Measure and Integration

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Module No. # 03

Lecture No. #11

Measurable Sets

Welcome to lecture 11 on Measure and Integration. In the previous lecture, we had defined, what is called an outer measurable subset. We had started looking at the properties of the outer measurable sets.

We will continue the study of properties of the outer measurable sets today and if time permits at the end, we will specialize the case, when the space is in the real line. So, let us recall what we have been doing. So, we were looking at properties of measurable sets. Let us just recall, what is an outer measurable set.

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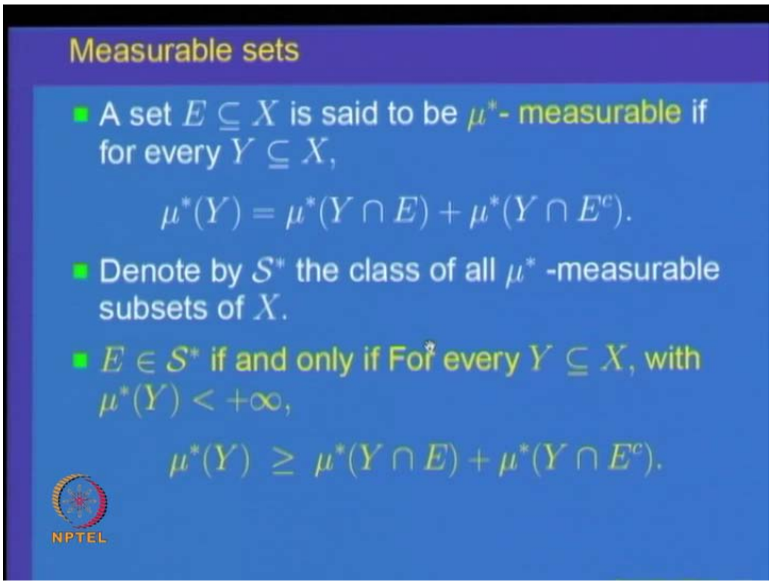

A subset E of X is said to be an outer measurable or mu star measurable. If mu star of any set Y is written, it is written as mu star of Y intersection E plus mu star of Y intersection E compliment. So this condition must be satisfied for every subset Y of X and then we said, let us denote by S star and the class of all mu star measurable sets. We gave an equivalent way of verifying, when a set is outer measurable. So, the condition is that a set E is measurable, if and only if every subset Y in X with mu star of Y is finite. We have the condition that mu star of Y is bigger than or equal to mu star of Y intersection E plus mu star of Y intersection E compliment.

So, instead of just saying that for every subset Y, this equality must be true. We have to only verify for those subsets Y of X for which mu star of Y is finite. Instead of equality, we have to verify only bigger than or equal to one-way inequality because the other way round is always true for mu star being countably sub additive.

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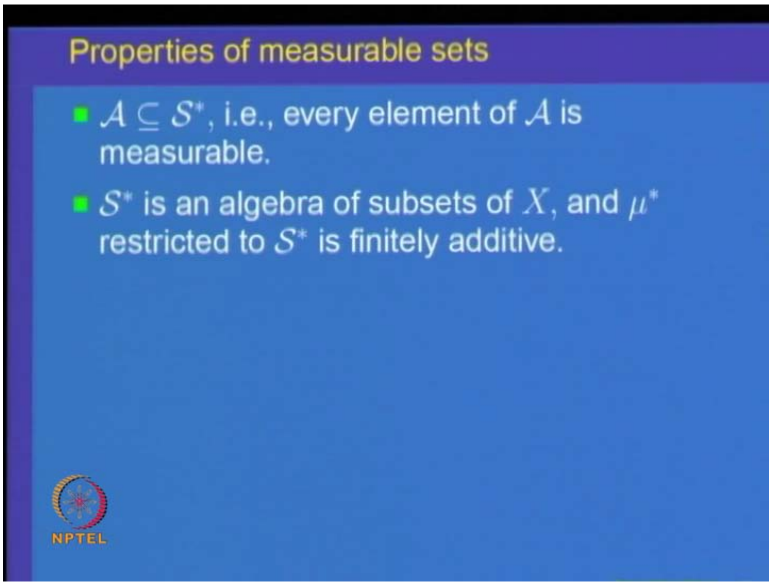

So, we will use this condition when we require. The first observation that we proved last time was that A is the given algebra on which the measure is defined. So, the first claim we proved is that every element in the algebra is also a measurable set. So, A is a subset of S star.

The second property, we were looking at was - if S star is an algebra of subsets of X and mu star restricted to S star is finitely additive. We had already observed that a set E is measurable, if and only if, its compliment is measurable. So, S star A is closed under compliments and we only have to verify that it is closed under unions and that proof working out in the last time and we had done it; let us just revise it again because we are going to need those inequalities.

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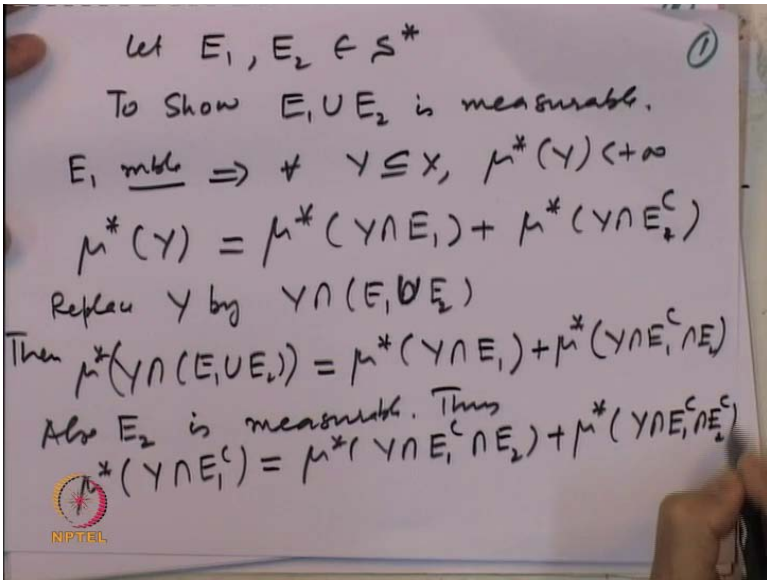

So, let E_1 and E_2 be measurable sets to show that E_1 union E_2 is measurable. E_1 is measurable and implies that for every subset Y contained in X. Let us have that special condition less than finite, we know that mu star of Y is equal to mu star of Y intersection E_1 plus mu star of Y intersection E_2, sorry E_1 compliment. This is true for every subset Y with that property. Let us replace, Y by Y intersection E_1 union E_2.

So, replace this Y so then we get so then what we have we have mu star of Y intersection E_1 union E_2 is equal to… so, Y is replaced by Y intersection E_1 union E_2, but E_1 is a subset of it. So, the first term is just mu star of Y intersection E_1 plus the second term becomes mu star of E_1 union E_2 intersection E_1 compliment. So, first term will give you only empty set, union Y intersection E_2 intersection E_1 compliment. So, E_1 compliment intersection E_2 and that is what we get by using the fact, E_1 is measurable and E_2 is also measurable. Thus, for every set Y, a corresponding equation holds for E_2 compliment, but we will replace Y by Y intersection E_1.

So, mu star of Y intersection E_1 compliment is equal to mu star of Y intersection E_1 compliment intersection E_2 plus mu star of Y intersection E_1 compliment intersection E_2 compliment. So, using the fact that E_2 is measurable, we have written mu star of Y intersection E_1 as the set intersection E_2 the set intersection E_2 compliment. Now, in these two equations, (Refer Slide Time: 06:14) look at this set this term Y intersection E_1

compliment intersection E_2 and that is also sitting here. So, we will compute the value of this and put it to that equation. So, let us do that.

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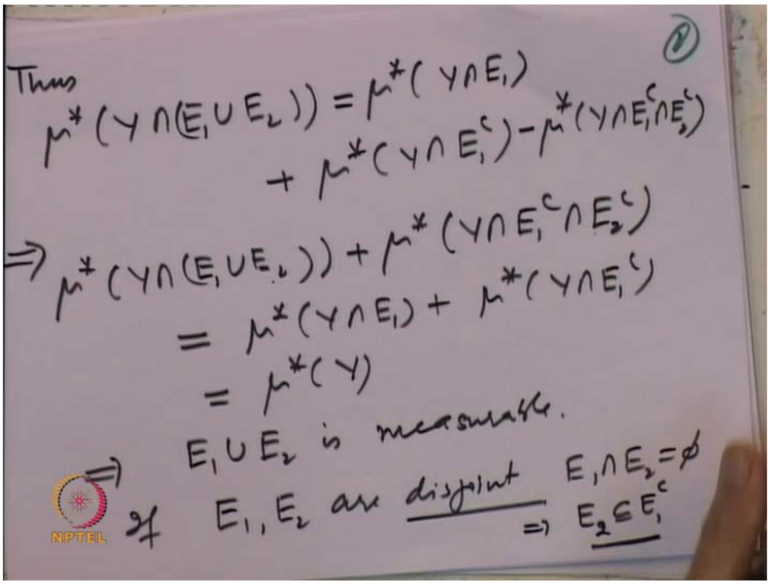

From this second equation, we put the value there and so we have got from these two equations (Refer Slide Time: 06:34). Thus, mu star of Y intersection E_1 union E_2 that is the left hand side of this equation. so that is equal to the first term, mu star of Y intersection E_1 plus Y intersection E_1 compliment intersection E_2 is equal to mu star of Y intersection E_1 minus that thing (Refer Slide Time: 07:04). So, mu star of Y intersection E_1 compliment minus mu star of Y intersection E_1 compliment intersection E_2 compliment. Now, one should note down here - we have taken one term on the other side. So, this is possible because all the sets involved have finite outer measure and this is the equation of real numbers.

So, we can take one term on the other side and in general that will not be possible, if Y, one of the terms is equal to plus infinity. So, the condition mu star of Y is finite is being used here. So, we get using the fact that E_1 and E_2 are measurable and we get this equation (Refer Slide Time: 07:53). From here, let us take this negative term on the other side and that implies mu star of Y intersection E_1 union E_2 plus mu star of Y intersection E_1 compliment intersection E_2 compliment is equal to mu star of this term; Y intersection E_1 and the second term plus mu star of Y intersection E_1 compliment. Now, using the fact that E_1 is measurable and this is same as mu star of Y. so, we have shown that for every subset Y with mu star of Y finite. It’s measure mu star of Y can be written as mu star of E_1 Y intersection E_1 union E_2 plus mu star of Y intersection E_1 compliment intersection E_2 compliment, but note this set is nothing but E_1 union E_2 compliment. So, this implies that E_1 union E_2 is measurable. Now, for the special case, if E_1 and E_2 are disjoint, it means E_1 intersection E_2 is empty set and that implies that E_1 is contained in E_2 compliment or E_2 is contained in E_1 compliment, either one is true. So, note this is true (Refer Slide Time: 09:53). In that case, let us go back and look at the first equation because E_1 and E_2 were measurable.

So, we had this condition. So, in this equation (Refer Slide Time: 09:58), this is true for every Y. So, let us replace this Y by E_1 union E_2 and that will give us the measure, mu star of union of E_1 union E_2. So, in this we are going to replace Y by E_1 union E_2.

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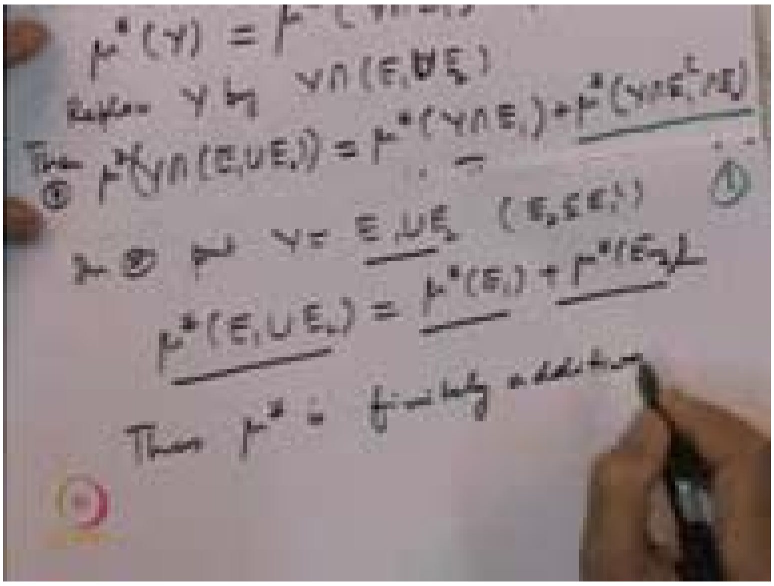

So, let us just put that equation here and look at what we are doing. So, in this equation, in the star we are putting Y equal to E_1 union E_2 and keep in mind they are disjoint. So, the left hand side will be mu star of E_1 union E_2 equal to right hand side. The first term is mu star of E_1 plus and in the second term; E_2 is a subset of E_1 compliment because E_1 E_2 are disjoint. This implies E_2 is a subset of E_1 compliment . So, that means this is nothing but, plus mu star of E_2 mu star of E_2. So, when E_1 and E_2 are disjoint, mu star of E_1 union E_2 is mu star of E_1 plus mu star of E_2. Thus, it means mu star is finitely additive.

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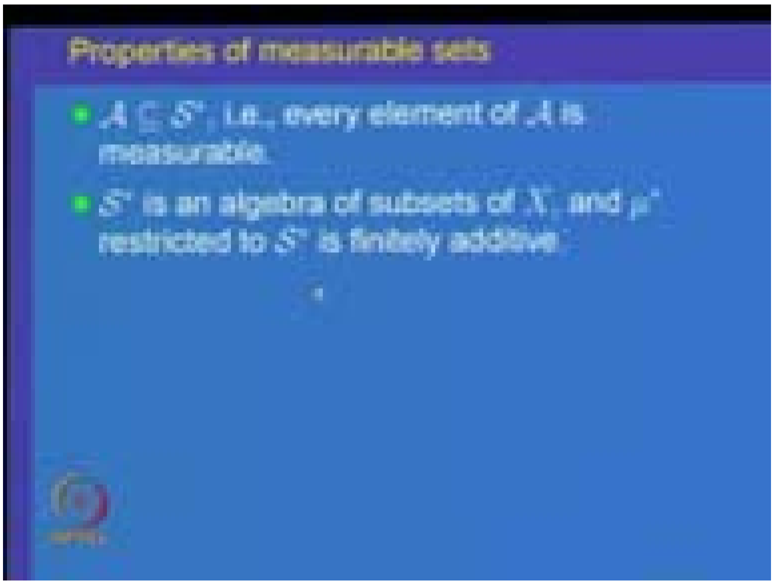

So, we have proved this property, namely: S star is an algebra of subsets of X and mu star restricted to S star is finitely additive. Next step is to go a bit further.

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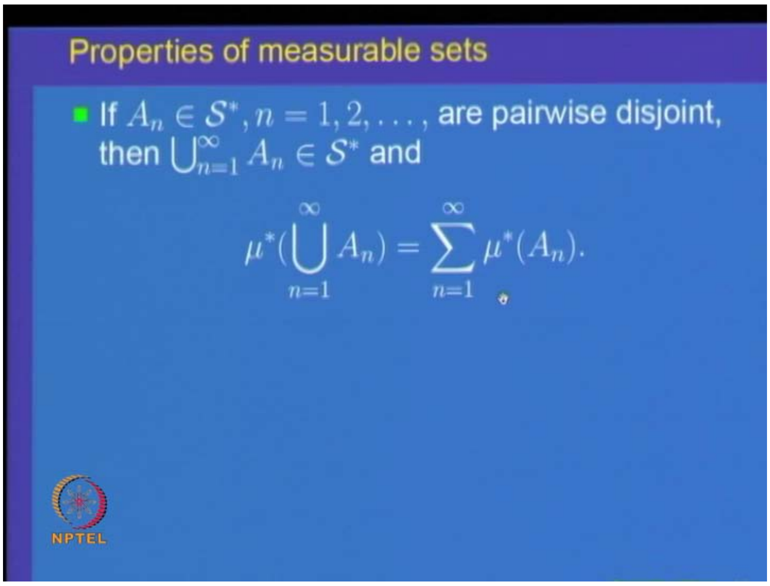

We want to prove that whenever you got a sequence of sets in S star, which are pairwise disjoint, then their union is also in S star. Mu star of the union is equal to summation of mu stars of A_n. That means, we are going to show that S star is closed under pairwise disjoint union of sets in even countably infinite and mu star is countably additive.

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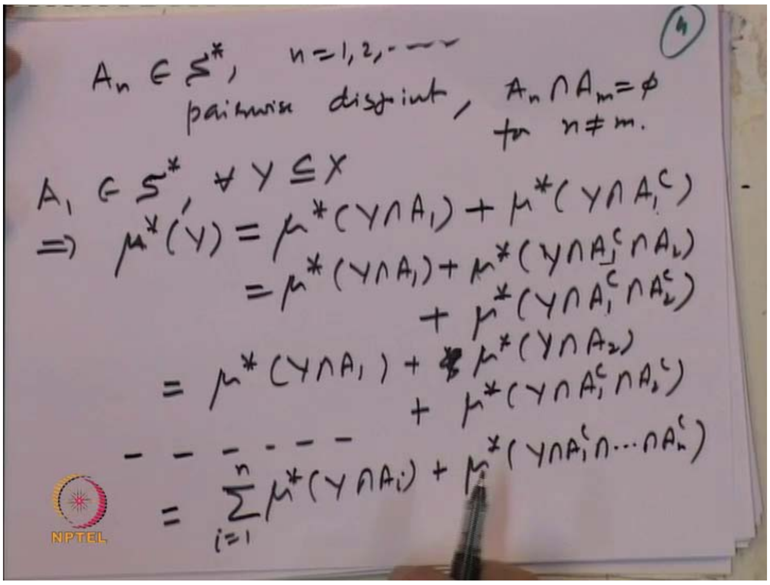

So, let us prove this property. Let us take A_n belong to S star n equal to 1, 2 and so on. Pairwise disjoint, that is, A_n intersection A_m is empty for n, not equal to m. so, we start A_1 belonging to S star. A_1 measurable implies that mu star of any set Y can be for every Y contained in X. I can write this to be equal to mu star of Y intersection A_1 plus mu star of Y intersection A_1 compliment.

Now, use the fact that A_2 is measurable. So, leave the first term as it is Y intersection A_1 plus A_2 is measurable. So, measure of mu star of this set can be written as mu star of A_1 compliment intersection A_2 plus mu star of Y intersection A_1 compliment intersection A_2 compliment. So, this term mu star of Y intersection A_1 compliment is written as mu star of Y intersection A_1 compliment plus intersection A_2 plus mu star of Y intersection A_1 compliment intersection A_2 compliment. So, here we have used the fact that A_2 is measurable. Now, observe that A_1 and A_2 are disjoint. So, A_2 will be a subset of A_1 compliment and this set is nothing but, Y intersection A_2. So, I get the first term same as mu star of Y intersection A_1. The second term is mu star of Y intersection A_2. The third term is mu star of Y intersection A_1 compliment intersection A_2 compliment. So, in the first, we used A_1 is measurable. In the second, we used A_2 is measurable and used A_1 and A_2 are disjoint. If we continue this process, after n steps we will have this is (Refer Slide Time: 15:01) equal to… The second step gives you, mu star Y intersection A_1 plus mu star of Y intersection A_2. So, after n steps this will have mu star of Y intersection A_i

i equal to 1 to n plus one term will be there, which is mu star of Y intersection A_1 compliment intersection up to A_n compliment.

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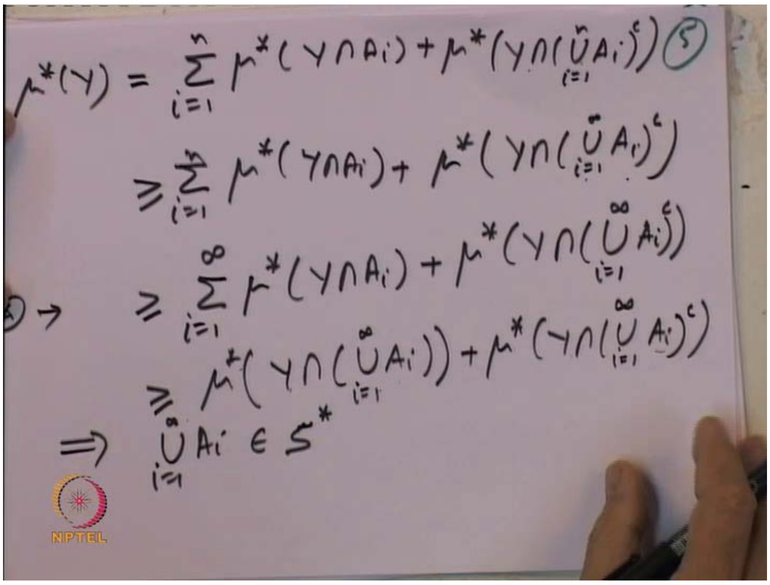

So, let us write this last term in terms of union. It is equal to summation of i equal to 1 to n mu star of Y intersection A_i plus mu star of Y intersection union A_i i equal to 1 to n compliments. So, this term is represented in terms of compliments of the unions. So, this is after n steps. For every n, we have got mu star of Y can be written as this (Refer Slide Time: 16:07) and now, it is true for every n. Here, I would like to write this union as 1 to infinity, if I do that, I will be make this set bigger.

Hence, the compliments will be a smaller set. So, if I replace this by Y intersection union i equal to 1 to infinity of A_i compliment. This set is smaller than this set (Refer Slide Time: 16:40). So, mu star of this will be bigger than mu star of this. So, if I write mu star, then this term is bigger than this term. So, this will be bigger than or equal to summation i equal to 1 to n and this term as it is. Y intersection A_i plus this (Refer Slide Time: 17:01). So, what we have done in the second term, where it has union 1 to n. I have taken union 1 to infinity and because of compliments this term will be smaller. So, instead of equality, I have got the inequality and this happens for every n. So, I can let n go to infinity and this will be bigger than or equal to summation i equal to 1 to infinity mu star of Y intersection A_i plus mu star of Y intersection union i equal to 1 to infinity A_i compliments. Now, mu star is countably sub additive.

So, the first term is bigger than or equal to mu star of Y intersection union A_i i equal to 1 to infinity. Second term, as it is mu star of Y intersection union 1 to infinity A_i's compliment. So, using the fact that for every n, A_n is a measurable set. We are able to say that mu star of Y is bigger than or equal to mu star of Y intersection the union A_i's plus mu star of Y intersection, the compliment of the unions. So, that implies that unions A_i 1 to infinity belongs to S star is a measurable set and not only that, we can say something more. So, in this equation star, let us put Y is equal to union of A_i's. So, what will we get?

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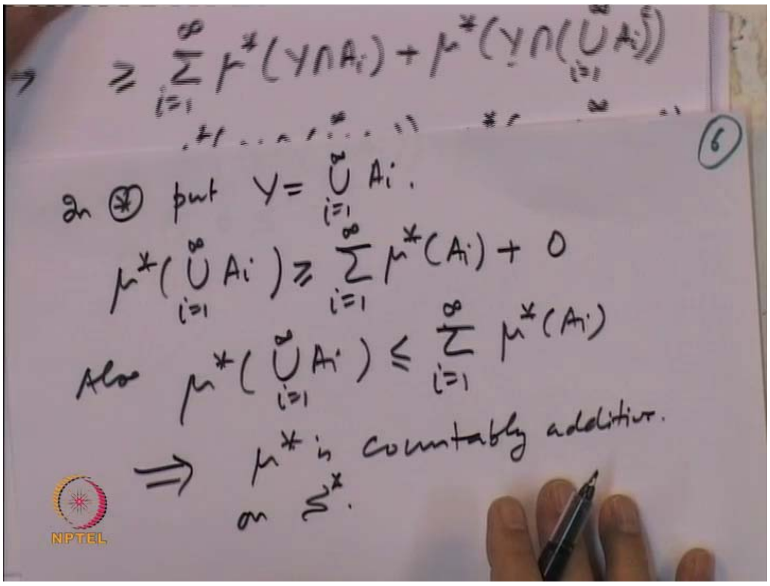

Let us do that substitution and see, what we get. So, in this equation; in star, take Y equal to union of A_i's. Then left hand side is mu star of union A_i 1 to infinity is bigger than or equal to summation i equal to 1 to infinity mu star of Y is union A_i plus this is union and compliment and that is empty set, mu star of that is equal to 0. So, that is equal to 0. So, what we get is- mu star of the union of A_i's is bigger than or equal this, by sub additivity. Mu star of the union A_i's is less than or equal to summation 1 to infinity mu star of A_i's. So, it implies that mu star is countably additive on S star.

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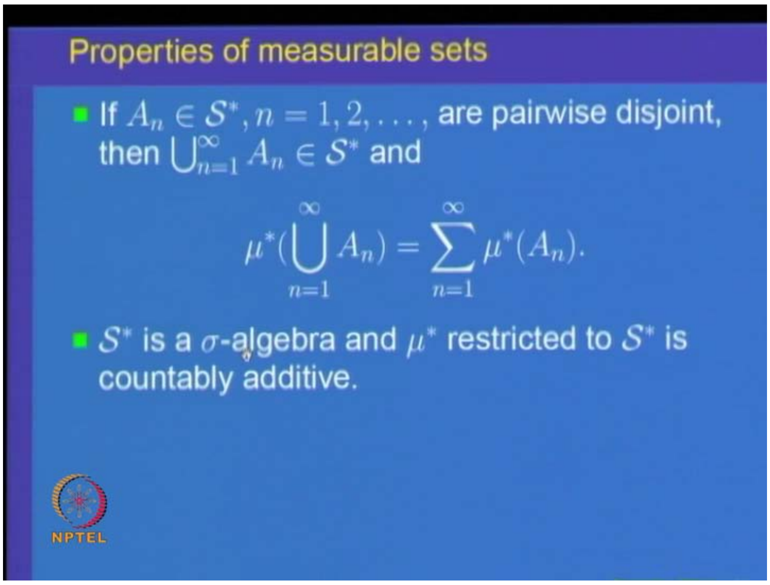

We have got the following property and what we have done till now is- S star, as a consequence of all these properties, we can say that S star is a class of all measurable sets is a sigma algebra of subsets of X and mu star on this is countably additive. So, we started with a measure mu on a algebra A of subsets of a set X. We defined an outer measure via this on all subsets of subset X. Then, we picked up a subclass namely: S star of sets, which are mu star measurable. we have shown that mu star, which in general is countably sub additive is actually countably additive on S star, the sigma algebra of measurable sets.

Why it is a sigma algebra? Because we already shown it is an algebra and it is closed under countable disjoint unions. So, any algebra, which is closed under countable disjoint unions is automatically a sigma algebra and we have shown this. It gives us a way of defining measures on ascending measures.

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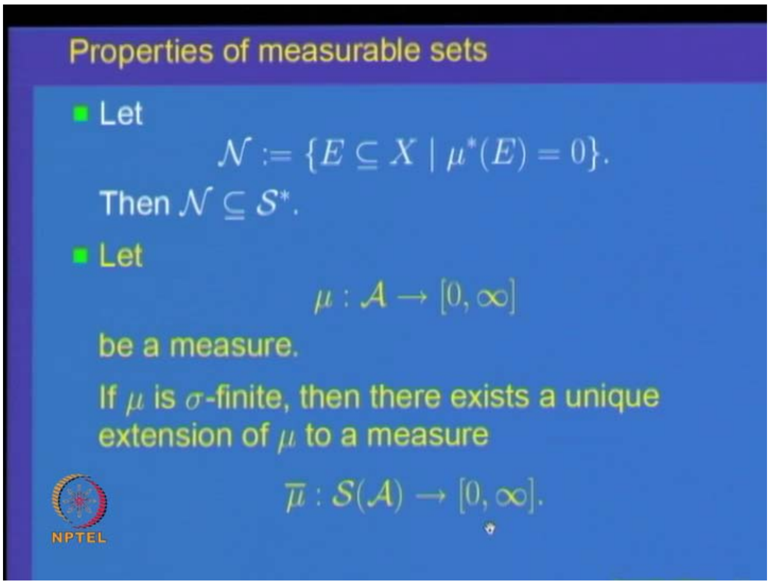

Before doing that, let us observe one more thing. Let us look at sets E in X, whose outer measure is 0. These are called sets of null outer measurable sets. So, the claim is every set, whose outer measure is 0 is automatically measurable.

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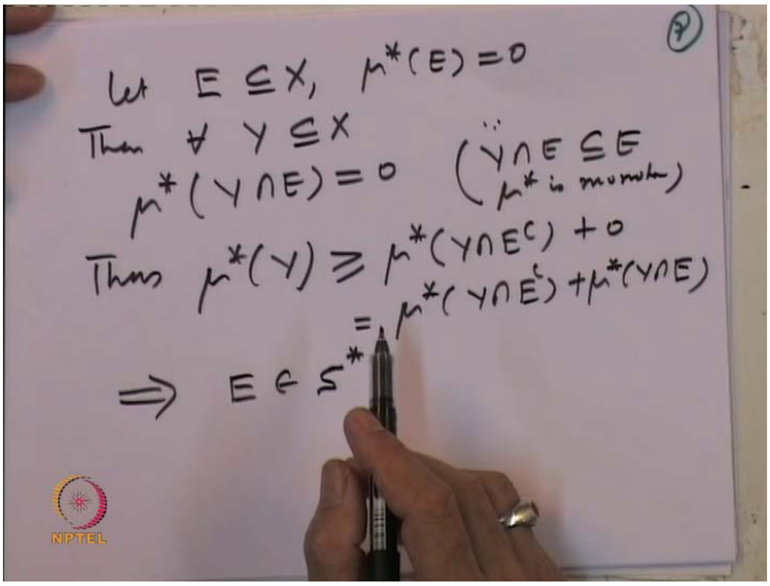

So, let us check that and let E be a subset of X. Mu star of E equal to 0, for every Y contained in X mu star of Y intersection E is equal to 0 because Y intersection E is contained in E because mu star is monotone. So, this is 0 and thus mu star of Y is bigger than or equal to mu star of Y intersection E compliment because again Y intersection E compliment is a subset of this. I can add 0 to it and that is equal to mu star of Y intersection E compliment plus mu star of Y intersection E. That is precisely saying that the set E is measurable.

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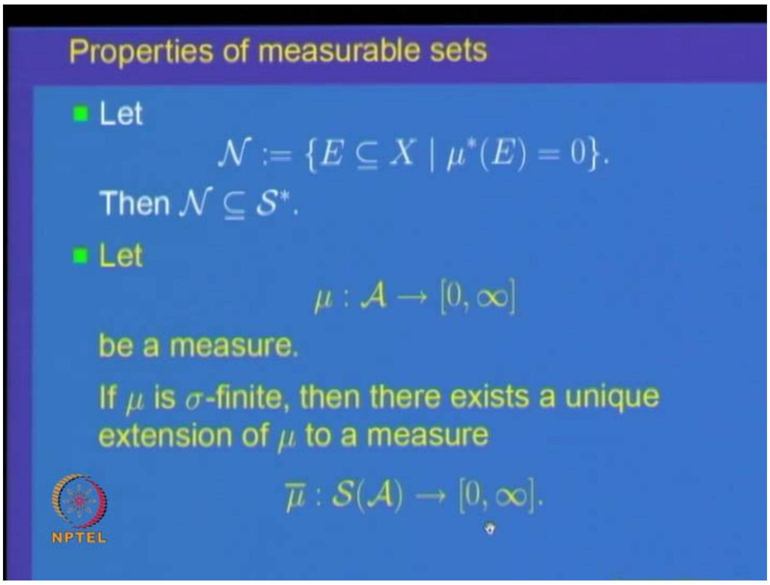

That shows the class of mu star null sets are also measurable. So, this class N is inside S star. So, let us summarize the process. Now, what we have? So, let us start with a measure mu on a algebra; a mu is a measure. A is a algebra of subsets of the set X, if mu is sigma finite, then there exists a unique extension of mu to the sigma algebra generated by A. How do we conclude that? The conclusion for that is as follows:

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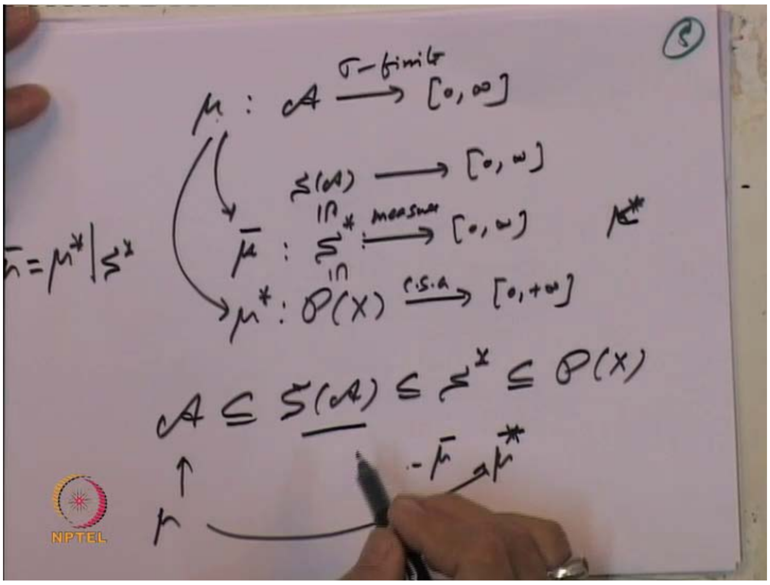

So, mu is on the algebra and it is sigma finite; given. So, we define outer measure mu star from it, which is defined on all subsets of X. It is countably sub additive. We picked up the class of measurable sets S star. So, if we restrict mu to this, let us call it as mu bar. There is a restriction of mu to the smaller class S star. Keep in mind, S star is the class of measurable sets. So, what is mu star? Mu bar is equal to mu star and restricted to S star. This is a measure (Refer Slide Time: 24:55), S star is sigma algebra and we know that this is an extension. So, from mu we come to mu bar, an extension of mu from the algebra to S star. Note: All sets in A are measurable. So, the sigma algebra is also inside here.

A is inside S of A, which is inside S star and which is inside all subsets of X. so, mu is defined here. We get mu star here and when we restrict, we get mu bar and that is same as mu bar on S of A. So, we get measure mu bar on S of A and that is same as mu bar on S of A. so, what is mu bar? Mu bar is the restriction of the outer measure mu star to the sigma algebra generated by A. That is inside the class of measurable set. So, it is a well defined measure because mu is sigma finite. Suppose, there was another extension by some other method to the sigma algebra, then by the uniqueness of measures on the sigma algebras, we know that there is only one possible extension and that we have already proved. In that case, an extension exist, if two measures agree on the algebra, they will also agree on the sigma algebra, provided they are sigma finite. So, uniqueness follows from that theorem.

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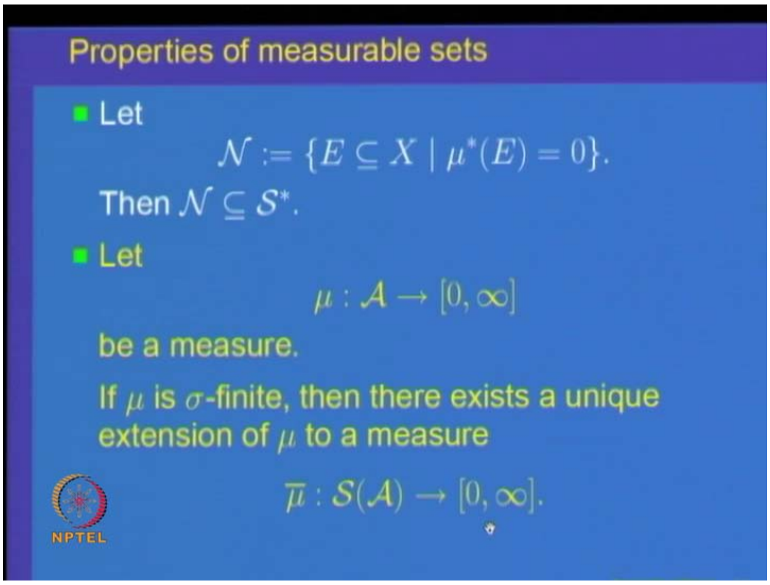

So, we have got, if mu is a sigma finite measure on an algebra, then we can extend it to the sigma algebra generated by it. This is the extension process. So, one has to start with a measure mu on an algebra. Recall, we already have extended it from a semi algebra to the generated algebra. Essentially, it says, if we have a measure on a semi algebra of subsets of a set X, the measure mu is sigma finite. Then, it can be uniquely extended to a sigma finite measure on the sigma algebra generated by that algebra.

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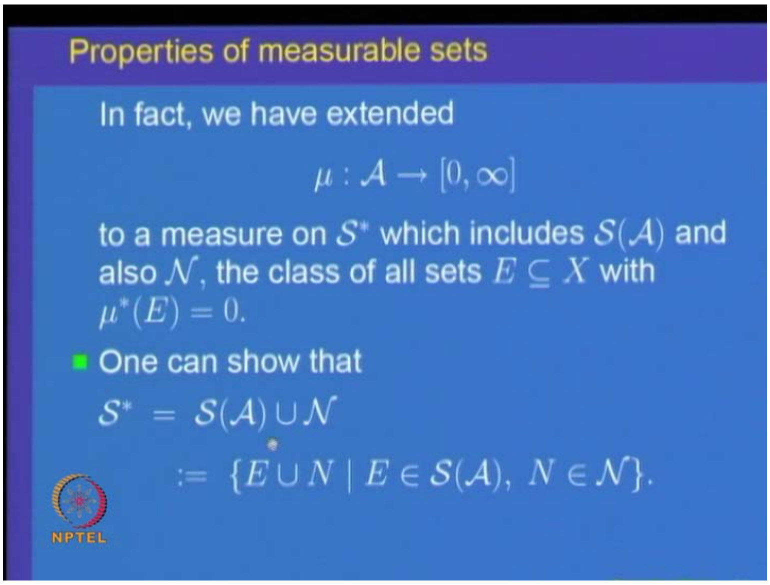

In fact, we have proved something more. We have actually shown that not only mu, which is defined on the algebra extends to S of A, the sigma algebra generated by it.

Actually, it extends to a class S star, which not only includes S of A. It also includes the class of mu star null sets, sets of outer measure mu 0. So, let us denote the class of the sigma algebra generated by S of A and the null sets by a new name. So, what we are saying is- one can show that this S star, the class of all outer measurable sets, which is sigma algebra and includes S of A.

It also includes N So, it includes this union and we are writing it as sets of the type. So, S A union of N is not the union of these two classes. It denotes sets of the type E union N, where E belongs to S of A and N is a null set. So, take sets, which are in the sigma algebra generated by A, adjoined to it any mu star null set. So, look at this new collection and one can show that S star is same as E union N. It involves two things: one is- this collection is a sigma algebra and the other is- this sigma algebra is same as S star. We will not go into the details of this, as they are slightly technical. We will assume this, but it gives us new notion. So, let us define that.

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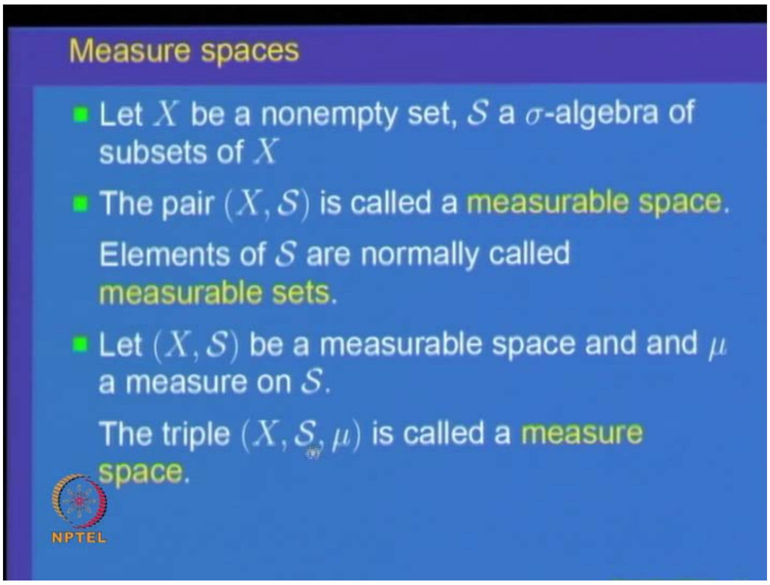

So, let X be a nonempty set. S, a sigma Algebra of subsets of the set X. The pair X, S from now onwards will be called as a measurable space So, a measurable space is a pair, where X is a set and S is a sigma algebra of subsets of it. This elements (Refer Slide Time: 29:30) of S normally are called measurable sets. Suppose, we are given a measurable space X, S and we are given a measure on the sigma algebra S. Then we get a triple X, S and mu and it is called a measure space. So, a measure space signifies a ordered triple, where the first element X is a set X, the second one is a sigma algebra of subsets of a set X and mu is a function defined on the sigma algebra taking non negative values and it is countably additive; it is a measure. So, this triple is called a measure space

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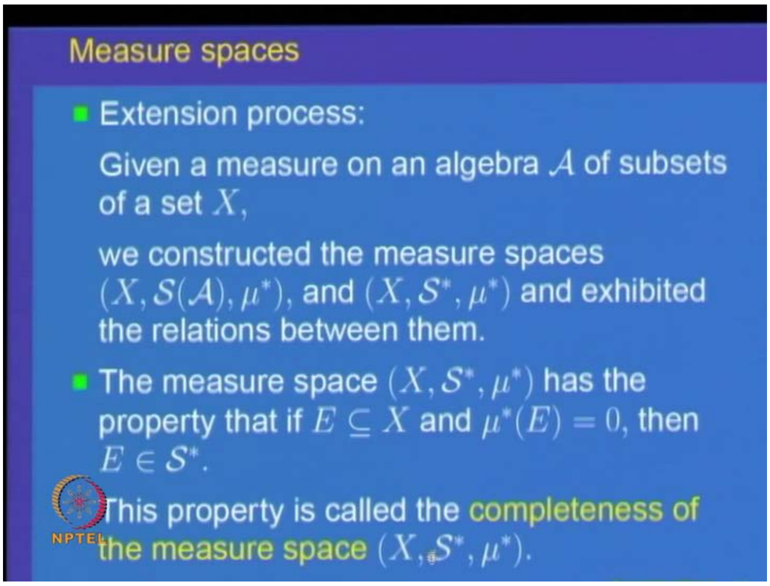

So, what we have done in our extension process? We can now summarize it as follows: Given a measure on a algebra, A of subsets of a set X. What we did? We constructed two measure spaces: one was- X S of A the sigma algebra generated by it mu star, which is the outer measure induced by mu. We know mu star on S of A is a measure and. we also have the measure space X S star and mu star. Mu star on S star is the class of all outer measurable sets.

So, we get these two measures spaces. Keep in mind, S of A is a subset of S star. We gave the relation between these two, namely: the measure space and this is in some sense we can say it is a bigger measure space because the sigma algebra S star is bigger than S of A. This measure space has a special property, namely: if we take any set E in X, mu star of E is 0, then E belongs to S star. So, for example, this is a very special thing. Suppose, you take any subset A of E, then by monotone property mu star of A also will be 0. So, that also will be inside S star. S star includes all mu star null sets. Such a measure, normally is called a complete measure space. So, our construction has given the measure space X S star mu star. It is a complete measure space, namely: all sets of outer measure 0 are elements of S star and that is a nice condition to have. We will see it later on. So, this is called a complete measure space. So, a complete measure space is a space such that the sigma algebra S star or sigma algebra includes all null sets, whose measure is 0.

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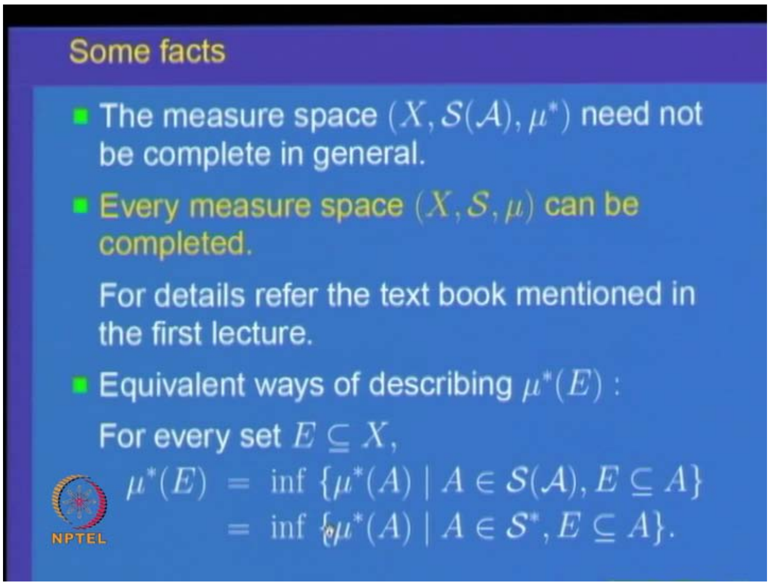

In general, a measure space need not be complete. For example, this measure space need not be complete. So, there is a theorem, which says every measure space X S mu can be completed. This process of completion of a measure space is a slightly technical one. The basic idea is- given a measure mu on a algebra S of subsets of a set X collect together all sets, whose outer measure mu star is 0 and adjoin them or add them to the sigma algebra S. That means, generate a new sigma algebra by taking S and the sets, which are null sets. So, that gives a bigger sigma algebra and on that bigger sigma algebra, one can show, we can extend that measure mu to the sigma algebra. New measure space becomes complete. So, the process is very much similar to looking at X S of A and mu star and these are the X S star and mu star. So, we will assume this theorem that every measure space X S mu can be completed. So, if you are interested in looking at the technical details for this, look at the textbook, which we mentioned in the first lecture, namely: An Introduction to Measure and Integration by me. So, we will leave these details for those who feel more interested in looking at the details. Next, we will give some equivalent ways of describing the set, mu star of E. Mu star of E can be also written as infimum of mu star of A, where A belongs to S of A and all E are inside A. So, look at all elements from the sigma algebra generated by A, which include that set E. Look at the mu star of A and take the infimum of them. So, in some sense, mu star of a set can be approximated by sets from elements of S of A and a similar result is true for elements, which are measurable sets. So, these are technical things and facts, which we will not prove. Most probably, we will not be using them in our course, but it is nice to know the relation between mu star of E and mu star of sets in the sigma algebra S of A and S star of A.

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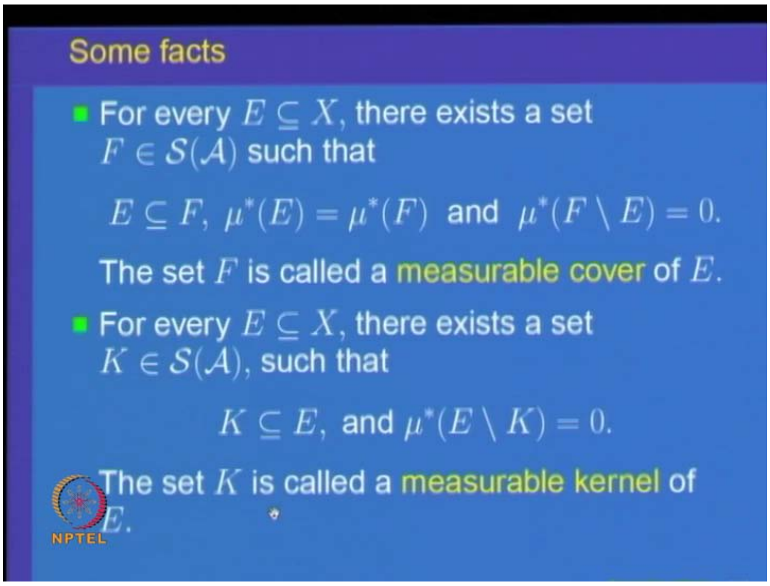

Here is another fact which again we will not be proving and most probably we will not be using, namely: that for every subset E in X, you can find a set in the sigma algebra S of A; the sigma algebra generated by A, such that the set E is a subset of F. So, F, which includes E and the outer measure of the two are same and that in turn implies that outer measure of F minus E is 0 So, essentially it says for every set E contained in X, there is a set in the sigma algebra S of A, such that the difference is got outer measure 0. Such a set is called a measurable cover of E because F covers E. and a similar result for a set that is inside. If E is in X, then you can find a set K inside E, such that the difference mu star of E minus K is 0. Such a set is called a measurable kernel of E So, given any set E, there is a cover by a measurable set. There is a smaller set inside, which is a kernel and difference is sets of a measure is 0. So, these things we will not prove.

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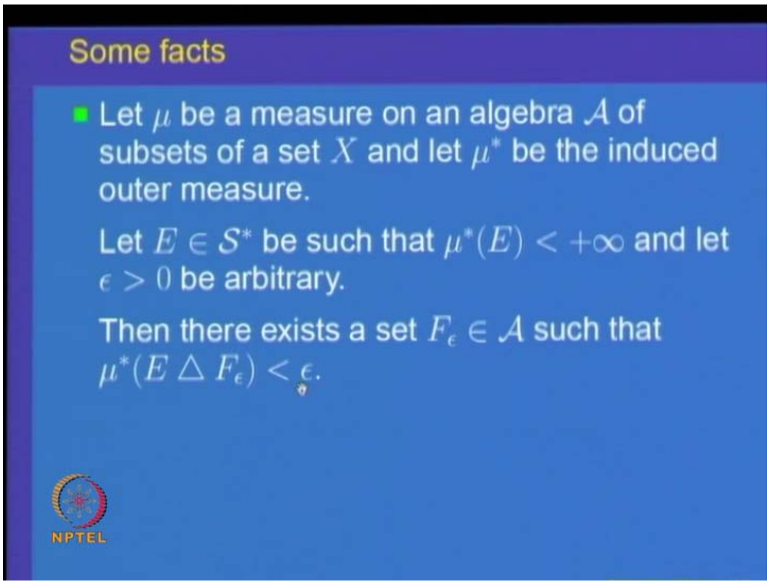

We will prove a result, which we will need later on. That relates the outer measure with measure of the set inside the algebra that we have started with. We start with a measure mu on algebra A of subsets of a set X. Let, mu star be the induced outer measure.

Suppose, we have got a set E, such that mu star of E is finite. This set need not be in the sigma algebra. So, take any set E, such that mu star of E is… so, we do not need this condition that E should be a measurable set. So, take any set, whose outer measure is finite and then given any epsilon. You can find a set in the algebra A, such that mu star of E symmetric difference that set F epsilon is less than epsilon. So, this is a very nice result which says any set of finite outer measure, as I said, this is I mentioned is not there; it is not needed for any - it is a typo - for any set of finite outer measure, you can find a set in the algebra, such that mu star of E symmetric difference. So, the measure of the symmetric difference is small.

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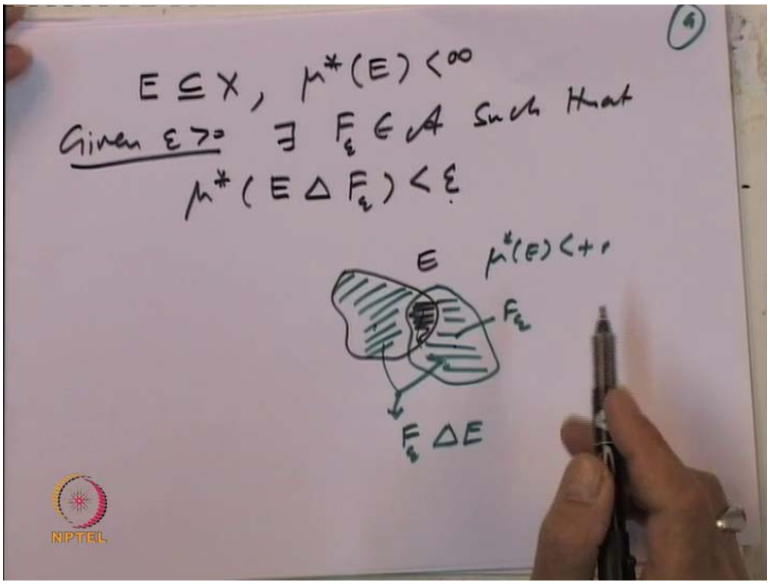

Let us look at a proof of this result. So, what we are saying is- let us take a set E contained in X with the condition that mu star of E is finite. It says, given epsilon bigger than 0, there exists a set F epsilon belonging to algebra, such that mu star of E symmetric difference with F epsilon is less than epsilon. Let us see, what we were saying. We were saying that, this is the set E and it says given a set E with the condition that mu star of E is finite.

I can find a set, call this as F epsilon such that, what is the symmetric difference? Symmetric difference is E minus and F minus. So, that is the portion and this portion (Refer Slide Time: 39:29) is F epsilon symmetric difference E. So, it says these are common portion and it says, outer measure of the sets, which are outside the common portion is small. Essentially, almost you can say that E and F epsilon are same.

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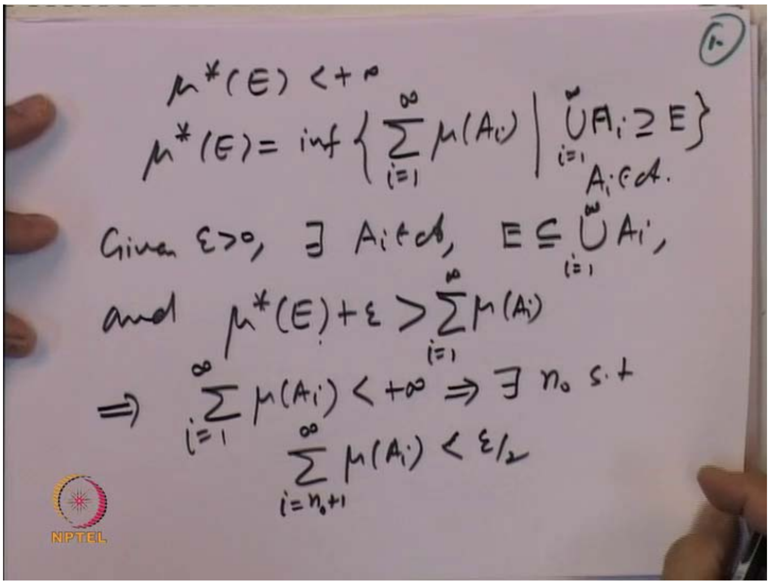

So, let us prove this property. Let us observe that mu star of E is finite. What is mu star mu star of E? If you recall, mu star of E is equal to infimum of sigma mu of A_i i equal to 1 to infinity, where this A_i is the union of A_i's cover the set E and A_i's in the algebra. So, this being finite, given: epsilon, a small quantity bigger than 0. There exists a covering for sets A_i belonging to the algebra, such that E is contained in union of A_i's and mu star of E, which is infimum plus the small number is bigger than sigma mu of A_i's and that is by the definition of the infimum; infimum is finite.

Note, because this is finite, it implies that the series i equal to 1 to infinity mu of A_i is finite. So, as a consequence of this, there exist some n_0, such that the tail of the series, n_0 plus 1 to infinity mu of A_i is less than epsilon by 2. That is because the series is convergent.

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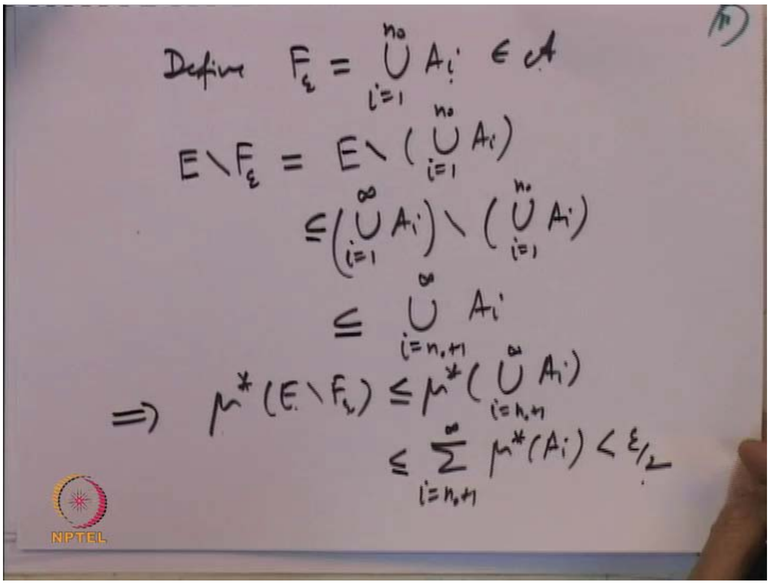

Once that is done, let us define the set F epsilon to be equal to union of A_i i equal to 1 to the stage n_0. Note: this set belongs to the algebra because it is a finite union of elements in the algebra and it belongs to the algebra. So, let us look at the set E minus F epsilon. What is that? That is E minus union i equal to 1 to n_0 A_i. Now, the set E is contained in union i equal to 1 to infinity A_i. This is contained in this minus union i equal to 1 to n_0 A_i.

So, I can say this is contained in union i equal to n_0 plus 1 to infinity of A_i. So that implies mu star of E minus F epsilon is less than or equal to mu star of the set union i n_0 plus 1 to infinity A_i and is sub additive. So, (Refer Slide Time: 43:30) this was subset of this. So, mu star of this is less than or equal to… by monotone property and by sub additive property this is less than or equal to sigma i equal to n_0 plus 1 to infinity mu star of A_i. If you recall, we have less than epsilon by 2. So, we get that mu star of E minus F epsilon is less than epsilon by 2

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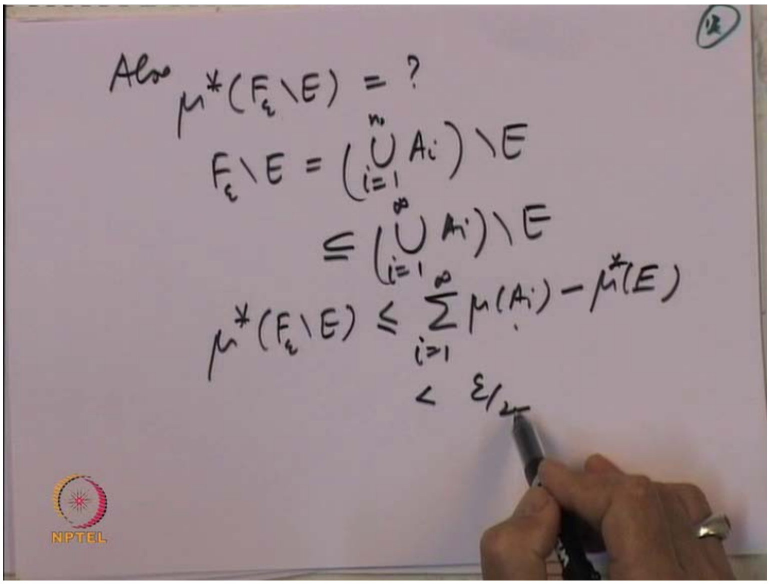

So, we get that mu star of E minus F epsilon is less than epsilon by 2. Let us also compute the measure of the other part, namely: We also want to compute mu star of F epsilon minus E. We want to compute, what is this equal to? F epsilon minus E is union i equal to 1 to n_0 A_i minus E. Note: This is a subset of union of i equal to 1 to infinity A_i minus E. E is a subset of this and that implies mu star of F epsilon minus E is less than or equal to mu star of this. So, that is, sigma i equal to 1 to infinity mu of A_i minus mu star of E.

If you recall the way we started, we had summation mu star of A_i. This relation (Refer Slide Time: 45:15) says, sigma mu of A_i minus mu of E is less than epsilon. So, we could have started with by epsilon by 2 and then we have got this is less than epsilon by 2. So, we are getting that mu star of F epsilon minus E is less than epsilon by 2.

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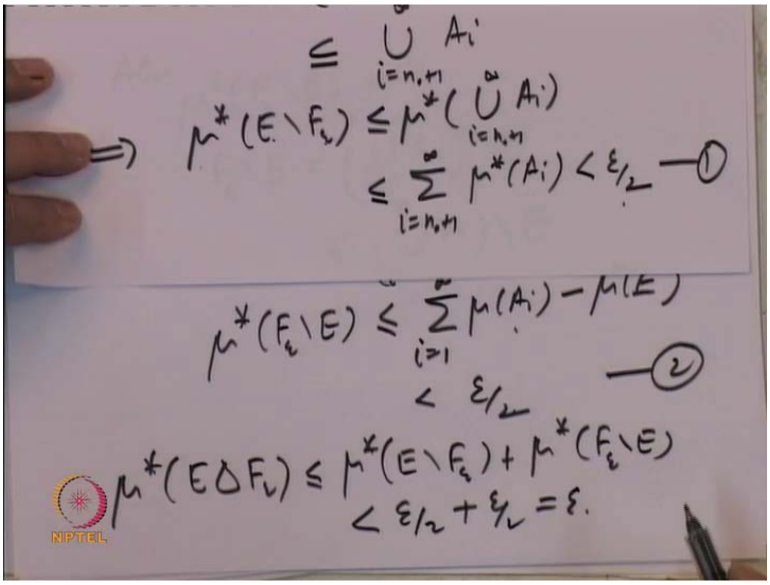

We have already shown that on that mu star of F minus E epsilon is less than epsilon by 2. So, putting these two together, we call this as 1 and call this as 2. so, by putting 1 and 2 together, mu star of E delta F epsilon is less than are equal to mu star of E minus F epsilon plus mi star of F minus E and both of them are less than epsilon by 2 plus epsilon by 2, which is equal to epsilon.

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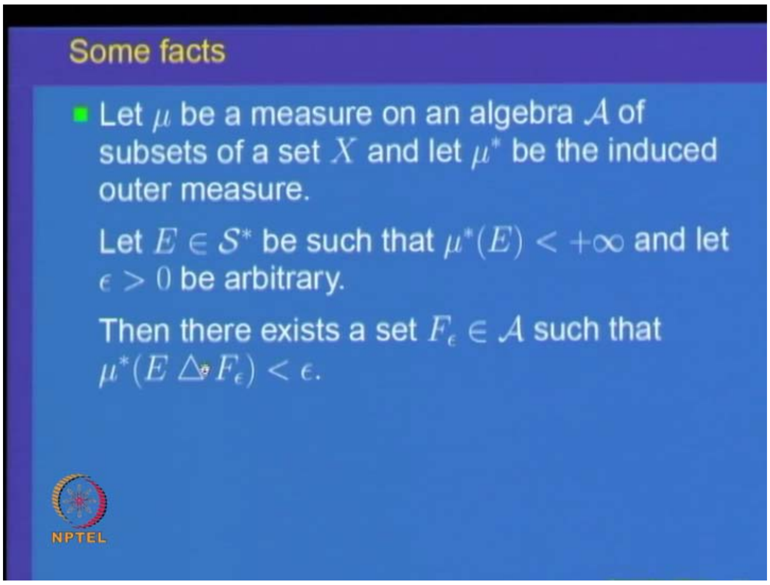

So, that proves the required property, which we wanted to prove, namely: that given epsilon bigger than 0. There is a set F epsilon, which is in the algebra A, say that mu star of E delta F epsilon is less than epsilon.

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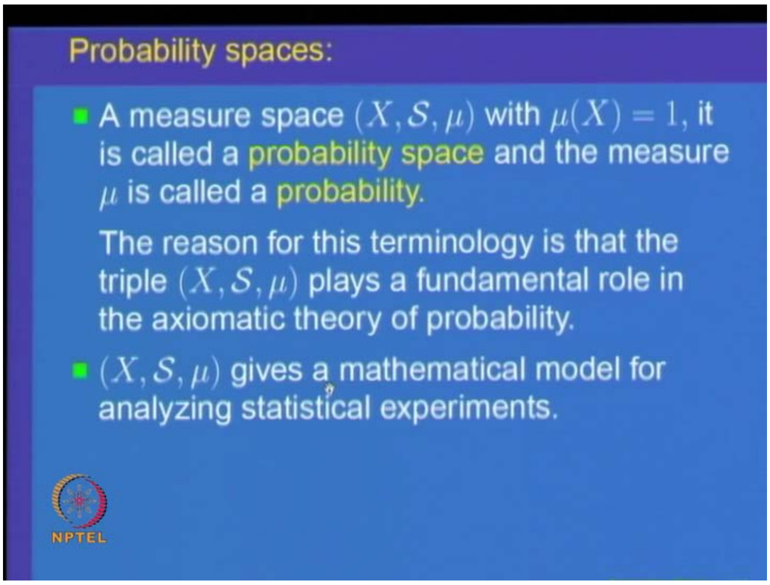

So, this is an approximation property, which we will be using later on to prove some facts. So, this is the process of extension theory. The process of extension theory gives us ways of constructing triples, which are measure spaces. At this point, it is worth mentioning there are measure spaces of importance in other subjects, called probability theory.

A measure space X, S, mu, where mu of X is 1, that is, totally finite measure and mu of the whole space is equal to 1 and is called a probability space. The measure mu is called a probability. So, a measure space, where mu of X is one; is called a probability space and mu is called a probability.

The reason for this terminology is such triples play a fundamental role in axiomatic theory of probability. Whenever you want to describe a phenomena - a statistical phenomena which depends upon some randomness, one has to construct a probability space to analyze it. So, this gives a mathematical model in the theory of probability to analyze statistical experiments.

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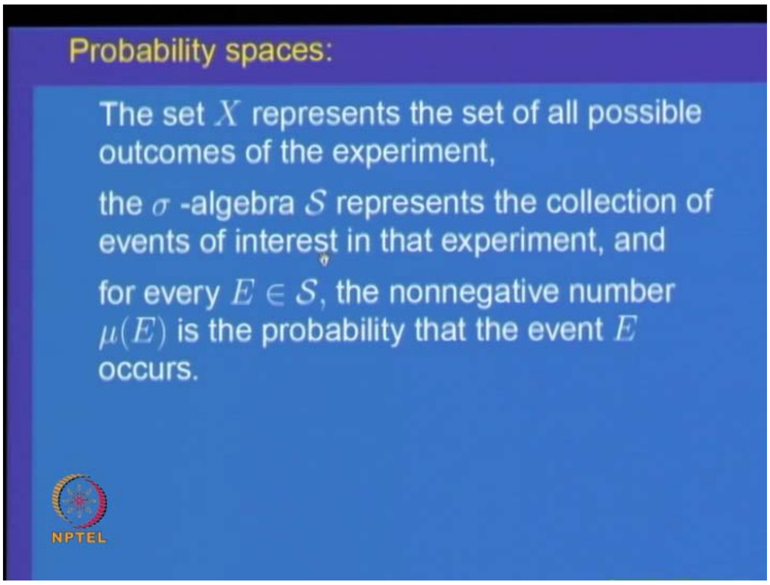

So, let me give you a few things more, the set X denotes in the triple X, S, mu. X represents the set of all possible outcomes of the experiment. For example, you are tossing a coin and all possible outcomes are head or tail. You are throwing a die and there are six possible outcomes: the number 1, 2, 3, 4, 5, 6. You are observing the temperature of a particular place every day at a particular time; the observation will be a real number.

So, in any particular experiment, all possible outcomes of that experiment constitutes a set. That is the set X and all the sigma algebra S, represents the collection of events of interest in that experiment. So, any subset of the set of outcomes in the experiment is called an event. So, for example, when you are tossing a coin, there are two outcomes possible: head and tail. If you look at the singleton h, that is, an event when you toss head can come or a tail can come. If you are throwing a dye, then the outcomes possible are 1, 2, 3, 4, 5, 6. Look at the subset 1, 3 and 5 of X, the set of all odd outcomes. So, when you throw, it is possible to find out whether that event has occurred or not. It means, whether the outcome was an odd number or not. So, that is a subset of set of all possible outcomes.

In general, when you want to describe its statistical experiment, one has to construct a class of subsets of that set X of interest that one requires because of mathematical considerations that class would be a sigma algebra. So, the sigma algebra represents the collection of events of interest in that particular experiment. Finally, for every event E of interest, you want to assign the possibility of that event happening or the probability of that event taking place.

So, a probability is a measure defined on the sigma algebra of all possible event of interest in taking non-negative values and of course, probability of the whole space, the chance of the whole space happening is 1 and probability of the empty set is 0. So, the probability is a set function defined on the collection of all events of interest. We want that to be a measure. So, that is the reason that the triple X, S, mu is called a probability space. It gives a mathematical model for analyzing statistical experiments, when mu of X is equal to 1.

So, in today's lecture, we have constructed measure spaces. From the next lecture onwards, we will specialize this measure space, when X is real line. It gives an important example of measure space and a measure called Lebesgue measure. So, we will do that in the next lecture. Thank you.