Elementary Numerical Analysis
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Lecture No. # 40
Q R Method

We are considering q R decomposition of n by n invertible matrix using reflectors. So, we will discuss this q R decomposition and then after that I want to consider approximation of a continuous function by polynomials in the 2 norm. So, that is known as least square approximation. We have already considered best approximation in the infinity norm. So, now this will be best approximation in the 2 norm. So, first we look at the q R decomposition of an invertible matrix.

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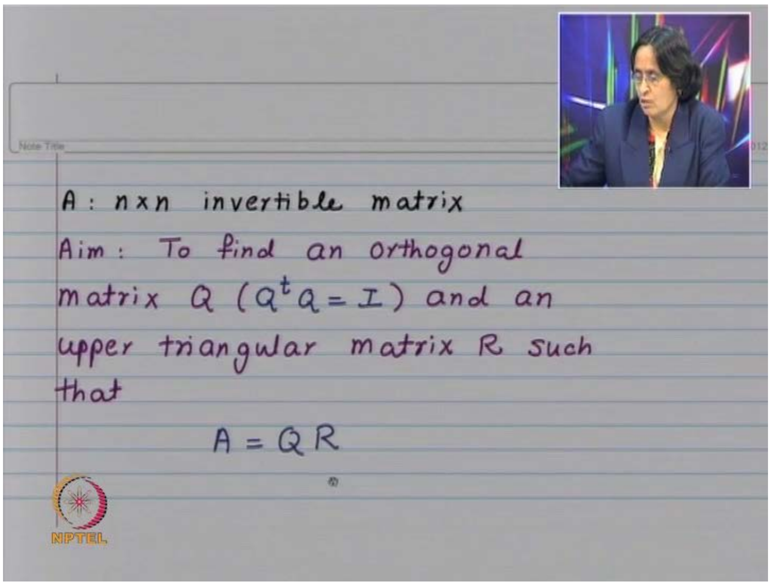

So, we have a to be an invertible matrix of size n and our aim is to find an orthogonal matrix q; that means, matrix which satisfies q transpose q is equal to identity and an upper triangular matrix R, such that a is equal to q into r. So, this we are going to achieve using reflectors.

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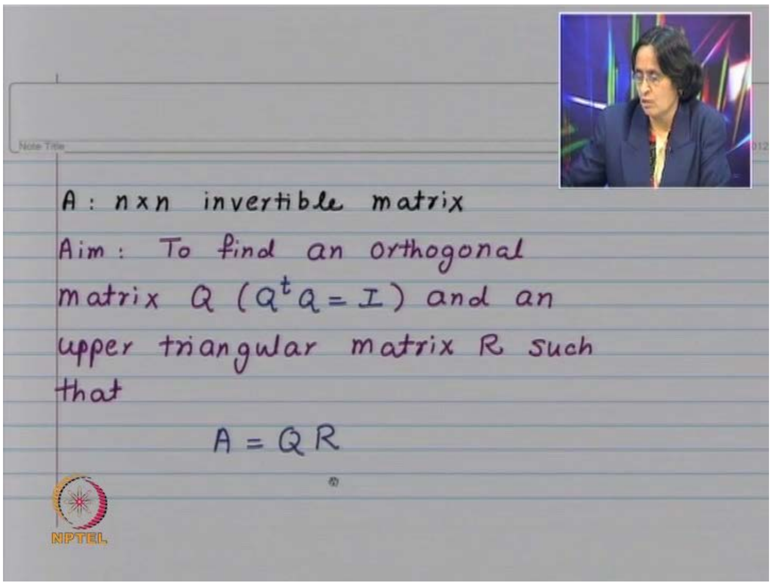

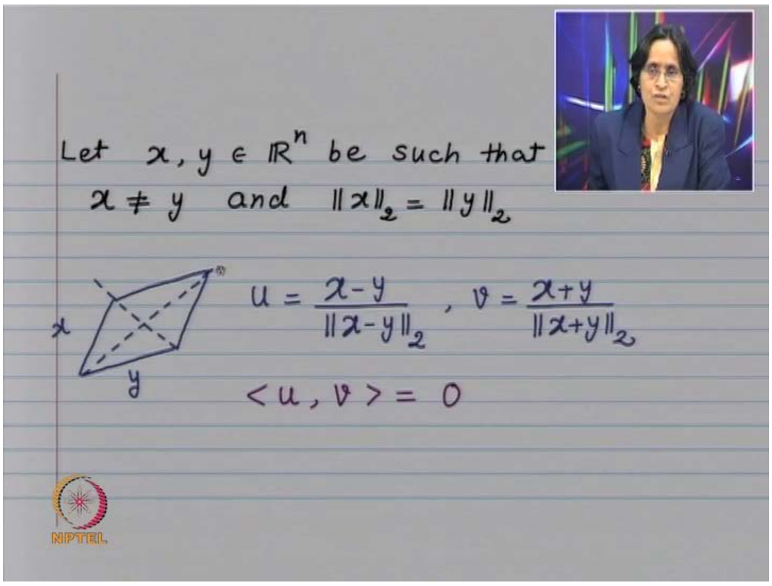

So, let me recall the reflectors, if you have 2 vectors in R n, such that x is not equal to y and the euclidian norm of x and y, they are the same. Then what we do is, we look at parallelogram with sides as x and y. The diagonals of this parallelogram, they will be given by vector x plus y and vector x minus y. We consider a unit vector in the direction of x minus y, which is given by u is equal to x minus y divided by its norm and v is a unit vector along the other diagonal. So, v is x plus y divided by its norm. These 2 unit vectors u and v, they are going to be perpendicular. So, inner product of u with v is equal to 0. What we want is a reflector, which will take vector x to vector y. So, we want orthogonal matrix q such that q x is equal to y. So, we look at the reflection in the line along the direction of v. The reflector q will be given by identity matrix minus 2 u u transpose.

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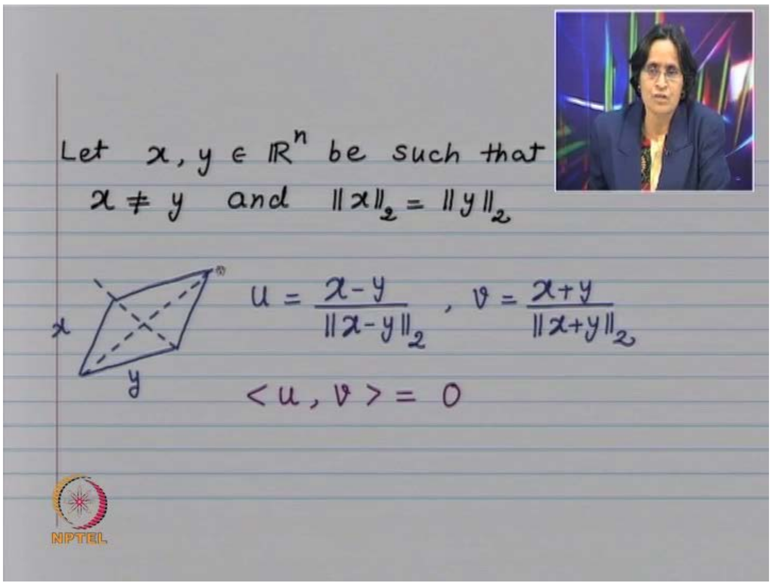

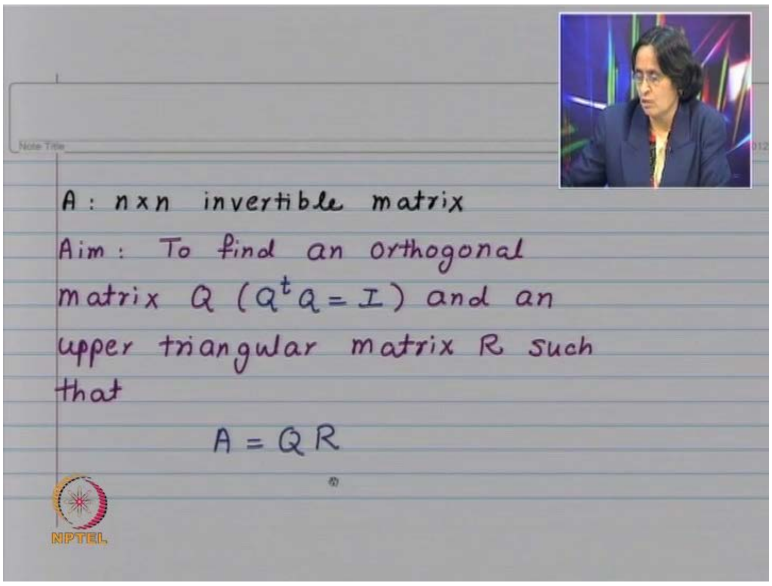

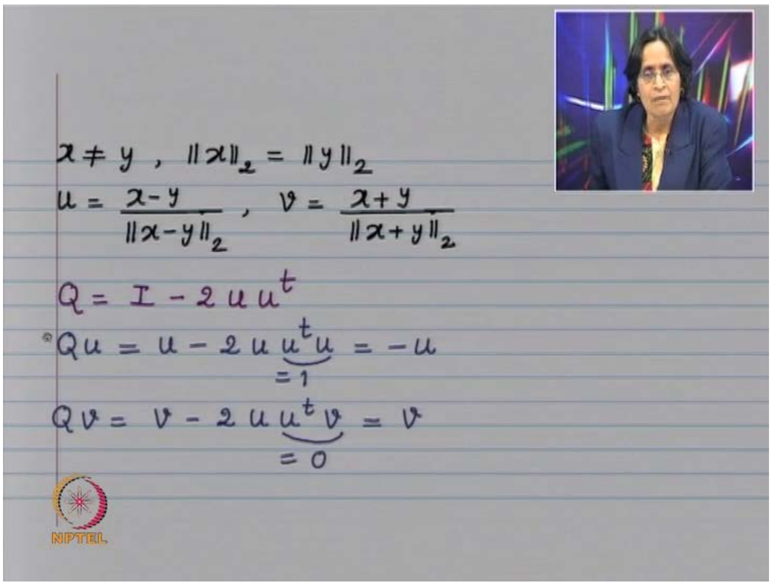

When you look at q into u, that is going to be u minus 2 u u transpose u. U being a unit vector u transpose u will be equal to 1. So, we have u minus 2 u, So, that is equal to minus u. On the other hand when I look at q of v, it will be v minus 2 u u transpose v, since u and v are perpendicular to each other, q of v will be equal to v. So, thus if you define u and v in this fashion and look at q to be equal to identity minus 2 u u transpose, it has got property, that q u is equal to minus u and q of v is equal to v.

Now, look at vector x. This we write as x plus y by 2 plus x minus y by 2. Q of u is equal to minus u. U is x minus y divided by norm of x minus y. And hence q of x minus y by 2 will be minus of x minus y by 2 because x minus y by 2 is perpendicular to vector u. Q of v is equal to v and hence q of x plus y by 2 will be equal to x plus y by 2 and thus we get q of x is equal to y.

We have achieved the fact that if x and y are vectors which are not equal with the same euclidian norm, then we can find q, such that q x is equal to y. Now, let us look at the properties of q. First of all q transpose is going to be equal to q. Q x is equal to y and consider q square. So, this will be identity minus 2 u u transpose multiplied by itself. So, when you multiply, you are going to have identity minus 2 u u transpose minus 2 u u transpose plus four u u transpose u u transpose. This u transpose u is going to be equal to 1 because u is a unit vector.

So, thus we have here four u u transpose, which will get canceled with this minus four u u transpose and you have q square is equal to identity. Thus q transpose is equal to q and q square is equal to identity. So, this matrix q is going to be orthogonal matrix. So, our q is the desired matrix which takes vector x to vector y and it is a orthogonal matrix.

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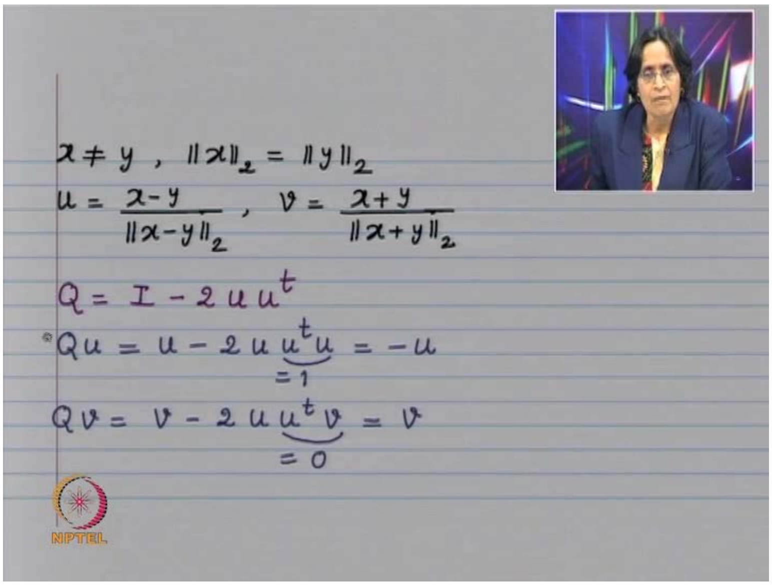

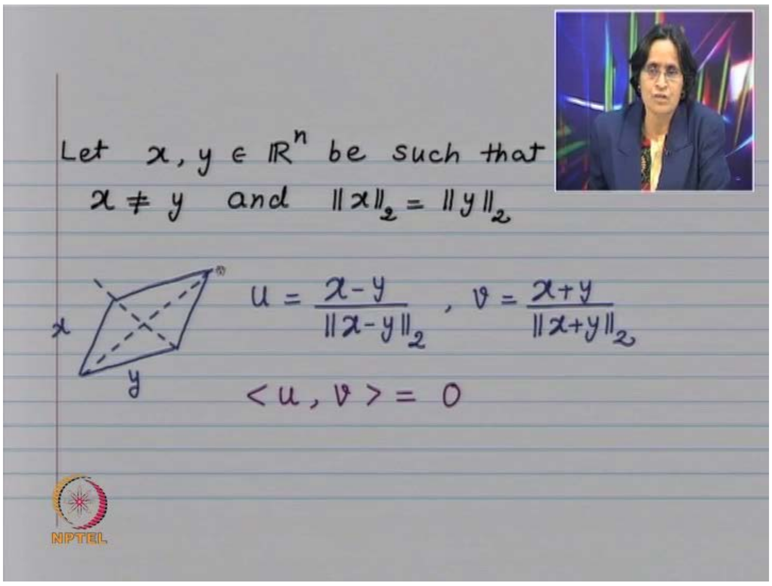

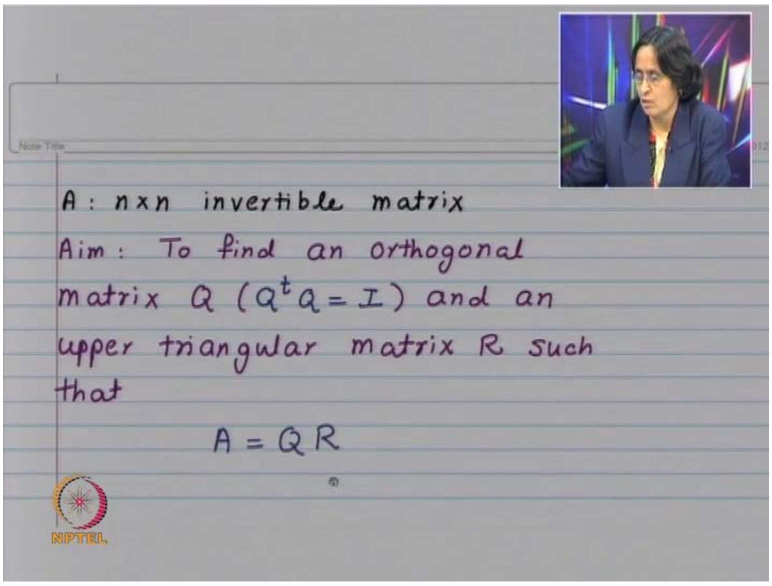

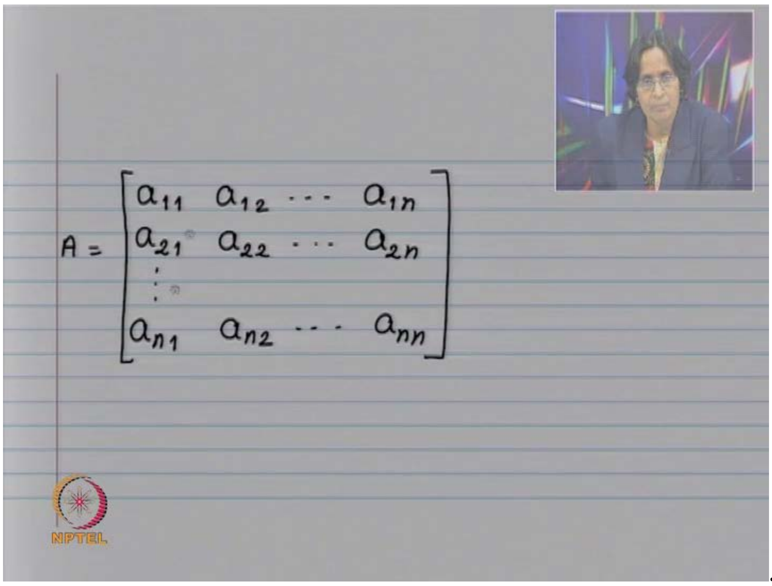

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Now, look at our n by n matrix. So, it is a invertible matrix. So, what we are going to do is, we are first going to look at the first column, a 1 1 a 3 1 a n 1. This column will be a non-0 column, because a is invertible. This column, what we want to do is, we want to reduce a to an upper triangular form. That means, we want to introduce zeroes in the first column below the diagonal. That is what we are going to do using reflectors. So, now we have got the first column. It is a non-0 vector. We want to convert it into a vector, which has only first entry to be non-0 and all other entries to be 0. Since the new vector should have the same euclidian norm as the original vector, the first entry, it should be norm of the first column or it should be minus of norm of the first column.

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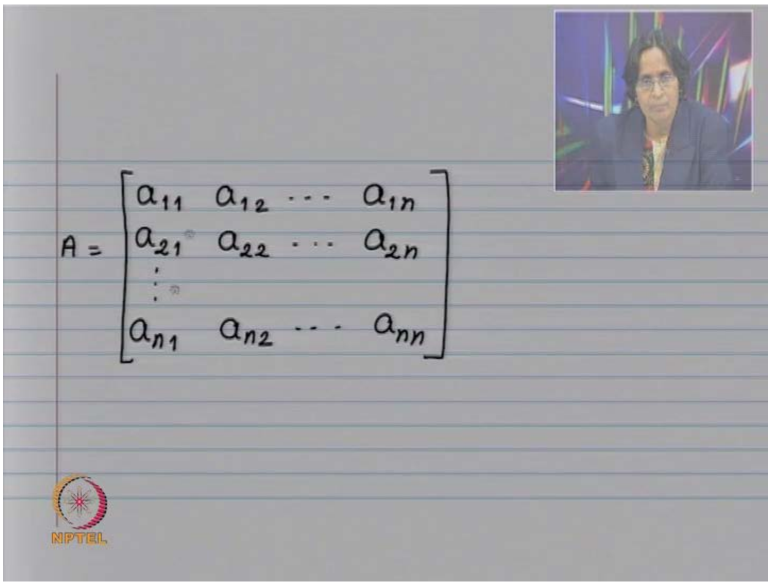
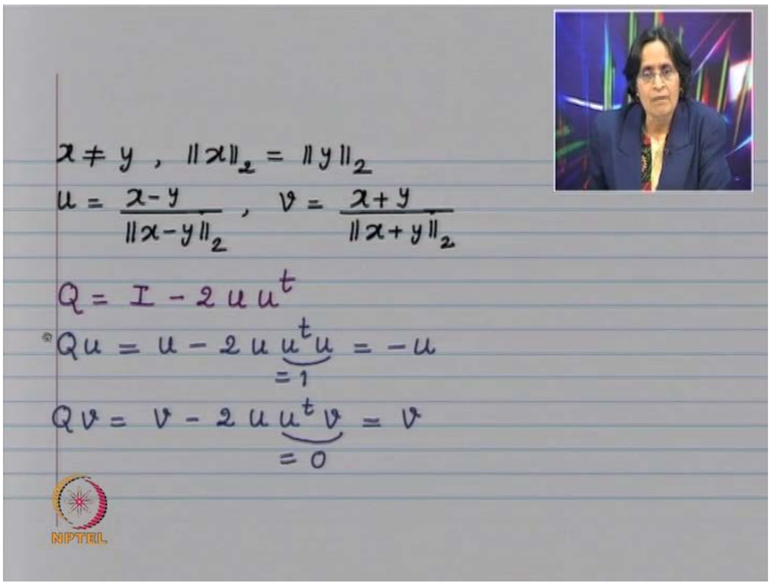
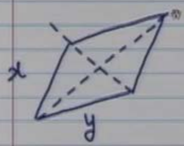




Let $x, y \in \mathbb{R}^n$ be such that $x \neq y$ and $\|x\|_2 = \|y\|_2$



$$u = \frac{x-y}{\|x-y\|_2}, \quad v = \frac{x+y}{\|x+y\|_2}$$

$$\langle u, v \rangle = 0$$

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A : $n \times n$ invertible matrix

Aim : To find an orthogonal matrix Q $(Q^t Q = I)$ and an upper triangular matrix R such that

$$A = QR$$

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$$\begin{array}{ccc} x & & y \\ \| & & \| \\ \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} & \longrightarrow & \begin{bmatrix} \sigma_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array}$$

$$\sigma_1 = \left(\sum_{i=1}^{n} a_{i1}^2 \right)^{1/2}$$

$$u = \frac{x-y}{\|x-y\|_2}$$

$$Q_1 = I - 2uu^t$$

$$\boxed{Q_1 x = y}$$

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So, here take vector x to be equal to a 1 1 a 2 1 a n 1 the first column, vector y to be sigma 1 0 0 0 where sigma 1 is nothing, but euclidian norm of x; that means, a 1 1 square plus a 2 1 square plus a n 1 square whole thing raise to half. Then euclidian norm of y is equal to norm of x, even if I write here minus sigma 1, then also this property will be satisfy. So, thus we have got 2 vectors, which I have got the same norm, then we know how to construct a orthogonal matrix q, such that q x is equal to y. So, sigma 1 is the norm of this vector, we look at u to be equal to x minus y divided by its norm and then q 1 is equal to identity minus 2 u u transpose, then q 1 of x is going to be equal to y. So, thus we can reduce the first column to a column of the form sigma 1 0 0 0.

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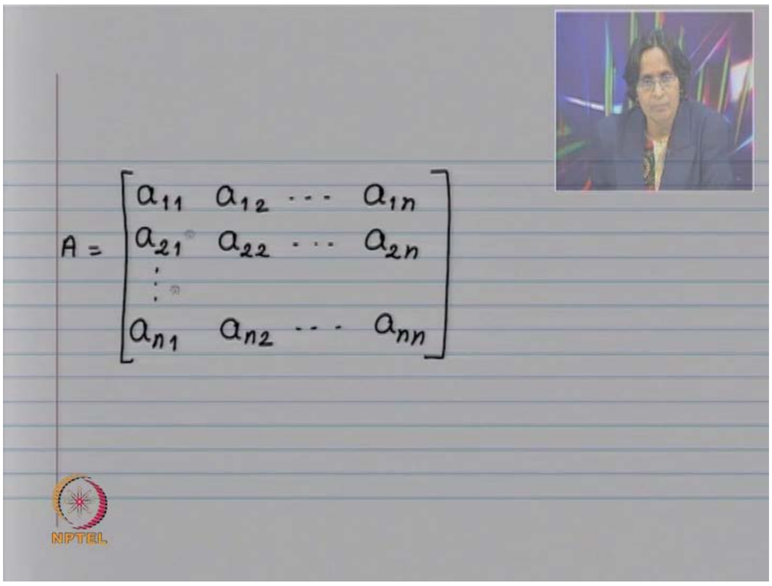
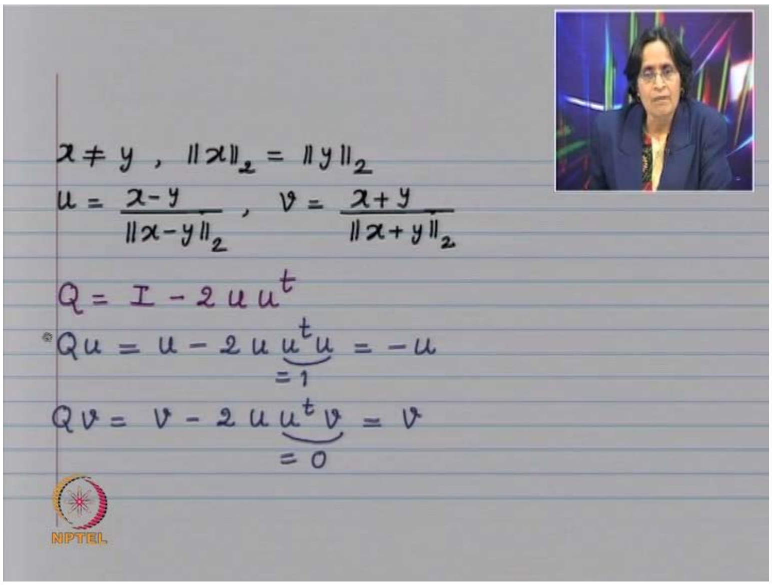
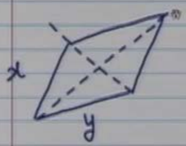




Let $x, y \in \mathbb{R}^n$ be such that $x \neq y$ and $\|x\|_2 = \|y\|_2$



$$u = \frac{x-y}{\|x-y\|_2}, \quad v = \frac{x+y}{\|x+y\|_2}$$

$$\langle u, v \rangle = 0$$

A : $n \times n$ invertible matrix

Aim : To find an orthogonal matrix Q $(Q^t Q = I)$ and an upper triangular matrix R such that

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$$\begin{array}{ccc} x & & y \\ \| & & \| \\ \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} & \longrightarrow & \begin{bmatrix} \sigma_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array}$$

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$$u = \frac{x-y}{\|x-y\|_2}$$

$$Q_1 = I - 2uu^t$$

$$\boxed{Q_1 x = y}$$

Now, let me look at the norm of x minus y. Norm of x minus y is given by the square of all the entries, sum it up and then its square root. So, it is going to be summation a I 1 square minus 2 times sigma 1 a 1 1 plus sigma 1 square. Sigma 1 square is the norm of x, so, it is summation a I 1 square and hence norm of x minus y will be equal to 2 sigma 1 into sigma 1 minus a 1 1.

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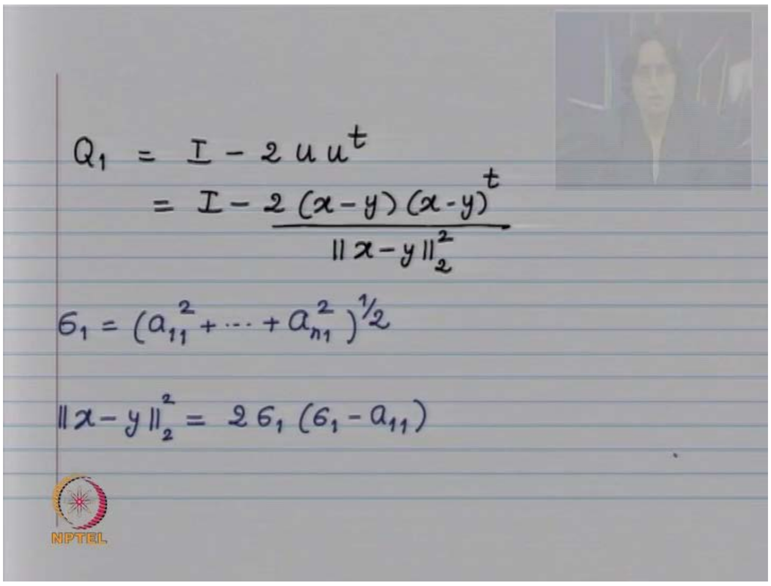
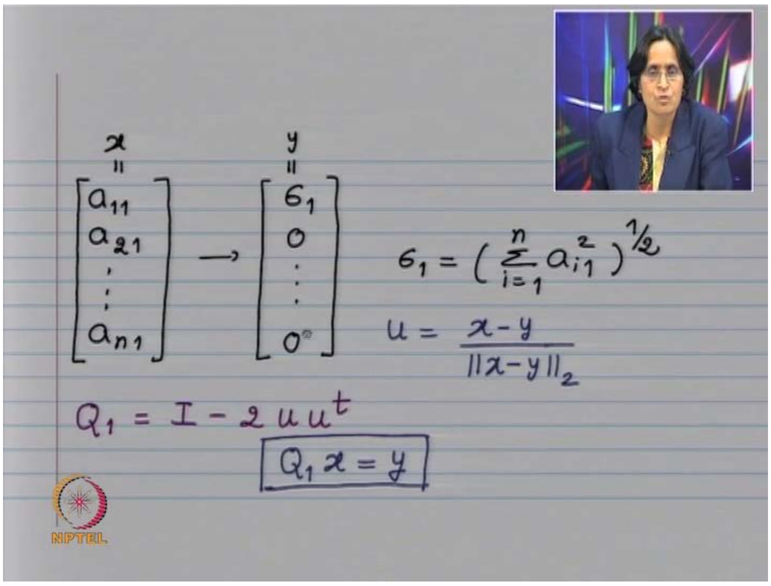




$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$x \neq y$, $\|x\|_2 = \|y\|_2$

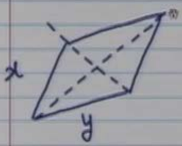
$$u = \frac{x-y}{\|x-y\|_2}, \quad v = \frac{x+y}{\|x+y\|_2}$$

$$Q = I - 2uu^t$$

$$Qu = u - 2u\underbrace{u^t u}_{=1} = -u$$

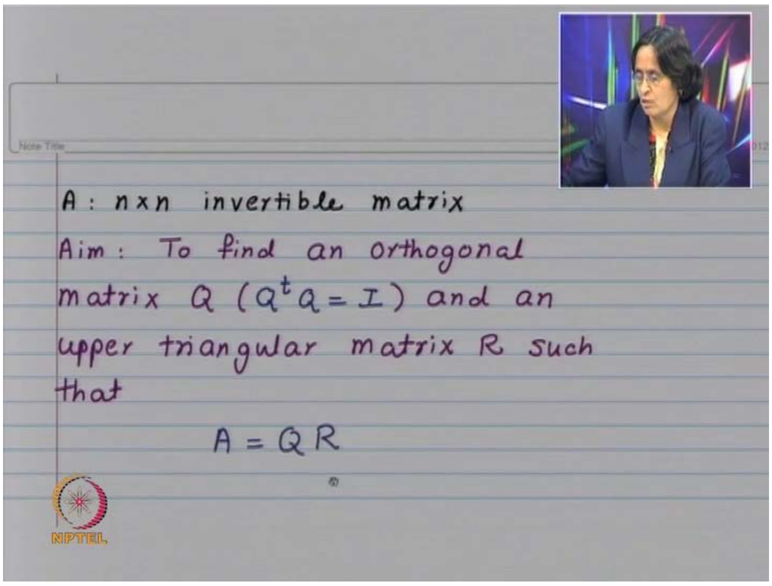
$$Qv = v - 2u\underbrace{u^t v}_{=0} = v$$

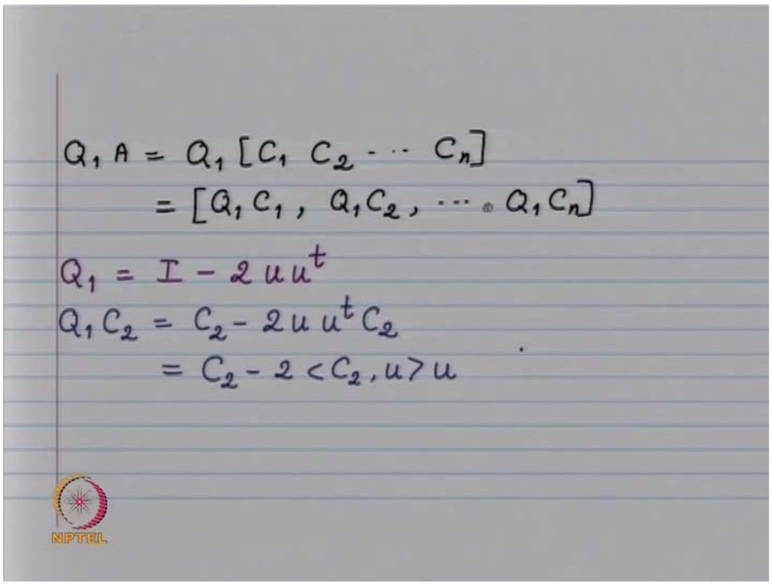
Let $x, y \in \mathbb{R}^n$ be such that $x \neq y$ and $\|x\|_2 = \|y\|_2$



$$u = \frac{x-y}{\|x-y\|_2}, \quad v = \frac{x+y}{\|x+y\|_2}$$

$$\langle u, v \rangle = 0$$





So, we have calculated, we need to calculate sigma 1, we need to calculate. Once you calculate sigma 1, norm of x minus y is given by this quantity. Then q 1 is identity minus 2 u u transpose. This we have calculated that is 2 sigma 1 sigma 1 minus a 1 1 and sigma 1 is this. So, now we are not going to calculate q 1 explicitly as a matrix. What we want to do is apply q 1 to a. So, we will look at the columns of a, we will call them as C 1 C 2 C n and we need to look at the action of q 1 on each column. So, we have q 1 a is equal to q 1 C 1 C 3 C n the columns of a, this will be q 1 C 1 q 1 C 2 q 1 C n. Q 1 C 1 is going to be vector sigma 1 and then 0 0 0.

Look at q 1 C 2, q 1 is identity minus 2 u u transpose and hence q 1 C 2 will be C 2 minus 2 u u transpose C 2. So, this is nothing, but inner product of C 2 and u. So, the second column will be given by original column C 2 minus 2 times, this inner product multiplied by vector u, and similarly for the other columns q 1 C 3 q 1 C 4 and q 1 C n. Then look at this q 1 into a. The first column is reduced to sigma 1 and then 0 0 0. The second third nth column, they will all get modified. So, I am denoting the modified entries of the matrix as a 1 2 super script 1 a 2 2 superscript 1 and so on. So, this is about the first column.

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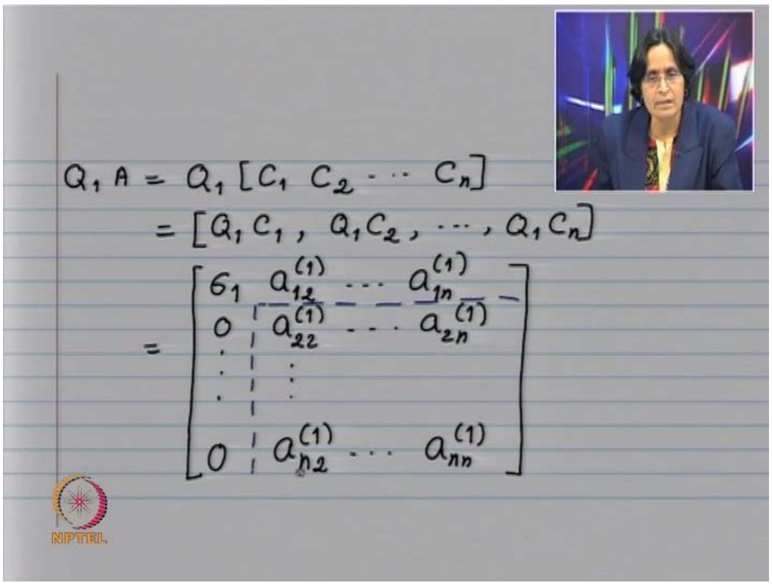

Now, what we want is, we will look at the second column, and in the second column, we want to introduce zeros below the diagonal. Now, in the process, we do not want our first column to be disturbed, because we have already achieved the desired form. So, what we will do is, we will look at this n minus 1 by n minus 1 sub matrix. And then we will look at the first column of that n minus 1 by n minus 1 sub matrix and find a orthogonal matrix of size n minus 1, which will reduce the first column of this smaller matrix 2 a vector of the form here it will be non-0 and rest of the things they are going to be 0.

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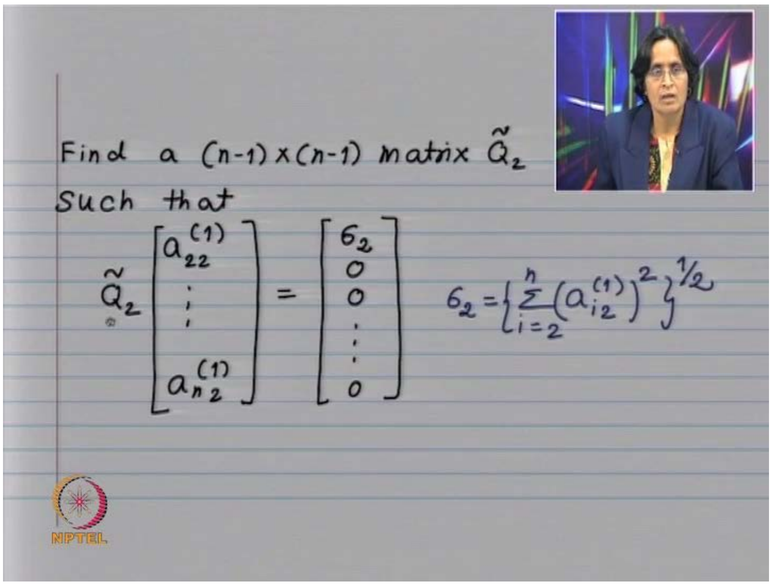

So, find n minus 1 by n minus 1 matrix q 2 tilde, such that q 2 tilde of this vector of size n minus 1. It becomes sigma 2 and then all the entries to be 0, where sigma 2 will be norm of this vector. So, this is q 2 tilde next what we do is, we add the entries to q 2 tilde and obtain a n by n matrix q 2. So, q 2 is going to be 1 here, this will be a row vector of length n minus 1, this will be column vector of length n minus 1 and then this will be q 2 tilde.

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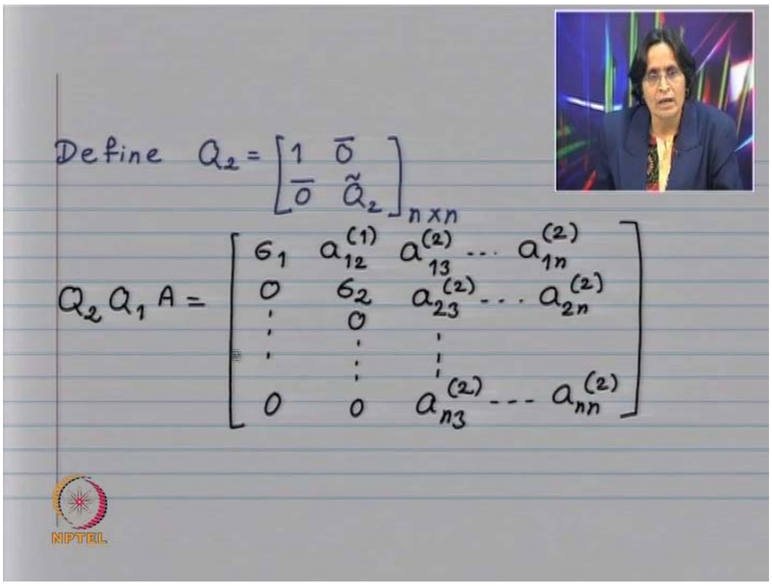

Then, when you consider q 2 q 1 a, because of the nature of q 2, the first row and first column of our original matrix will not be changed and you will get here sigma 1 remaining entries 0, here sigma 2 and remaining entries 0 and then so, on. Then we will go to a third column. So, we will look at a matrix of size n minus 2 and using the same idea we continue. So, like that we will find matrices q 1 q 2 q n minus 1, such that when you pre multiply a by this matrix, what you get is upper triangular matrix R. Each of q I is going to be a symmetric matrix and its square will be identity. So, each q I will be orthogonal matrix.

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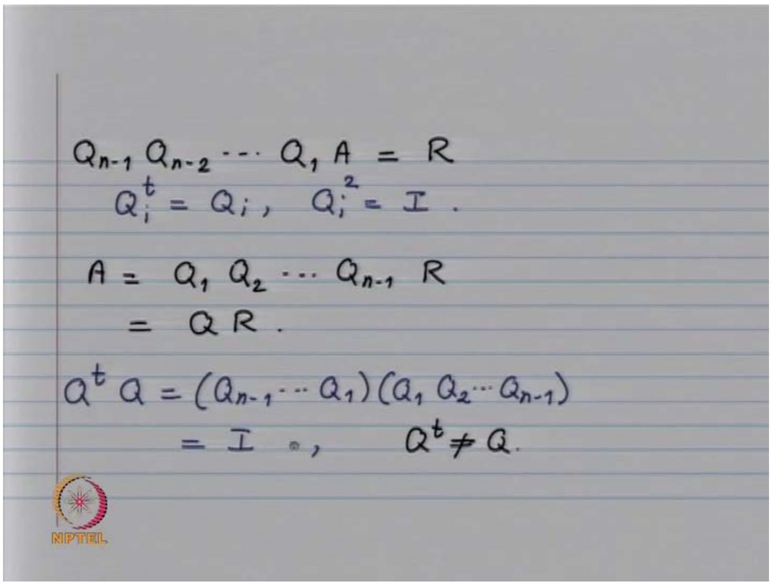

So, q I square is identity; that means, inverse of q I is equal to q I and hence from here, I will get a to be equal to q 1 q 2 q n minus 1 into R. Look at this product, that is going to be our matrix q. Since q 1 q 2 q n minus 1 they are orthogonal, their product also will be orthogonal, here we had q I transpose is equal to q I, but when you take their product, it will not be a symmetric matrix, but what we want is, we want q to be orthogonal. So, q transpose q will be, when you take the transpose, you change the order. So, it will be q n minus 1 q n minus 2 q 1 and then q 1 q 2 q n, q 1 square will be identity then q 2 square will be identity and then, so on.

So, that is how we get q R decomposition of a matrix a, where a is invertible and q is going to be an orthogonal matrix, R is going to be an upper triangular matrix. Such a decomposition is not unique, but then for the uniqueness, we can impose some conditions on the diagonal entries of the matrix R. For example, we if we say that R should be such that, all diagonal entries they are bigger than 0, then the q R decomposition with this additional condition is going to be unique. Now, we are going to look at an example of a 2 by 2 matrix and we want to find its q R decomposition.

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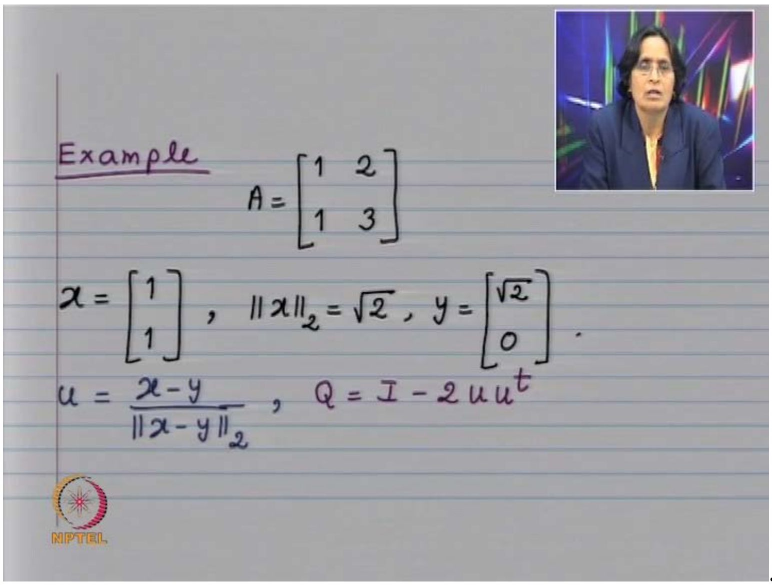

So, a is matrix with first column to be 1 1 and second column to be 2 3. The determinant of this matrix is going to be equal to 1 and hence it is a invertible matrix. The first column I denote by x. So, x is equal to 1 1. Its norm is going to be root 2. Define y to be equal to vector root 2 0. So, thus x and y, they have got the same norm. Next look at u is equal to x minus y divided by norm of x minus y and q to be equal to identity minus 2 u u transpose. We have seen that such a matrix q will be such that q x is equal to y. And I said that we are not going to calculate q explicitly, what we need is its action on the columns, but since it is a illustrative example of size 2, let us calculate the what q looks like.

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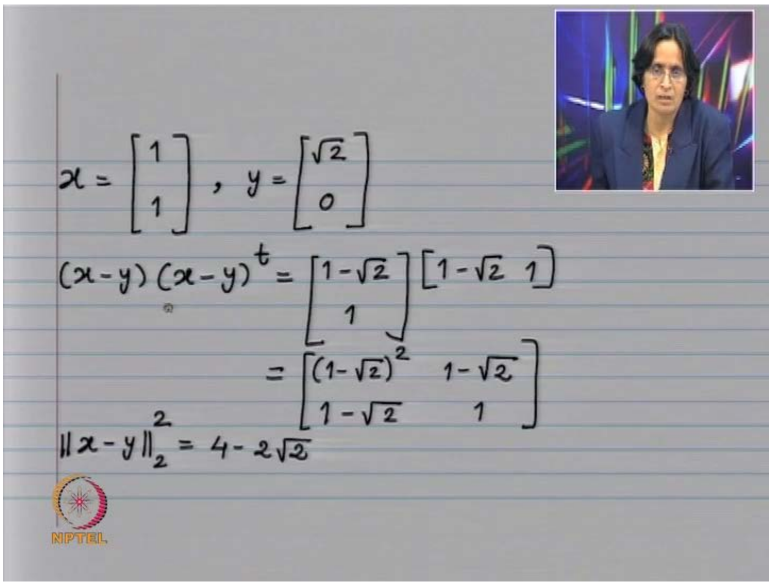

So, here is our vector 1 1 y is vector root 2 0, when you consider x minus y into x minus y transpose. So, this will be vector 1 minus root 2 1 and then transpose will be a row vector 1 minus root 2 1. So, take the multiplication. So, this will be 1 minus root 2 square 1 minus root 2 1 minus root 2 and then 1. So, thus and also norm of x minus y square, it is going to be equal to 4 minus 2 root 2. So, this is this matrix and norm is going to be 4 minus 2 root 2, because we are going to divide by this norm here.

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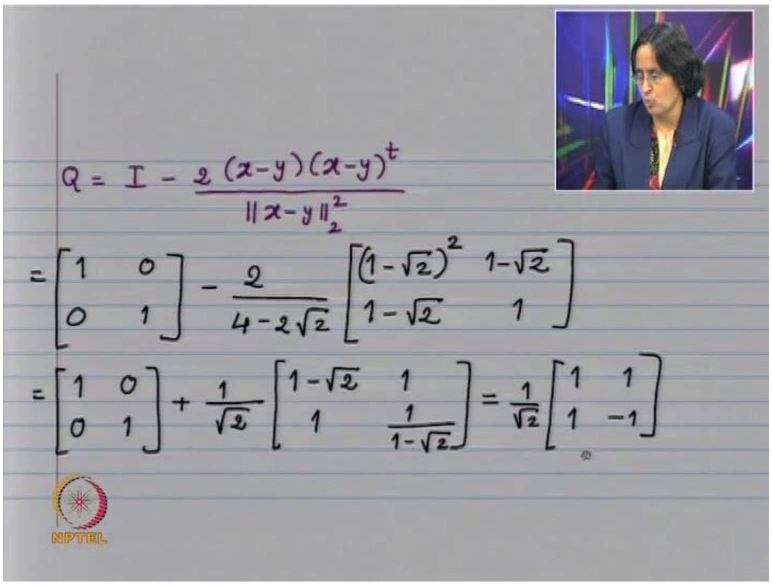

Now, q is equal to identity matrix minus 2 x minus y x minus y transpose upon norm of x minus y square, this is the identity matrix 2 4 minus 2 root 2 was norm of norm square of this x minus y and this matrix, we have seen that it is 1 minus root 2 square 1 minus root 2 1 minus root 2 1. So, now one can simplify and then see that q is equal to 1 by root 2 1 1 1 minus 1. Notice the columns of q, they have norm to be equal to 1 and they are perpendicular to each other. So, this is our q, and now let us look at r.

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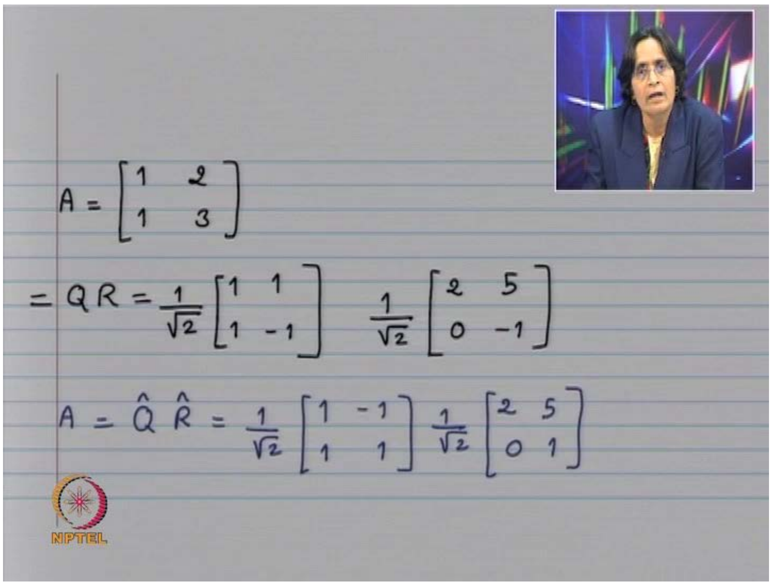

So, a is 1 2 1 3 the columns of q are nothing, but the vectors, column vectors of a ortho normalized. So, we have got this and then one can check that q into a is going to give us this matrix r. So, it becomes an upper triangular matrix 1 by root 2 2 0 5 minus 1. Now, since q transpose is equal to q inverse, a is equal to q into R. So, this is our original matrix, this we have written as a product of orthogonal matrix q and then upper triangular matrix r.

Now, we were saying that the diagonal entries of R should be positive. Here we have got this entry to be negative, but then it can be adjusted with the entries of q. So, q into R is here. So, if I consider q cap and R cap, where q cap is 1 by root 2 1 1 minus 1 one and R cap to be this, then q cap is also orthogonal matrix and R cap is upper triangular matrix. So, this is q R decomposition using the reflectors and now we are going to look at the q R method, we had already defined it. So, I just want to state the method again and its convergence.

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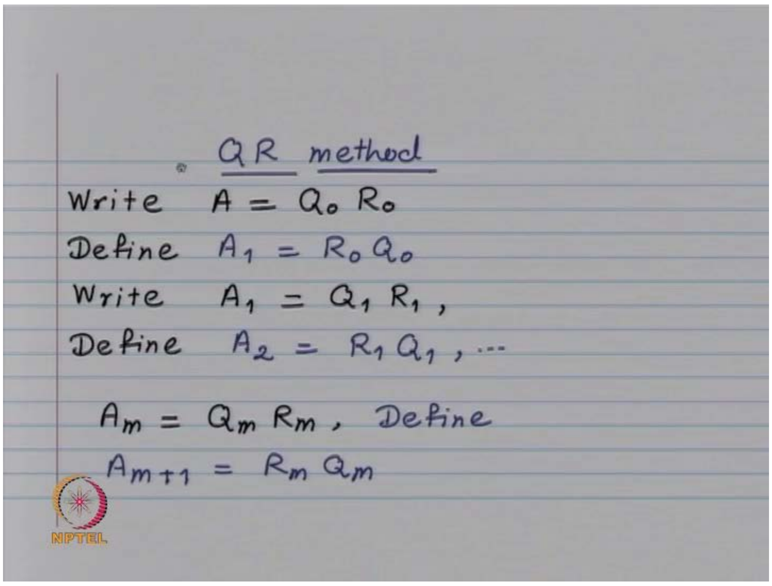

So, the q R method consists of writing a as q 0 into R 0. So, a is the our starting matrix, we find its q R decomposition and then we define a 1 to be q 0 and R 0 multiplied in the reverse order. Then find the q R decomposition of this new matrix. Once you find this q and R, you multiply them in reverse order, matrix multiplication is not commutative, so, you are going to get a different matrix in general. Like that when you reach a m, then find its q R decomposition and then a m plus 1 is equal to R m into q m. So, you see in the q R method, one needs to calculate this q R decomposition at each stage and that is why we wanted some efficient way of doing the q R decomposition.

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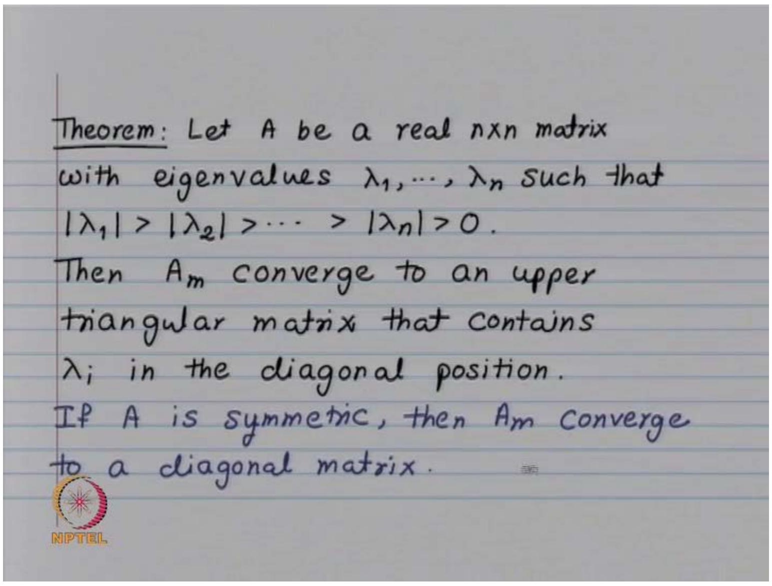

So, here is the theorem, here is a sufficient condition for convergence of q R method. So, let a be a real n by n matrix with Eigen values lambda 1 lambda 2 lambda n, such that mod lambda 1 bigger than mod lambda 2 bigger than mod lambda n bigger than 0. The matrix is real. So, its Eigen values they are either real or they are going to be complex conjugating pairs. But because of this condition that no 2 Eigen values they have the same modulus, all Eigen values they are going to be real. Then a m converges to an upper triangular matrix that contains lambda I in the diagonal position.

If a in addition is symmetric, then a m converge the sequence a m converges to a diagonal matrix. Symmetric real matrices, they are going to be diagonalizable. And in general, if you have a matrix a, then we know that there exist a similarity transformation which will convert a to a upper triangular matrix. So, this is what we try to achieve iteratively in the q R method. If this condition is not satisfied, then the iterates in the q R decomposition, they may not converge.

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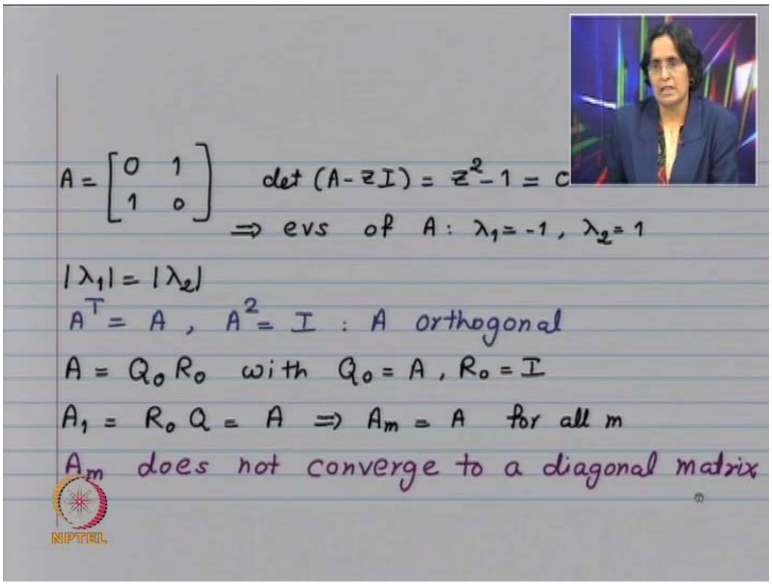

So, here is an example, look at matrix a 2 by 2 matrix with entries as 0 1 1 0, then the Eigen values they are given by minus 1 and 1. So, that means, 2 Eigen values they will have the same modulus. A is a symmetric matrix and a square is equal to identity. So, a itself is a orthogonal matrix and hence its q R decomposition can be q 0 is equal to a and R 0 is equal to identity, but in that case, I will get a 1 to be equal to R 0 into q 0 which is same as a. So, that means, all the iterates they are going to be equal to the original matrix a, and in this case a m s they do not converge to a diagonal matrix.

So, this is about the q R method, we could not prove the convergence of q R method, but that is beyond the scope of the first course on a elementary numerical analysis. Now, what I want to do is, I want to look at least square approximation of a continuous function by using polynomial. Now, before that let me just mention that the q R decomposition, it can be used to find solution of system of linear equation.

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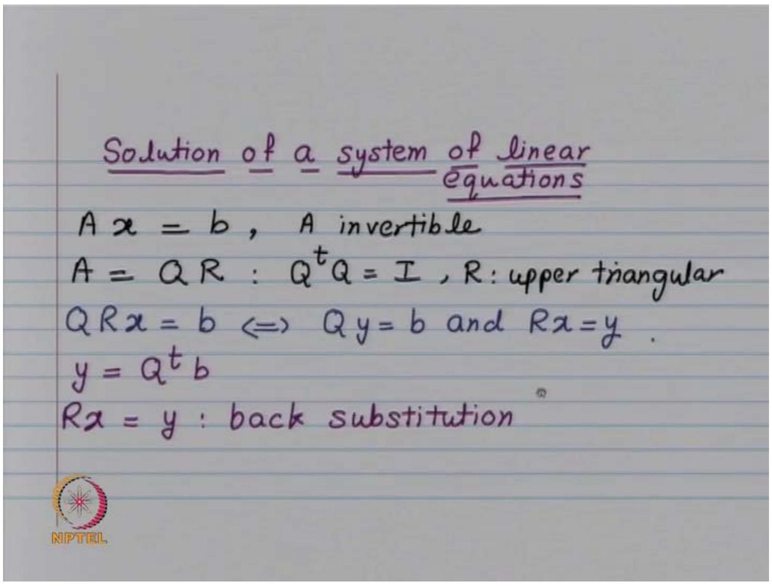

So, we have got a system a x is equal to b, where as a is invertible matrix, we have written a is equal to q into R, where q is orthogonal and R is upper triangular. So, the original system becomes q R x is equal to b, this is equivalent to 2 systems q y is equal to b and R x is equal to y. So, first solve this that is nothing, but y is equal to q transpose b and then solve R x is equal to y by back substitution.

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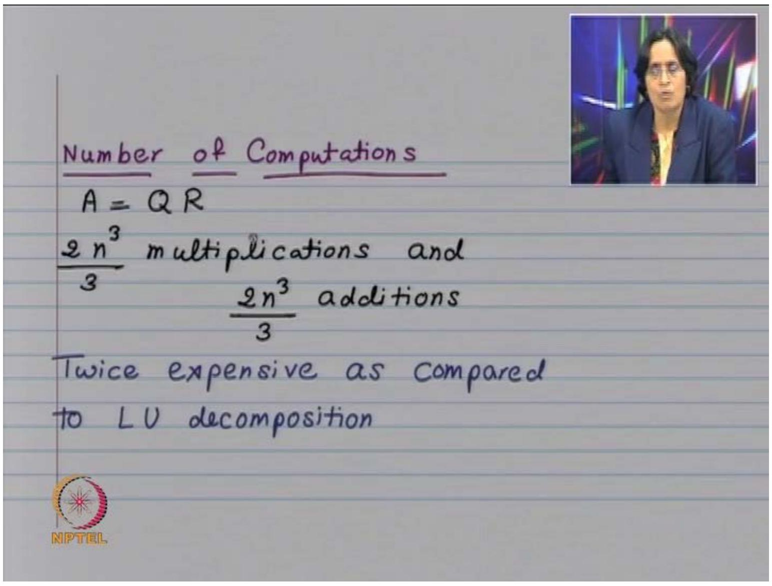

However the number of computations for q R decomposition they are going to be of the order of 2 n cube by 2 multiplications and 2 n cube by 3 additions, which is twice as expensive as the as compared to the L U decomposition. So, that is why one does not use q R decomposition for solution of system of linear equations.

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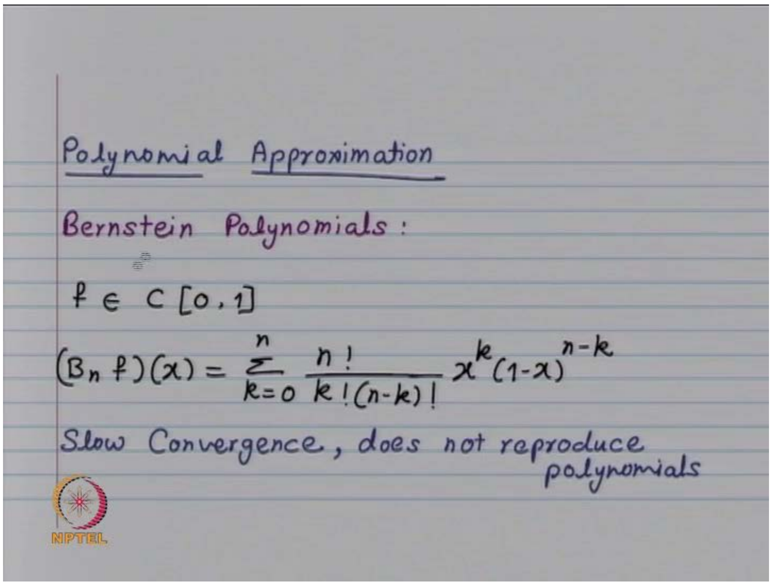

Now, let us look at the polynomial approximation. We had looked at Bernstein polynomials and then the disadvantage of Bernstein polynomials was slow convergence and it does not reproduce the polynomial. So, that is why what we did was, we looked at the best approximation.

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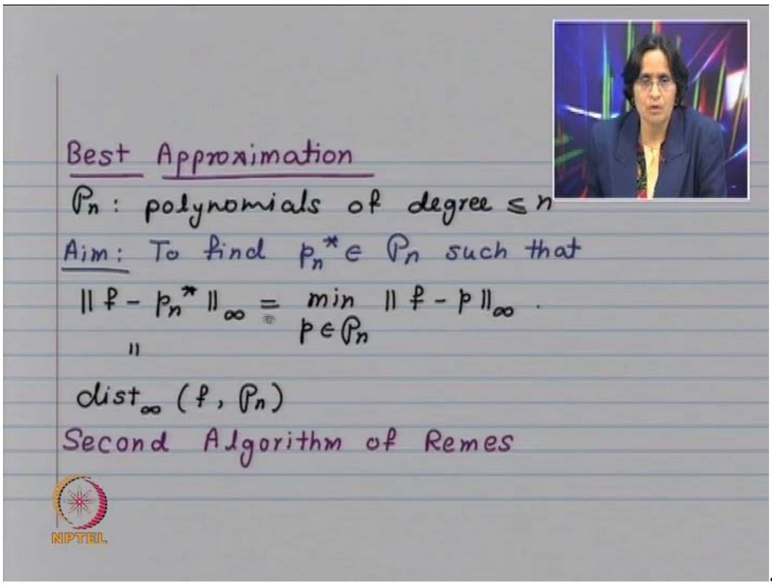

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Now, in the best approximation, our aim is to find p n star, such that the error in the maximum norm or the infinity norm that is minimize. So, that means, we are trying to do the best, as for as the error is concerned, but then there exists a unique best approximation p n star, but in order to find is, we need a iterative method. So, that is why the best approximation, it was not advisable or we did not consider the best approximation, now what I want to do is, I want to consider the best approximation, but instead of infinity norm in the 2 norm.

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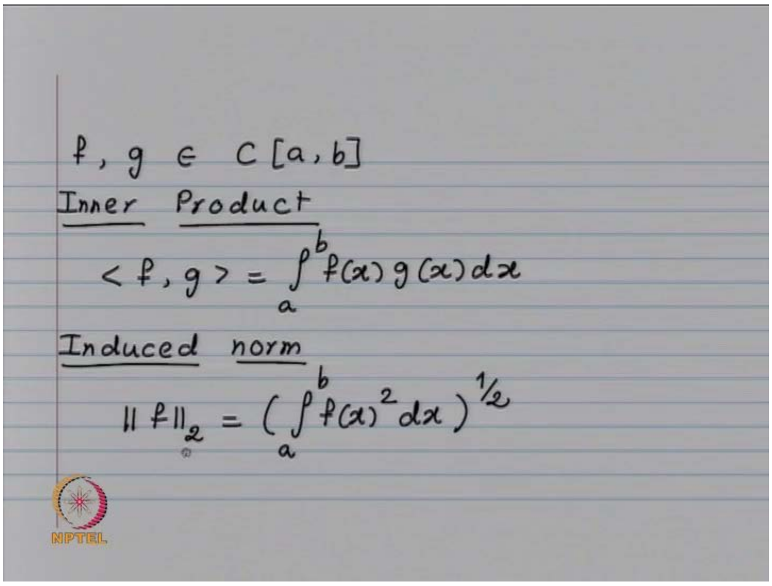

So, our space is C a b, on that we have got this inner product, inner product of f and g as integral a to b f x into g x d x. Take f and g to be real valued functions. This inner product, it induces a norm for elements of C a b and that is integral a to b f x square d x whole thing raise to half. So, that is the 2 norm and now we are going to look at the best approximation from the space of polynomials in the 2 norm.

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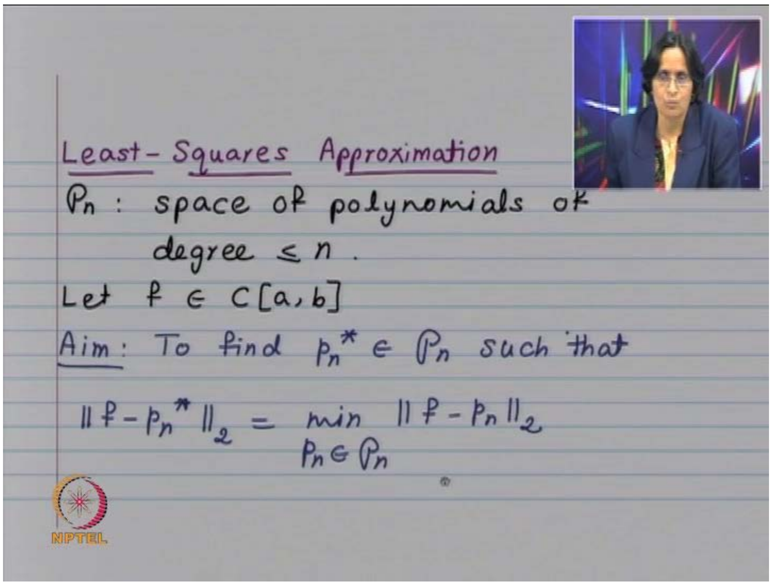

So, that is known as the least-squares approximation that p n is the space of polynomials of degree less than are equal to n. F is a continuous function we want to find a polynomial of degree less than or equal to n, such that norm of f minus p n star is its 2 norm is equal to minimum of norm of f minus p n 2 norm when p n varies over script p n. We have to show the existence of such a best approximation p n star and then the way to find such p n star. So, for that purpose, we are going to use what are known as Legendre polynomials. So, look at the functions 1 x x square x cube and. So, on.

So, that is linearly independent, apply gram Schmidt Roth normalization process to it, then you will get the Legendre polynomial. So, q 0 q 1 q 2, these are going to be Legendre polynomials with the property, that q I is polynomial of exact degree I and they are mutually perpendicular. So, inner product of q I with q j is 1, if I is equal to j and 0 if I not equal to j. The gram schmidt orthonormalization process has the property that span of 1 x x square x raise to n. If I look at the first n plus 1 functions here, then that is going to be same as span of first n plus 1 Legendre polynomials q 0 q 1 q 2 and. So, on.

So, span of q 0 q 1 q n will be same as span of 1 x x raise to n and that is nothing, but the space of polynomials of degree less than or equal to n. Q I is a polynomial of degree I, they are they form a Roth normal set and hence linearly independent. So, q 0 q 1 q n will be a basis for the space of polynomial. 1 x x square x raise to n, that is also basis for space of polynomials of degree less than are equal to n. So, here is another basis. So, when I look at a polynomial p n of degree less than or equal to n, I can write uniquely as a linear combination of q 0 q 1 q n. So, thus alpha 0 alpha 1 alpha n, these are scalars.

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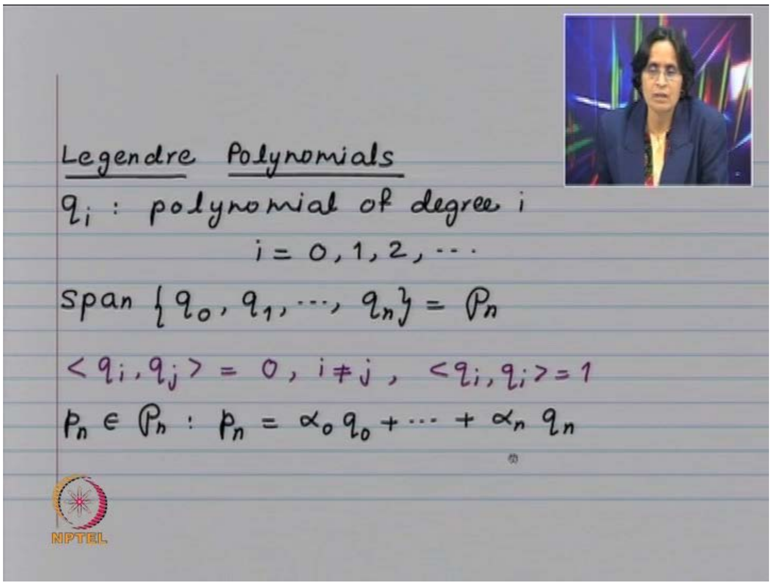

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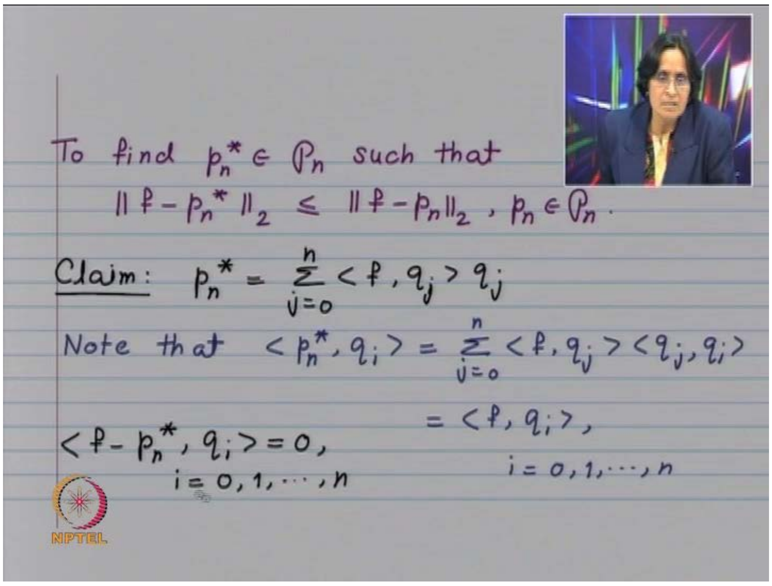

Now, look at our claim is that the best approximation in the 2 norm is going to be given by summation j goes from 0 to n, inner product of f with q j q j, f is a given continuous function. Q 0 q 1 q 2 etcetera, these are the Legendre polynomial. So, look at this p n star and we want to show that norm of f minus p n star is less than or equal to norm of f minus p n. So, we are showing the existence and then show that it is the best approximation. If I take inner product of p n star with q I, then by linearity of inner product in the first variable, it will be summation j goes from 0 to n f comma q j q j comma q I.

Now, this will be 1 only when j is equal to I and hence it is inner product of f with q I for I going from 0 1 up to n. So, thus f minus p n star q I will be 0, for I is equal to 0 1 up to n. Q 0 q 1 q n these are basis for the polynomials of degree less than o equal to n, and thus f minus p n star will be perpendicular to each polynomial of degree less than or equal to n and it is this property we will use to show this inequality.

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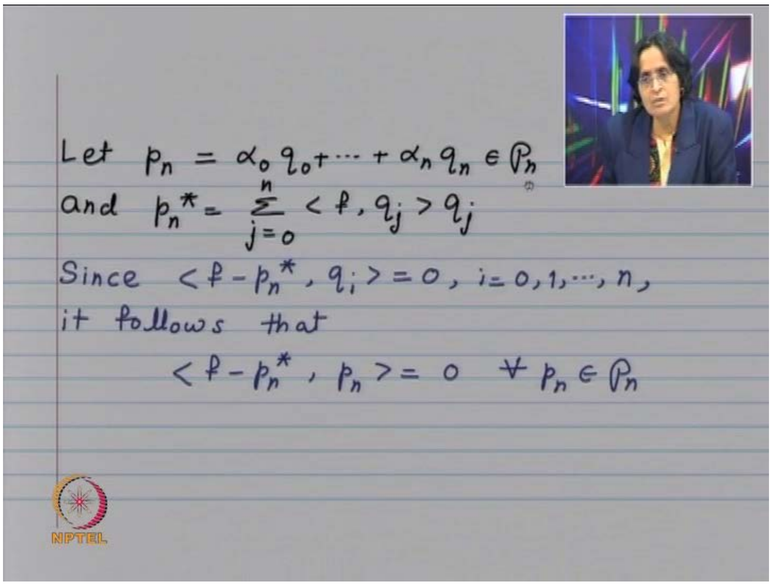

So, f minus p n star is perpendicular to each polynomial of degree less than or equal to n. Since q 1 q 2 q n they form a basis for sequence of for a space of polynomials of degree less than or equal to n. Consider f minus p n norm square, add and subtract p n star. So, f minus p n star plus p n star minus p n, f minus p n star is perpendicular to each polynomial of degree less than or equal to n.

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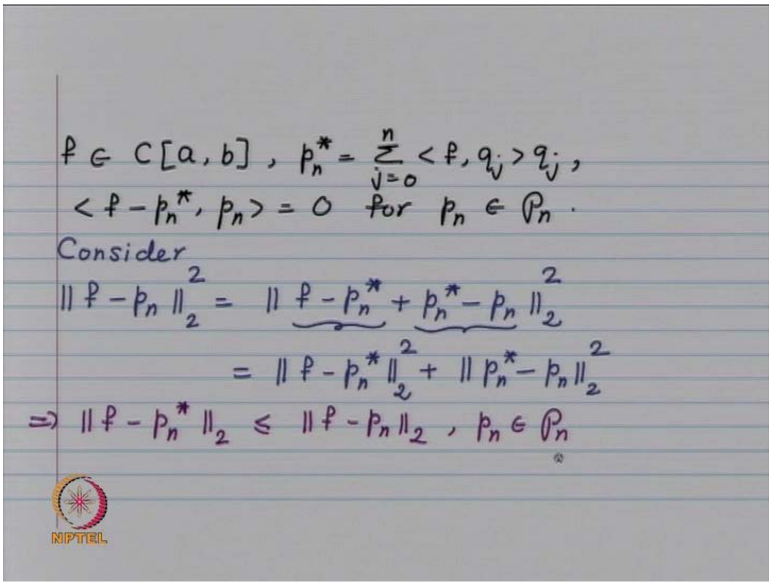

So, it will be perpendicular to this polynomial. And hence by Pythagoras theorem, it will be norm of f minus p n star square plus norm of p n star minus p n square, and thus norm of f minus p n star is less than or equal to norm of f minus p n, p n belong into script p. So, unlike in the case of best approximation by polynomials in the infinity norm in the case of best approximation in the 2 norm, we can find the best approximation explicitly. In case of infinity norm, we needed to go to a iteration process.

So, thus we have considered polynomial approximation of a continuous function. So, there are various ways. So, one was Bernstein polynomial approximation, then there was the best approximation in the infinity norm, then approximation by interpolating polynomials and now approximation best approximation in the 2 norm. Now, all these approximations, they have some desirable properties, some not so, desirable properties. Now, what I am going to do is, I am going to recall what all results we have proved in this course.

So, our course, this is the last lecture. So, now I want to recall what all things we did briefly. I have already talked about the approximation of continuous function by polynomials in various ways and then we started with the interpolating polynomial. Many of the topics in this course, they were based on this interpolating polynomial. We proved existence of and uniqueness of interpolating polynomial by using Lagrange functions or Lagrange polynomials, but then such a definition is not recursive; that means, if we find a polynomial of degree n and then add 1 more interpolation points, then we have to do all the work again. So, that is why we looked at the divided difference form or the Newton's form of the polynomial.

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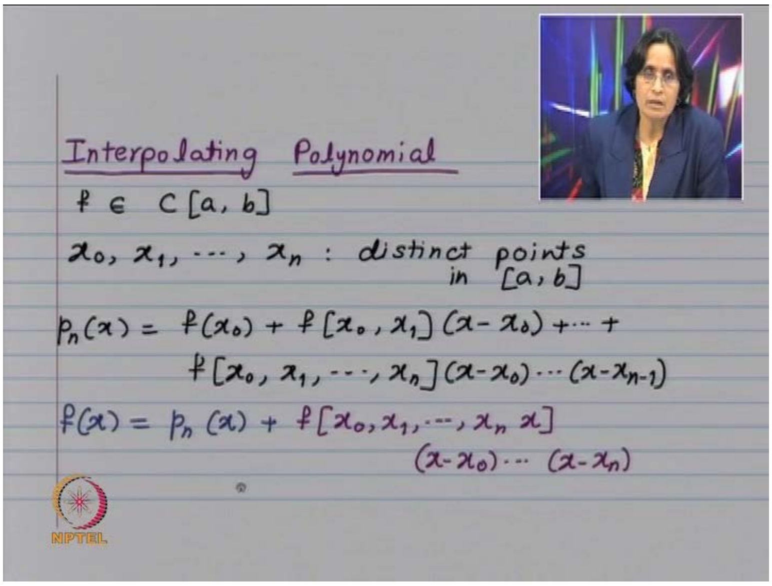

So, then Newton's form is given by, you have got x 0 x 1 x n to be distinct points in interval a b, then there is a unique interpolating polynomial of degree less than or equal to n. That polynomial is given by this form. This is known as Newton's form and if from p n to p n plus 1, I have to go I have to just add 1 more extra term. The error in the interpolating polynomial, it is given by f x minus p n x is equal to divided difference based on x 0 x 1 x n x and then multiplied by x minus x 0 x minus x n.

So, this was a very important formula, because when we use this polynomial approximation for various problems, then we need to know what is the error involved. Now, the first topic we considered was numerical integration, all continuous functions they are Riemann integral, but when it comes to finding the integral, it is not easy, for some functions, yes, but otherwise the definition using Riemann sums is not of much use in numerical analysis, when we want to calculate the integral.

So, now integral a to b f x d x will be approximately equal to integral a to b p n x d x. This p n x if you write it in the Lagrange form; that means, summation f x I l I x, where l I x is this polynomial of degree n, then integral a to b p n x d x, it is given by summation w I f x y where w I is integral a to b l I x d x. So, thus integral a to b f x d x is approximately equal to summation w I f x I, i goes from 0 to n. Choices of n and of the interpolating point, they give raise to varies numerical quadrature rules and the rules which we have considered the basic rules, these are the midpoint rule, when you are considering the constant polynomial with the interpolation point to be the midpoint, trapezoidal rule when you consider the approximation by linear polynomial with interpolation points to be the end points or Simpson's rule when you consider approximation by quadratic polynomials with 3 interpolation points as the 2 end points and the midpoint and these are all special cases of Newton cotes formulae, where you sub divide your interval into n equal parts and take your interpolation points as the n plus 1 partition points.

Now, once we got basic rules, then we considered composite rules. So, composite rule is divide your interval a b into smaller sub intervals and then on each sub interval, apply a numerical integration. We also considered Gaussian integration, where we started with integral a to b f x d x to be approximately equal to summation w I f x I and treated w I the weights and x I the nodes as unknowns to achieve the maximum exactitude. The way we did numerical integration, for the numerical differentiation we use the same idea, that polynomials are infinitely many times differentiable. So, consider a interpolating polynomial and then derivative of it will give you an approximation to derivative of a function. And this later we used for finite difference method for the solution of differential equations.

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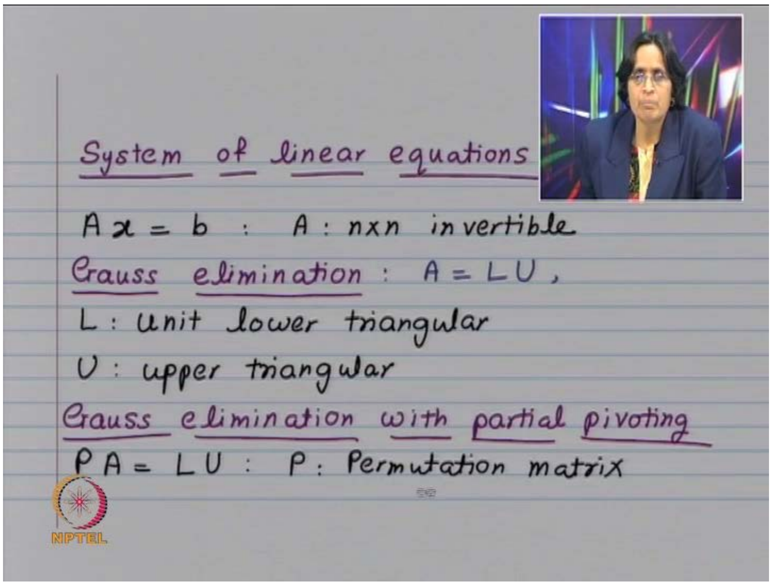

Then important topic was system of linear equations. So, Ax is equal to b with the assumption that a is n by n invertible matrix. First we considered gauss elimination method, this gauss elimination method is equivalent to L u decomposition of a matrix a, where l is unit lower triangular matrix; that means, the diagonal entries are equal to 1 and u is upper triangular. Next we considered gauss elimination with partial pivoting and in that case, it is equivalent to L u decomposition of not matrix a, but matrix p into a, where p is a permutation matrix; that means, the matrix obtained from the identity matrix by finite number of row interchanges. If your matrix a is positive definite, then you have got what is known as colicky decomposition. We have L u decomposition for a matrix a under certain conditions.

One of the conditions are in fact, necessary and sufficient condition is, look at the principle minors, if they are all not equal to 0, then you can write a as l into u, if a is positive definite, then we can write it as g into g transpose where g is going to be a lower triangular matrix. So, this will need half the number of computations as compared to l u decomposition, but it will be possible only for positive definite matrices. So, we have the colicky decomposition of a positive definite matrix as a is equal to g g transpose, we also considered iterative methods for solution of a x is equal to b and those were the Jacobi and gauss-sidle method.

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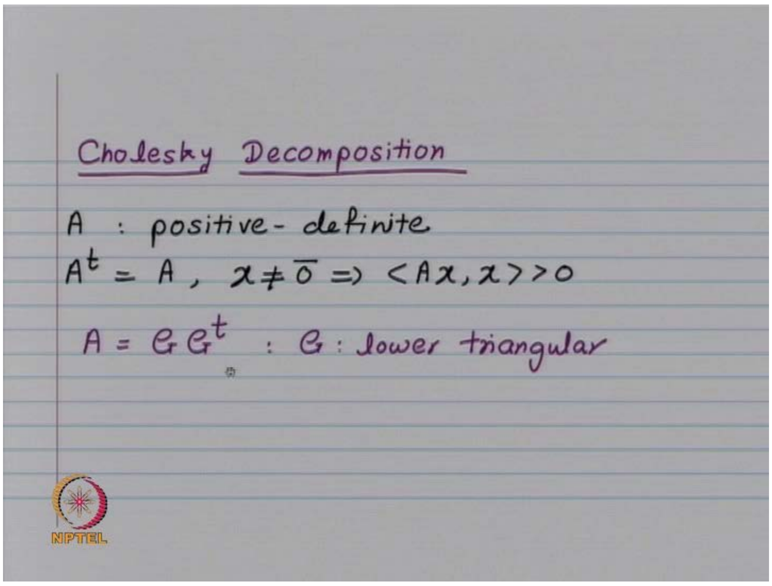

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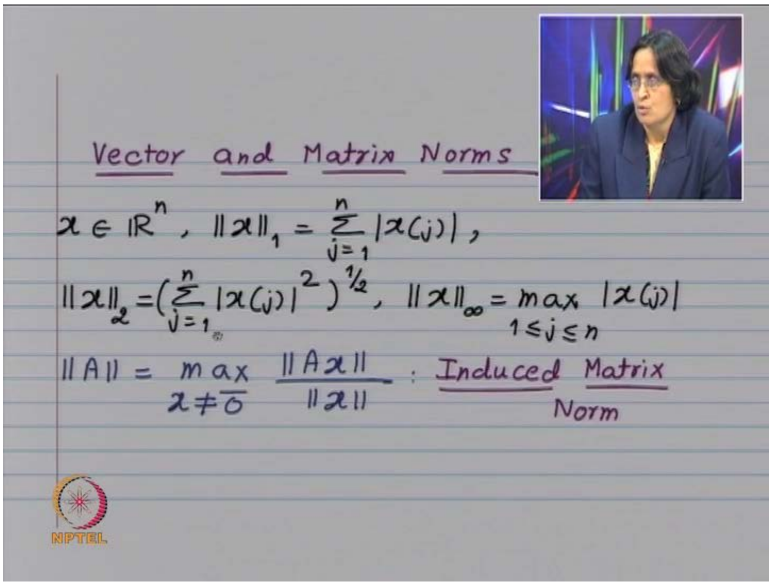

.We looked at vector and matrix norms. So, for the vector, we considered mainly 1 norm 2 norm and infinity norm. 2 norm is the well-known Euclidian norm. Once you fix vector norm, then we define induced matrix norm as norm a is equal to maximum norm a x by norm x x not equal to 0. And then corresponding to 1 vector norm and infinity vector norm we have got a formula for norm a in terms of its value of its entries a I j. For norm a 2 we have to be satisfied only with an upper bound we want to solve a x is equal to b, but then because of the finite precision of computers, instead of a x is equal to b, we will be solving a nearby system. There will be, instead of a there will be a plus delta a, instead of the right hand side b, you will have b plus delta b and instead of x the computed solution will be x plus delta x.

One wants to know what is the error between x and x cap. The exact solution and the computed solution. So, relative error is norm of x minus x cap by norm x some vector norm and then we showed that this will be less than or equal to the error in the coefficient matrix error in the right hand side and in that what comes into crucially into picture is the condition number norm a into norm a inverse. For a x is equal to b, we looked at the iterative methods which were Jacobi and gauss-sidle methods.

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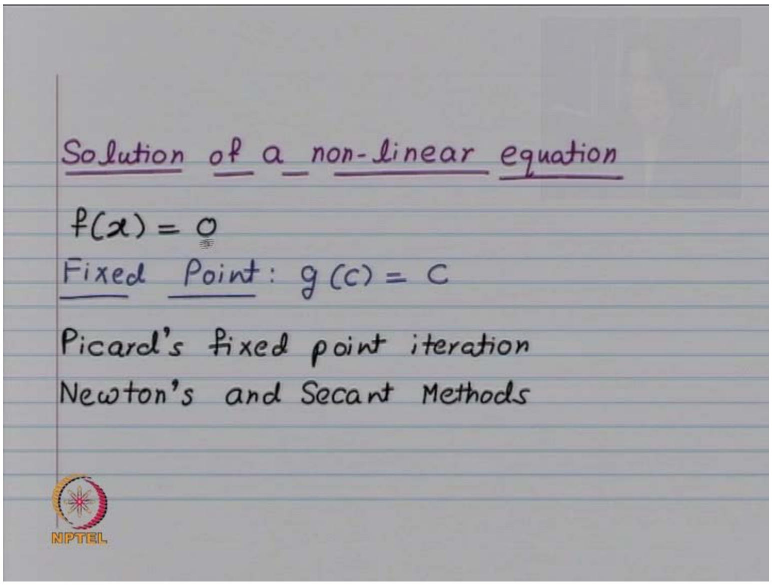

Then we wanted to look at the solution of non- linear equations f x is equal to 0, this is related to finding a fix point of a method g of C is equal to c. So, we considered Picard's, it fixed point iteration and in detai,l the Newton's method, secant method also regular falsi method for finding 0 of a function.

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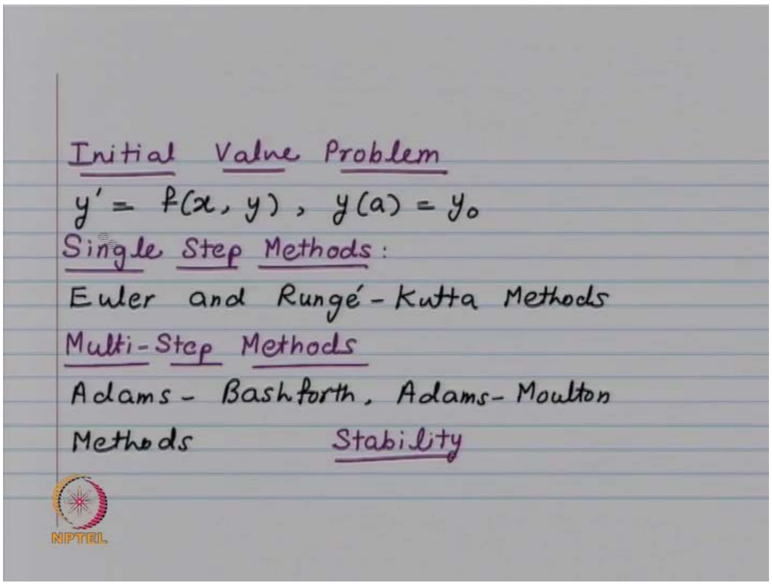

Differential equations, we looked at initial value problem, here there were 2 types of methods, single step methods such as Euler and range kite methods, these are classical methods and multi-step methods such as Adams-Bashforth and Adams-Moulton which are relatively of recent origin method and the important thing is the stability of the methods which we are considered in detail.

Then we looked at the boundary value problem and for the boundary value problem the method which we considered was the finite difference method, where the derivatives are replaced by finite differences. So, that finishes our course and it was a pleasure to give this course. So, thank you.