Measure and Integration

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Module No. # 03

Lecture No. # 10

Outer Measure and Its Properties

Welcome to lecture number 10 on measure and integration. Let us recall that in the previous lecture, we started looking at the notion of outer measure. Let us recall, how the outer measure was defined and what are the properties the outer measure.

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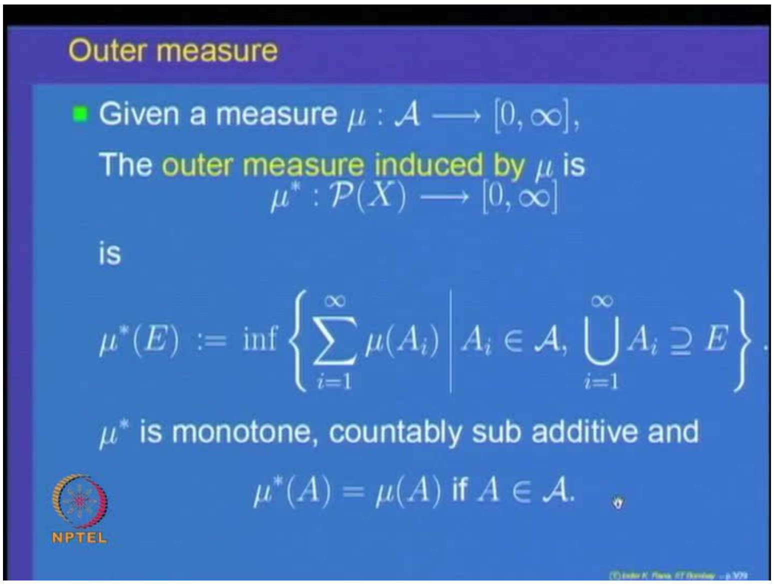

Given a measure mu on algebra A of subsets of a set X. The outer measure induced by this measure is a set function defined on the class of all subsets of the set X. So, it is a function defined by the power set of X; of course, taking non-negative values. It is defined as for a set E in subset of X, look at a countable disjoint - a countable covering of the set E by elements of the algebra A.

Now, look at the size, the measure of the set A I, so that is mu of A i. Add the measures of all the sets A i, so that this union covers, so that gives you a number, which we can think of as approximate measure of the set E. Look at the infimum for all such possible coverings of E. So, mu star of E is the infimum of all summation mu of A i such that union of A i is cover E. We proved properties of this set function - the outer measure namely mu star is monotone, it is countably sub additive. On the sets, in the algebra, mu star is same as A. So, mu star extends the measure mu, but it is only monotone and countably sub additive.

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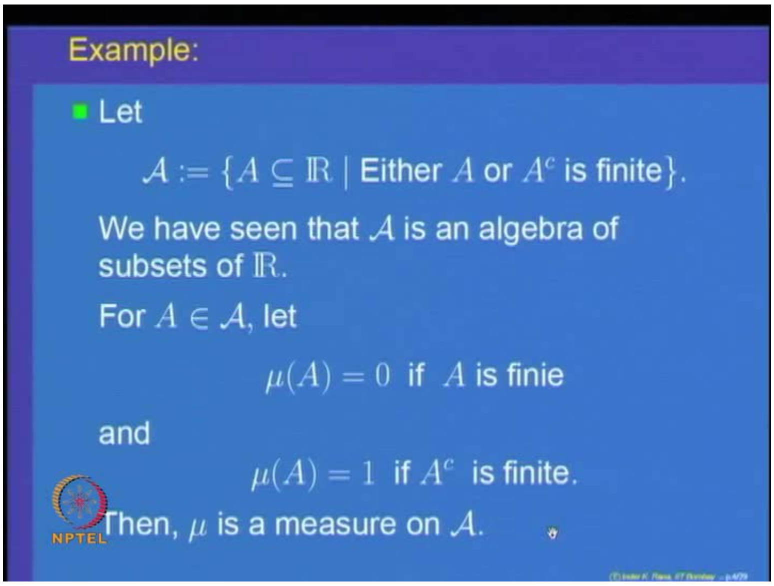

Let us look at an example of this outer measure. In this example, we will start with the collection A of all subsets of the real line, which are either finite or their complements are finite. Recall, in the beginning of the lectures, we have shown that this collection A is algebra of subsets of A. So, this collection A forms algebra of subsets of the set R.

Let us define A set function mu on this; mu of the set A is equal to 0, if the set is finite. If the set in the algebra is not finite, then you know complement is finite. So, in that case, we define mu of A to b equal to 1 if A complement is finite. We have also checked for this example that mu is a measure on this algebra A. I would strongly say that you try to prove it yourself once again that this mu is a measure - that is mu on the algebra A is countably additive.

Let us denote by mu star - the outer measure induced by mu on all subsets of the real line. So, mu star is the outer measure given by this particular measure.

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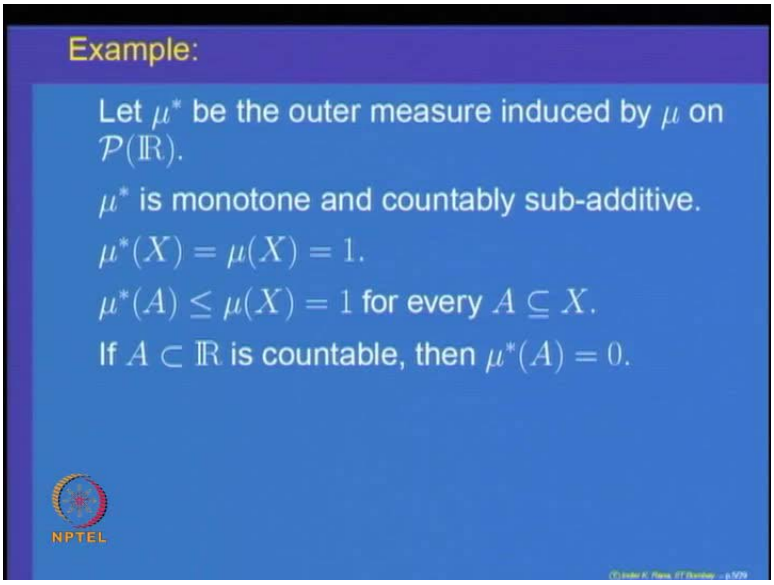

We want to correct, find some properties of this mu R of this outer measure mu. If you recall, just now we said, outer measure always is monotone and it is countably sub additive. So, these properties are true for any outer measure, in particular to this outer measure also.

We want to do something more; let us note that mu star of X the whole space is same as mu of X and that is equal to 1. Because, mu star extends, X belongs to the algebra. Actually, X complement is empty set, which is finite. So, by definition, mu star of X should be equal to mu of X, which is equal to 1.

If A is any subset of X and mu star being monotone, so we know that mu star of A is less than or equal to mu star of X and that is less than that is equal to 1. So, mu star of every set is going to be between 0 and 1. So, this is a property we have reduced from the general facts that mu is monotone and mu star of the whole space is equal to 1.

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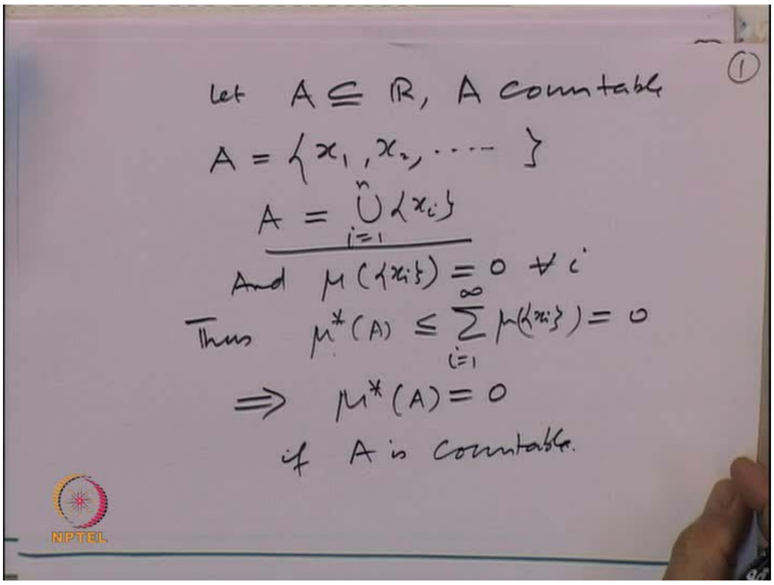

We want to show that if A is a countable set in real line, then mu star of A is equal to 0. Let us see how do we show that? Let us take a set A. So, let A contained in R A countable; that means, I can write A as a sequence x 1, x 2 and so on. So, I can write A is actually equal to union of singletons x i, i equal to 1 to n. By definition, mu of the singleton x i, it is a finite set, so that is equal to 0 for every i.

Thus, mu star of A, which is less than or equal to summation mu of x i, this is one covering measure of each one of them being equal to 0, so this is equal to 0. So, mu star of A is less than or equal to 0, we know it is always bigger than or equal to 0. That implies that mu star of A equal to 0, if A is countable.

We have shown that for a countable subset in the real line, the outer measure which we define is going to be 0, whenever A is countable.

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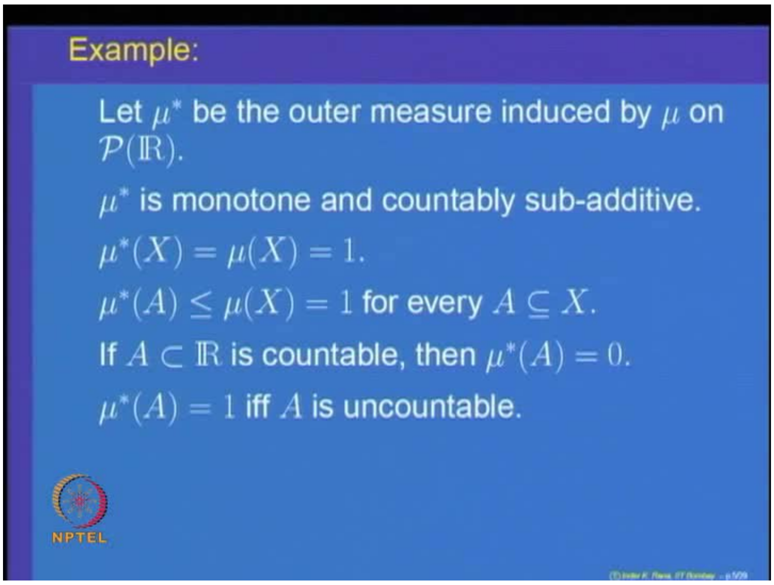

Let us go further and look at some other properties of this outer measure. We want to show that mu star of A is equal to 1 if and only if A is uncountable, so let us prove that.

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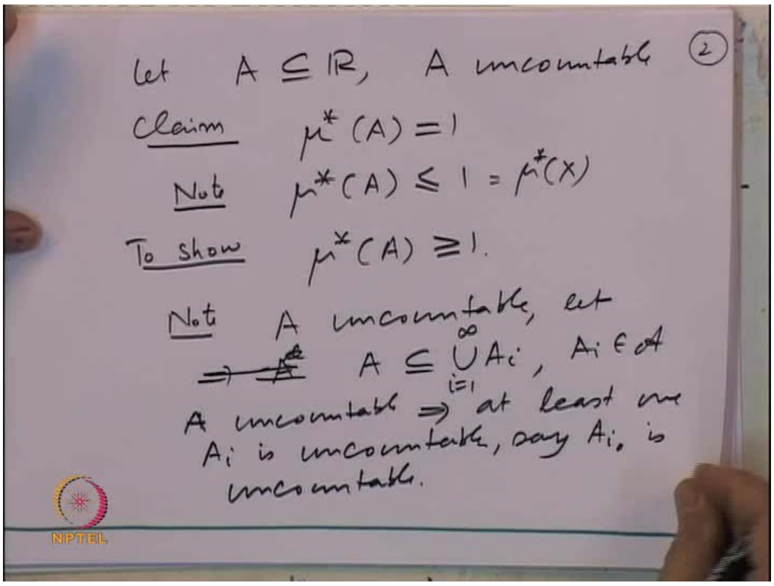

Let A contained in R, A uncountable; claim mu star of A is equal to 1. So, note mu star of A is less than or equal to 1 that is obviously right. That is by the definition we said, mu star is monotone, which is equal to mu of x, anyway mu star of x. So, mu star of A is always less than or equal to 1. To show that mu star of A is also bigger than or equal to 1, let us note A uncountable. Look at A is uncountable, so what can you say about a complement of the set. If A is uncountable, let us take a covering of A i, i equal to 1 to infinity. Let us take a covering, where A i belongs to the algebra A.

Now, note, in this covering, A uncountable implies at least one A i is uncountable. So, this is the observation, which is going to be crucial. A is a sub set of unions A i, which are in the algebra. If A is uncountable, then each one of them cannot be countable, because if each one of them is countable, then countable union of countable sets will be countable (Refer Slide Time: 08:50). So, A will be countable. That means, there is at least one A i, which is uncountable; say A i naught is uncountable.

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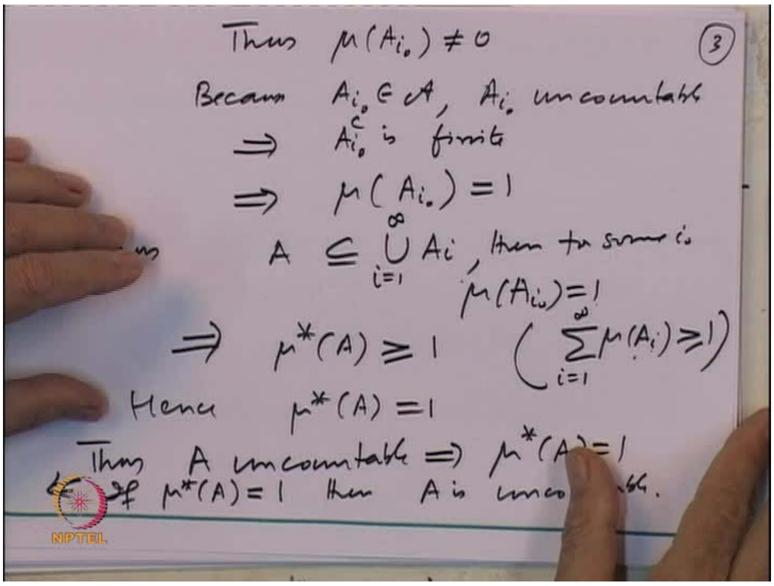

But that will be what? A i naught is uncountable, means we have got that mu A i naught is uncountable, so mu of A i naught. What we can we say about that? So that cannot be equal to 0; can we say that cannot be equal to 0? Let us see. One of the A i naught is uncountable, so that means, it is not finite.

This is because, A i naught belongs to the algebra A that is a crucial thing. So that means, either A i naught is finite or it is complement is finite, but we know that A i naught is uncountable. So that means A i naught uncountable, so that implies that A i naught complement is finite. Because, it belongs to the algebra, so either A i naught has to be finite or its complement has to be finite. So, this is finite that implies that mu of A i naught by definition is equal to 1.

Thus, what we have shown? If A is contained in union A i 1 to infinity, then for some i naught mu of A i naught is equal to 1. So, that automatically implies the fact that mu star of A - any covering will have at least one of the elements, so implies this is bigger than or equal 1. Because, mu star of A is the infimum of - because mu star of A the summation i equal to 1 to infinity mu of A i is going to be bigger than or equal to 1, because A i naught is the 1, at least one of the term is 1.

For every covering, A i of A sigma mu of A i is bigger than or equal to 1, so the infimum has to be bigger than or equal to 1.Hence, mu star of A is equal to 1. We have proved thus A uncountable implies mu star of A is equal to 1. The converse is obvious, because, conversely, if mu star of A is equal to 1, then A is uncountable, because for countable sets we already shown mu star of A is equal to 0.

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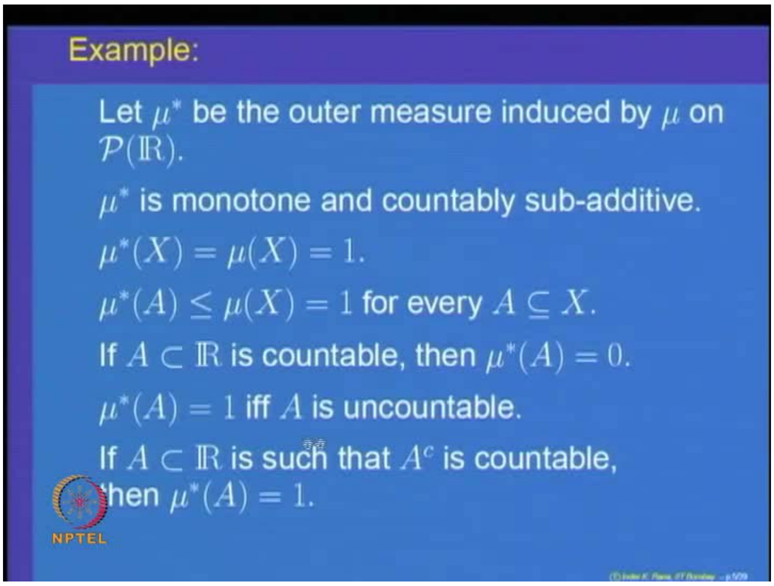

So, A is uncountable if and only if we have shown, we have characterized all sets for which outer measure is going to be equal to 1. We have shown the fact that mu star, the outer measure induced by the measure that we are looking at, namely mu, if mu of A is equal to 0, whenever A is finite and mu of A is equal to 1, when A complement is finite. If we look at the outer measure induced by this measure, then that has the property that mu star of every countable set is equal to 0. mu star of a set is equal to 1, if and only if, the set is uncountable, so that is the property that we have. So, mu star of A is equal to 1 if and only if, A is uncountable.

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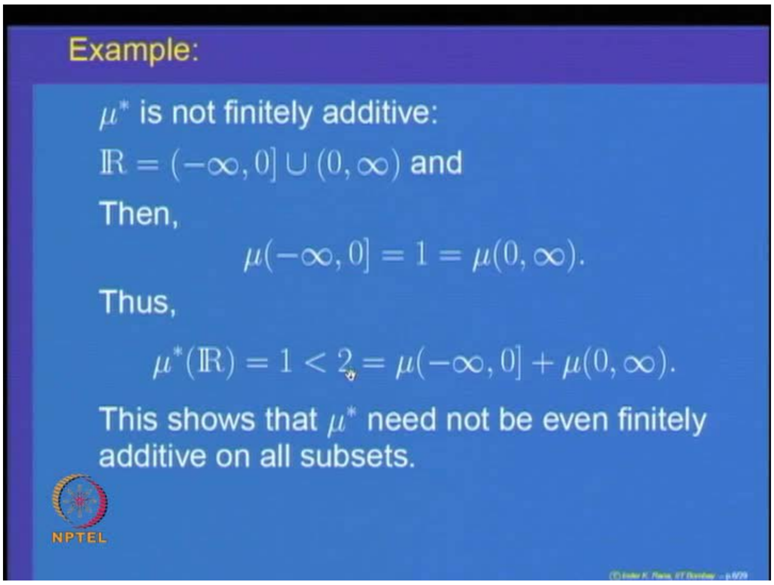

Now, we already know that mu star is countably sub additive. We want to know is mu star is additive. So, we will show that is not the case that mu star is not even finitely additive. To show that let us observe that the real line, I can decompose into two disjoint sets, minus infinity to 0, 0 close, union 0 to infinity.

The real line is written as a union of two subjects. Now, the outer measure of the set minus infinity to 0 that is uncountable set, so that is equal to 1 and the outer measure of 0 to infinity are also equal to 1.

So, outer measure of both of these sets is equal to 1, their union is R. So, we get mu star - mu star of R is equal to 1, which is strictly less than 2 equal to some of the outer measures of each one of them. So that says mu star of R is strictly less than mu star of minus infinity to 0 plus mu star of 0 to infinity. So, this should be mu star and this also should be mu star. So that says that mu star - the set functions mu star is not even finitely additive.

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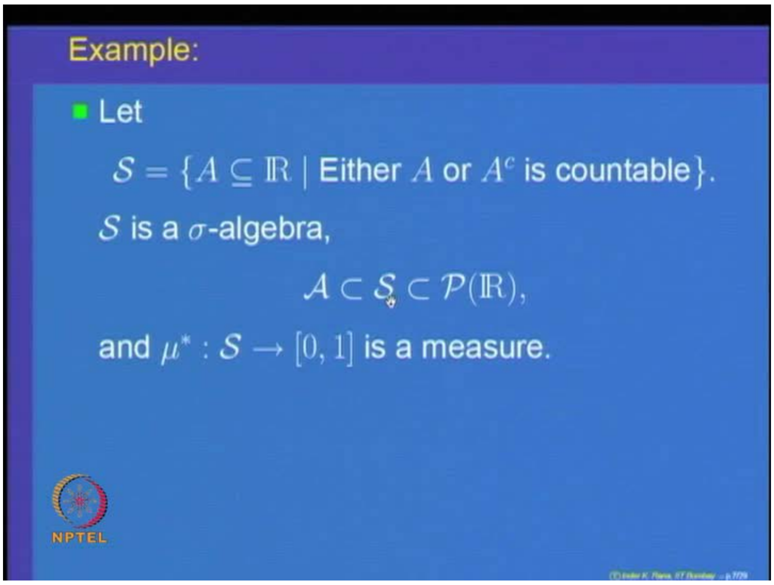

But, let us observe more facts about this outer measure. This outer measure on all subsets is not finitely additive, but it has some nice property. It is countably additive on a sub class. So, let us look at the sub class to be S, which is A sub sets of R - all those subsets of R, where either A or A complement is countable.

So, keep in mind, A or A complement is countable. We had shown that this collection forms sigma algebra. Our given algebra A, which was the collection of all sets for which A or A complement is finite, obviously is a subset of this class S of subsets which A or A complement is countable and that is of course a sub set of all subsets.

Algebra A contained S - this sigma algebra S, and which is inside of PR. It is actually - we have also shown that mu star, so what is mu star on S? If a set A is countable, then mu star of A is 0. If it not countable, but then A complement is countable, then mu if A complement is countable, then the set A cannot be countable, it has to be uncountable and for uncountable sets mu star of A is equal to 1.

So, mu star restricted to S is the set function, which is mu star of a set A is 0 if A is countable and mu star of a set A complement of A is equal to 1 if its complement is countable. That we have already shown is a measure on the class of all sets in the sigma algebra S.

Given that measure in our example - given the measure on the algebra A, when we define the outer measure, which is defined on all subsets of PR, is not even finitely additive, but if we restricted to the collection of sets S, which is A or A complement countable, then on that class it is a measure and it extends. So, the given measure on the algebra does not extend to all subsets, but at least it extends to a sub collection of all subsets and that includes the original algebra.

This is a situation which we are going to see very common in our extension process. So, we are starting with a measure mu on algebra. Just now we said in the process of extension let us define outer measure on all subsets. Here is an example, which says on all subsets outer measure may not be even finitely additive.

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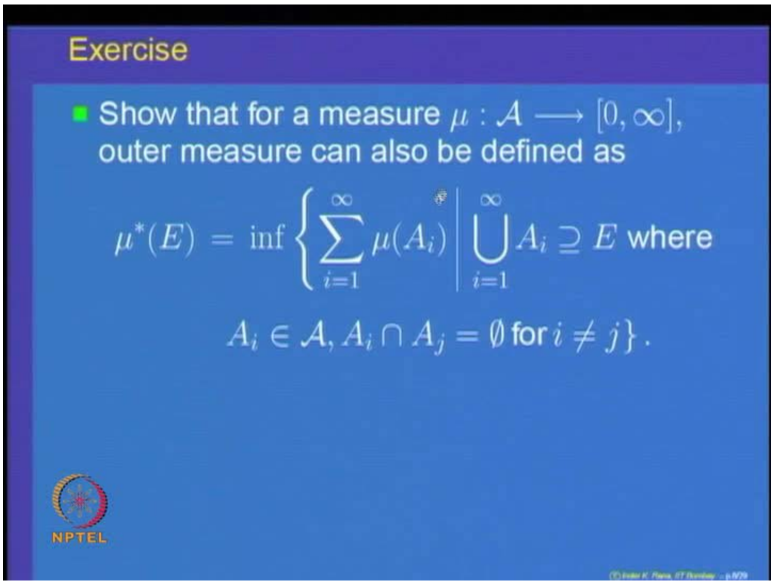

At least this example says that we can probably find a subclass of all subsets of that set X, which includes the given algebra and on that probably it is countably additive. So, the problem is to look for a collection of subsets on which it is going to be countably additive.

Before we go over to that process, let me give you an exercise, which all of you should try. Namely, if mu is a measure on algebra, then we define the outer measure as infimum. Look at all coverings and look at summation mu of A I, there was no condition input on the sets A i. The exercise says if you take only those coverings of E by elements of the algebra, which is pair wise disjoint and then take the infimum over only such coverings that also will give you the outer measure.

The exercise is that outer measure for a set E can be defined in terms of countable disjoint coverings of E. Namely, take coverings of E by elements of the algebra, the elements of the covering are pair wise disjoints. Look at the sum mu of A i and take the infimum over such covering.

So, infimum is taken over all countable disjoint coverings. In the original definition, we did not put this condition. In the exercise, both of these are same. The answer lies on the simple observation that when a measure mu is defined on algebra and you look at the union of elements in the algebra, any union can also be written as a union of pair wise disjoint sets. Union of elements of the algebra can be represented in terms of pair wise disjoint sets that fact we have used earlier also. Using that you can try to prove this exercise.

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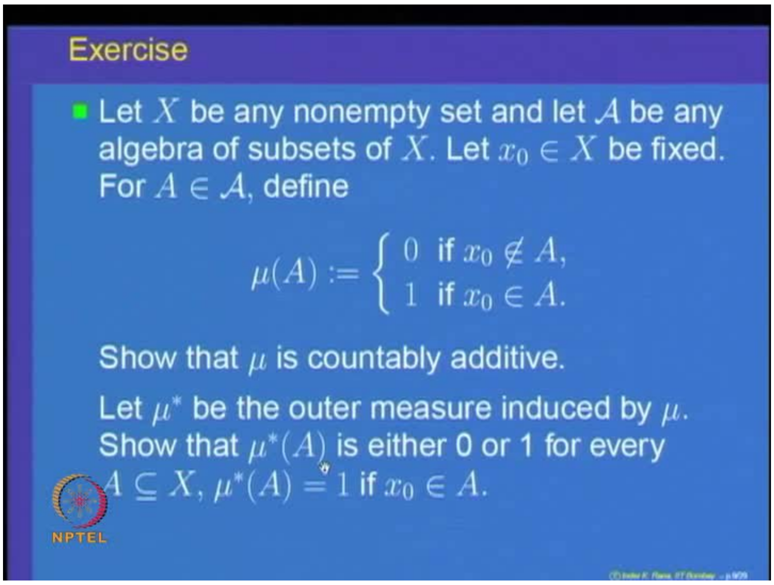

Here is another exercise, which you should try to prove, to get familiarize with the concept of outer measure. Let us take X and a nonempty set, A is algebra of subsets of a set X and let us fix any element x 0 in x - any arbitrary element. Now, given any subset A in the algebra, either x naught will belong to A or x naught will not belong to A - two possibilities.

So, if x naught does not belong to A that particular element that you have fixed does not belong to A put mu of A - define mu of A to be equal to 0 and define mu of A equal to 1 if x 0 belongs to A. So, whenever x 0 is in the measure of that set is 1, otherwise it is equal to 0. Show that this is a countable additive set function, namely it is a measure, because your empty set is automatically 0.

We would like you to characterize, show that the outer measure - look at the outer measure induced by this measure. Show that the outer measure has the property that outer measure of every set is either 0 or 1 again. Outer measure of a set is equal to 1, if the element x naught belongs to A that is a property of mu also. So, we want you to show that mu star of A is equal to 1 if x 0 belongs to A.

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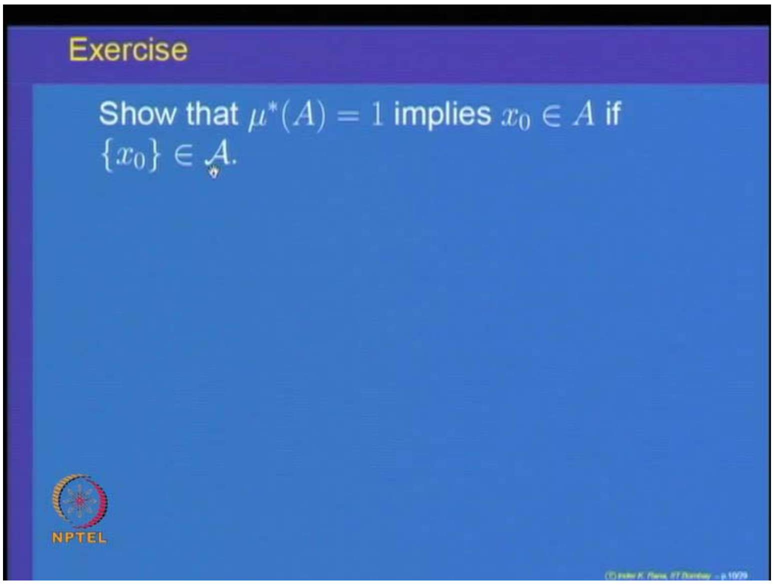

Can you say that the converse is true, namely we would like you to also show that mu star of A is equal to 1 implies x 0 is belongs to A, but you need the condition that if x 0 belongs to A, because x 0, the singleton x 0 may not be the algebra, so that makes the difference. So, look at this exercise and try to prove the fact asks for. That will help you to understand what is an outer measure? How does the outer measure of a set change with given conditions?

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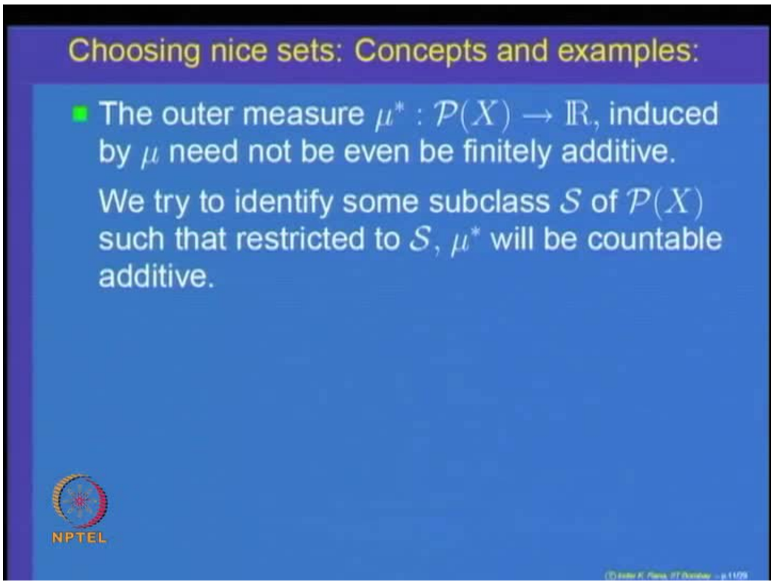

Now, let us look at the problem. You are given a measure mu on algebra A of subsets of a set X. You had defined the outer measure, which in general is countably sub additive. How to pick up sets, how to pick up those subsets of X such that mu star restricted to them will become countably additive as add point in the previous example that we discuss?

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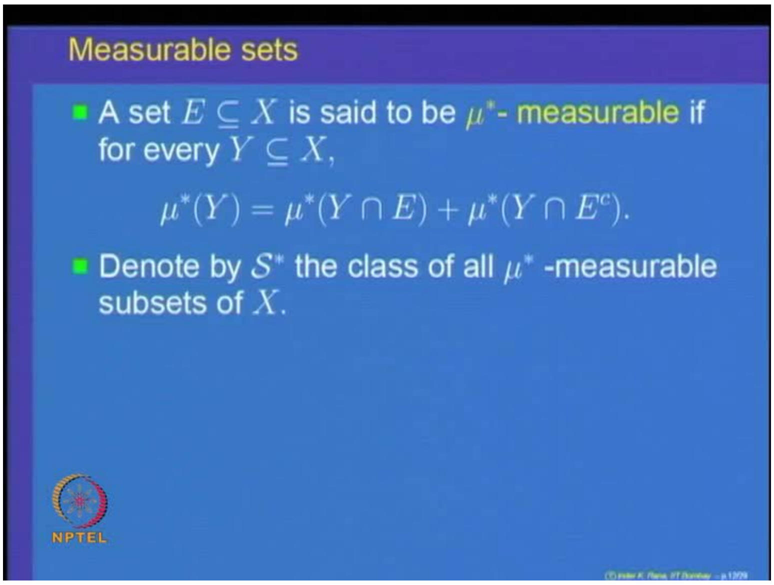

For that what is called the concept of a measurable set. A subset E of X - so mu on the algebra is fixed, mu star is defined via that measure mu. So, mu is fixed, we are saying that a set E is mu star measurable; mu star is the outer measure induced by that measure is called measurable if for every Y in X mu star of Y can be written as mu star of Y intersection E plus mu star of Y intersection E complement.

So, be careful. We are saying E is measurable, so E measurable means, take the set E, which you want to test whether it is measureable or not. So, divide any set Y into two parts Y intersection E and Y intersection E complement, then measure of that two pieces should add up to give you the size of a mu star of Y.

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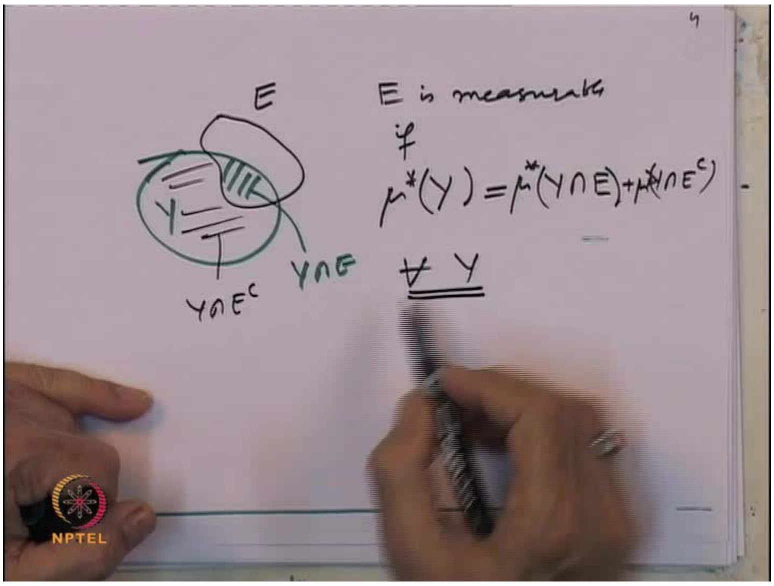

This is the picture here that you have got. This is my set E; I want to check whether it is measurable or not, so take any set Y. So, this is my Y that gives me this piece, which is Y intersection E and this is the part that is Y intersection E complement.

So, the requirement is, we are saying that E is measurable. Using E, Y is the set, cut it into two parts Y intersection E and Y intersection E complement. So, they are two disjoint pieces of Y, we should have mu star of Y is equal to mu star of Y intersection E plus mu star of the other part.

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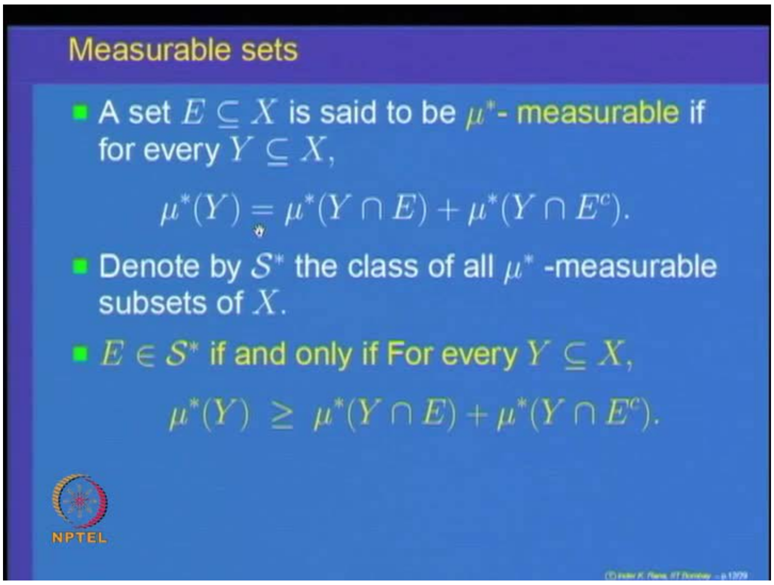

That should happen for every Y; it should happen for every Y that is important. We say that E is measurable, for any subset Y divide Y into two parts, Y intersection E and y intersection E complement. We want measure outer measure of Y, should be sum of these two outer measures. So that we say set E is measurable.

Let us denote by S star, the collection of all mu star measurable subsets. So, whenever set E satisfies this condition, we say that is measurable. Let us put all the measurable sets in a collection and call that as S upper star. So, S upper star is the collection of all mu star measurable subsets of X.

Here is one observation that a set is measurable if and only if mu star of Y - definition says, it should be equal to the sum of the pieces, but it is enough to say that mu star of Y is bigger than or equal to mu star of Y intersection E plus mu star of Y intersection E complement. That is, because for every set Y; Y is equal to Y intersection E plus Y intersection E complement and mu star is always sub additive, so that means, mu star of Y is always less than or equal to mu star of Y intersection E plus mu star.

The inequality less than or equal to is always true, because mu is monotone, so to verify whether a set is measurable or not one has to check only that mu star of Y should be bigger than or equal to mu star of Y intersection E plus mu star of Y intersection E complement, for every set Y. So, only bigger than or equal to has to be checked.

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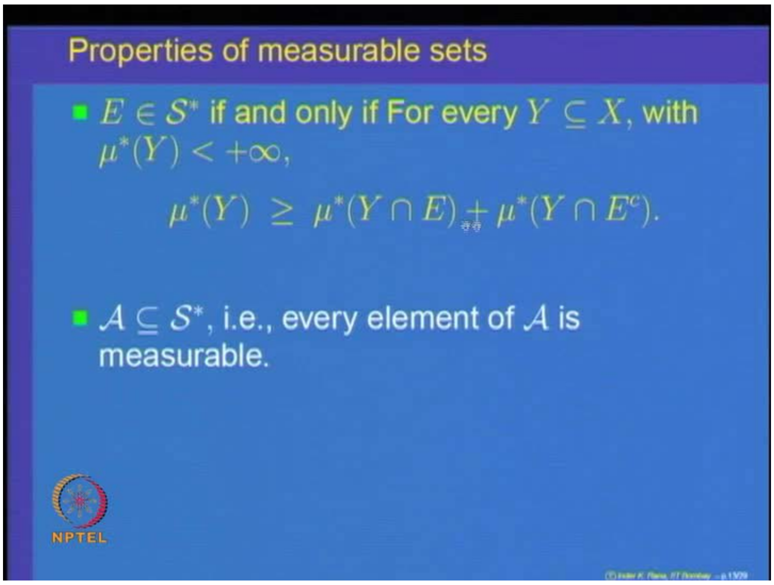

We have to check only bigger than or equal to. In case, mu star of a set is infinity, then this inequality is obvious. That means one has to check this inequality only for sets, for which mu star of Y is finite. So, that gives us simplification saying that a set E is measurable, if and only if for every subset Y of X, with the property that mu of mu star of Y is finite. One has to verify that mu star of Y is bigger than or equal to mu star of Y intersection E plus mu star of Y intersection E complement, for every subset Y, with mu star of Y finite.

So, this is the condition we are going to use again and again, to prove or check whether a set E is mu star measurable or not. We are going to now understand the properties of this class S star. What are the properties of this collection of measurable sets?

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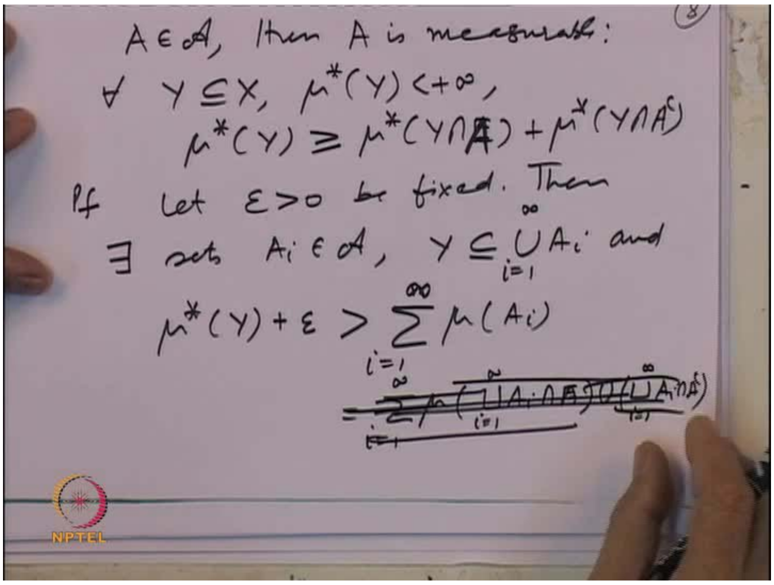

The first observation is that every set in the given algebra is measurable; that means, if A belongs to the algebra A, then this condition is always going to be true. So, Let us verify that. Let us show that if A belongs to the algebra, then A is measurable. That is for every Y is subset of X with mu star of Y finite. We should have mu star of Y is bigger than or equal to mu star of Y intersection A, so Y intersection A plus mu star of Y intersection A complement. So, this is what we have to show? Let us look at the proof of this.

Now, we are going to use the fact that mu star of Y is finite. mu star of Y finite means that it is a infimum of some th quantities. So that crucial definition - what is definition of infimum, we are going to use?

Let epsilon greater than 0 be fixed, then by definition of infimum, there exist a covering, so their exist sets A i belonging to A with the fact that the set Y is covered by union of A i, i equal to 1 to infinity. mu star of Y, which is infimum plus a small number does not remain the infimum, so it is bigger than or equal to mu of A i, i equal to 1 to infinity. So, here, we are using the fact that mu star of Y is infimum and that is a finite quantity.

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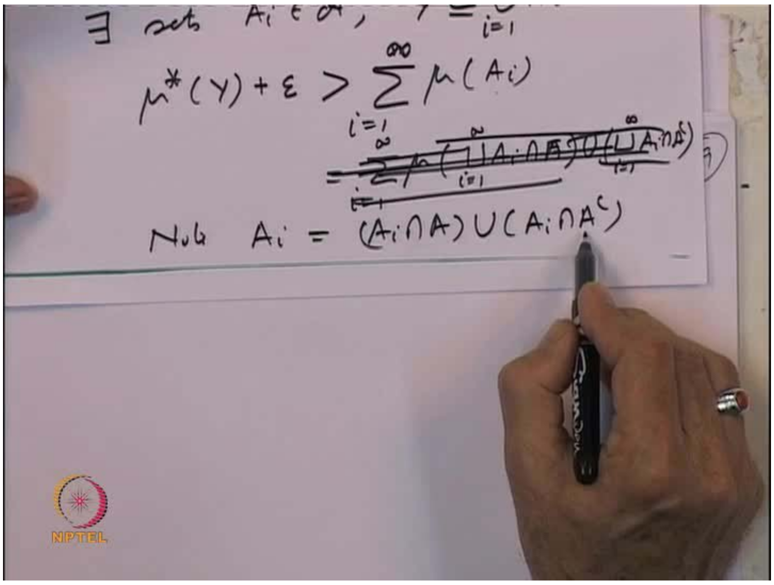

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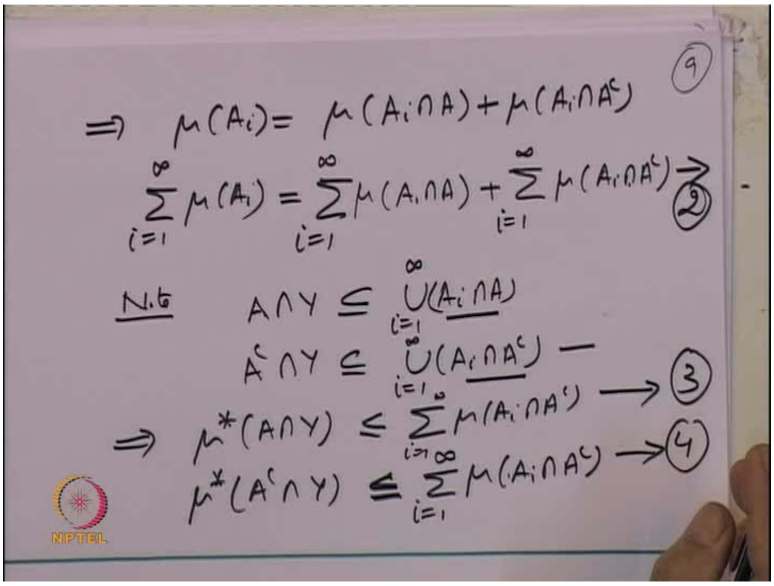

Now, A is in the algebra, so I can write this as - this is equal to sigma I equal to 1 to infinity mu of union A i intersection A, the set is A1 to infinity union of the sets, union i equal to 1 to infinity A i intersection A complement. So, what I am saying is – sorry, not the union, this is wrong. So, let me just simply write it as. Let us observe what we are saying. We are saying, because of this each set A I - so let us note; A I - I can write as A i intersection A union A i intersection A complement . These are two disjoint sets and mu is a measure, all the A i, A everything is in the algebra. So, using the fact that mu is a measure, I can right it as - so implies that mu of A i is equal to mu of A i intersection A plus mu of A i intersection A complement.

Now, mu of A i intersection A complement. So that means summation mu of A I, i equal to 1 to infinity is equal to summation i equal to 1 to infinity mu of A i intersection A plus summation i equal to 1 to infinity mu of A i intersection A complement. Now, let us note that the set A intersection Y is covered by union of A i intersection A, i equal to 1 to infinity, because Y is covered by union of A I, so A intersection Y is covered by this. A complement intersection Y is covered by union of i equal to 1 to infinity A i intersection A complement.

These are sets in the algebra, because A belongs to the algebra; that is the crucial thing. This will imply that mu star of A intersection Y is less than or equal to summation of mu A i intersection A complement i equal to 1 to infinity. mu star of - the second one gives me A complement intersection Y is also less than or equal to - using this summation i equal to 1 to infinity mu of A i intersection A complement.

Look at this equation, look at this equation, and look at this equation (Refer Slide Time: 33:20). So, summation mu star of A i if bigger than this sum and that sum is bigger than or equal to mu star of A intersection Y. This sum is bigger than or equal to mu star of A intersection Y, A complement intersection Y. We had mu star of Y plus epsilon was bigger than this summation - so that summation.

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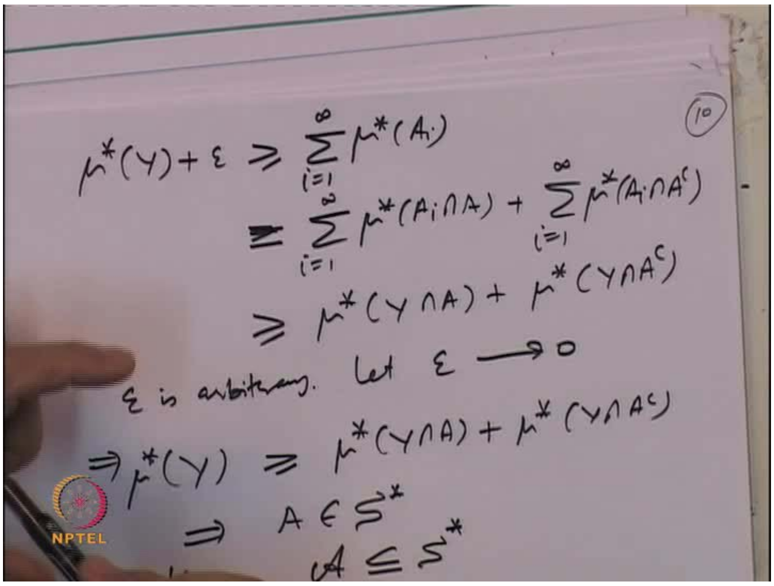

Putting these three equations together, so if you call that earlier equation as 1, call this equation as 2, call this equation as 3 and call this equation as 4, then putting all these four equations together what we have is the following. That mu star of Y plus epsilon, which was bigger than or equal to summation mu star of A I, i equal to 1 to infinity that is equal to summation of mu star of A i intersection A, i equal to 1 to infinity plus 1 to infinity mu star of A i intersection A complement. That is bigger than or equal to mu star of Y intersection A plus mu star of Y intersection a complement.

Now, epsilon is arbitrary, so let epsilon go to 0. This inequality will be still maintained, will imply that mu star of Y is bigger than or equal to mu star of Y intersection A plus mu star of Y intersection A complement. That will imply that A belongs to S star that is A is a measurable set.

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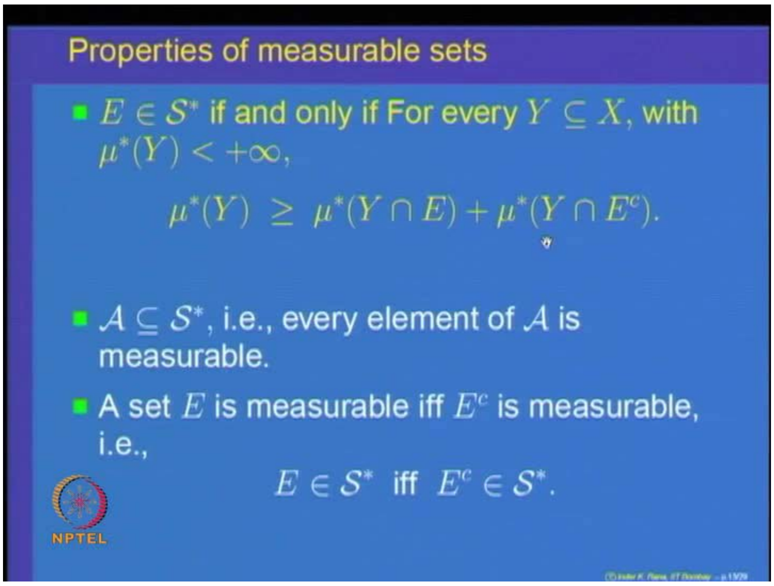

Hence, we have proved that the algebra A is included in the collection S star. That is what we wanted to prove, so this is the proof of the fact that the algebra A is contained in S star, every element of A is measurable.

The next property that the class of measurable sets is closed under complementation, namely, if E is measurable, then E complement is also measurable. That is obvious, because in this current area if you want to check, if E is measurable, then this is what we required. To check E complement is measurable, the same thing is required, because this will become E complement and E complement of complement is E. So, it is the same criteria, same equation to be verified.

Obviously, because the definition is inbuilt, E and E complement symmetric with respect to E and E complement. That says the set E is measurable if and only if, its complement is measurable or the collection S star of measurable sets is closed under complements.

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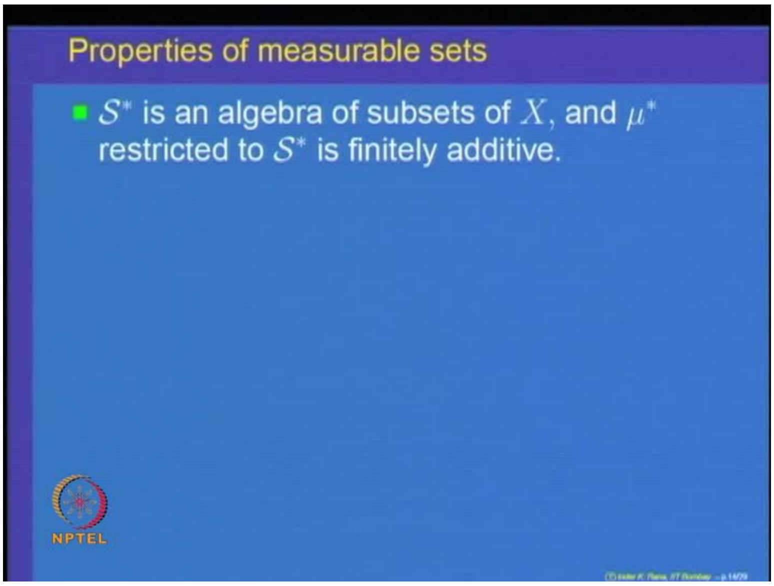

Next, we want to check the property. So, the collection of all measurable sets one, it includes the class of all subsets in the original algebra A. We want to check that now; it is algebra of subsets of X. That means a mu star restricted to S star is finitely additive. So, two things we want to check, one S stat is algebra and mu star restricted to S star is finitely additive.

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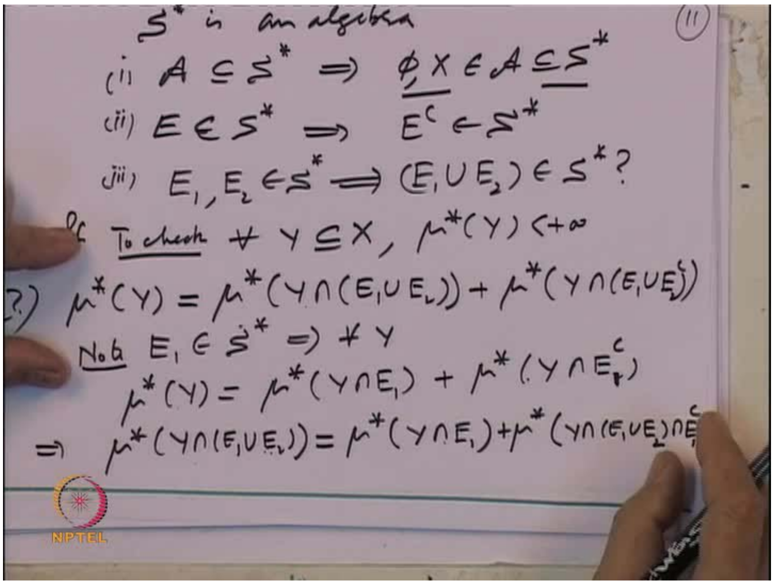

Let us see what we have to check for these? First of all, we want to check that S star is algebra. We have already shown A is inside S star, so that implies the empty set and the whole space that belong to A and hence A is in s star. So, empty set and the whole space belong to it.

We just now observed that E belonging to S star implies E complement belongs to S star. If E is measurable, E complement is measurable that also we have checked. Let us check the third property, namely if E1 and E2 belong to S star, we want to check - this implies E1 union E2 also belongs to S star. That means union of measurable sets is again measurable. So, this is what we want to check.

Let us look at a proof of this. To check that E1 E2 is measurable, we have to check - to check for every Y contained in X, mu star of Y finite. We have to check that mu star of Y can be written as mu star of Y intersection the set that is E1 union E2 plus mu star of Y intersection E1 union E2 complement. So, this is the property that we have to check.

So what we will do is, we will compute each one of the term and show it is equal to mu star of Y. For that we start - note E1 is measurable, so that implies mu star of Y, we can write it as mu star - implies for every Y, mu star of Y is mu star of Y intersection E1 plus mu star of Y intersection E2.

Now, this is important that this happens for every Y, so you can change Y according to my requirements. What I want to do is I will change this Y to Y intersection E1. So, I want to compute Y intersection E1 union E2, so let us change this Y to that.

So that implies that mu star of Y intersection E1 union E2 is equal to - here I should replace Y by Y intersection, so mu star of Y intersection E1 intersection E1 union E2, but E is the subset of E1 union E2, so that is just Y intersection E1; is that clear. Because, if I replace Y by Y intersection E1 union E2, then this intersection with E1 is just Y intersection E1 plus - what is the second thing, let us write, mu star of – sorry, this one is E1 complement, I am sorry we made a mistake saying it is measurable - Y is mu star of Y intersection E1 plus mu star of Y intersection E1 complement.

Now, when we replace Y by Y intersection E1 union E2, so this is same as this plus the second term is Y intersection E1 union E2 intersection E1 complement.

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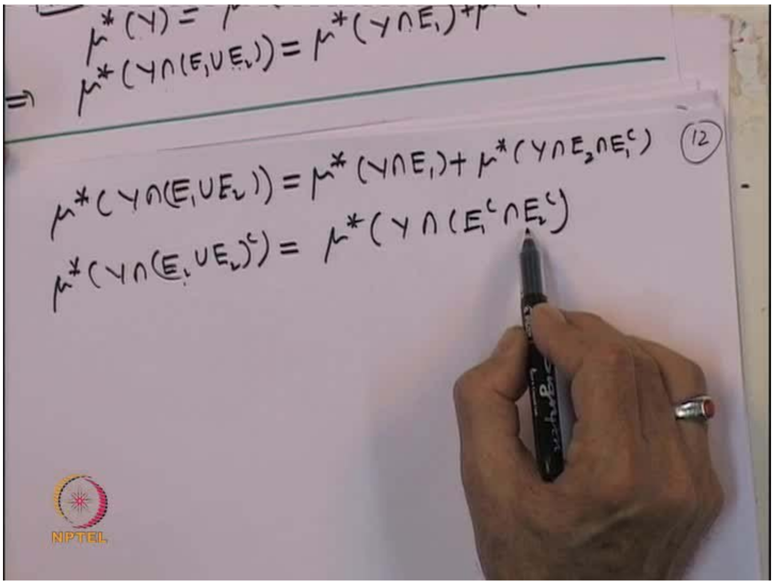

Let us simplify that. So, what we have got is the following that mu star of Y intersection E1 union E2 in the left hand side is equal to mu star of Y intersection E1 plus - what is this, now E1 union E2 intersection E1 complement, so when I take E1, E1 complement that is going to be empty set, so this set is nothing but mu star of Y intersection E2 intersection E1 complement. So, we have computed mu of Y intersection E1 union E2 to be equal to this.

Now, I also want to compute what is mu star of Y intersection complement of this. So, what is the complement of this? E1 union E2 complement, so what is that going to be? That is going to be mu star of Y intersection -we are using our De Morgan's laws for set theory, this is E1 complement intersection E2 complement.

So, I want to compute mu star of E1 complement intersection E2 complement. How can we compute that? Recall saying that E1 was measurable, we had that so you may replace Y by Y intersection E2 complement, then I will get the required set here.

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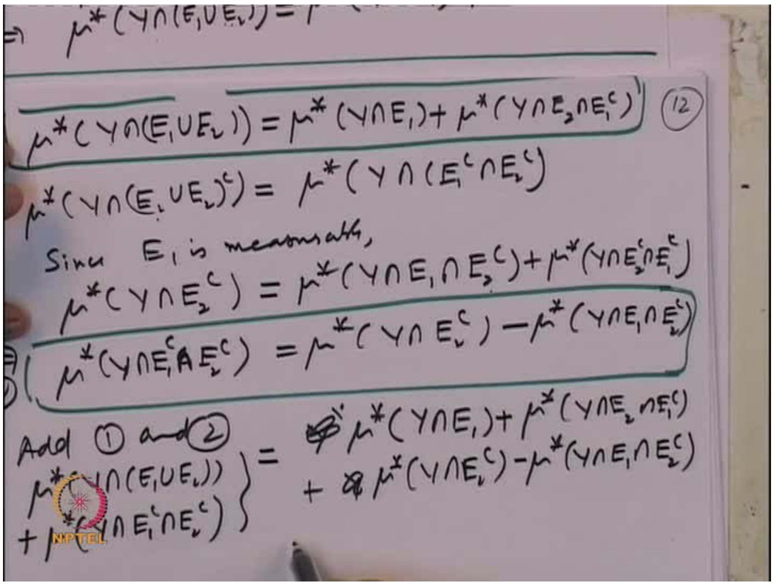

So, use this equation, since E1 is measurable, we have mu star of Y - we will just keep it here to follow - so mu star of Y intersection, instead of this we want Y intersection E2 complement. So, let us look at Y intersection E2 complement is equal to mu star of Y intersection E1 intersection E2 complement plus mu star of - what will be this set, Y intersection E2 complement intersection E1 complement. So, that is what we will have, so this is what I wanted.

Let us observe, in this equation, all the numbers are real numbers, because of the assumption that mu star of Y is finite. So this is a subset, so this finite, this is finite, this is finite and all are finite numbers, so I can interchange them. I can take one term on the other side, if required (Refer Slide Time: 43:50).

Let us do that, so from here we compute, implies mu star of Y intersection E1 complement intersection E2 complement, this set is equal to mu star of Y intersection E2 complement minus - take it on the other side, it is mu star of Y intersection E1 intersection E2 complement. So, we have gotten the required quantities. We wanted what is mu star of Y intersection E1 union E2, so that is lying here (Refer Slide Time: 44:49). We wanted that is lying here - the second term.

Let us add these two terms, so add it. Call this equation as 1, call this equation as 2, add 1 and 2 and that will give you that mu star of Y intersection E1 union E2 plus mu star of Y intersection E1 complement intersection E2 complement, so this is equal to - there we have got mu star of Y intersection E1 plus mu star of Y intersection E2 intersection E1 complement plus Y mu star of Y intersection E2 complement minus mu star of Y intersection E1 intersection E2 complement. So, this is what we have got and we want to check that this should come out to be equal to mu star of Y.

Let us again try to use, so this is mu of intersection E2 complement here and that is E1 intersection E2 complement. Let us observe, till now we have not used anywhere the fact that E2 is measurable. So, let us try to use that fact that E2 is also measurable and so that we can simplify this quantity.

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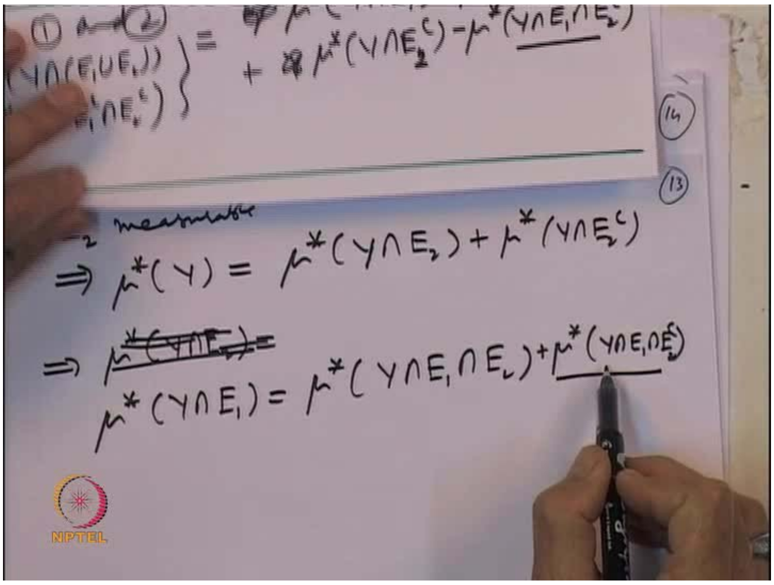

Now, observe E2 is measurable implies the following fact; we want to simplify this. So, let us look at, what is going to be Y intersection E2 and Y intersection E2complement.

So, E2 measurable means, for every Y we have got mu star of Y is equal to mu star of Y intersection E2 plus mu star of Y intersection E2 complement, because of measurability.

Now, I want to use this to compute one of the terms here. Let us replace Y by Y intersection E2, so that implies - I can replace this by mu star of Y intersection E2, will be equal to that will not give us anything. Let us replace this by Y intersection E1, so implies mu star of Y intersection E1 is equal to mu star of Y intersection E1 intersection E2 plus mu star of Y intersection E1 intersection E2 complement.

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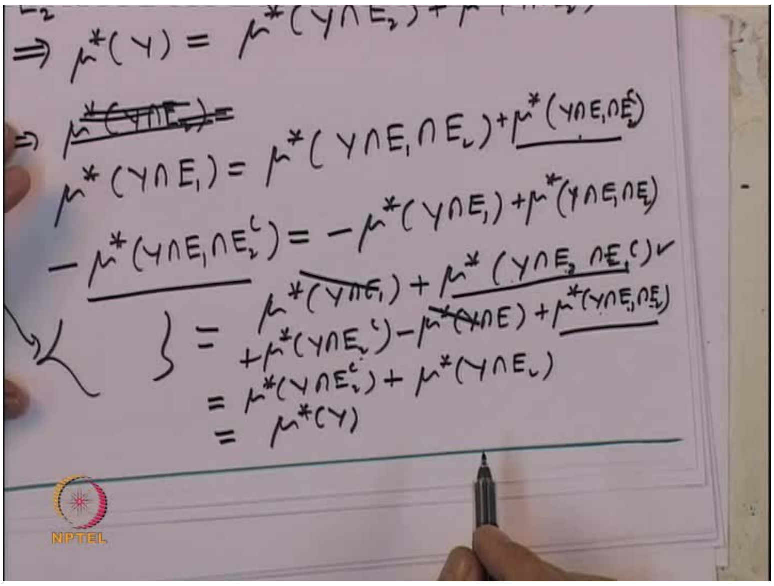

What is mu star of Y intersection E1 intersection E2 complement? That term is here. So that we want with a negative sign, so if I take it on the other side; that means minus mu star of Y intersection E1 intersection E2 complement is equal to - I bring it on the other side that is minus mu star of Y intersection E1 plus mu star of this term, which is Y intersection E1 intersection E2.

Now, this is what we have reached here. So, this is the value that I was looking for (Refer Slide Time: 49:16). Let us put in this value, so this required quantity I will just take it here, is equal to mu star of Y and here is minus mu star of Y, so those two terms will cancel out. Let me just write that is mu star of Y intersection E1 plus mu star of Y intersection E2 intersection E1 complement that we already had, plus mu star of Y intersection E2 complement and minus mu star of from here, Y intersection E plus mu star of Y intersection E1 intersection E2.

Now, these two terms cancel out. What we have left with this, so this is equal to mu star of Y intersection E2 intersection E1 complement and Y intersection E2 intersection E1. Look at these two terms, so these two terms with this Y intersection E2 intersection E1 complement plus Y mu star of Y intersection E1 and E2; that means, these two terms are nothing but mu star of Y intersection E2 and one term is here. So, this is mu star of Y intersection E2 plus - what I am saying is this plus this term is nothing but mu star of Y, so this is complement mu star of Y intersection E2, is that clear? This terms as it is (Refer Slide Time: 49:40).

Now, look at the fact that E1 is measurable, so mu star of Y intersection E2 is mu star of Y intersection E2 intersection E1 complement plus mu star of Y intersection E1 intersection E2 and now, once again using the fact that E2 measurable that is equal to mu star of Y.

We have proved the required condition that mu star of Y is equal to mu star of Y, so we have proved that this is mu star of Y is equal to mu star of Y intersection E1 union E2 plus mu star of Y intersection E1 complement intersection E2 complement.

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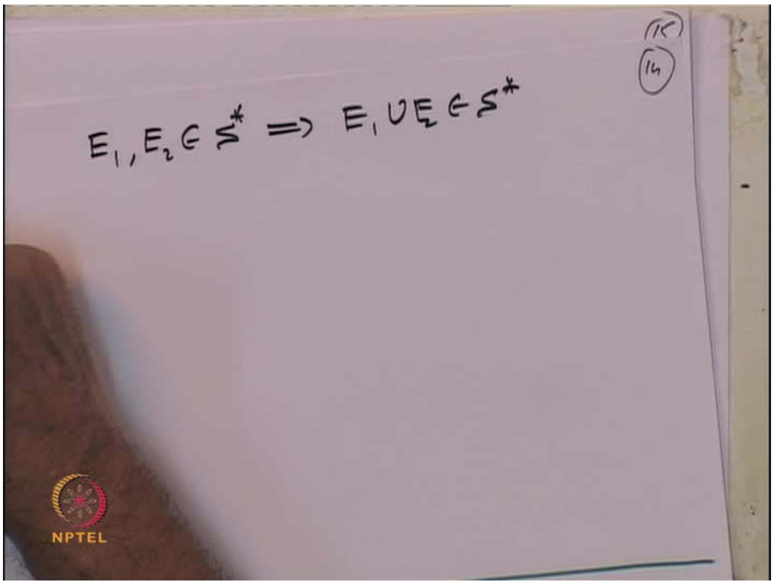

That means we have proved the fact that S is algebra of subsets of the set X. What we have shown is, E1 E2 belonging to S star implies E1union E2 also belong to S star.

Here, let me just comment that this proof looks a bit technical, but it is not so difficult. E1 measurable gives you one condition that mu star of Y is equal to something. E2 measurable gives you mu star of Y is equal to something. Now, these sets Y are arbitrary. We have given E1 and E2 are measurable means, mu star of Y is equal to mu star of Y intersection E1 plus mu star of Y intersection E2 E1 complement.

So, you can change this Y to Y intersection E1, Y intersection E2 and so on. So, write down what are equations which are given; write down the equation, the equality that we proved and just manipulate, this is only a simple algebra, which is required.

Today, what we have done is, we have looked at, we have defined the concept of what is called a measurable set for an outer measure mu. We have shown that the original elements of the algebra are already measurable sets and the class of all measurable sets forms algebra. So, we will continue the analysis of this class S star in our next lecture; thank you.