Measure and Integration

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Module No. # 04

Lecture No. # 12

Lebesgue Measure and Its Properties

Welcome to lecture twelve on Measure and Integration. If you recall, last time we looked at the extension of a measure from algebra to the sigma algebra generated by it and slightly beyond the class of all outer measurable subsets.

Today, we are going to look at some special applications of this; a particular case of that extension theory for the real line. That is the topic for today's discussion namely, Lebesgue Measure and Its Properties.

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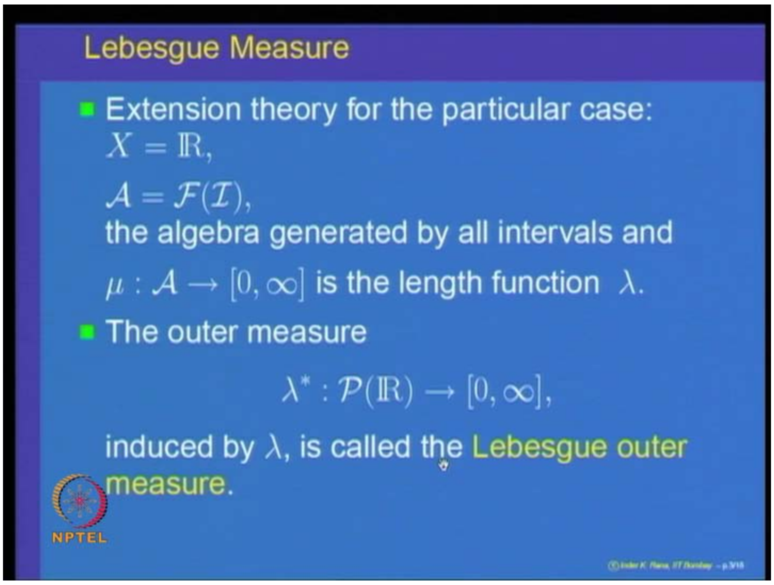

For the extension theory, we are going to apply it for the case X is equal to real line.

The set is the real line. The algebra A is the algebra generated by all intervals in the real line and mu on this algebra is the length function that we had defined. We had seen that the length function on the algebra generated by all intervals is a countably additive set function.

The outer measure induced by this length function, which is denoted by lambda star is on all subsets of the real line and that is called the Lebesgue outer measure. So, the outer measure induced by the length function is called the Lebesgue outer measure.

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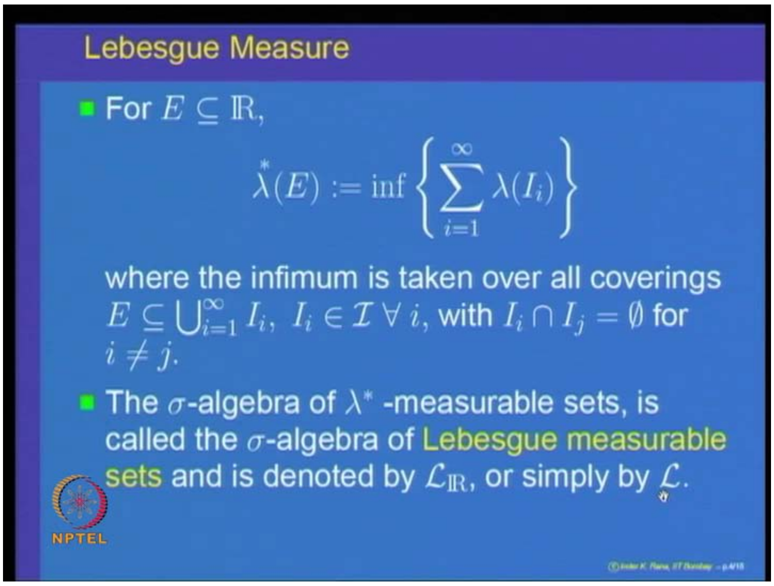

Let us just look at what is the Lebesgue outer measure for a subset of the real line. If you recall, we defined it as outer measure of a set E is… look at all possible coverings of a set E by elements in the algebra.

Here, the algebra being the algebra generated by intervals, it is finite disjoint union of intervals. So, we can write this outer Lebesgue measure as the infimum over summation lambda of the intervals I_i where, the intervals I_i's form a covering of the set E and these intervals are pair-wise disjoint.

So, lambdas all of E is the infimum of the sums of the lengths of the intervals which form a covering of E. We can take these intervals to be disjoint because if not, we can make them as disjoint. So, that is Lebesgue outer measure for a set E.

The class of all Lebesgue outer measurable sets; lambda star measurable sets is called the sigma algebra of Lebesgue measurable sets.

The sets that are outer measurable with respect to lambda star is called the sigma algebra of outer measurable or Lebesgue measurable sets and is denoted by L suffix R. Just to indicate, L for the Lebesgue and R for the real line. In case there is no confusion, we will denote L_R by simply L. So, this is the class of all Lebesgue measurable sets.

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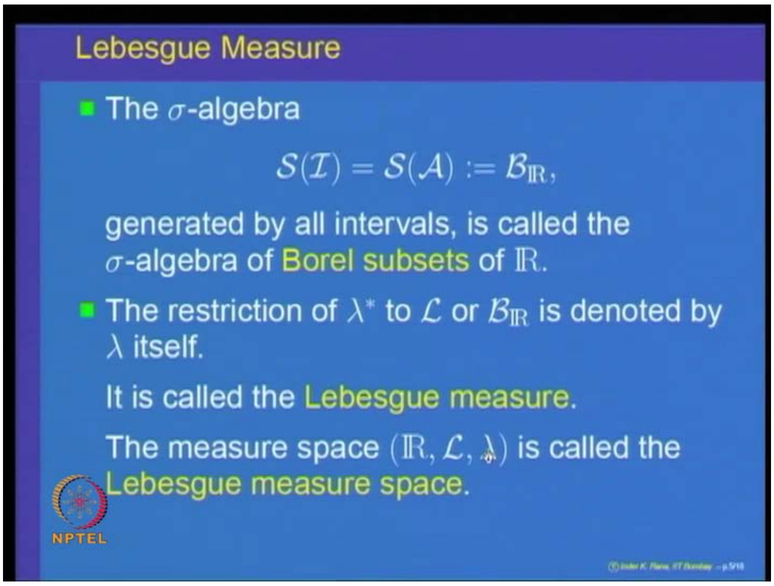

If you recall, we had also defined the sigma algebra of Borel subsets of real line and that was the sigma algebra generated by all intervals. A being the algebra generated by intervals, the sigma algebra generated by finite disjoint union of intervals is same as the sigma algebra generated by all intervals. That is same as the definition of the Borel sigma algebra of the real line.

We had already seen these properties. So, the length function in particular is also defined for all Borel subsets because the sigma algebra generated by A is inside the class of all outer measurable sets that is L.

So, we have got that S of A, that is, a Borel sigma algebra is inside the class of all Lebesgue measurable sets. So, for all Borel subsets, the notion of length is defined. This is called the Lebesgue measure.

Let us just summarize what we are saying. We are saying that the extension theory when applied to the particular case of the real line gives us the notion of length for a class of subsets of the real line, which are nothing, but the class of outer Lebesgue measurable sets. That includes the class of all Borel subsets. So, that also gives us the notion of length for all Borel subsets of the real line.

The triple are Lebesgue measurable sets, the length function as extended by the extension theory. This triple is called the Lebesgue measure space. So, the extension theory applied to the real line gives us the notion of the Lebesgue measure space and it extends the notion of length from intervals to the class L of all outer Lebesgue measurable sets.

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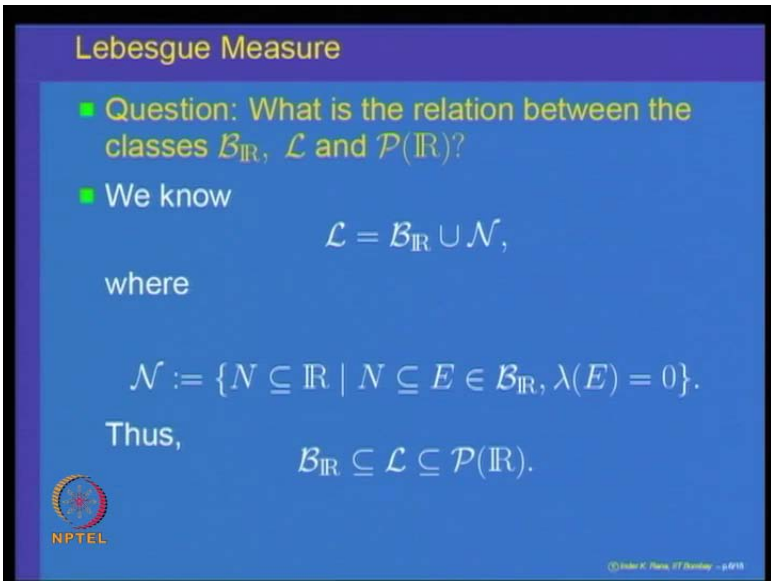

Let us recall that the sets, Borel subsets form a subset of the class of all Borel sets, which is a sub class of the class of all Lebesgue measurable sets. Of course, Lebesgue measurable set is a sub class of all subsets of real line.

The question is, can we say something more regarding these three classes namely Borel subsets, Lebesgue measurable sets, and P R.

Let us observe which we have done during outer measures that the Lebesgue measurable sets are characterized by the Borel subsets of the real line union the null sets. So, what are the null sets? Sets in R; subsets of R such that N is contained in a Borel set of measure 0.

Equivalently, one can also define it as sets of outer Lebesgue measure 0. So, B_R is a subset of L.

We know that outer measure 0 sets are also measureable. So this… and we said that this class is nothing but… This forms the sigma algebra and that is equal to the Lebesgue measurable sets.

That means, the B_R union N is equal to L. So, all null sets are part of L, but we want to characterize what is the relation between B_R and L and what is the relation between L and P R.

At present, we only know that the Borel sets are subsets of all Lebesgue measurable sets, which is a subset of P R.

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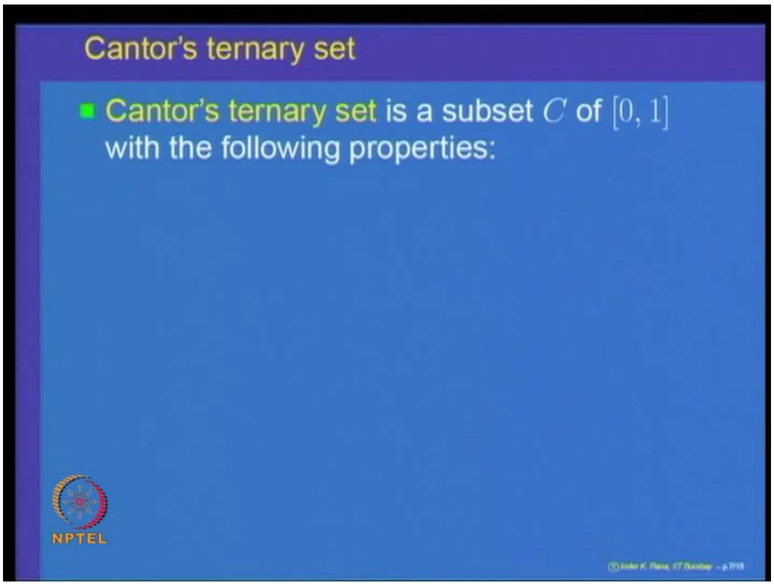

To say something more, we need to look at what is called a special subset of the real line called Cantor's ternary set. So, we are going to discuss and spend some time on a special subset of real line, which is called Cantor's ternary set.

Cantors ternary set is an example of a set that has very nice properties and it is useful both from the topological point of view as well as measure theoretic point of view. So, let us look at what is called Cantor's ternary set.

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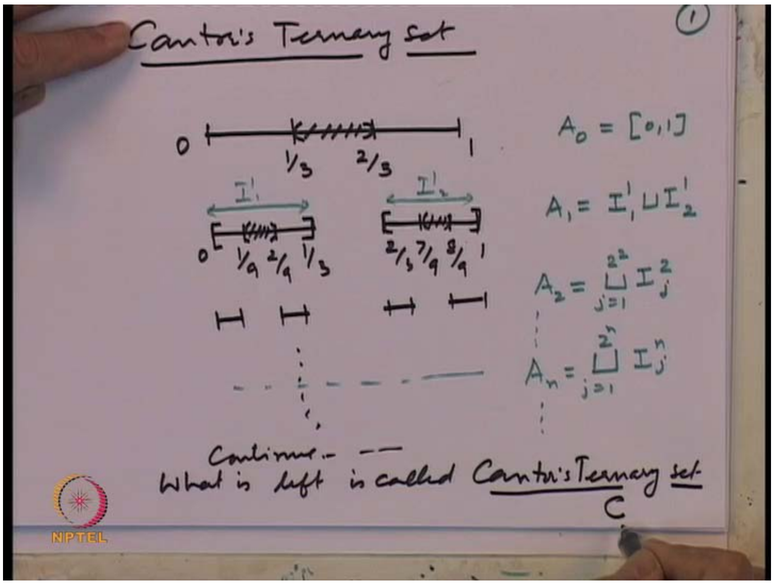

Ternary; It is called Cantor's ternary set because it was given by the mathematician George Cantor; first defined by George Cantor. Ternary set, which are because it involves ternary expansions of real numbers.

What we are going to do is a construction. We are going to construct a Cantor's ternary set. As step one, let us look at the interval 0 to 1.one. I will first describe this process of Cantor's ternary set construction and then, we will analyze its properties. So, what is the first step? The step is, divide this into three equal parts. So, that is 1 by 3 and 2 by 3. Remove the middle open portion. So, this open portion is removed from the interval 0, 1.

What does it give? It gives us two pieces – 0 to 1 by 3 and from 2 by 3 to 1. So, it gives us two closed intervals.

At the first step; at the first stage, having removed the middle one third of the closed interval 0, 1 and middle one third open interval, we get these two. Now, we repeat that process again with these two sub intervals. So, from each of these sub intervals, remove the middle one third portion; that is, middle one third is 1 by 9 and 2 by 9, and here, the middle one third will be equal to 7 by 9 and 8 by 9 (Refer Slide Time: 09:50).

So, this is the middle one third portion, which we are going to remove at the second stage. That will give us four sub intervals and we will continue this process. Eventually, something will be left. So, continue.

Question is, what is left? What is left is called Cantor's ternary set. Let us analyze and let us denote this set, the Cantor's ternary set by the letter C. How do we mathematically construct this? So, that is a question.

For that, let us start with the first stage that is A_0, that is, the closed interval 0, 1. After having performed the first stage, I write what is left as A_1. That consists of two disjoint intervals – 0 to 1 by 3 and 2 by 3 to 1. So, it consists of two disjoint intervals. Let us write them as, at the first stage one union the second one first stage the second one.

(Refer Slide Time: 11:30) This portion is the first interval and this portion is the second interval. That is, $I\,1_{\,1}$ and this one is $I\,1_{\,2}$. At the second stage, we will be left with four disjoint closed intervals. Let us write them as union I second stage j; j equal to 1 to 4. So, that is going to be 2 to power 2.

Let us see what it will be at the nth stage. If you continue (( )) this process, how many intervals will be there at the nth stage? So, there will be intervals… How many of them? We start with 1. At the next stage 2, at the next stage 4, and so on. So, there will be disjoint intervals j equal to 1 to 2 to the power n. There will be 2 to the power n closed subintervals of 0, 1. Let us write them as $I\,n_{\,j}$. So, these are the intervals.

What is A_n? A_n is the union of those intervals which are left at the stage (( )) at the nth stage; what we want? As we continue this process, we want what is C? So, mathematically, how do we write C?

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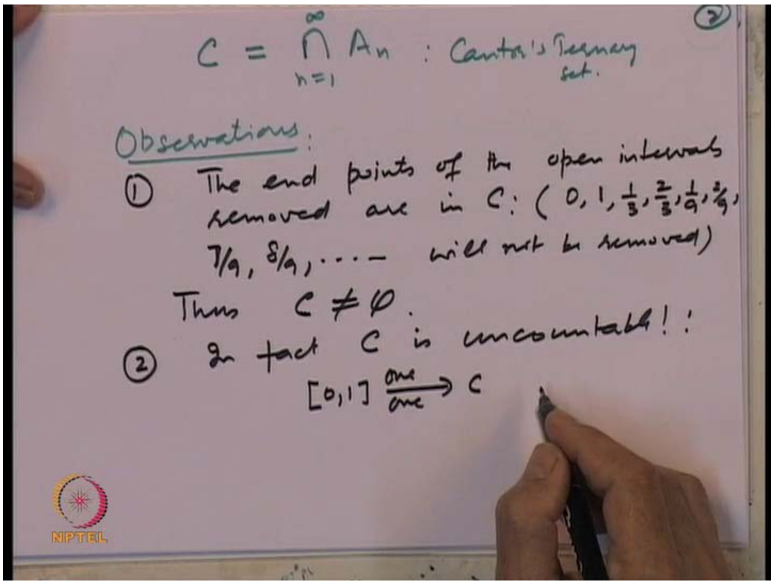

We can write it as the Cantor set, C. We can define it as an intersection of A_n's where, n is equal to 1 to infinity. Each A_n is a subset of the previous one. So, eventually, let us write what is left as an intersection of all these A_n's. So, this is what is called Cantor's ternary set.

Let us make some observations about this Cantor's ternary set. The first observation is that the end points of the open intervals removed are in C. Say, for example, 0; 0 is not going to be removed and 1 is not going to be removed. At the first stage, we removed the open middle one third. So, 1 by 3 is not going to be removed and 2 by 3 is not going to be removed. At the next stage, 1 by 9 will not be removed and 2 by 9 will not be removed. Similarly, 1 by 3, we have already listed. Then, 7 by 9 will not be removed, 8 by 9 will not be removed, and so on.

For example, these points will not be removed. They will stay in this process of removing middle one third open interval from each sub interval at every stage. So, that means… Thus, the class C the set C is a non empty set. It is non empty. So, that is the first observation. There is something left behind.

The second observation that we want to show… In fact, C is uncountable; that it is an uncountable set. How do you prove that C is uncountable? What we are going to do is, we are going to define a map from the closed interval 0, 1 to C. To prove this, we will define a map, which is one one.

We will define a one one map from 0, 1 to the Cantor's ternary set. We will prove that the cardinality of the set C is at least as much as 0, 1 and C being a subset of 0, 1, it cannot be more than that of 0, 1. So, cardinality of C will be same as cardinality of 0, 1.

That may seem a very strange observation to you that from C, we have removed… From the interval 0, 1 we have removed so many pieces and still what is left is as much as the points in 0, 1. So, these are the properties of infinite sets. Actually, they are the characterizing properties of infinite set. Interval 0, 1 is an uncountable set and from that, we are removing sub intervals and still what is left behind is as much as 0, 1.

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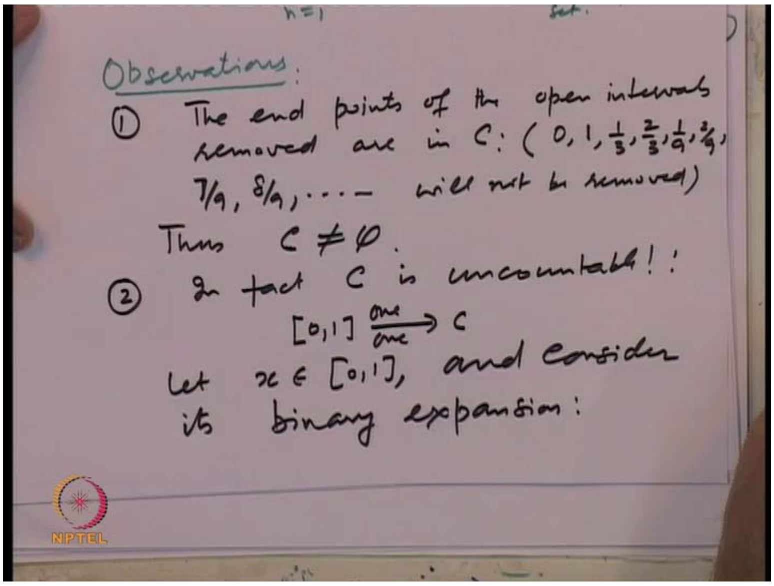

Let us prove this fact, namely, there is a one to one map for this. For this. Let us start… Let us take a point x that belongs to 0, 1 and consider its binary expansion. What is the binary expansion? The binary expansion of a point in 0, 1 is written as;

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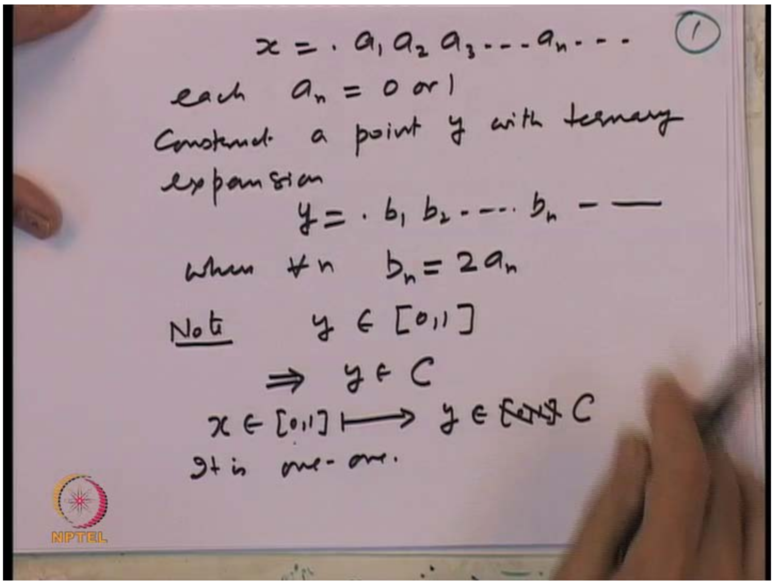

X can be written as point $a_1a_2a_3\ldots a_n$ and so on; where, each a_n is equal to 0 or 1. So, that is a binary expansion of every point. Essentially, the idea is that the interval 0, 1 can be divided into two parts – name first part as 0, second part as 1 and C at each stage where it lies. So, that is, 0 1 1.

Let us assume… See there are two different ways of writing say for some points, there are two different ways of writing binary expansions. So, we will fix one of the ways and say there is a unique binary expansion for every point in 0, 1. So, we will fix that binary expansion process.

Now, what we do is the following… Construct a point y with ternary expansion b… (( )) So, y is equal to point b_1b_2 and b_n; where, for every n, b_n is nothing, but two times a_n.

In the binary expansion, look at the nth place – either it will be 0 or 1. Double it and call that as b_n. So, b_n is twice as much as a_n. So, each b_i is either going to be 0 or it is going (Refer Slide Time: 19:00) to be two. So, this is the ternary expansion. Note that y belongs to 0, 1 because it is dash… It is dot $b_1b_2b_3$ and so on; no integral part. So, it is going to be part of… It is a point in 0, 1 and it has… In the ternary expansion, the only numbers that come are 0, two times a_n; a_n is 0 or 1, or it is 0 or 2.

In the ternary expansion of y, which is in 0, 1, only 0 or 2 appear and that implies that y belongs to C because in the construction of a Cantor's ternary set, we have removed the middle one third. So, in the ternary expansion, the number 1 is not going to appear. So, each one is… So, this is a part of… So, this is the observation we make.

Starting with a point x that belongs to 0, 1 with binary expansion a_1, a_2, and a_n, construct a point y; send it to the point y. So, this x is sent to the point y, which is… Again in 0, 1. In fact, it belongs to… So, let us write more specifically it belongs to C.

We have got a map from 0, 1 to C and the claim is that this map… (Refer Slide Time: 20:32) This is… It is one-one. That is obvious because for every point x, we have got this binary expansion a_1, a_2, a_3, and a_n; the unique binary expansion. So, if you take two different points x_1 and x_2… Let us try to write this mathematically that this is…

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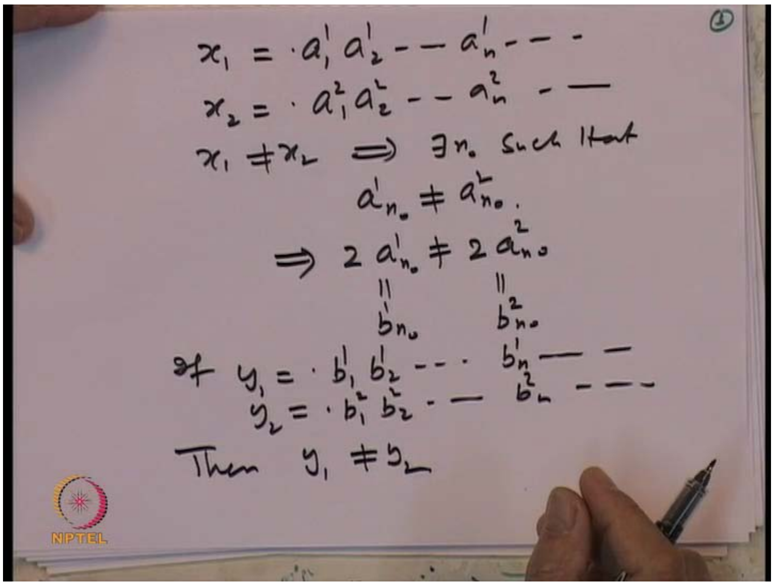

Let us take a point x_1 with binary expansion a 1 $_1$, a 1 $_2$, a 1 $_n$, and so on. Let us take another point with unique binary expansion that we have fixed the methodology. So, a 2 $_1$, a 2 $_2$, and a 2 $_n$, and so on.

X_1 not equal to x_2. So, if x_1 is not equal to x_2 that implies there exists some stage n_0 such that a 1 $_{n0}$ will not be equal to a 2 $_{n0}$ and that implies that two times a 1 $_{n0}$ will not be equal to two times a 2 $_{n0}$. This is b 1 $_{n0}$ and this is b 2 $_{n0}$. So, that means y_1… If we have y_1 that is, point b 1 $_1$, b 1 $_2$ up to b n $_n$ and so on.

Y_2 is the other point; the image of x_2 that is, b 2 $_1$, b 2 $_2$, b 2 $_n$, and so on, then so… If this is so (Refer Slide Time: 22:18), then y_1 is not equal to y_2.

(Refer Slide Time: 22:31) So, that means this process of sending x…; taking x with binary expansion is this and constructing y with ternary expansion is this. So, if you send x to y, this gives us a map from 0, 1 to C, which is one-one and hence….

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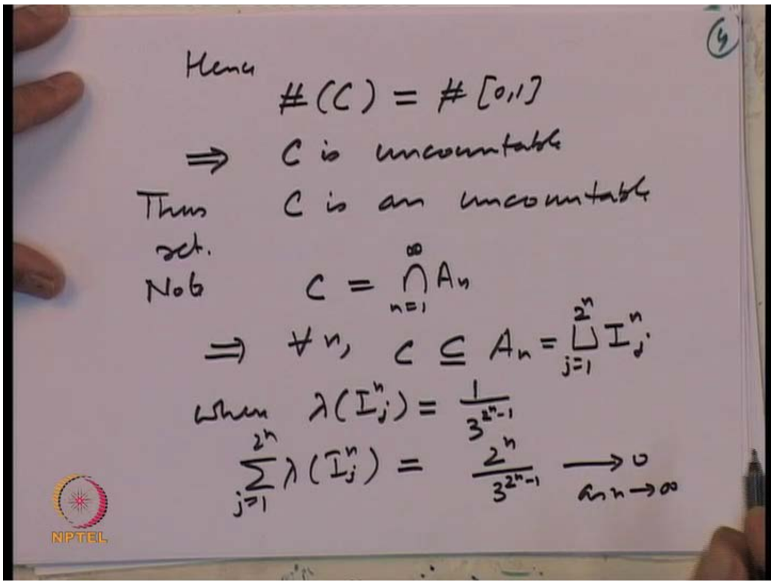

This implies as a consequence. Hence, the cardinality of C is same as cardinality of 0, 1.

If you recall, the cardinality of 0, 1; that is… Let us write this implies that C is uncountable because 0, 1 is uncountable. Thus, C is an uncountable set. This follows from our construction that C is an uncountable set.

In fact, now, let us try to calculate. Note that C, which is equal to intersection of A_n's implies that for every n, C is a subset of A_n. What was A_n? That was a disjoint union of intervals I n j; j equal to 1 to 2 to the power n.

At the nth stage, what will be the length of each…, where the length of each I n $_j$. What is the length of the intervals that are left at the nth stage? That is, one over… Let us look at the construction. At the first stage, (Refer Slide Time: 24:24) when we removed two… At A_1, two intervals were left – each of length 1 by 3. So, this is 1 by 3, this is 1 by 3, this is 1 by 3, and this is 1 by 3. So, four intervals at the second stage of length 1 by 3.

At the nth stage, how many 2 n intervals of each of length… How many will be left? 2 to the power n intervals – each of length how much? So, here, the length of each I n $_j$. At the second stage, it is 1 by 3. The second stage is 1 by 3 and nth stage will be 1 over 3 to the power 2 n minus 1. So, that will be the length of each one of them. There are 2 to the power n of them. So, what is the total length?

Sigma lambda of I n $_j$; j equal to 1 to 2 to the power n, that is, 2 to the power n intervals – each has got the same length. So, divided by 3 to the power 2 to the power n minus 1. Observe that this number goes to 0. That is, 2 to the power n by 1 over 3 to the power n; that goes to 0 as n goes to infinity. So, that means what? That means C can be covered by…; for every n, by 2 to the power n intervals whose length is this (Refer Slide Time: 26:11) and that can be made as small as they want.

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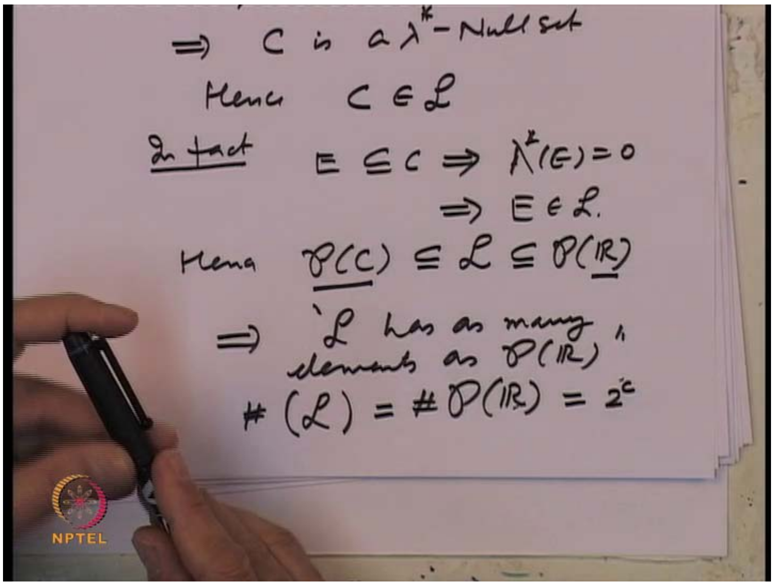

So, that means the outer Lebesgue measure; lambda star of C is equal to 0 because what is lambda star of E? It is a infimum of the sums of the intervals that cover the set C and here, we have shown that C is contained in A_n, which is a finite disjoint union of intervals. The total length of these intervals is becoming smaller and smaller. So, that is length of C. So, that implies that C is a lambda star null set. Hence, C belongs to Lebesgue measurable set and not only that C is (( )). In fact, what we know is something more – that if E is any subset of C, then that implies that lambda star of E is also equal to 0 because lambda star is (( )) and that implies that E also belongs to L. Hence, all subsets of C; power set of C is a sub class of Lebesgue measurable sets.

Of course, Lebesgue measurable sets are a subset of power set of real line. Now, but C is uncountable and R is uncountable. So, what does this imply? That means, this implies that L has as many elements as P of R. So, what is the meaning of this – has as many elements as P of R? That is same as the cardinality.

If you know what is cardinality; cardinality of L is same as the cardinality of the power subset of real line and if you know that the cardinality of real line, which is… We are going to call as cardinality of continuum, which is denoted by small letter c and this is denoted by 2 to the power c (Refer Slide Time: 28:36).

If you look at… So, what does that prove? That proves that if you look at from the cardinality point of view; if you look at how many elements are there in the class of all Lebesgue measurable set, then it says cardinality of Lebesgue measurable sets is as much as the cardinality of all subsets. So, if you look from the cardinality point of view, you cannot say that the class of all Lebesgue measurable sets is a proper subset of the class of all subsets of the real line, but that does not also imply that all subsets of real line are Lebesgue measurable. So, the questions still remains undecided whether the class of all Lebesgue measurable sets is a proper sub class of all subsets of real line. To decide this question is a bit difficult and that relates to some fundamental questions in set theory.

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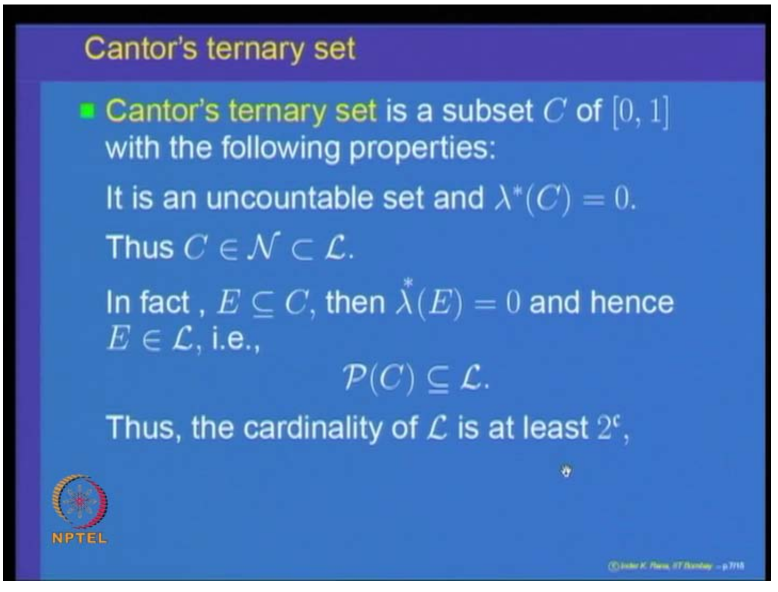

Let us look at… What we have shown just now, let us just recapitulate that lambda star of C is equal to 0 and that says that power set of C is a subset of L. Hence, there are at least as many elements in L as 2 to the power c. So, that is the cardinality of the continuum.

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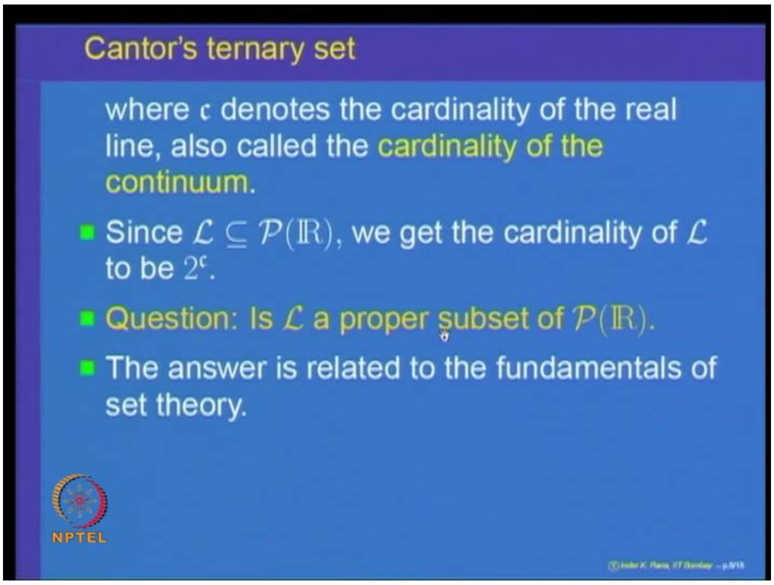

So, we get the cardinality of L and power set is same; both have got same cardinality. So, the question still remains is L a proper subset of P R? If you recall the answer to this question, A is related to some of the fundamental questions in set theory.

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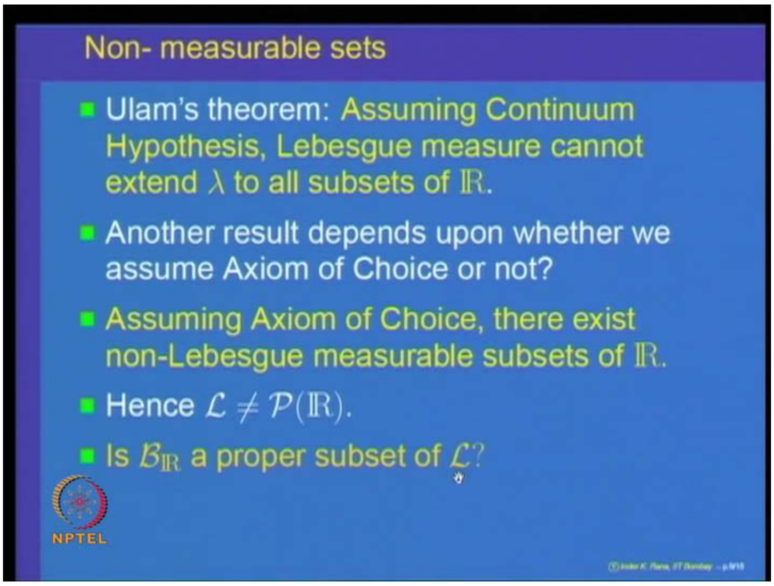

If you recall, we proved what is called Ulam's theorem. We did not prove it really, but we mentioned what is called Ulam's theorem and I said that one can read the proof of this in the text book that we have mentioned. That statement of the Ulam's theorem says,

– Assuming continuum hypothesis, Lebesgue measure cannot be extended to all subsets; all of real line.

There is something called continuum hypothesis in set theory. I will not explain at this stage what is continuum hypothesis because we will be going straightly off stream, but it is what the mentioning here that the set theory is based on certain Axioms. (( )) So, whatever modern mathematics we are doing is based on Axiomatic set theory. There is a… which has some kind of some Axioms on which we are we can deal with set theory, but there is something called continuum hypothesis, which relates to the subsets of real line and so on. That is not part of the Axioms of set theory and that is why it is called continuum hypothesis. Some people believe in continuum hypothesis and do mathematics according to that and some people do not believe in it.

If you assume continuum hypothesis, then Ulam's theorem says that you cannot extend; that means not all subsets of real line are measurable. Another result which one can use and which is again not part of the Axiomatic set theory is the following, which says that supposing you assume what is called Axiom of Choice. Axiom of Choice is another Axiom that is not part of the Axiomatic set theory. One can either accept it as part of set theory and do mathematics or do not accept as part of it and do mathematics. So, mathematicians those who accept Axiom of Choice are supposed to be doing what is called non constructing mathematics because there are some existence theorems, which assume Axiom of choice helps in proving some theorems that are existence in nature. For example, proving that every vector space has a basis requires the need of using Axiom of Choice. You cannot prove it if you do not assume Axiom of Choice. There are many results in mathematics, which are which use Axiom of Choice and which are not true if you do not assume Axiom of Choice.

What is Axiom of Choice is basically saying; very heuristically saying given a non empty collection of non empty sets you can pick up one element from each set and form a new set. So, it is how sets can be constructed when the sets are not indexed by a family that is finite in number essentially.

It says – given any indexed family of non empty sets and that indexing set is also non empty, you can pick up one element from each one of these sets and form a new set. So, using this one can show there exist sets in the real line that are not Lebesgue measurable.

So, we will prove this result. Assuming Axiom of Choice, there exist non-Lebesgue measurable sets in real line. Let us prove the existence of non measurable sets by assuming Axioms of Choice.

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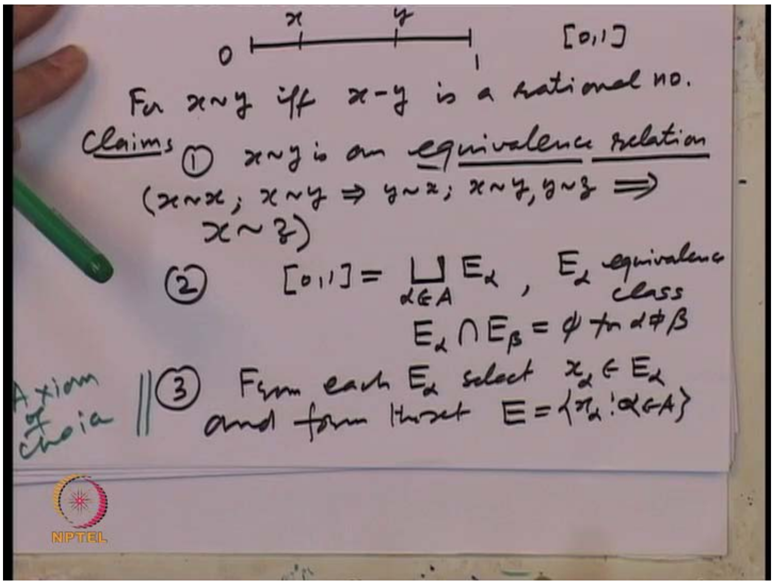

Let us start. What we are doing is existence of non-measurable sets. That is what we are discussing. We want to construct a subset of the real line that is not Lebesgue measurable. To start with, consider once again the interval 0 to 1. This is the interval 0 to 1.

On this, I am going to define a relation. For x related to y if x minus y is a rational number. For x and y, take two points x and y in 0, 1 and say that they are related with each other if and only if x… their difference is a rational number.

The first observation claims (( )) I will just write claims. One; that this x related to y is an equivalence relation. What does equivalence relation mean? It means it is reflexive, symmetric, and transitive. What is reflexive? x related to x; that is obvious because x minus x is 0 and that is a rational number.

Secondly, if x is related to y; that means x minus y is a rational number. So, the difference the negative of that, that is, y minus x also is a rational number. So, that implies that y is related to x. If x is related y, then y is related x. That is called symmetry; that the relation is symmetry.

The third one is… Let us put x is related to y and y is related to z. x related to y means x minus y is rational and y related to z means y minus z is rational. If it is a difference that implies that x minus z is a rational. So, that implies that x is related to z. It is an equivalence relation; it is (( )) It is a reflexive, symmetric, and transitive. Every equivalence relation given on a set partitions the set into equivalence classes. So, that is the basic idea; that 0, 1 can be partitioned into equivalence classes by this relation.

So, that implies…

Second; that implies… Let us write that 0, 1 can be written as a disjoint union of equivalence classes. So, let us write it as E_α; alpha belongs to some indexing set and let us call it as A; E_α equivalence class. Recall equivalence class means E_α intersection E_β is empty for alpha not equal to beta. That is why I have written it as a union with this sign. That means equivalence classes – they cover 0, 1 and they are disjoint. So, that is the partition of the set on which equivalence classes are defined. So, that is. The third step is from each $E\alpha$, select some element x_α and form the set. Let us call it as E, which is x_α; alpha belongs to the indexing set A.

What we are saying is, using this equivalence relation partition (( )) interval 0, 1 into equivalence classes. From each equivalence class, pick up one element; exactly one element x_α. Select one element x_α. Choose one element x_α from each equivalence class and put them together in a box; call that E and claim is that E is a set. (Refer Slide Time: 39:05) This is the place where we are using Axiom of Choice. That means E_α is a collection of… non empty collection of non empty sets. From each, we can pick up one element and form this set. This is possible only if we assume Axiom of Choice. So, here is the place where we are using Axiom of Choice. So, from each equivalence class we have picked up one element and constructed a set E.

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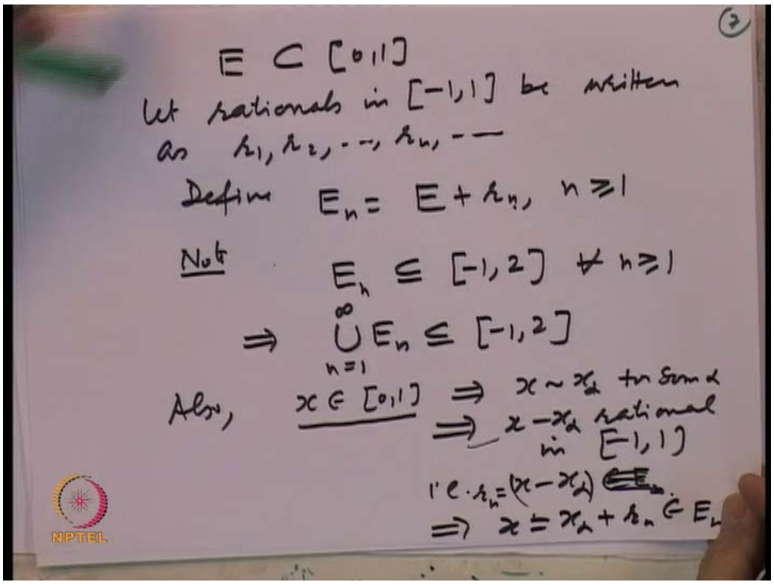

Obviously, this set E is a subset of 0, 1 because each equivalence class is a subset of 0, 1 and from each, we have picked up one element. So, this is a subset of 0, 1.

Let us write… Let rationals in minus 1 to 1 be written as r_1, r_2, r_n and so on. Rationals in the interval minus 1 to 1 is a countable set. So, they can be enumerated; they can be written in the form of a sequence. We are not saying r_1 is smaller than r_2 or anything, but we are just giving a numeration of the rationals. They are countably many, so we can write them as a sequence and construct define a set E_n, which is E plus r_n; n bigger than or equal to 1. Construct a set En. This

Let us observe where is this set E_n. E is in 0, 1 and each r_n is between minus 1 to 1. So, what can we say about the set E plus r_n? E can be 0 to 1 and r_n could be minus 1 to 1. So, that means each one of them is the subset of minus 1 to 2.

At the most, this sum can become minus 1, where the elements of E_r are smaller. Smallest one is 0 and the possibility here is minus 1. The largest possible is r_n is equal to 1 and E also element is 1. So, 1 plus 1 is 2. So, for every n, E_n is a subset of 0, 1 of minus 1 to 2. So, this implies that the union of E_n's is also contained in minus 1 to 2. That is one observation. Also, if I take x that belongs to 0, 1; if I take an element x in 0, 1, that implies x is related to x_α for some alpha because the equivalence classes cover 0, 1. So, every element x in 0, 1 has to belong to one of the equivalence class. Say it belongs to E_α, which means it is related to x_α the element that we have picked. So, that implies that x minus x_α is a rational. X minus x_α related means the difference is rational and where will that rational be?

X is in 0, 1 and x_α is in 0, 1. (Refer Slide Time: 42:42) This is a rational in minus 1 to 1 because both could be 1. That means, that is, x minus x_α belongs to E_n because if it is a rational in minus 1 to 1, that must be equal to some r_n. That means x is equal to x_α plus r_n and that means it is in E_n. What we are saying is, for every x in 0, 1, x minus x_α belongs to… So, that implies that x is equal to x_α plus r_n and that belongs to E_n. So, x belongs to E_n.

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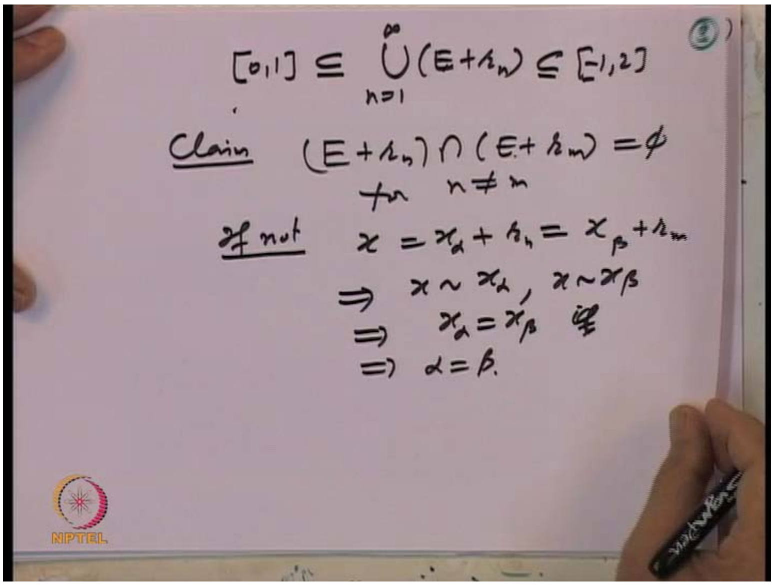

The second observation is that 0, 1 is inside the union of E_n's. So, that is what we have got. This construction we have got is following that 0, 1 is contained in union of E plus r_n, that is En; n equal to 1 to infinity and that is contained in minus 1 to 2. We have used Axiom of Choice in this construction of the set E.

Now, here is one observation that… Let us move on to observe claim that these sets E plus r_n intersection E plus r_m are disjoint sets for n not equal to m. To prove this, let us take an element x, which is common. So, if not x belongs to E plus r_n, that means x is equal to x_α plus r_n, and it is also equal to… It is also in E plus r_m so, it is also equal to some e beta plus r_m. That implies x is related to x_α and x is related to x_β. That this x is related to (( )) That implies either x_α is equal to x_β if… That should be same and that is possible implies that alpha is equal to beta. (Refer Slide Time: 45:25) If alpha is not equal to beta, then this is not possible. So, that says… That means that these two sets are disjoint.

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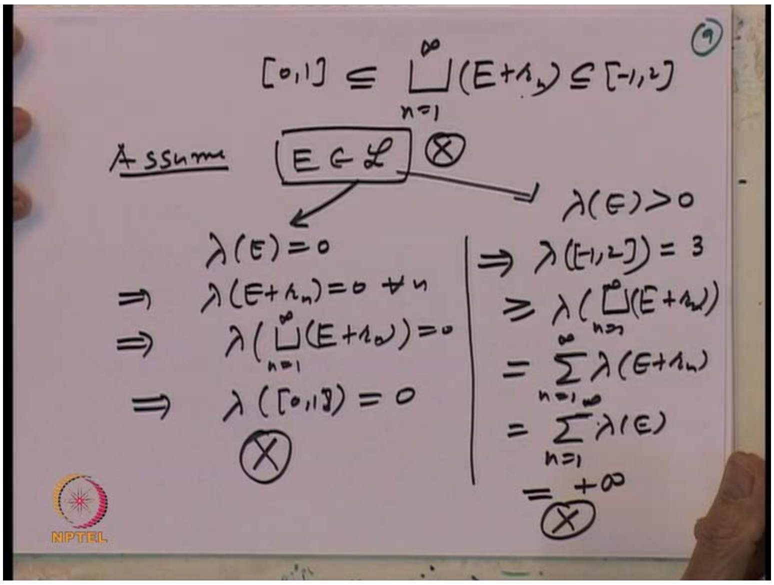

This is what we have got. As a consequence, let us write this as that 0, 1 is contained in a disjoint union of E plus r_n; n equal to 1 to infinity and that is contained in minus 1 to 2.

Till now, we have not done anything except we defined a equivalence relation and using Axiom of Choice, we constructed a set E and this as this property. Now, suppose assume that E is Lebesgue measurable, then there are two possibilities – one, Lebesgue measure of E is equal to 0, but that implies Lebesgue measure of E plus r_n is equal to 0 for every n because Lebesgue measure is translation-invariant and that implies that the Lebesgue measure of the union E plus r_n is equal to 0. That implies… Because 0, 1 is inside this, that means Lebesgue of 0, 1 is equal to 0, which is a contradiction because Lebesgue measure of 0 to 1 is equal to 1.

The second possibility is that the Lebesgue measure of E is strictly bigger than 0, then that implies Lebesgue measure of minus 1 to 2; this closed interval (Refer Slide Time: 47:17) is bigger than or equal to Lebesgue measure of this union because that is subset of it. That is equal to sigma lambda of E plus r_n and that is equal to sigma lambda of E because for every n it is same. This being a positive quantity, added infinite number of times, this is equal to plus infinity.

This is again a contradiction because lambda of minus 1 to 2 is actually equal to 3 and 3 equal to infinity is a contradiction. (Refer Slide Time: 48:00) Either case, this assumption cannot be true. So, this is a set, which is in 0, 1 and which is not measurable.

What we have shown is the following that (Refer Slide Time: 48:16) if we assume Axiom of Choice, then there exist non-Lebesgue measurable sets in the real line. Without Axiom of choice or without continuum hypothesis, it is not known that you can construct subsets of the real line, which are not measurable non-Lebesgue measurable.

In fact, there is a theorem, which says that the condition that assume Axiom of Choice… Actually, if you put this (Refer Slide Time: 48:46) as an Axiom in set theory that every subset of the real line is Lebesgue measurable; if you take that as an Axiom and if your set theory Axioms are already consistent, then adding this new Axiom to your set theory will not make any difference; it will still leave is consistent.

The distance of non-measurable sets get related to fundamental questions in set theory. So, on this side, we will leave it as it is saying that if you either assume continuous hypothesis or you assume Axiom of Choice, then there exist sets, which are not Lebesgue measurable.

Let us tend to the other side. Can we say that the Borel sigma algebra, the Borel subsets of real line form a subset of this form a sub class of Lebesgue measurable sets. What is the relation between these two? Can we say that the Borel sets form a proper subset of the class of all Lebesgue measurable sets?

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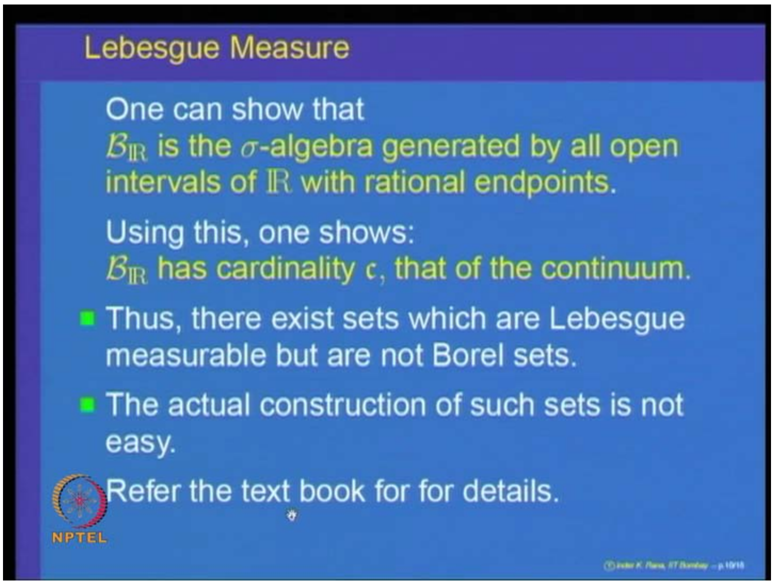

One can show… We will not prove most of the things here because they are slightly technical. First observation is that the Borel sigma algebra of the real line, which is the sigma algebra generated by all intervals is the same as the sigma algebra generated by all open intervals because one can show that every open set in real line is a countable union of open intervals, that is, using the basic topology in the real line. So, topological properties of real line come into play and not only that, in fact you can take open intervals with only rational end points. If you generate the sigma algebra by them, that is same as the Borel sigma algebra. This needs a proof. We will not prove it, but indicate what is involved here.

The Borel sigma algebra… This is a countable family – open intervals with rational end points. You take a countable family of intervals and generate the sigma algebra and that is B_R. One can show that the cardinality of this process of generating is exactly equal to c.

Using these properties one shows… (Refer Slide Time: 51:05) Using this construction one shows that the cardinality of the sigma algebra of Boral sets is same as that of c that of the continuum and that is called the real line, whereas the cardinality of the Lebesgue measurable sets was 2 to the power c. That means there exist sets. So, looking at the cardinality says that there exist sets, which are Lebesgue measurable, but are not Borel sets, but construct… actual construction of these sets is not very easy. It is possible to construct such sets, which are Lebesgue measurable, but are not Borel sets. They are called Analytical sets / Analytic sets. For that, we refer the (( )) our text book for more details. Those of you who are interested; they should refer the text book for more details.

What we have shown today is that in the special case of the extension theory, we get the (( )) of the length function on a class of sets, which are called Lebesgue measurable sets, which include the Borel sigma algebra of subsets of the real line. The cardinality of the Lebesgue measurable sets is same as the cardinality of all subsets.

If you make some assumptions like continuum hypothesis or Axiom of Choice, you can show existence of sets, which are not Lebesgue measurable; otherwise, you cannot show. There is no such proof known. On the other side, the Borel sigma algebra has got cardinality c, which is much stricter strictly less than the cardinality of Lebesgue measurable sets.

We will continue looking at the properties of Lebesgue measurable sets viz-a-viz, open sets, close sets, and the group (( )) on real line in the next lecture. Thank you.