that all these statements are equivalent. So, if one of them is true, then the other two statements are also true. So, that gives you a characterization of Lebesgue measurable sets in terms of open sets or G delta sets.

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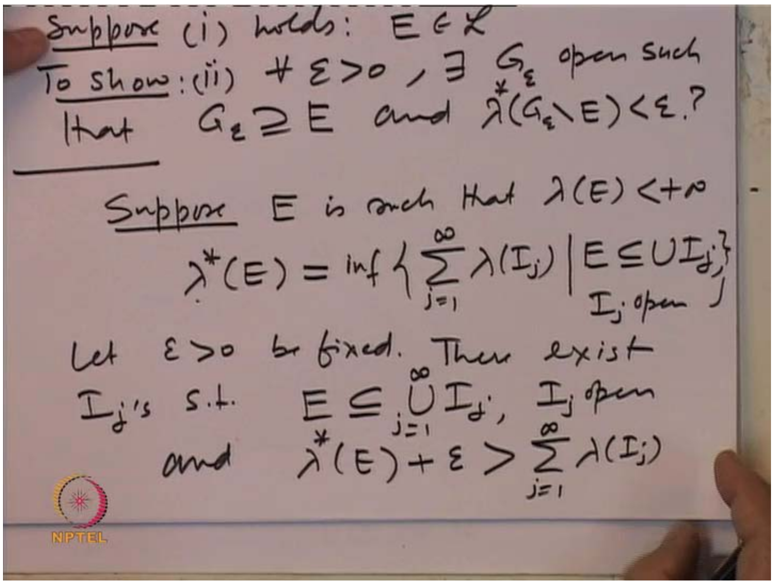

Let us prove this theorem. We will start with looking at… Let us assume; suppose the statement (i) holds, that is, E is Lebesgue measurable.

To show (ii), that is, for every epsilon bigger than 0, there exists a set G epsilon open such that G epsilon includes the set E and the Lebesgue outer measure of G epsilon minus the set E is less than epsilon. So, this is what we have to show.

Let us start with something… (Refer Slide Time: 27:16) This is regarding the outer measure. So, let us start looking at the set E. First, suppose E is such that; E is Lebesgue measurable; let us suppose that Lebesgue measure of E is finite.

What is Lebesgue measure of E? Recall that Lebesgue measure of E is same as its outer Lebesgue measure, which is same as infimum over sigma lambda of I_j; j equal to 1 to infinity, where the set E is covered by union of I_j's intervals and each I_j open. Recall that we had made an observation that in the definition of Lebesgue outer measure, you can assume that all the intervals involved are open.

Now, let us fix… be fixed and since (Refer Slide Time: 28:22) this number is finite and it is infimum, by the definition of infimum, there exists a covering. So, there exist intervals I_j's such that E is contained in union of I_j's; each I_j open and we have got the property namely, lambda star of E, which is same as lambda of E because it is measurable plus the small number epsilon is bigger than sigma lambda of I_j; j equal to 1 to infinity.

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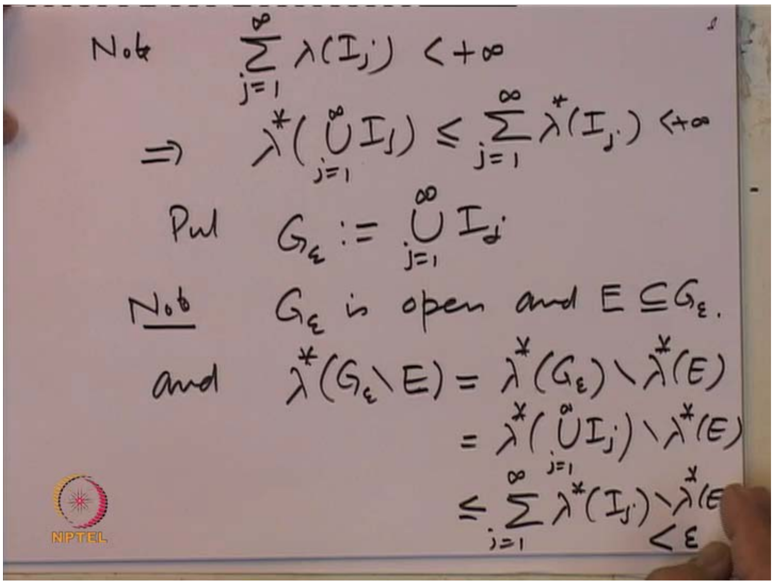

Note that (Refer Slide Time: 29:23) lambda star of E is finite, so this is finite quantity. So, that implies sigma lambda of I_j; j equal to 1 to infinity is finite. So, I have got all the sets. This implies that if I look at lambda star of union I_j; j equal to 1 to infinity, that will be less than or equal to summation j equal to 1 to infinity lambda of I_j by the sub additive property of the length function and that is finite (Refer Slide Time: 29:58). So, this set union of I_j's is a set of finite outer measure.

Let us define; put G epsilon to be the set, which is union of I_j's. Now, let us note that first of all G epsilon is open. Why it is an open set? Because each I_j is an open interval. So, a countable union of open intervals is an open set; is open and E is inside union of I_j's. So, E is contained in G epsilon. So, we have got the required property, that is, we have got a cover of E by an open set. Let us note that and what is the difference? Lambda star of G epsilon minus E.

(Refer Slide Time: 30:57) Now, note that G epsilon is an open set and it is a Borel set. So, it is a Lebesgue measurable set. E is given to be Lebesgue measurable and it is a subset of it. We have just now observed that everything is finite. So, we can say this is equal to lambda star of G epsilon minus lambda star of E (Refer Slide Time: 31:24). Here, we are using the finite additivity property of the length function. Lambda star of G epsilon, that is, same as lambda star of union I_j because G epsilon is union I_j minus lambda star of E. That is less than or equal to sigma by countable sub additive property; 1 to infinity, lambda star of I_j minus lambda star of E.

By our choice of intervals I_js, if you recall, (Refer Slide Time: 32:06) this was the choice. That means, lambda star of this summation minus this is less than epsilon, which is less than epsilon. So, lambda star of G epsilon minus E is less than epsilon. So, what does that mean? That means we have proved the required property when lambda star of E is a finite set.

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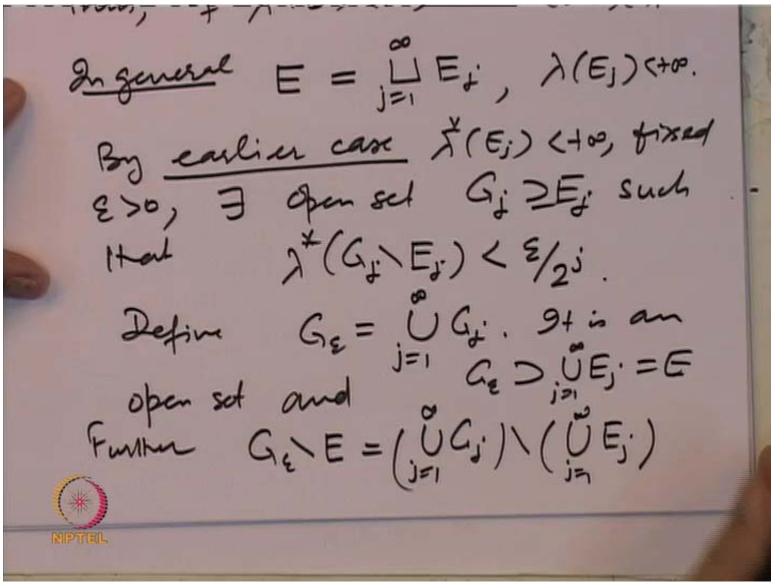

Thus, if lambda star of E is finite, then (i) implies (ii). Let us remove this condition. Here is an important step; we should keep in observation that we first proved a property about the length function for sets of finite measure. Now, we are going to extend this using the fact that lambda is sigma finite.

Whenever one wants to prove a property about the length function or about sigma finite measures, many a times it is easier to prove it when the under laying set is of finite measure. Then, extend it to general sets of sigma finite measure. That is what we are going to do now.

In general, the set E, which is Lebesgue measurable may not have a finite Lebesgue measure, but the Lebesgue measure being sigma finite, you can write E as a disjoint union of sets E_j; j equal to 1 to infinity such that lambda of each E_j is finite. It is a measurable set and it is finite.

Now, by the earlier case, because lambda star of E_j is finite, for fixed epsilon bigger than 0, there exists open set – call it G_j, which includes E_j such that lambda star of G_j minus the set E_j is less than the small number epsilon, but we are going to write less than epsilon to the power 2 to the power j. Soon you will see why we are doing that; because we are going to make it small for each piece and we are going to add up these pieces.

(Refer Slide Time: 34:57) Now, define the set G epsilon to be equal to union of G_j; j equal to 1 to infinity. Then, G_j is open; it is an open set because it is a countable union of open sets G_j's and G epsilon includes… G_j includes E_j; so, includes union of E_j's, which is equal to E. So, it is an open set, which includes E.

Now, let us look at the difference. Let us observe further that G epsilon minus E; what is that equal to? That is union of G_js minus union of E_js. That is the definition and that is how we constructed.

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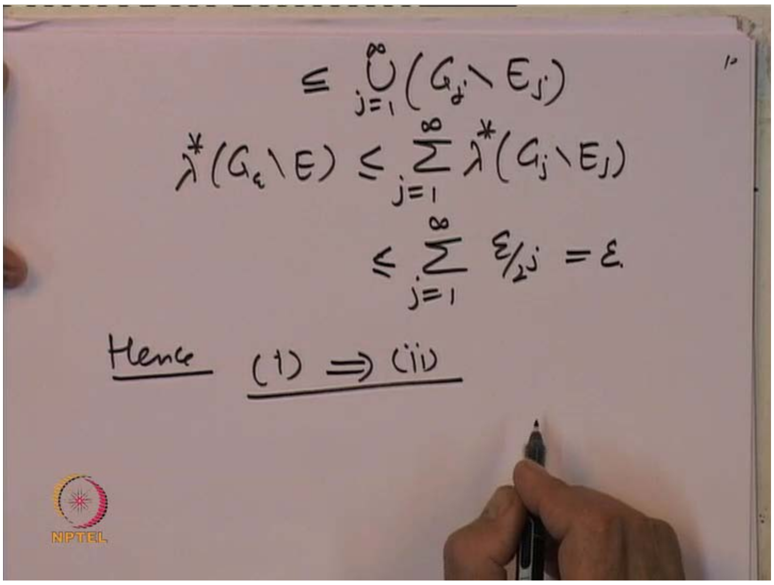

Now, here it is a simple set theoretic property namely, (Refer Slide Time: 36:09) that this is a subset of union j equal to 1 to infinity G_j minus E_j. So, this is a simple set theoretic that union of G_js minus union of E_js is a subset of union of G_j minus E_j. Once that is verified; which is easy to verify, we get that lambda star of G epsilon minus E is less than or equal to sigma j equal to 1 to infinity lambda star of G_j minus E_j, which by our choice is less than epsilon (( )) to the power j. So, this is less than or equal to summation j equal to 1 to infinity epsilon by 2 to the power j, which is equal to epsilon.

So, that proves the second property completely; in the general case also. Hence, what we have shown is that (i) implies (ii) namely, if E is Lebesgue measurable, then given epsilon, (Refer Slide Time: 37:17) we can find an open set G epsilon such that the outer measure of this is small; is less than epsilon.

Now, let us go to the second step of the verification. We have (Refer Slide Time: 37:31) verified the first step namely, (i) implies (ii). Now, let us verify that (ii) implies (iii).

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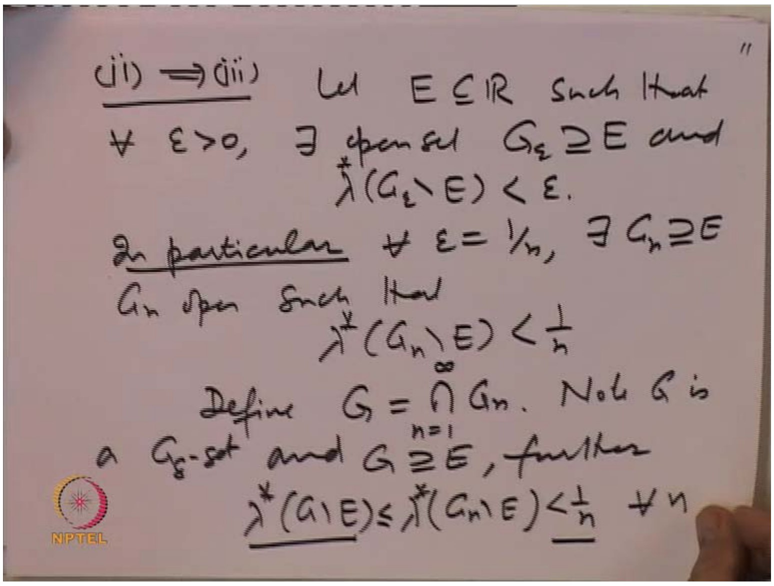

Let us assume (ii) holds, so (ii) implies (iii). We are given a set E, so given (ii) means – Let E be a subset of real line such that for every epsilon bigger than 0, there exists an open set G epsilon, which includes E and lambda star of… E is just a set, so we cannot write lambda now. So, it is lambda star of G epsilon minus E is less than epsilon.

That is what is given to us and we have to construct a set such that the difference has got Lebesgue measure zero. So, the obvious way is, (Refer Slide Time: 38:33) make this epsilon small and small. In particular, that says for every epsilon equal to 1 by n, there exists G_n includes E; G_n open such that lambda star of G_n minus E is less than 1 by n.

We have specialized this (Refer Slide Time: 39:06) given condition for each epsilon equal to 1 by n. We got an open set G_n, which includes E. Now, because we want its smallest, we want to let this become smaller and smaller, so it says the following.

Define G equal to intersection of G_n; n equal to 1 to infinity. So, what is G? G is an intersection of open sets. Note that G is what is called a G delta set. G delta set, by definition is an intersection of open sets and the set G includes E because each G_n includes E, so G also includes E.

Further, lambda star of G minus E. Note that (Refer Slide Time: 40:10) G minus E is a subset of G_n minus E because G is an intersection. So, G minus E is a subset of G_n minus E. By monotone property, lambda star of G minus E is less than lambda star of G_n minus E, which is less than 1 by n. So, lambda star of G minus E is less than 1 by n for every n.

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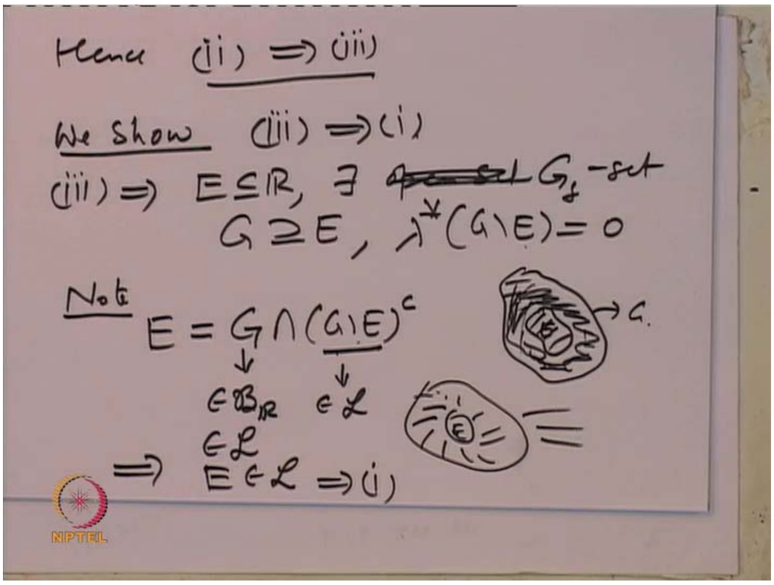

So, that implies the fact that lambda star of G minus E is equal to 0. Hence, we have shown that (ii) implies (iii).

Now, let us conclude the proof by showing… So, we show that (iii) implies (i). So, what is (iii)?

(iii) implies that for E is a subset of R, there exists a G delta set; G includes E and lambda star of G minus E equal to 0. So, (Refer Slide Time: 41:45) here is the set E, that is the set E inside and this is the set G, which covers it. So, the remaining part has got measure zero, but note; what is E?

E is same as… you take the set G; this is the full set (Refer Slide Time: 42:06) and intersected with the complement of the outer portion. So, this is the complement. Look at the complement of this. So, intersection G minus E complement.

(( )) a simple observation because what is G minus E complement? This inside portion is… Let me just draw a picture again. (Refer Slide Time: 42:33) This inside is E and outside is G. So, this shaded portion is G minus (( )) What is complement? Complement is the outside portion here and E. When you intersect it with G, you get E. So, E is nothing, but G intersection G minus E complement. (Refer Slide Time: 42:53) This is a G delta set and a G delta set is an intersection of open sets. So, this set belongs to B_R and hence, this set also belongs to L. So, it is Lebesgue measurable.

(Refer Slide Time: 43:07) This set G minus E has got outer measure zero. It is a Lebesgue measureable set because all sets of outer measure zero are Lebesgue measurable. So, G minus E also belongs to L. Intersection of two Lebesgue measurable sets is Lebesgue measurable. So, this implies that E belongs to L. So, (iii) implies… So, this implies (i).

This proves completely the fact that the three properties (Refer Slide Time: 43:43), that E is Lebesgue measurable is equivalent to saying for every epsilon bigger than 0, there is an open set G epsilon such that E is a subset of G epsilon and lamda star of G epsilon minus E. The difference is outer measure small. That is equivalent to saying that for the set E, there exists a G delta set covering it such that the difference has got measure zero. So, this gives us a characterization of Lebesgue measureable sets in terms of open subsets of a real line.

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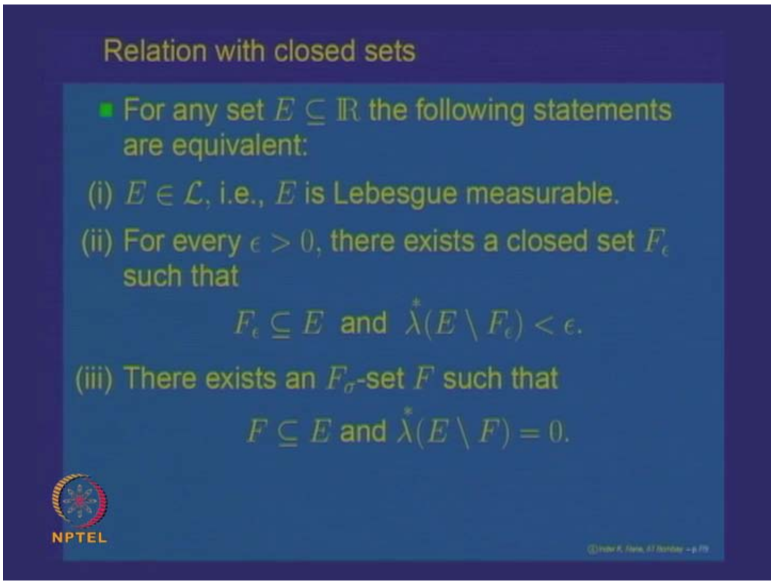

A corresponding characterization of Lebesgue measurable sets is obtained in terms of closed sets. Let us state that also and prove it. Let us look at the next… that for any set E in R, the following is true namely, E is Lebesgue measurable; is equivalent to saying for every epsilon bigger than 0, there is a closed set inside E such that Lebesgue measure of the difference is small. That is equivalent to saying that there is an F sigma set. So, what is an F sigma set?

An F sigma set is nothing, but a set, which can be expressed as a countable union of closed sets. So, there exists in an F sigma set F inside E such that lambda star of E minus F is equal to 0. Let us quickly prove this and this will use the earlier characterization.

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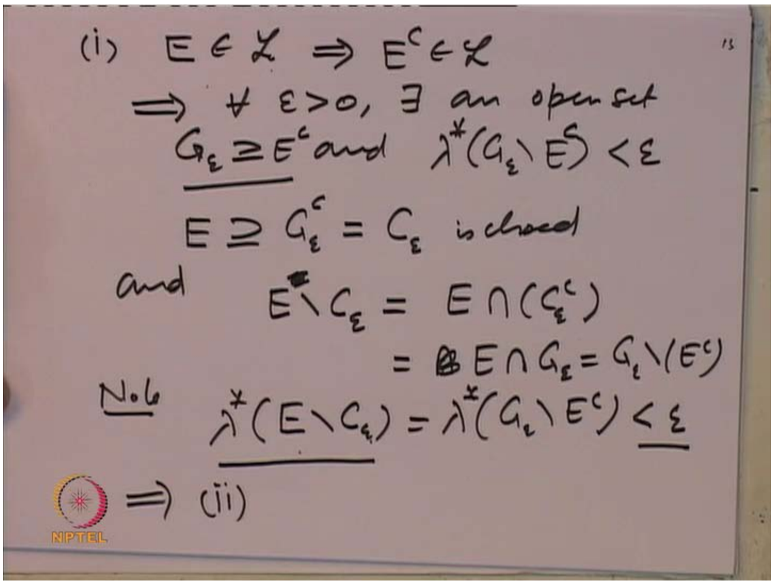

Suppose (i) holds E belongs to Lebesgue measurable sets. E belongs to Lebesgue measurable sets implies there is an open set, which covers E with difference of measure small, but we want close sets. So, let us observe E belongs to L also implies that E complement belongs to L because these Lebesgue measurable sets is a sigma algebra. So, if E is Lebesgue measurable, complement is also measurable. So, this implies…

By what have proved just now, E complement is Lebesgue measurable. For every epsilon bigger than 0, there exists an open set G epsilon such that this includes E and outer Lebesgue measure of G epsilon minus E is less than epsilon. Just now we proved this fact.

(Refer Slide Time: 46:25) This is of E complement. We are applying the previous; just now proved result for E complement. Now, if E complement is inside G, that means E includes G epsilon complement. Note that G epsilon is an open set. So, its complement is a closed set. Let us call it as C epsilon. So, C epsilon is closed.

It includes E and we want to find what is the Lebesgue outer measure of E minus C epsilon. What is that? (Refer Slide Time: 47:07) That is E intersection C epsilon complement by set theory and that is same as E intersection G epsilon. So, E minus C epsilon is E intersection C epsilon complement and C epsilon complement is nothing, but G epsilon. So, (Refer Slide Time: 47:46) this is G epsilon intersection E.

We want to conclude now. Note that Lebesgue outer measure of E complement… (Refer Slide Time: 48:02) So, Lebesgue measure of… We had E minus… What is this? (Refer Slide Time: 48:10) This, we can also write it as G epsilon minus E complement because E complement complement will be E. So, E minus C epsilon is same as outer measure of G epsilon minus E complement and that is less than epsilon.

What we have shown is, if E is Lebesgue measurable, then there is this (Refer Slide Time: 48:37) closed set inside it such that the outer measure is less than epsilon. So, implies (ii) holds. So, we have proved (i) implies (ii).

(Refer Slide Time: 48:50) We have just used the fact that a set is open if and only if its complement is closed and applied the previous criteria. So, we have proved (i) implies (ii).

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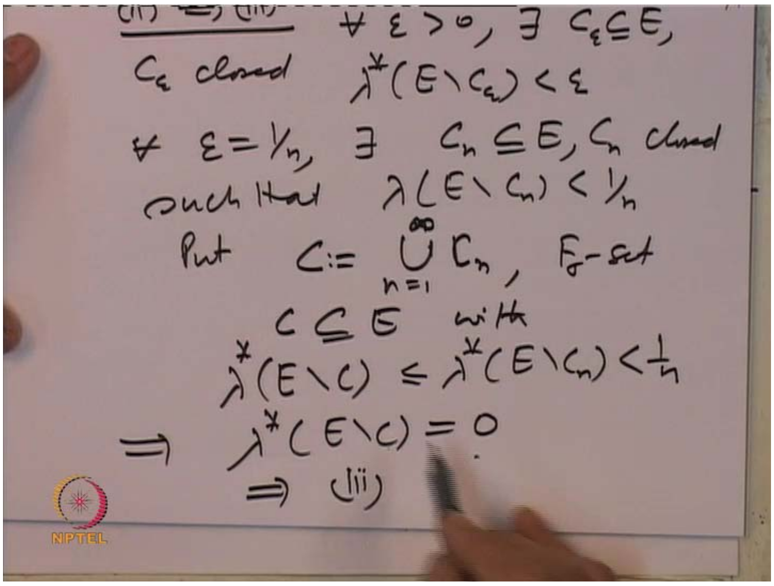

Let us look at (ii) implies (iii) and that is once again for every epsilon, what is given that there exists a closed set C epsilon closed inside E; C epsilon closed with the property that outer measure of E minus C epsilon is less than epsilon.

In particular, for every epsilon equal to 1 by n, there exists a set C_n, which is inside E, C_n closed such that Lebesgue measure of E minus C_n is less than 1 by n. Technique is same, so put C equal to union of these sets C_n. (Refer Slide Time: 49:58) This is 1 to infinity. So, this is an F sigma set because it is a countable union of closed sets.

Each C_n is inside E, so this set C is also inside E with the property that lamda star of E minus C; C is the union of all the C_ns. If I take only one of them, C will be a subset of E minus C will be a subset of that. So, it is less than or equal to lambda star of E minus C_n, which is less than 1 by n. So, implying lambda star of E minus C is equal to 0 and that proves. Hence, (iii) holds.

(Refer Slide Time: 50:47) Given, for every epsilon, there is a closed set inside it. We have shown that there is a closed set inside E with difference… There is a…; not a closed set; an F sigma set with measure zero.

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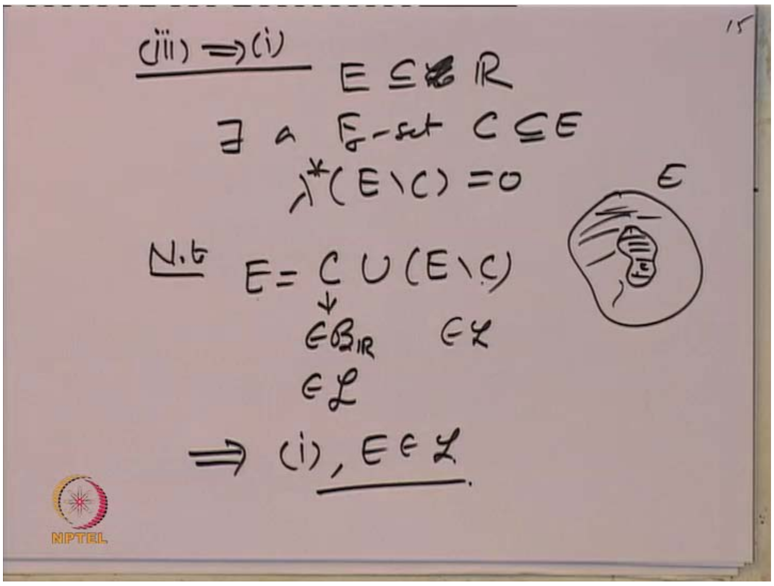

Finally, let us prove that (iii) implies (i). So, E is a subset of real line E with the property that there exists an F sigma set C contained in E with the property that lambda star of E minus C is equal to 0.

(Refer Slide Time: 51:27) Here is the set E and I have got a set C inside it. So, this is a set C inside it such that the difference has got measure zero. Now, note that the set E can be written as C union E minus C. So, it is this portion outside and union the inside portion. This is an F sigma set, so it is a Borel set and hence, it belongs to a Lebesgue measurable set.

The set E minus C is a set of measure zero. So, that is a Lebesgue measurable set. So, E is a union of two Lebesgue measurable sets. This implies (i), that is, E is Lebesgue measurable set. We have proved the third property (Refer Slide Time: 52:17) namely, (ii) implies (iii) and we have just now proved that (iii) implies (i).

The Lebesgue measurable sets are very nicely connected with topological nice sets namely, open sets and closed sets. So, every Lebesgue measurable set, can be covered by an open set such that difference has got Lebesgue measure small and which is equivalent to saying a set is Lebesgue measurable. That is also a characterization.

Similarly, a set E is Lebesgue measurable if and only if we can find a closed set inside it such that the difference has got measure small. So, Lebesgue measure is a very nice measure on the real line and it is defined on all Lebesgue measurable sets. In particular, it is defined for all Borel sets and it is translation invariant. It is compatible with the group structure and it also has nice properties with respect to the topological structure namely, with respect to open sets and closed sets.

With that we conclude our analysis of Lebesgue measure and Lebesgue measurable sets. In the next lecture, we will start looking at functions on measurable spaces and their properties.

Thank you very much.