Measure and Integration

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Lecture No. # 1
Introduction, Extended Real Numbers

Good morning. My name is Inderkumar Rana. I am Professor in the Department of Mathematics, IIT, Bombay. I will be taking you through this course on Measure and Integration. This is a course which is normally taught at masters level M.Sc. in Mathematics, and sometimes in departments like Physics, Electrical Engineering also. So, let us go through the basic objectives of this course. This course is called Measure and Integration. This also goes by various names such as Real Analysis, Advance Real Analysis and so on.

The aim of this course - the objectives that will we covering in this course are as follows:

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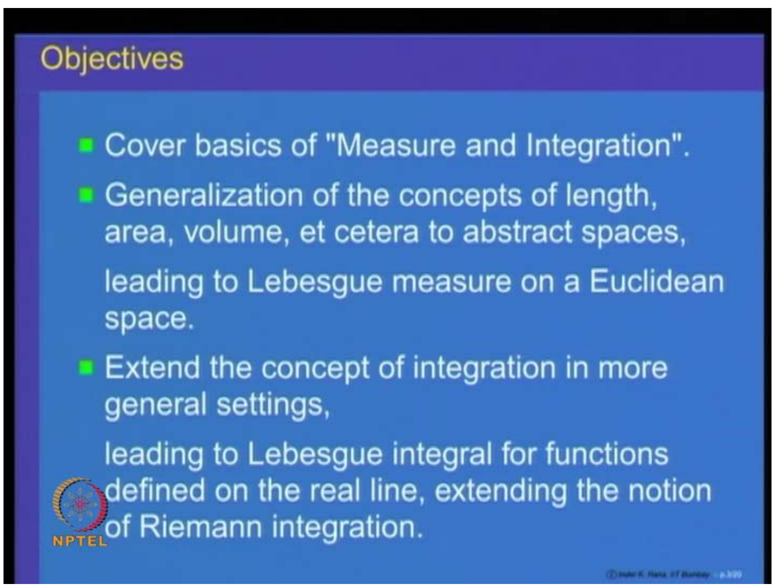

So, the aim is to generalize the concept of length, area, volume, etc., to abstract spaces. That leads to the notion of Lebesgue measure on Euclidean spaces and general concept of measures on general spaces. Then, also we will extend the notion of integration which is normally done in

UG levels called Riemann integration, to more general settings. That leads to the notion of Lebesgue integral and other notions of abstract integration. So, these are the basic sort of outline of the course that we are going to follow.

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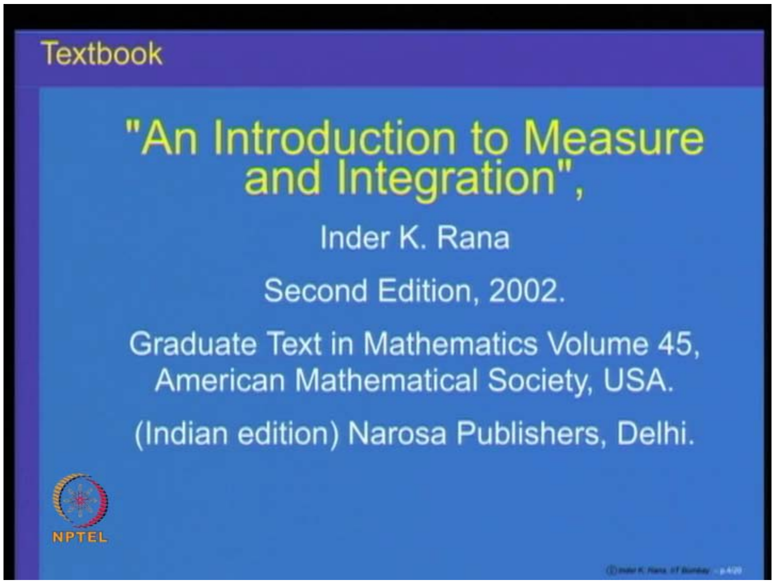

We will be following the text book, "An Introduction to Measure and Integration" written by me. This is published jointly by Graduate Text in Mathematics by an American Mathematical Society. Indian edition of this is available through Narosa Publishers, New Delhi.

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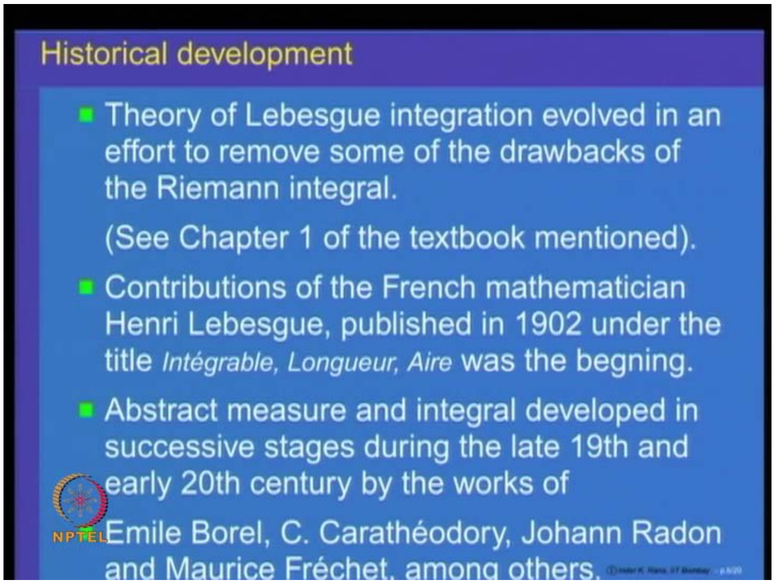

So, why is Lebesgue integration needed and what is the need for extending the notion of Reimann integration? There are some problems, drawbacks of Reimann integration. To study about them, you should look at chapter 1 of the text book I have mentioned. We will not have time to go through these drawbacks of Reimann integration and how efforts which are made to remove these drawbacks led to the development of Lebesgue measure, Lebesgue integration and so on.

So, for these, we refer chapter 1 of the book. Historically, this was developed by the French Mathematician Henri Lebesgue, who published as a part of his PhD thesis in 1902 Integral, Longueur, Aire. Then this was developed further into the abstract spaces by various mathematicians in 19th and 20th century, some of them being Emile Borel, Caratheodory, Radon and Frechet, among others.

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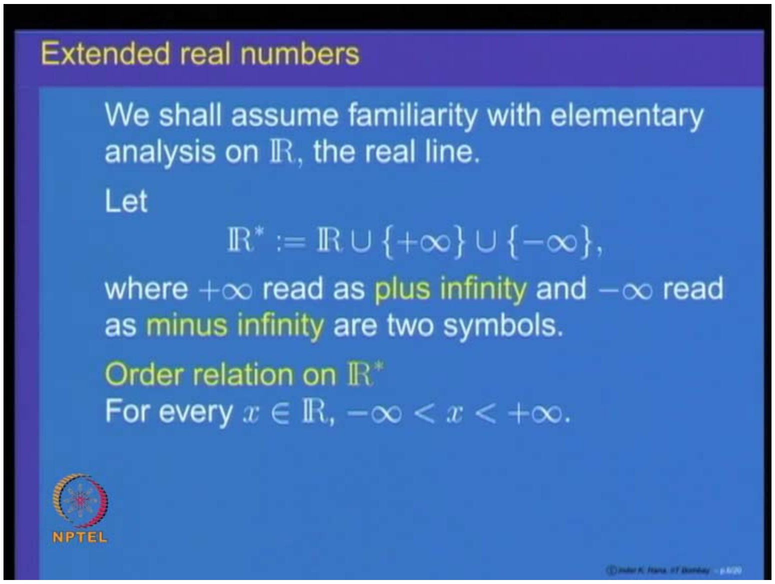

So, to start with there are some prerequisites for this course, which we shall assume and we will hope that you have gone through an elementary course, first course in Real Analysis and you are familiar with the properties of the real line, what is real line, what are called open intervals, closed intervals, what is a topology on the real line and what are called complex subset of real line. So, basic course on Real Analysis is going to be assumed throughout this course.

So, if you have difficulty in this, look up some elementary book on first course on Real Analysis and go through these topics, so that you are you are not left behind and you are able to understand what are the things we are going to discuss, concepts we are going to discuss.

Then, there is one - the basic space of course is real line, but there is a notion of what is called extended real numbers, which we are going to use in our course. Since this is not normally discussed in most of the text books or in courses in Real Analysis, we will go through some of these concepts on extended real numbers.

So, first of all, what is this set of extended real numbers? (Refer Slide Time: 05:00)

The set of extended real numbers denoted by R star is the set of real numbers to which we adjoin 2 new symbols; one is called plus infinity and the other is called minus infinity. Now, once we adjoin these 2 new symbols to the set R, we get the extended set R denoted by R star.

Now, as you all know, the set of real numbers have got algebraic operations of addition, multiplication - there is an order on it. So, when we add these 2 new symbols to them, we would like to define how these 2 new symbols, these 2 new objects, behave with the respect to the original order structure, the original operation of addition, multiplication and so on. So, we are going to define what are called operations of additions, multiplication, and order on the set of extended real numbers.

The first one is the order relation. So, we are going to assume or we are going to say that, for every real number x in R lies between the 2 new symbols minus infinity less than x strictly less than plus infinity. So, this is how the new symbols plus infinity and minus infinity behave with respect to the order structure. So, minus infinity in R star is the smallest element and plus infinity is the largest element in R star as far is the order is concerned; for real numbers, the same original order stays.

Next, let us look at the algebraic operations on R star.

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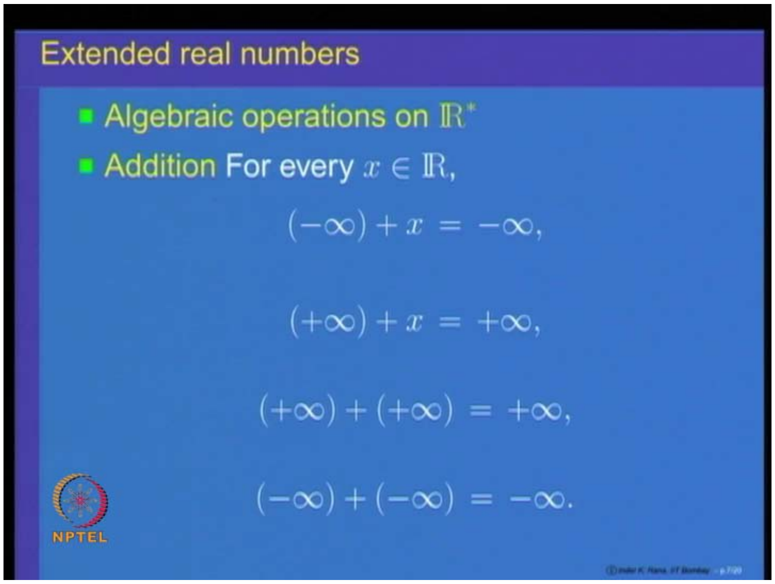

So, for real numbers x and y, we already know what is x plus y, but for infinity and minus infinity - the 2 new symbols, how are these operations defined? here are the… (Refer Slide Time: 07:00). So, for every x belonging to R, if we add minus infinity to x, we should get minus infinity. So, that is the rule. We are specifying how does minus infinity behave with respect to addition of real numbers. Similarly, plus infinity plus x is equal to plus infinity; whatever be x, positive or negative, when added to minus infinity, you get minus infinity, and when added to plus infinity, you will get plus infinity.

Now, how does infinity plus infinity added to itself? What is the outcome?

It says plus infinity plus plus infinity is plus infinity. Minus infinity plus minus infinity is minus infinity. Let us specify that plus infinity plus minus infinity is not defined. So, these are the only for relations among addition; addition of plus infinity minus infinity with respect to x, plus infinity with itself and minus infinity with itself.

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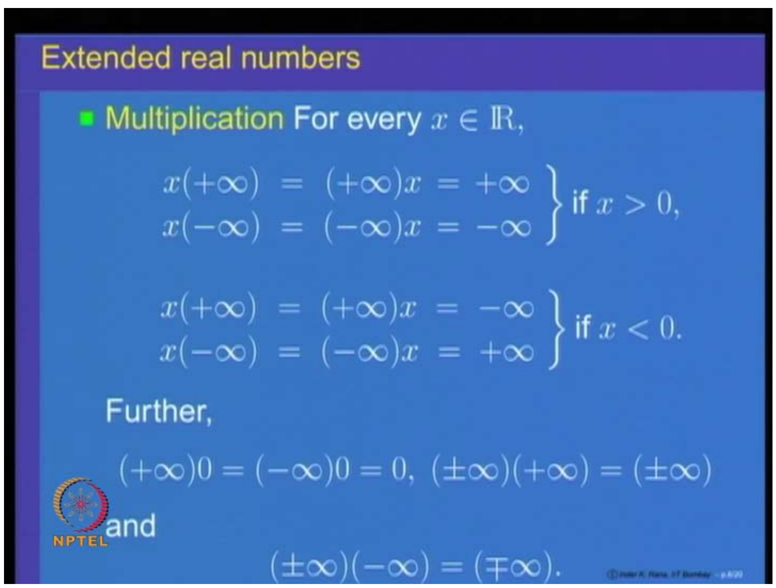

Next comes, the rules for multiplication:

For every real number x, x into plus infinity is equal to plus infinity into x is plus infinity, if x is non-negative. Similarly, x multiplied by minus infinity is same as minus infinity multiplied by x is equal to minus infinity; again, under the condition that x is bigger than 0; more or less, we are following the rules of multiplication for real numbers.

Similarly, if x is negative, we have x multiplied by plus infinity or plus infinity multiplied by x is equal to minus infinity, the sign changes of infinity.

Similarly, x multiplied by minus infinity is equal to minus infinity multiplied by x is equal to plus infinity, if x is less than zero.

So, depending upon whether x is bigger than zero or x is less than zero, the rules for multiplication are as specified. Of course, if x and y are real numbers, the multiplication between x and y is same as that of real numbers. So, these are the rules for multiplication.

Of course, there is specific element; this particular element called 0 in the real numbers - how does that behave with respect to plus infinity and minus infinity?

Here are the rules:

For plus infinity into zero is same as minus infinity into zero is equal to zero; that is same as for real numbers also; x multiplied by zero, whether positive or negative is always equal to zero.

Of course, if I multiply plus infinity with itself, the answer is plus infinity and if minus infinity is multiplied with plus infinity, the answer is minus infinity. So, plus minus infinity multiplied by plus infinity is plus minus infinity and similarly plus minus infinity multiplied by minus infinity is minus plus infinity, the sign changes of the outcome.

So, these are the rules for addition, multiplication and order structure on the set R star, which is nothing but the real numbers along with two new symbols, plus infinity and minus infinity.

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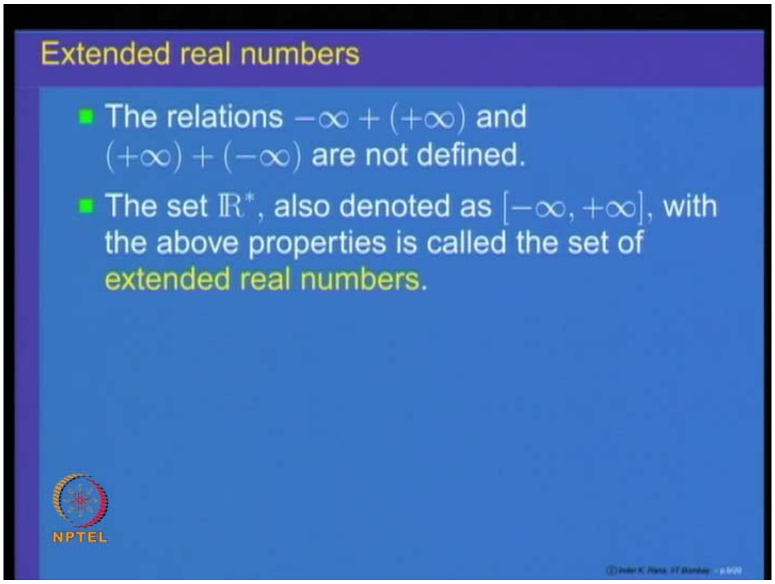

So, once again, let us specify that the relations minus infinity plus plus infinity and plus infinity plus minus infinity are not defined. So, with these rules, we get the set R star of extended real numbers, which is also denoted by this square bracket minus infinity comma square bracket plus infinity. So, that is essentially something like saying, the real numbers are denoted by the open kind of interval minus infinity to plus infinity.

If you close it upon both sides, that is the notation news for extended real numbers. So, once you are familiar with the order, familiar with the addition and multiplication on the extended real numbers, we can look at the notion of sequences in real numbers, and also the notion of supremum and infimum on subsets of extended real numbers.

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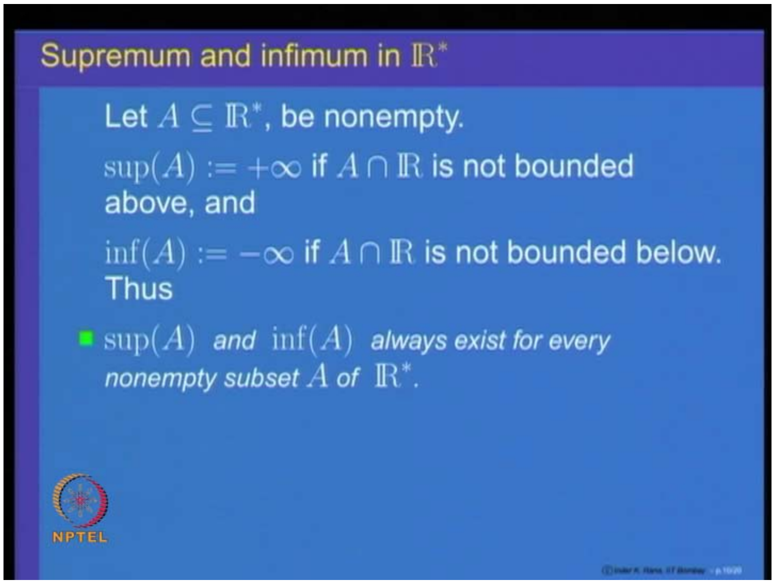

So, let us first look at A is a sub set of extended real numbers and let us assume A is nonempty set. Now, there is a possibility that A is sub set of real numbers only. Then, we know that the completeness property of real number says, if the set A is bounded above, it must have least upper bound or namely the supremum.

Now, in the case A is a sub set of R star is a sub set of extended real numbers, that means there is possibility of minus infinity or plus infinity being a part of it. Suppose if it bounded above, then it has to be a subset of real numbers and supremum will exist.

If it is not bounded above, that means plus infinity is going to be a part of it. So, we will define the supremum of A to be equal to plus infinity, if A as a part of R is not bounded above. Similarly, we will define the infimum of the set A to be equal to minus infinity, if A inter section R is not bounded below.

So, what we are saying is - in the sub set of extended real numbers, a set which is bounded above or bounded below does not matter, we do not have to say that. So, every subset, nonempty subset of extended real numbers will always have supremum and will have always have infimum. Of course, this supremum will be equal to plus infinity if z a is not bounded above, and infimum will be equal to minus infinity if it is not bounded above. So, it is a very nice situation; every sub set has supremum as well as infimum.

Now, similar conditions will hold or similar result will hold for limits of sequences in R star.

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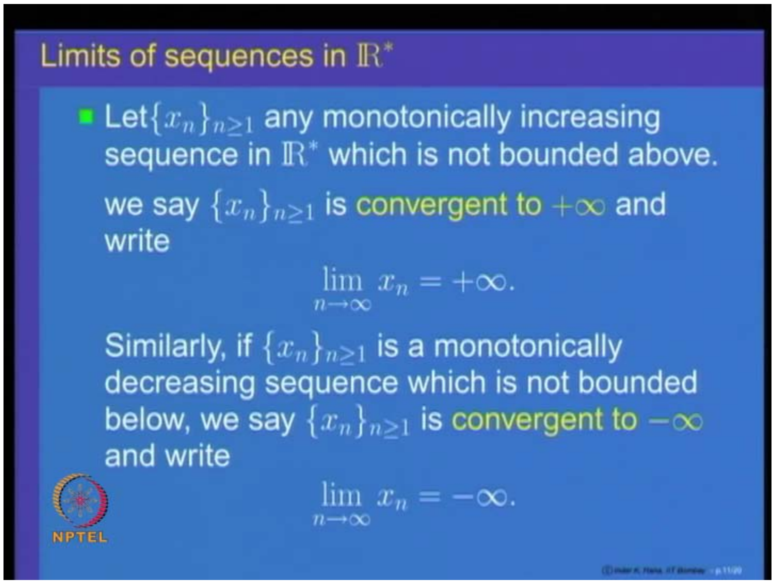

So, let us look at a sequence x_n which is monotonically increasing and which is not bounded above. If you recall, as a sub set, as sequence in real numbers, if a sequence is monotonically increasing and is bounded above, then it must be convergent.

Now, if is a sequence is (( )) monotonically increasing and it is in R star, it is a sequence of extended real numbers, and not bounded above, that means plus infinity is going to be an element of it. So, if it is not bounded above, we will say sequence is convergent to plus infinity and write this as equal to plus infinity.

Essentially also, we would say, when x_n is sequence of real number which is monotonically increasing and not bounded above, in that case also, we write the limit to be equal to plus infinity. For a sequence of real numbers, it is only symbolic way of saying that, a monotonically increasing sequence not bounded above converges to plus infinity. But, as a sequence in R star, it converges to an element of R star namely to as infinity.

Similarly, the sequence x_n in R star which is monotonically decreasing; if it is not bounded below, we say, it converges to minus infinity and write this as limit n going to infinity x_n is equal to minus infinity.

So, this is how we will analyze sequences in R star which are monotonically increasing or monotonically decreasing.

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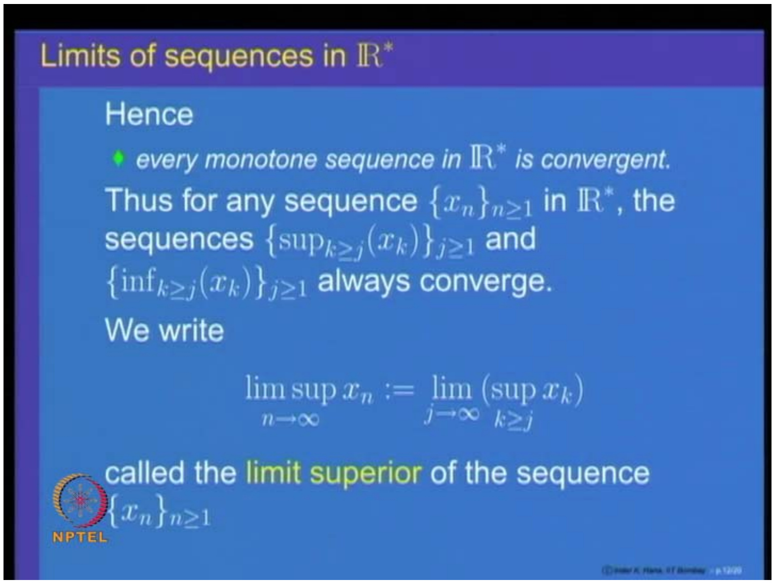

Similar concepts can be developed for series in R. So, let us just say two things about sequences: Because every monotone sequence is convergent, if I look at the sequence, given any sequence, look at the supremum of that sequence, from the stage j onwards. So, supremum k bigger than or equal to j of x_k; then, that gives a new sequence and that sequence will always converge. Similarly, the infimum from k bigger than or equal to j x_k will also converge, because, these are monotone sequences and monotone sequences in R star always converge.

So, limit of the supremum, supremum k bigger than or equal to j for that is denoted by limit superior of x_n. Similarly for the infimum k bigger than or equal to j, x_k is called the limit inferior of the sequence.

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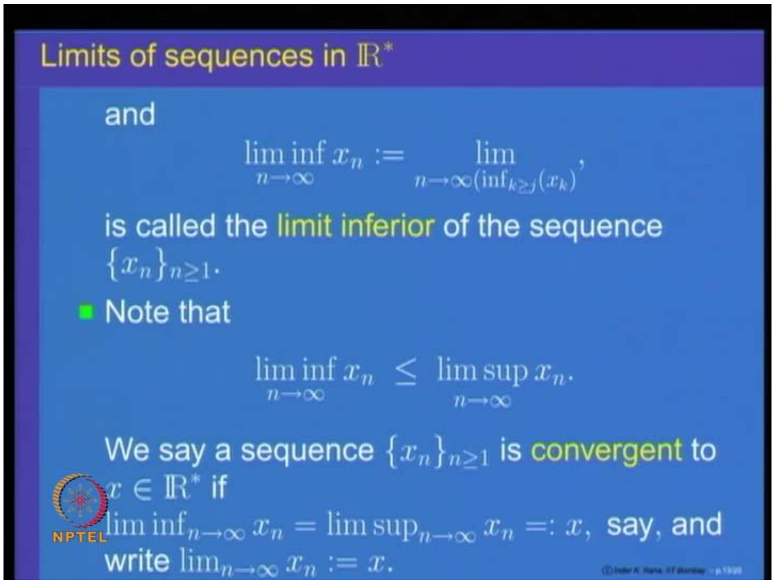

So, in general, we all know that limit inferior is always less than or equal to limit superior and the sequence will converge when limit inferior is equal to the limit superior, even in the case of sequences in R star. So, this is how sequences behave.

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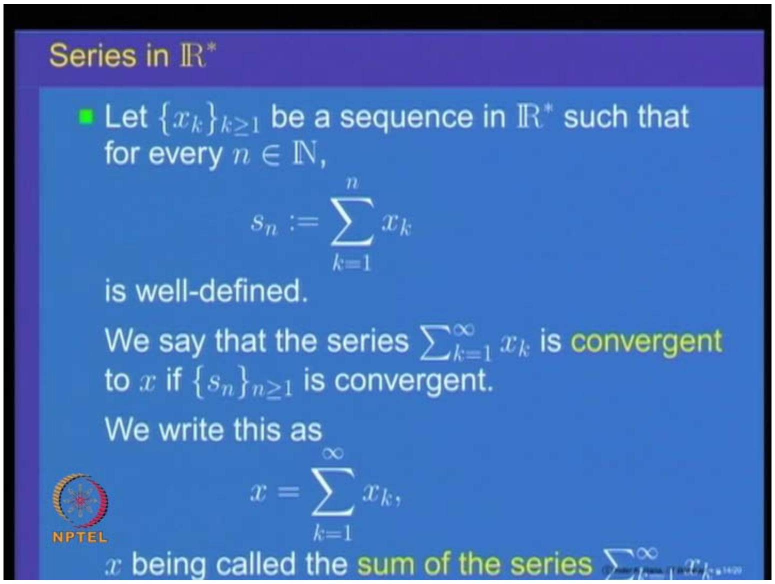

Now, let us look at sequence. From sequences, let us go to concept of series. Suppose $x_{k,}$ k bigger than or equal to 1 is sequence in R star, then, let us look at the partial sums of this sequence $s_{n;}$ that is, the sum of first n terms of the sequence that is denoted by $s_{n,}$ which is summation k equal to 1 to n $x_{k.}$ So, for every n this is well defined.

One can ask whether this sequence is convergent or not in R star. So, if this series is convergent in R star, then, it means if the sequence is convergent in R star, the sum of partial sums in the sequence of partial sums - if it is convergent in R star, we say that the series is convergent and the limit is called the sum of the series. So, this is basically same as that real line; only keep in mind how sequences behave in the real line.

So, with this basic discussion about what is the basic space of the extended real numbers we are going to deal with, we start with a proper concept in our subject Measure and Integration.

The first few concepts are going to be discussions about class of subsets of a nonempty set.

So, we are going to look at some collection of subsets of a given nonempty set x with certain properties. This collection of subsets, which we are going to call as semi algebras, sigma algebras and monotone classes are various classes which play an important role later on in our subject.

So, let us start with looking at what is called a semi-algebra of subsets of a set X.

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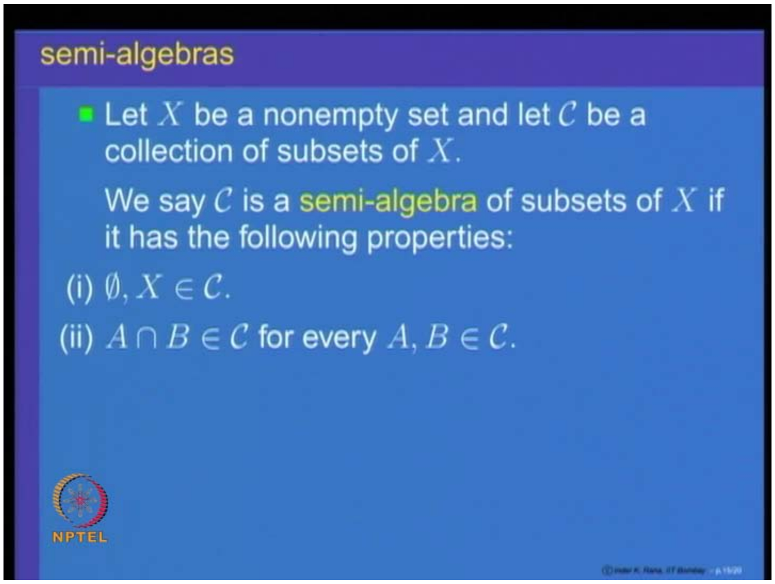

Let X be a nonempty set and let C be a collection of subsets of that set X. We say that the class C is a semi-algebra of subsets of X, if it has the following properties, this collection C has the following properties:

One: The empty set and the whole space are members of this class X. So, the first property desired of C is that, the empty set and the whole space X are members of this class.

The second one is that, this class is closed under intersections; that means, if A and B are two elements of this collection, then the intersection of these sets A and B should also be a member of the class C. So, this class C is closed under intersection; that is the second property.

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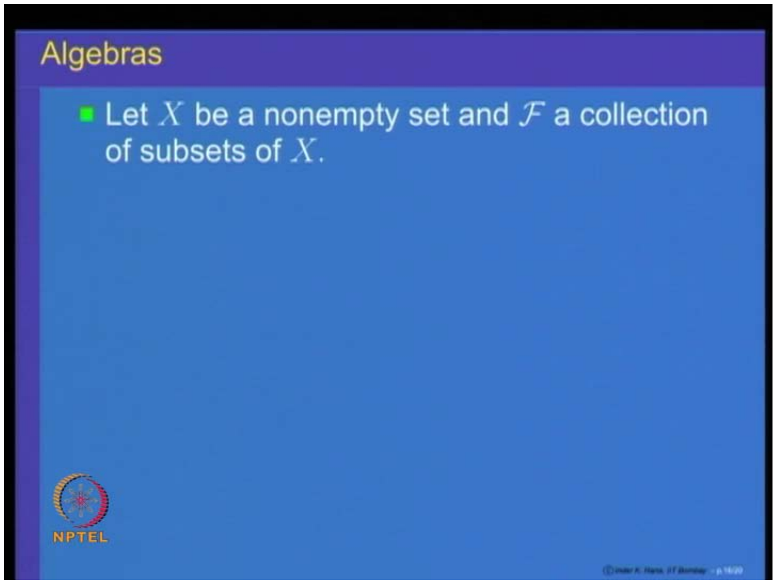

There is a third property which we will describe soon. That is saying that, this class need not be closed under compliments, but will require additional properties. So, let me write that property and explain because, it is best understood when it is written.

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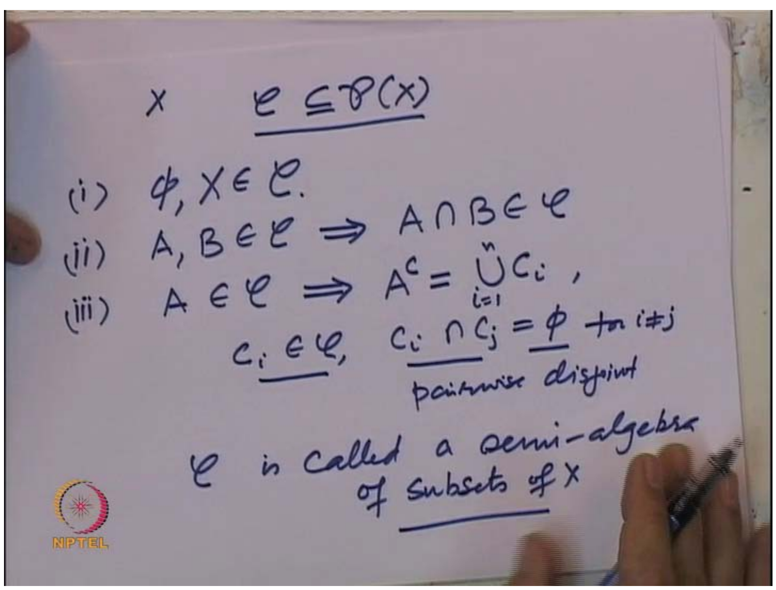

So, X is nonempty set, C is collection of subsets of the set X. The first property we said empty set and the whole space belong to C and second property was - if A and B belong to C, then that implies A intersection B belong to C and third property which is very crucial is that, if A belongs to C, then that implies, the set, look at the set A compliment - that need not be in C, but we want to say, you can write C as union of elements c_i finite number of them, i equal to some 1 to n such that c_i's are elements of C and they are pair wise disjoint; c_i intersection c_j is empty.

So, let us just go through these concepts again. A collection C in P X having the following properties:

one: The empty set - the whole space are elements of it.

If A and B belong to it, then, the intersection of these 2 sets namely A intersection B is also an element of this collection C.

The third property is - if A is a subset of C, then A compliment, the compliment of this set in X of course, should be representable as well as union of c_i's i equal to 1 to n, where the c_i's are elements of c and they are pair wise disjoints. So, this property that c_i intersection c_j is nonempty for i not equal to j. We just say, they are pair wise disjoint. So, such collection C is called a semi-algebra of subsets of X.

Let us look at some examples, to get familiarized with this notion of semi-algebras.

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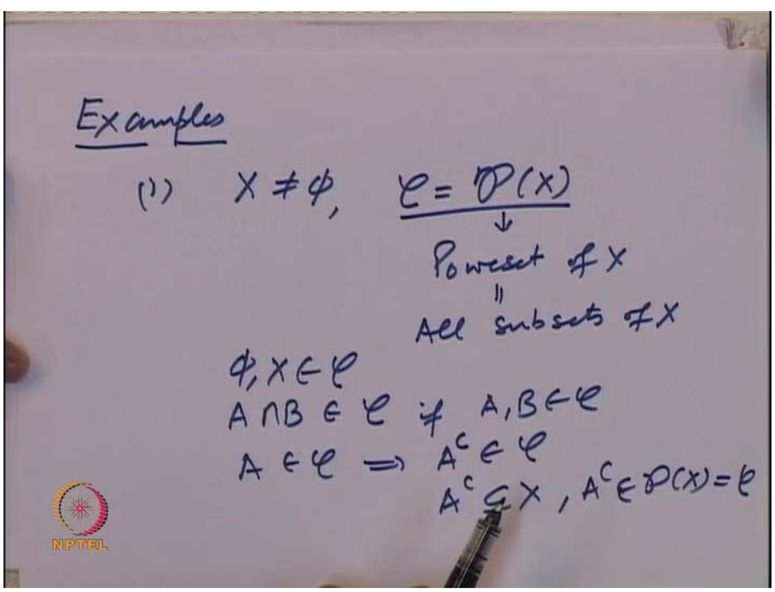

So, let us take X - any nonempty set. Let us take the collection C to be equal to all subsets of X; so, P of X. What is P of X? That is a power set of X. So, normally this is called the power set of X which is same as all subsets of X. So, C is the collection of all subsets of X.

Do you think it is closed under…. So, do you think phi and X belong to C? Obviously, it is a collection of all subsets. So, phi and X belong. Obviously, A intersection B also belongs to it if A and B belong to C. Because, if A and B are subsets of it, then, naturally A intersection B also is a subset. In fact, if A belongs to C, then that implies A compliment also belongs to C because, A compliment itself is a subset of X. So, A compliment belongs to P X, it is a subset of X, so belongs to P X which is C. So, the collection of all subsets of the set X is an example, which is an obvious example of a semi-algebra of subsets of a set X.

Let us look at some more examples. This was an obvious example. So, let us look at the second example.

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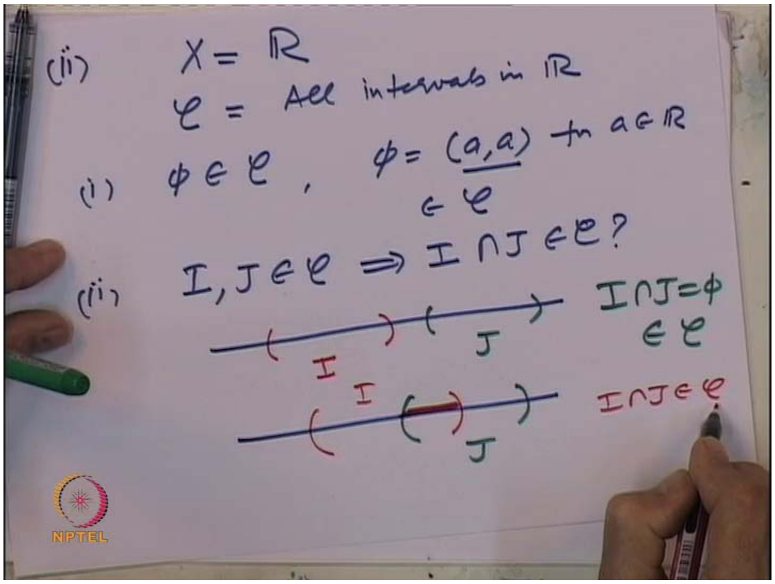

Let us look at X equal to real line and let us take the collection C, all intervals in R. So, we are looking at the collection of all intervals in R- that is the collection C.

So, first property - is empty set a member of C? Well, here we will have to understand, of course yes. One way of looking at it is - empty set can be written as the open interval, a comma a, for any point a belonging to R and that is a interval; so, that belongs to the collection C.

Second property: Let us take 2 intervals I and J belong to C. So, does this imply the intersection I, intersection J belong to C? So, that is the question. That means, if I and J are two intervals, can i say, I intersection J is also an interval? Let us check in the picture.

Let us take 2 intervals I and J. So, let us say, here is the interval I and here is the interval J (Refer Slide Time: 25:28). So, one possibility is that they are disjoint from each other. So, here I intersection J is empty, hence, belongs to C.

What is the other possibility? Other possibility is let us take the intersect. So, here is my interval J (Refer Slide Time: 25:56) and here is my interval I (Refer Slide Time: 25:59). Then what is intersection of these two? So, this is I and that is J (Refer Slide Time: 26:05), that is my intersection of I and J, and clearly, that is also, I intersection J is also an interval. So, it belongs.

So, two cases: case one - when there is disjoint empty set, intersection is empty set and belongs to C. If they overlap, then the overlap itself is again an interval and that belongs to C. So, it is quite clear that this collection of all intervals is closed under intersection also.

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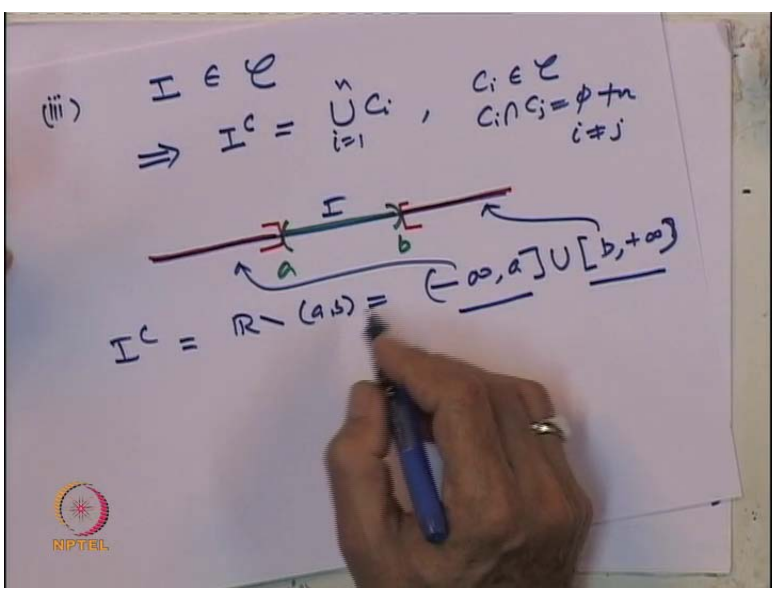

Let us look at the third property which is crucial, and that says, if I take an interval I, the third property that we want to verify is - if I is an interval, I should look at the compliment of that interval and should be able to write it as union of c_i i equal to 1 to n, where c_i belong to C, and c_i intersection c_j is empty for i not equal to j.

So, once again, let us look at an interval. So, let us look at an interval say, open interval a and b (Refer Slide Time: 27:23). So, that is my interval. So, what is going to be the compliment of this? In fact, the compliment of this looks like two paces, one is this side, other is this side. so this pace and this pace (Refer Slide Time: 27:36). So, for this, I can write that, the compliments R minus a b is equal to minus infinity to a in close union b to plus infinity. So, this is this part and this is this part and this is this part (Refer Slide Time: 28:00).

So, if I take an interval I, that is I compliment, where I is the interval a to b, then its compliment is a disjoint union of 2 elements.

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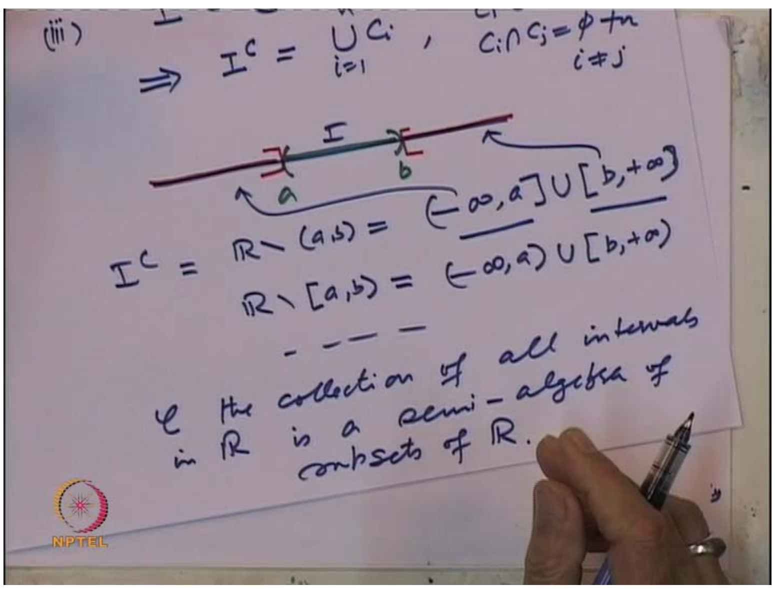

Similar cases we will follow. If for example, a is left open or right close, let us write R if it is of the type this (Refer Slide Time: 28:27), then I can write this as this point a is enclosed. So, minus infinity to a, the compliment will be this (Refer Slide Time: 28:35) open here and union b to plus infinity. Similarly, the other cases. So, I will say and you write down yourself. So, we verified C, the collection of all intervals in R is semi-algebra of subsets of real line.

Let us look at some more examples. Let us look at example number 4.

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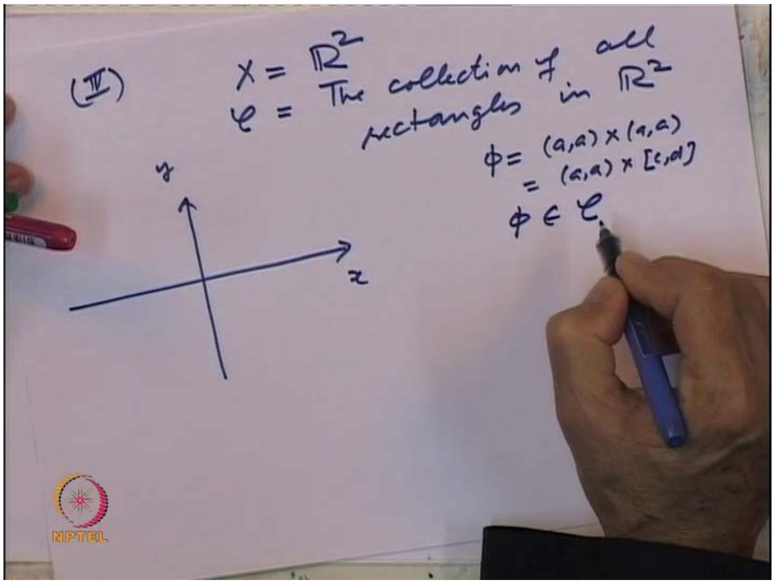

Let us take the set X to be equal to R 2 and C is the collection of all rectangles in R 2. So, let us just look at the picture and try to understand. So, here is R 2(Refer Slide Time: 29:50). Can I say empty set is a rectangle? Of course, empty set can be written as, a comma a cross a comma a, if you like. It does not matter. You can also write as, a comma a cross c comma d. In both cases, it is the empty set or any other such representation. So, empty set is an element of the collection of all rectangles in the plane.

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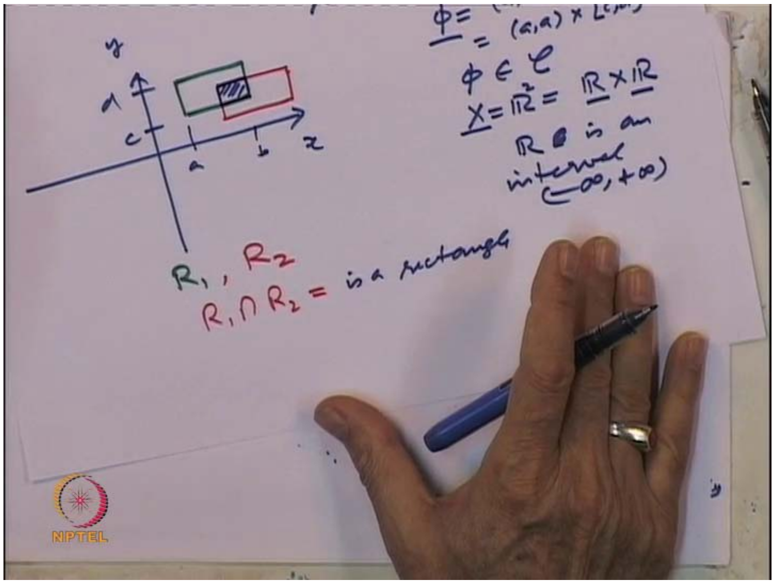

What about the whole space X that is R 2? of course that is R cross R and R is a rectangle sorry R is an interval. So, R is an interval which we write normally as minus infinity to plus infinity. So, empty set is an element of it, the whole space is an element of it.

Let us take 2 rectangles and see (Refer Slide Time: 31:00) whether the intersection of these two rectangles is also…. Let us take one rectangle here and another rectangle. The possibilities are - they do not intersect; in case they do not intersect, then there is nothing to prove because, the intersection is an empty set which is already a rectangle.

Let us take a rectangle which intersects with earlier rectangle. So, we are taking 2 rectangles say R_1 and R_2. We want to check whether R_1 intersection R_2 is a rectangle or not a rectangle. The picture is quite clear that R_1 intersection R_2 is this rectangle. So, R_1 intersection R_2 is a rectangle.

One can write down formal proof by writing this to be equal to a, b, c and d and so on (Refer Slide Time: 32:00), but that is not necessary, once we understand from the picture that intersection of two rectangles is again a rectangle.

Of course, let us verify the third property - can I the represent the compliment of a rectangle as a finite disjoint union of rectangles.

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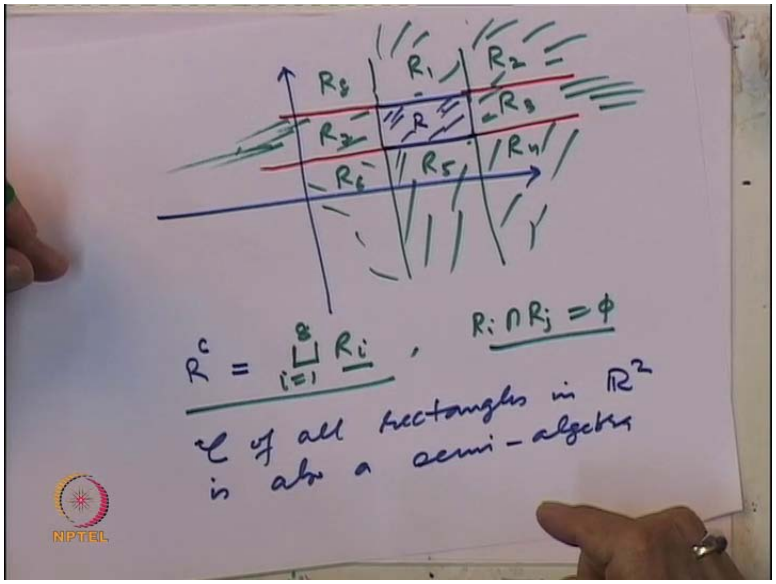

Let us take a rectangle. So, let us take a rectangle in the plane. So, this is a rectangle R, and I want to write R compliment. I want to see, what does it look like. Can I represent this as a finite disjoint union of rectangles, again?

Well. Let us obviously in the picture I can try to do as the following: I can draw lines passing through the sides (Refer Slide Time: 33:00) and I can draw another line passing through this. Then, it is quite clear that this is compliment of R. So, this was the set rectangle R (Refer Slide Time: 33:20) and its compliment is nothing but a rectangle R_1, a rectangle R_2, rectangle R_3, rectangle R_4, rectangle R_5, R_6, R_7 and R_8. Of course, these are rectangles R_1,R_2, R_3, R_4, R_5, R_6, R_7 and R_8.

So, this is R_5 (Refer Slide Time: 33:48) and this is, this part is.. (Refer Slide Time: 33:52). There are many ways of… So, I am looking at these whole infinite. I can look at this whole infinite, this side and this corner as a rectangle and this part (Refer Slide Time: 34:00) as a rectangle.

So, I can write it as union of R_i, i equal to 1 to 8, where R_i intersection R_j is empty. So, it is a matter of writing down the details, that depending on R whether which part of boundary is included or excluded. Accordingly, I can make these rectangles R_i’s to be disjoint.

So, this is true that the compliment of a rectangle in the plane is also a rectangle. So, that means what? It says that the collection of C of all rectangles in R_2 is also a semi-algebra of subsets of X.

We have given lot of examples of objects which are semi-algebras of subsets of X.

Now, let us go to a next stage of understanding, extending this concept of semi-algebra, to what is called an algebra of subsets of the set X.

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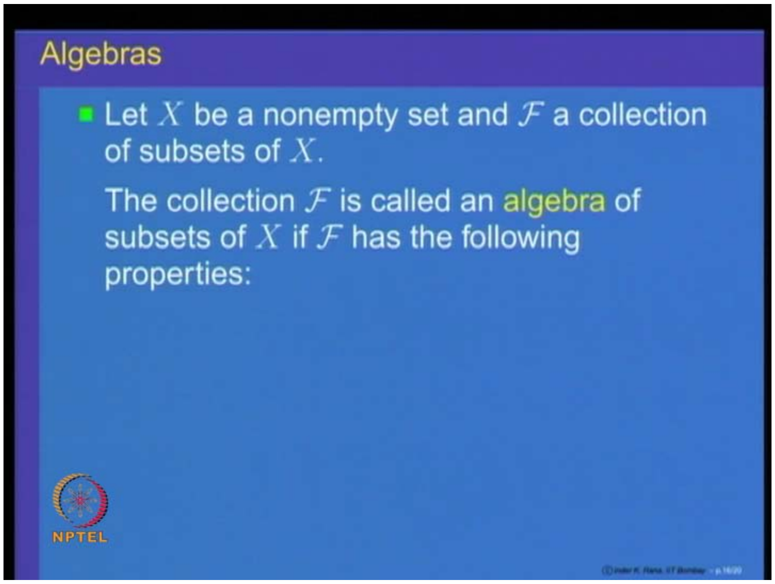

So, let X be a nonempty set and a collection F of subsets of X with the following properties:

One: Like semi-algebra, the empty set belongs to it, the whole space belongs to it, and of course, there is another property. So, let us better write this.

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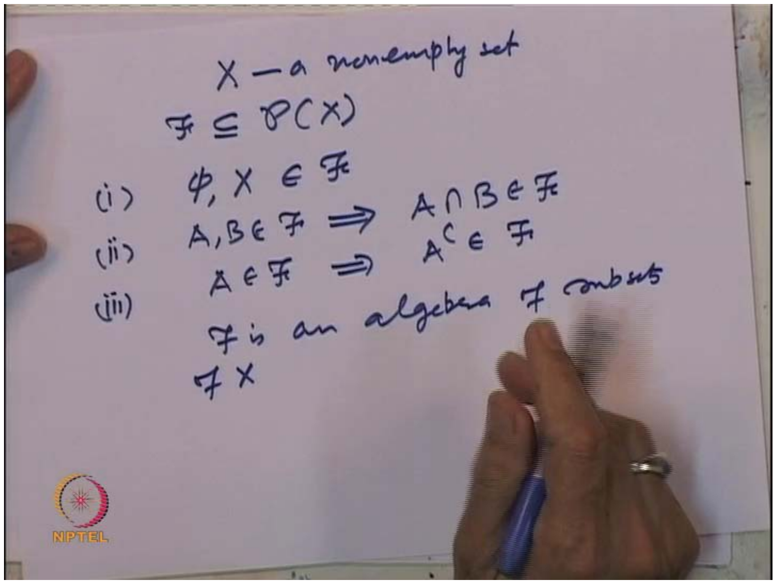

X is a nonempty set; F is a collection of subsets of X with the following properties:

One: If empty set and the whole space belong to F like that in semi-algebra;

Secondly we are going to look at the intersection property - if A and B belong to F, their elements of F, then that implies their intersection also belongs to F.

Of course, third property namely, in the case of semi-algebra, if I take element f in F, then its compliment need not be F, but we were able to represent finite disjoint union of elements of that class. But in algebra, in the new concept, we are demanding - this implies, A compliment also belongs to F. In these cases, we say F is an algebra of subsets of X. So, this is called an algebra of subsets of X.

So, how does the algebra differ from a semi-algebra?

This property (Refer Slide Time: 37:12) that is always true for algebra as well as semi-algebra. This property (Refer Slide Time: 37:17) true for algebra as well a semi-algebra. This property (Refer Slide Time: 37:21) may not be true for a semi-algebra. In that case, we will call as we said, A compliment is a finite disjoint union of elements of F, and here (Refer Slide Time: 37:37) we are saying, A compliment itself is an element of F.

So, let us look at some examples of this again, to understand.

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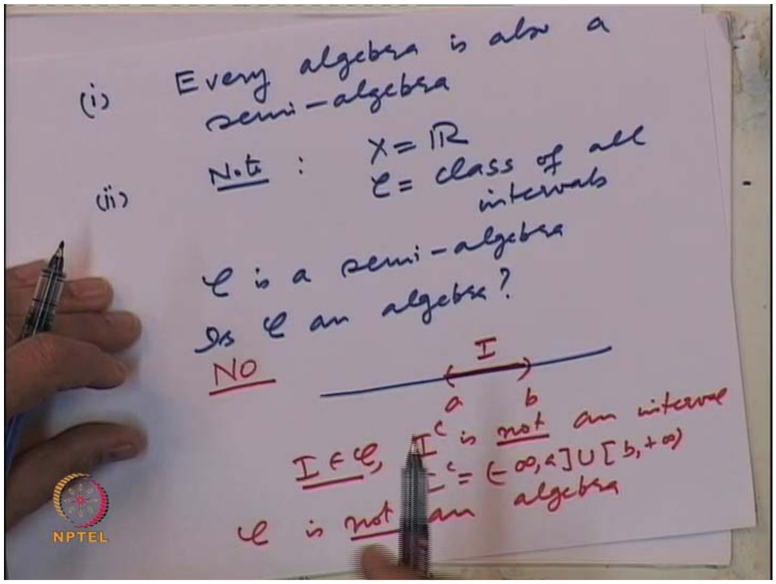

So first observation: Of course, every algebra is also a semi-algebra, because, the third property that we looked at namely in a semi-algebra, one would like to have A compliment to be a disjoint union of elements of F. In an algebra, it is itself in F; so is much stronger. So, every algebra is also a semi-algebra.

Let us note, when X was equal to R and C is equal to class of all intervals, we showed that C is a semi-algebra.

That question, “is C an algebra?” Obviously, the answer is no.

For example, I can take an interval, any non-degenerate interval say, a to b (Refer Slide Time: 39:00). Let us take this interval a to b that is by I. So, I belongs to C, but I compliment is not an interval, because, I compliment is nothing but minus infinity to a union b to plus infinity.

So, when I belongs to C, I’s interval is compliment. It need not be an interval, in general. So, that implies that this collection C is not an algebra.

So, let me emphasize again. Here, property one says, every algebra is also a semi-algebra (Refer Slide Time: 40:00) and this (Refer Slide Time: 40:02) says, every semi-algebra need not be an algebra; means, there are examples of collection of subsets of sets. For example, in the real line, the collection C of all intervals is a semi-algebra, but it is not an algebra.

So, the collection of subsets is an algebra, is a much stronger property says concept than that of a semi-algebra.

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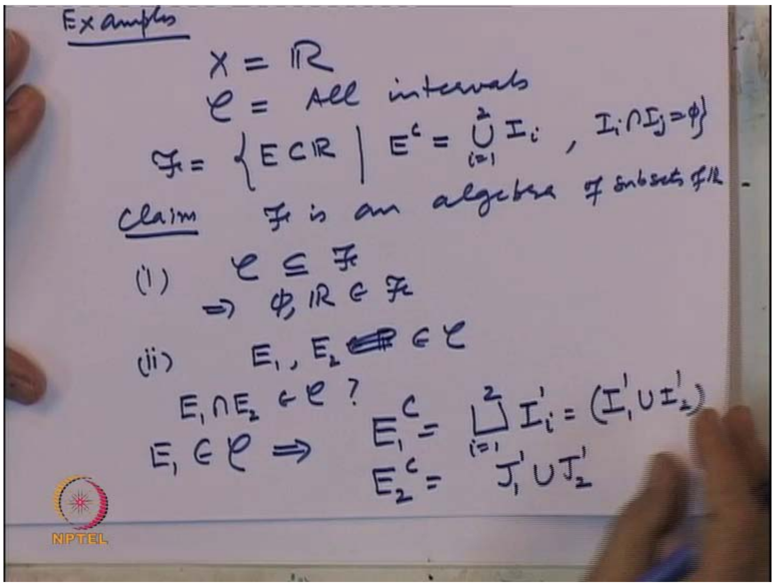

Let us look at some more examples. Let us look at the example, same example of X is real line. C is all intervals. You know that this collection is not an algebra; it is a semi-algebra.

Let us write, F to be the collection of all subsets of the real line, such that look at this collection of all intervals. It was not algebra because, the compliment of an interval need not be an interval, but it looked like it is a union of two disjoint intervals. So, let us write, where E such that E compliment is equal to a union of intervals I_i, i equal to 1 to 2, where I_i intersection I_j is equal to empty.

So, what we are saying, look at all those subsets of real line, which can be represented as disjoint union (Refer Slide Time: 42:00) of two intervals. So, claim F is algebra. We claim that, this is algebra of subsets of R.

Let us first observe, make some observations namely C is a sub set of F because, if I take an interval, then it has this property, namely it is a disjoint, its compliment is a disjoint union of two intervals. So, C is part of subsets of X. So, this as a consequence implies empty set in the whole space belong to F.

Let us look at the second property. Let us look at two elements (Refer Slide Time: 43:00). So, let us call E_1 and E_2 belong to R. oh sorry, E_1 and E_2 are elements of C.

You want to check whether E_1 intersection E_2 belong to C or not. So, what is E_1? Because E_1 belongs to C implies E_1 compliment can be written as a disjoint union. So, this square bracket normally indicates that I am writing something as disjoint union. So, $I_1$1, I equal to 1 to 2, or let us just simply write it as, this union of two intervals $I_1$1 union $I_2$1, where both of these are disjoint. Similarly, two compliments can be written as a disjoint union. Let us call $J_1$1 union $J_2$1 because, J_1 and J_2 are disjoint.

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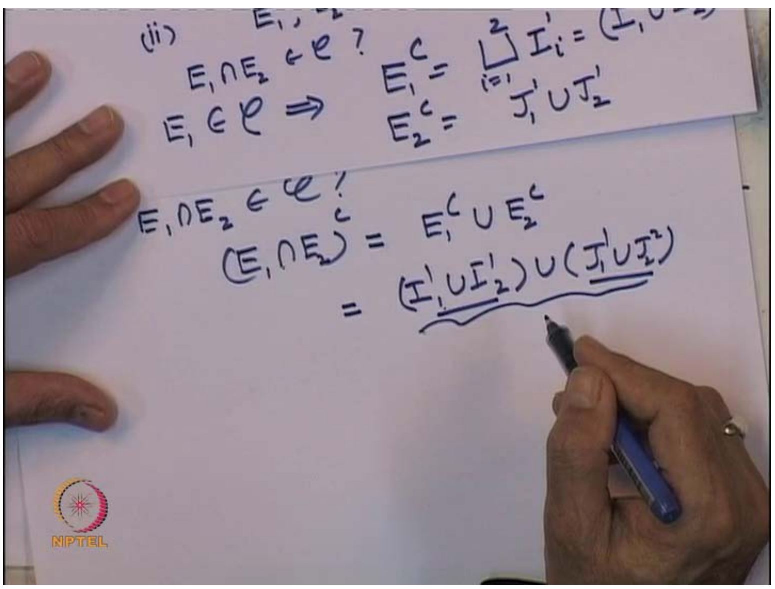
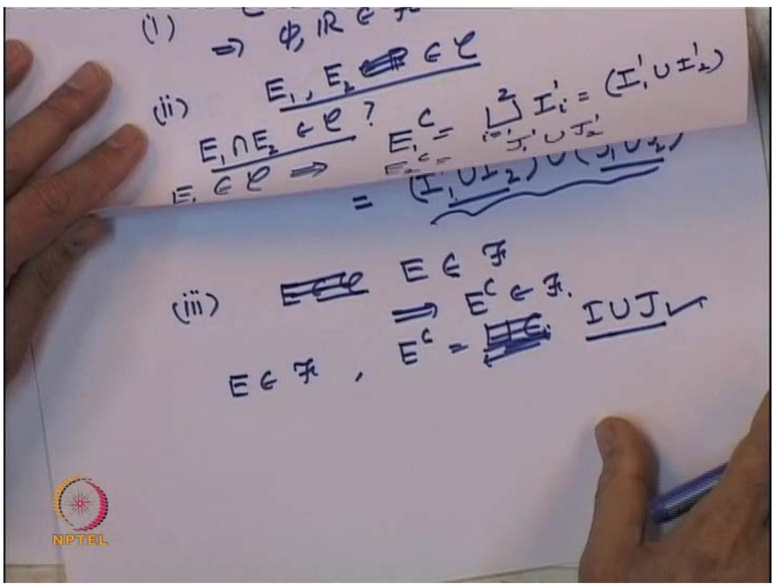


Now, we want to look at E_1 intersection into E_2. We want to look at E_1 intersection E_2 and we want to check whether this belongs to C or not. That means what? I should look at E_1 intersection E_2 compliment, and try to represent that as a union of two intervals. So, this is equal to E_1 compliment (Refer Slide Time: 45:00), there is intersection. So, by De Morgans law that becomes E_2 compliment.

Now, this is same as E_1 compliment is nothing, but $I_1$1 union $I_2$1 union E_2 compliment; that is, $J_1$1 union $J_2$2. From here, these two are disjoint, these two are disjoint, but all of these 4 may not be disjoint. (Refer Slide Time: 45:37) So, this set of ideas seems not leading us to claim that F is an algebra subsets of X.

So, let us modify our arguments.

(Refer Slide Time: 46:13)



Instead of checking this second one - whether the intersection belongs to it (Refer Slide Time: 46:00), let us look at the third property that we want there - that is true or not. So what is the third property? That property said, if a set E belongs to C.

We want to check algebra. Let us look at E belongs to F; does that imply E compliment belongs to F?

So, let us take a set E belonging to F. Now, what is E compliment? E compliment looks like a finite disjoint union of elements because, E belongs to it, it is a disjoint union of two elements. So, E compliment is equal to E compliment is I union J.

So it seems to say that, if I can show that the collection of finite disjoint union (Refer Slide Time: 47:12), this F is close under union, then I may be through. So, we modify all our arguments again and see, how do we proceed. This is how, one does not get all the time, a polished proof in mathematics; one has to modify the arguments. So, I modify my arguments to prove that F is algebra as follows:

So, the first step; let us keep in mind what we are trying to do.

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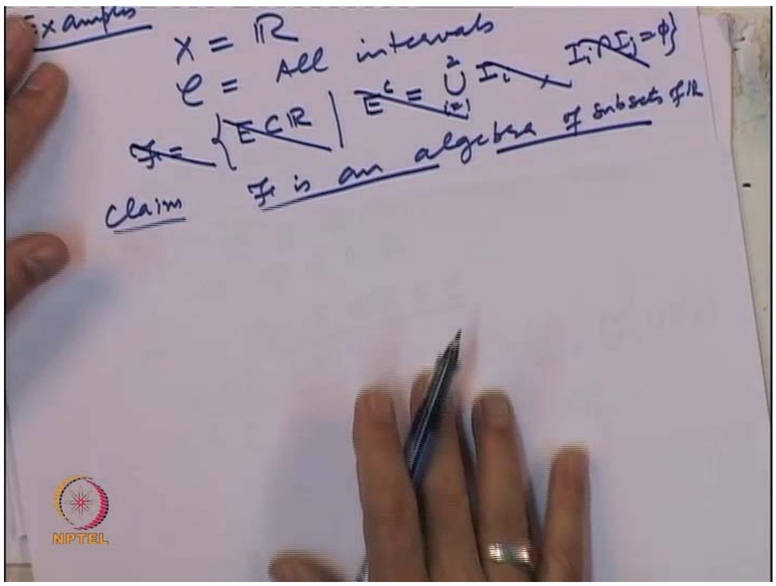

So, here is a collection F of subsets of X. Now, subsets of real line which are union of two of them, but that seems to complicate the issue. So, instead of this, let us modify this definition of F itself and let us look at the modified version of this example.

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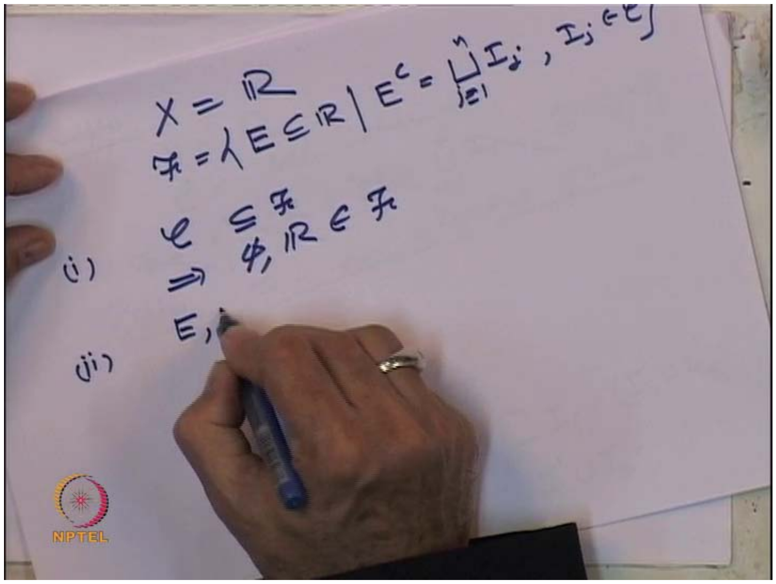

X is real line. Let us look at F - the collection of all those subsets E of R, such that, instead of seeing E compliment is union of two disjoint intervals, let me just write, E compliment is equal to a finite disjoint union of intervals I_j j equal to 1 to n, where Ij is R intervals and they are pair wise disjoint, and that is already indicated by writing this square bracket.

Now, let us observe, keeping in mind our previous arguments that C, the collection of all intervals is a part of F. So, that implies that empty set in the whole space are members of F.

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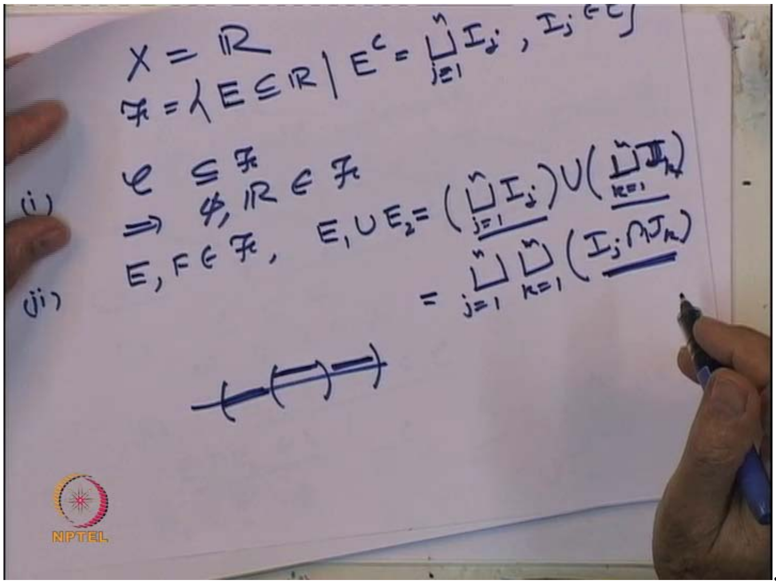

Second: Now, if E and F belong to this collection, then what is E?

E is a disjoint union of intervals. So, let us write E_1 union E_2 as, E_1 is a disjoint union, so let us write E_1 as union of I_j, j equal to 1 to n disjoint union of J_k, k equal to 1 to m.

Now, this collection of sets intervals are disjoint, this collection of intervals are disjoint, but all of them may not be disjoint; that does not matter much. (Refer Slide Time: 49:47). I can write this as union over j equal to one to n, union over k equal to one to m of I_j intersection J_k.

So, what I am doing is I am intersecting. So, the basic property is, if two intervals are not disjoint, then I can write them as union of disjoint paces. So, this collection of intervals, (Refer Slide Time: 50:22) the union of interval which may be over lapping, but I can intersect one another and write this as a disjoint union. So, these pairs of intervals will be disjoint; that implies, this will imply that if E and F belong to F, then that implies E union F also belongs to F. So, this collection of finite disjoint union of intervals is closed under unions.

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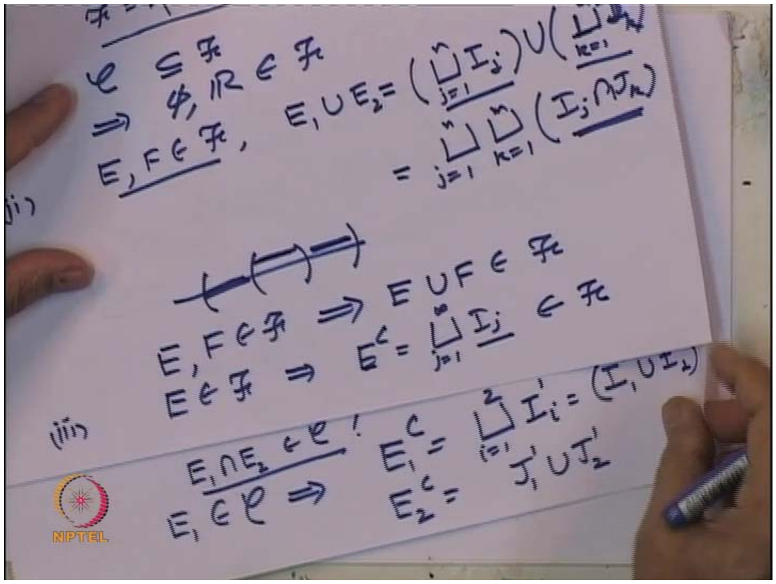

Let us write finally that, if E belongs to F, then that implies by definition, E compliment is disjoint union of intervals and each Ij is a interval, so, it belongs to F. Just now, we proved that it is closed under unions. So, this implies this also belongs to F.

So, the collection of sets of the real line which are finite disjoint union of intervals have the property; C is a sub set of it if; E and F belong to it. Then, it is closed under unions and also closed under compliments.

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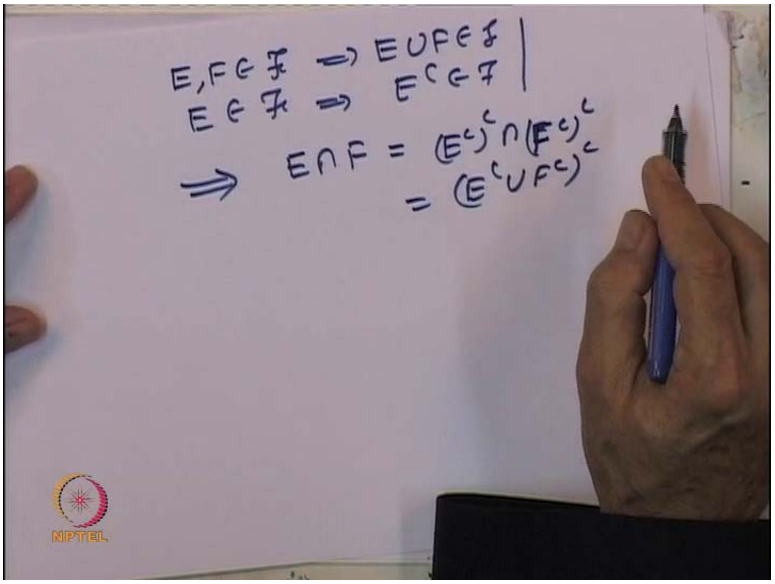
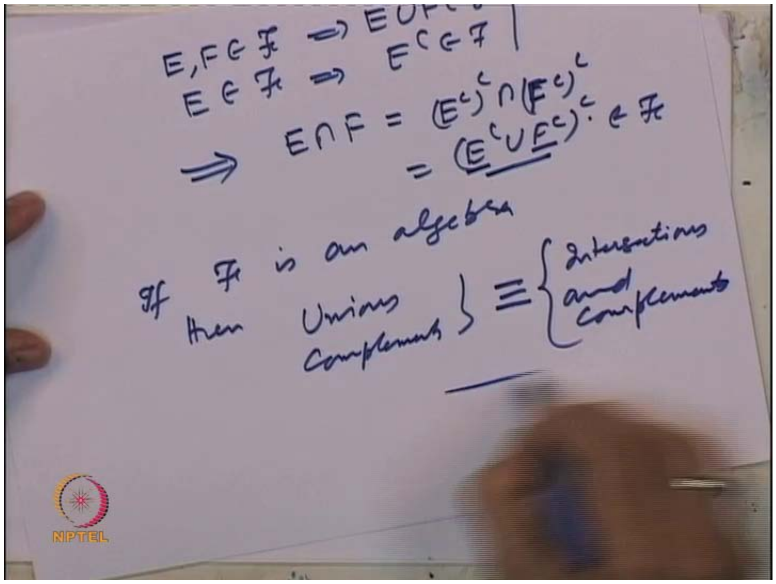


Now, it is a simple matter for us to check that these two properties - whenever a collection of sets is closed under unions E f belonging to F we showed, implies E union f belongs to it and also (Refer Time: 52:00) E belonging to F implies E compliment belongs to F, but these two properties imply that E intersection F also belongs because I can write this as, E as E compliment compliment union because, what is compli[ment] if you like this intersection of E F compliment compliment and that I can write as E compliment union F compliment compliment.

Simply it is just saying that, because this is De Morgan laws, this will give you E compliment compliment intersection.

Now E belongs to F. So, this belongs to F, this belongs to F, this union belongs to F and compliment belongs to F. So, this belongs to F (Refer Slide Time: 52:50).

(Refer Slide Time: 52:58)



So, basically in an algebra, if F is an algebra, then saying that it is closed under unions and compliments is equivalent to saying it is closed under intersections and compliments.

So, let us just summarize what we have done today.

We started by looking at our course, Measure and Integration and a set the underlying set of real numbers need to be extended to a larger class namely the set of extended real numbers. There we defined the notion of order, addition and multiplication and analyzed how sequences, series do and supremum of sets behave there. Then, we started at looking at collection of subsets of a set X with some properties.

First thing we looked at was - what is called semi-algebra of subsets of X, namely it is a collection of objects subsets of the set X with the property empty set and a whole space belong to it; it is closed under an intersection and the compliment of any set in this collection is representable as finite disjoint union of elements of that collection again. A typical example was that of all intervals in that real line.

Then we looked at slightly stronger concept namely the algebra. An algebra of subsets of a set X which we defined as the collection with the properties - empty set in the whole space belong to it it is closed under intersections and also closed under compliments. Then we made a remark - every algebra is a semi-algebra. The collection of intervals in the real line form a semi-algebra, but does not form an algebra.

We showed how to construct an algebra out of these intervals namely we looked at the collection of all finite disjoint unions of this intervals, and that collection we proved - it is an algebra of subsets of it.

So, we will continue analyzing such a collection of objects in our next lecture.

Thank you.