Elementary Numerical Analysis

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Lecture No. # 38
Inverse Power Method

In our last lecture, we have defined power method for calculating eigen vector associated with dominant eigen value; then, if the matrix is invertible and if the eigen value which has got lowest modulus - so, if modulus of lambda n is strictly less than modulus of the remaining eigen values, then we can apply power method to the inverse and then obtain approximation to eigen vector associated with the least eigen value.

Thus, we can approximate the eigen vector associated with the largest eigen value in modulus or the least eigen value in modulus.

Now, what about some intermediate eigen value? For that, we need to have some approximation to say lth eigen value and then we can extend our power method; that is known as inverse power method, which we are going to consider today; after defining inverse power method we are going to consider what is known as q r decomposition of a matrix.

We have considered various decompositions of matrix - l u decomposition, then Cholesky decomposition and today we are going to consider decomposition of a into q into r, where q is going to be orthogonal matrix and r is going to be upper triangular matrix.

Once we consider this q r decomposition we can define q r method for finding eigen values of matrix. Then, we are going to consider relation between q r decomposition of matrix and Gram-Schmidt Orthonormalization process of the column vector.

So, this is roughly the plan of today's lecture. Let us look at the power method - recall it.

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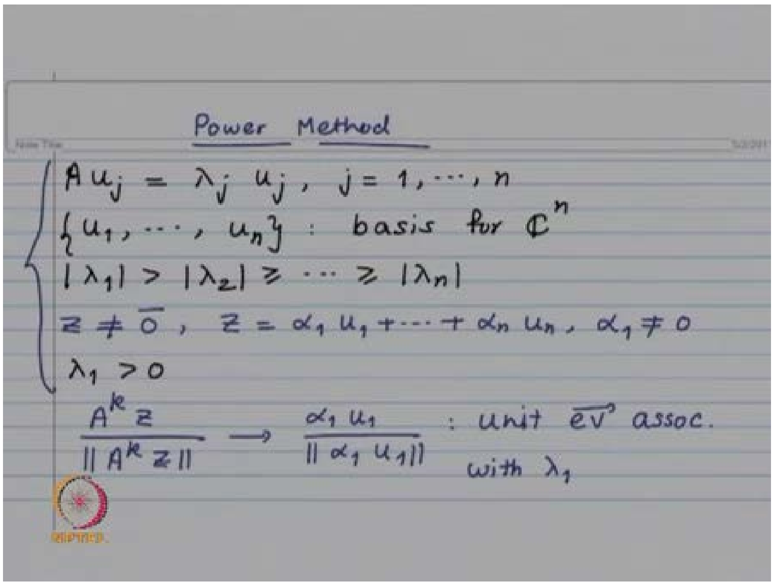

Our A is n by n matrix; our assumption is that eigen vectors u 1, u 2 and u n form a basis for c n.

Eigen values lambda j (s) are arranged in this manner and assumption is mod lambda 1 bigger than mod lambda 2 bigger than or equal to mod lambda n; these are our assumptions.

Not all matrices will have a basis of eigen vectors, but then we have seen that if the matrix is normal then there is a basis of eigen vectors; if our matrix A has n distinct eigen value, even then it has a basis of eigen vectors. So, this restriction of eigen vectors forming a basis is not too restrictive - there is big class of matrices for which it will be satisfied.

Thus our assumption is A should have a basis of eigen vectors and then it should have a dominant eigen value; that means, modulus of lambda 1 bigger than mod lambda 2 bigger than or equal to mod lambda 3 bigger than or equal to mod lambda n.

Afterwards we can have greater than or equal to; but, the eigen value which is biggest in modulus should be a simple eigen value and it should be strictly bigger than moduli of the remaining eigen values.

Then, we start with a non-zero vector; now, this non-zero vector can be expressed as a linear combination of u 1, u 2 and u n because u 1, u 2 and u n form a basis.

Then, we assume that our z - the vector arbitrary vector, which we are choosing, its component in the direction of u 1 is not equal to 0; so, alpha 1 is not equal to 0.

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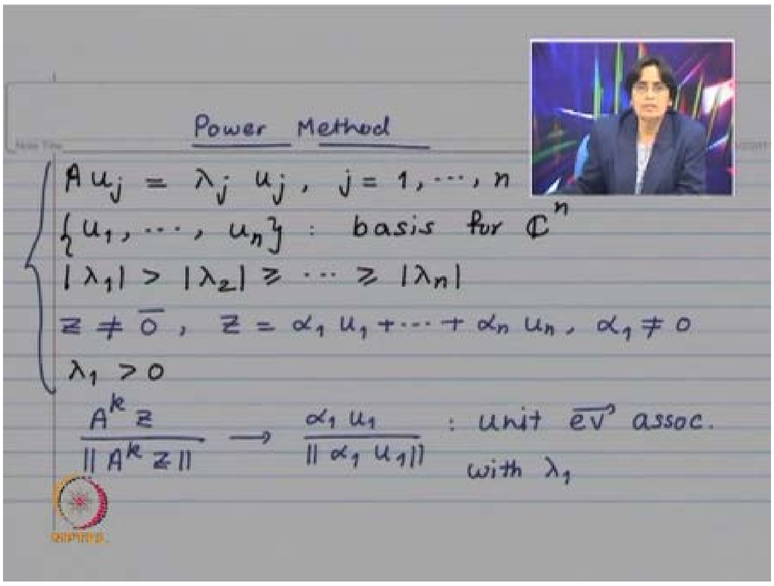

With these assumptions and lambda 1 to be bigger than 0 - that means, lambda 1 is real and positive. So, we formed A raised to k z divided by norm of A raised to k z.

Under these assumptions, A raised to k z upon norm of A raised to k z converges to alpha 1 u 1 by norm of alpha 1 u 1.

This is going to be unit eigen vector associated with eigen value lambda 1; it is easy to calculate these iterates - you start with a arbitrary vector z not equal to 0; find these iterates - if your matrix a is a sparse matrix, that means a lot of zeroes; then, this will not be computationally too expensive and this is going to converge to a unit eigen vector corresponding to the eigen value lambda 1; this is the power method.

Now, we want to look at extension of this method, which will allow us to find eigen vector associated with intermediate eigen value.

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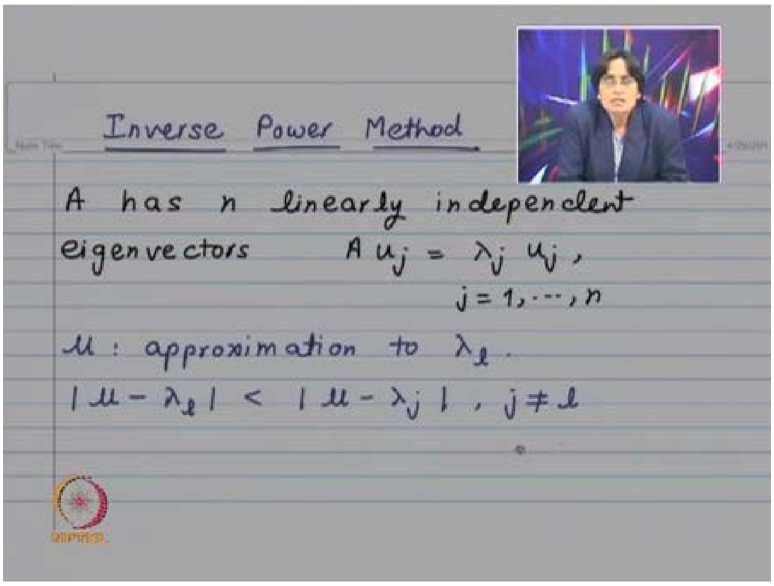

Again, our assumption is that A has n linearly independent eigen vectors - A u j is equal to lambda j u j, j is equal to 1 to up to n and mu is given to be an approximation to lambda l; some lth eigen value in between - it need not be the largest eigen value or the least eigen value in modulus.

Since mu is approximation to lambda l its distance from lambda l is going to be smaller than distance of mu from the remaining eigen values. So, mod mu minus lambda l is less than mod of mu minus lambda j, j not equal to l; now, our eigen values - these are roots of characteristic polynomial.

Now, for the characteristic polynomial we can find an interval in which our eigen value lies; for example, we had considered bisection method - we have got a polynomial, you find two numbers or two real numbers where the polynomial has opposite sign; that interval is going to contain a root of the polynomial; so, you go on subdividing it getting a smaller and smaller interval. So, in such a manner or by some other method we can have an approximation to eigen value lambda l.

Starting point of our inverse power method is - your given mu an approximation to lth eigen value, and our aim is to find approximation to an eigen vector corresponding to this lth eigen value lambda l.

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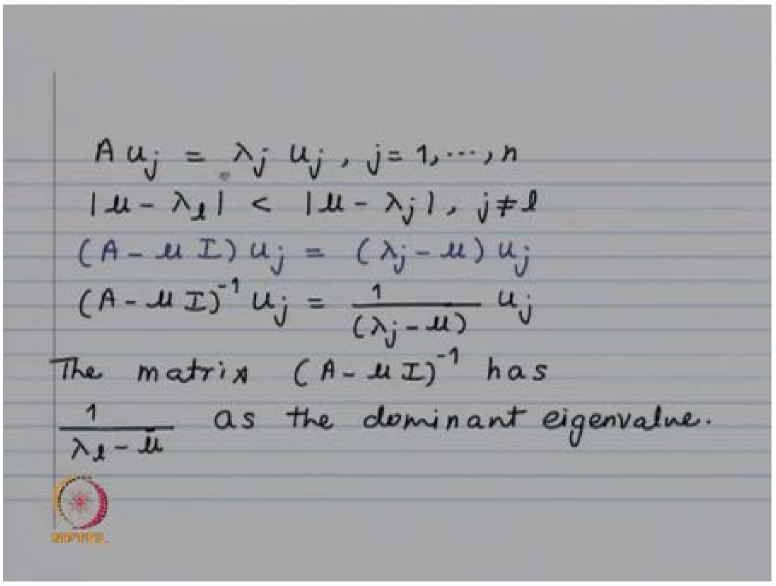

So, we have A u j is equal to lambda j u j - j is equal to 1 to up to n, mu being an approximation to lambda l, modulus of mu minus lambda l is less than modulus of mu minus lambda j, j not equal to l.

Now, from here A u j is equal to lambda j u j gives us A minus mu i u j is equal to lambda j minus mu u j; assume that mu is not equal to lambda l; that means, it is not an eigen value.

So, A minus mu i will be invertible and then from this relation we obtain A minus mu i inverse u j to be equal to 1 upon lambda j minus mu u j.

I apply A minus mu i throughout - A minus mu i inverse throughout; left hand side will be u j is equal to lambda j minus mu A minus mu i inverse u j and take lambda j minus mu on the other side.

Now, look at this - we have got modulus of mu minus lambda l to be less than modulus of mu minus lambda j, j not equal to l; if I take the reciprocal 1 upon mod of mu minus lambda l will be strictly bigger than 1 upon modulus of mu minus lambda j, j not equal to l, which will mean that 1 upon lambda l minus mu this is going to be a dominant eigen value of matrix A minus mu i inverse.

The idea now, is to apply our power method to this A minus mu i inverse; in power method, we need dominant eigen value; now, our A minus mu i inverse has dominant eignen value - 1 upon lambda l minus mu.

So, modulus of 1 upon lambda l minus mu is strictly bigger than 1 upon modulus of lambda j minus mu.

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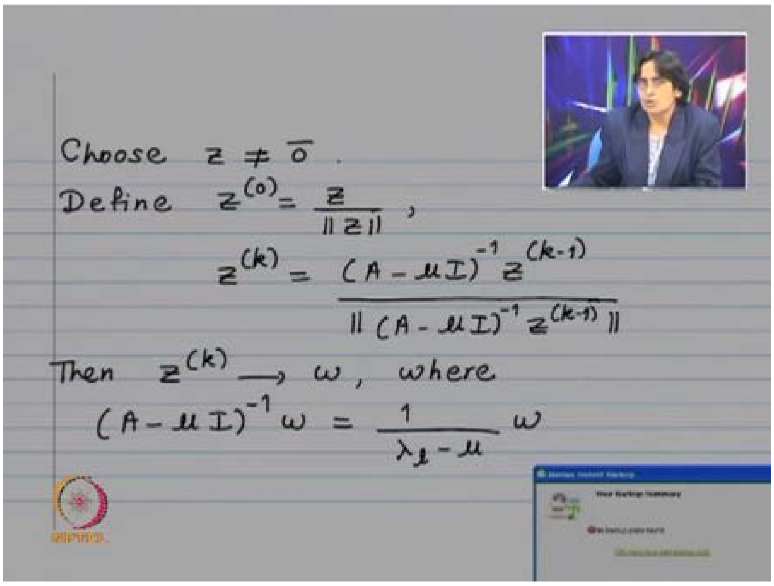

Here are our iterates - choose z not equal to 0 vector, an arbitrarily vector define z 0 to be equal to z upon norm z and then z k - the kth iterate, will be A minus mu i inverse z k minus 1 divided by its norm.

It is exactly the power method, which is applied to the matrix A minus mu i inverse; then, this z k is going to converge to, say, vector w; this vector w will be eigen vector associated with matrix A minus mu i inverse and 1 upon lambda l minus mu being the eigen value. Now, what we are interested in is eigen values of matrix A and associated eigen vectors.

We have obtained approximation to an eigen vector w or we have obtained - yeah - approximation to an eigen vector w, which is eigen vector of a minus mu i inverse. But, this w is also going to be eigen vector of a associated with eigen value lambda l.

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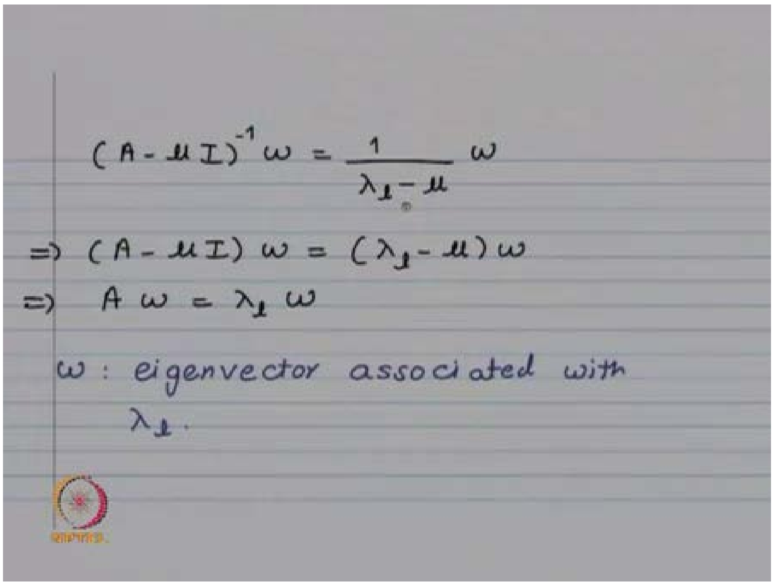

A minus mu i inverse w is equal to 1 upon lambda l minus w; that will mean that A minus mu i w is equal to lambda l minus mu into w; this mu w will get cancelled and then you are left with A w is equal to lambda l w.

So, w is eigen vector associated with lambda l. Thus, when we define the iterates in the inverse power method they involve matrix A minus mu i inverse. You get an approximation and you get the limiting vector as w; this w is going to be eigen vector of a associated with eigen value lambda l.

If lambda 1 is the dominant eigen value - power method - in that case, we obtained eigen vector associated with lambda 1; when you apply power method to matrix a inverse we obtained approximation to eigen vector associated with lambda l.

If mu - an approximation to eigen value lambda l is available, then we obtain approximation to eigen vector associated with lambda l, where lambda l can be - it is an intermediate eigen value. You need to have approximation mu available.

Here, there can be some problem or we want to see whether they is going to be a problem; what is the problem? We are saying that mu is going to be an approximation to lambda l; lambda l is eigen value, so, our a minus lambda l into i is not an invertible matrix; now, when you are going to have mu to be a better and better approximation to your eigen value lambda l, a minus mu i inverse can be ill conditioned.

Then whether this is going to pose a problem - now we will see that it does not matter in this particular case.

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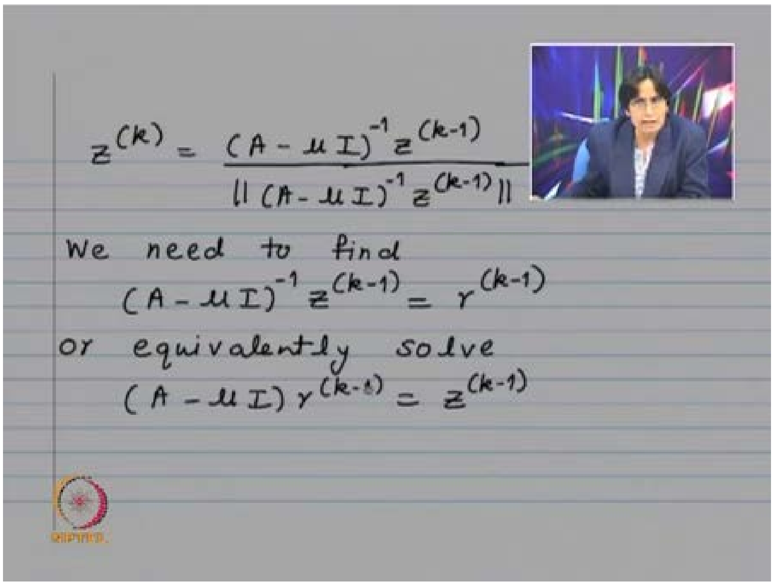

We have, in the inverse power method, we need to find z k to be equal to A minus mu i inverse z k minus 1 divided by norm of A minus mu i inverse z k minus 1.

Now, we are not going to calculate A minus mu i inverse. So, this A minus mu i inverse z k minus 1 will be obtained by solving a system of linear equations A minus mu i r k minus 1 is equal to z k minus 1 we want to calculate this. So, i am denoting it by r k minus 1.

Now, this r k minus 1 is going to be the solution of the system A minus mu i r k minus 1 is equal to z k minus 1; z k minus 1 is known, it is coming from the iterations step. So, we need to calculate this r k minus 1 and then divide by its norm so that you get the next iterate.

What I was saying - whether we are going to have some problem about the stability.

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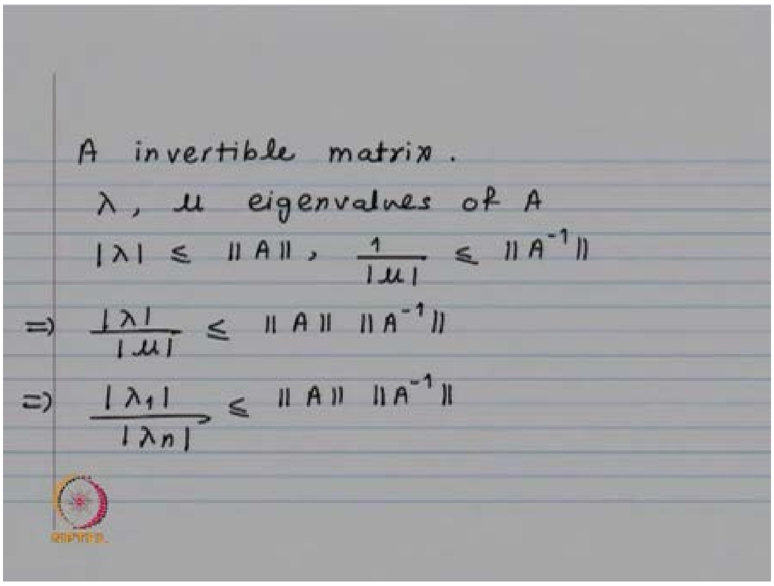

Here, if A is an invertible matrix and lambda and mu are eigen values of A, then modulus of lambda is less than or equal to norm of a; it follows from our basic inequality.

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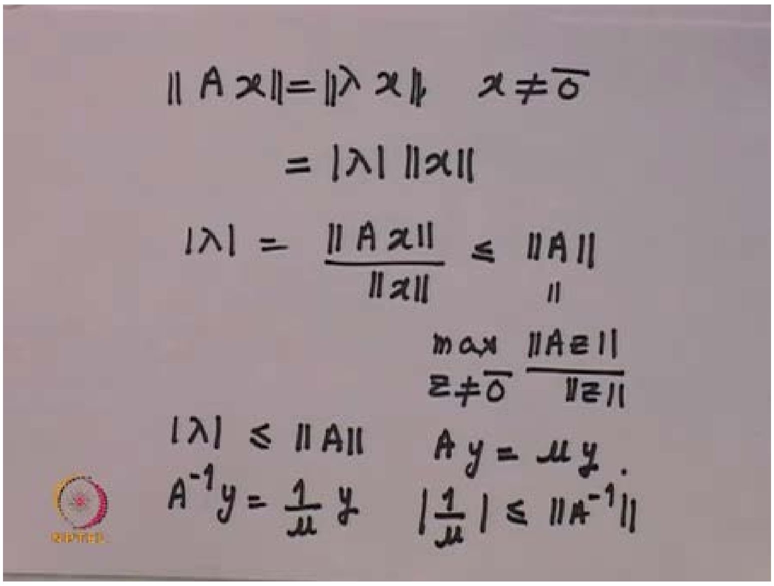

You have got - suppose you have A x is equal to lambda x, x not equal to 0 vector; then, take norm of both the sides . So, norm of A x is equal to norm of lambda x; this will be modulus of lambda times norm x by property of norm.

Hence, mod lambda will be equal to norm A x divided by norm x and this will be less than or equal to norm of A, where norm of A is maximum of norm A z divided by norm z; z not equal to 0 vector.

Thus, we have got mod lambda to be less than or equal to norm of A; if you consider A y is equal to mu y - another vector, another eigen value mu and associated eigen vector. If A is invertible, A inverse y will be equal to 1 upon mu times y; hence, again using similar inequality you get modulus of 1 by mu to be less than or equal to norm A inverse.

So, the condition number norm A into norm A inverse will be bigger than or equal to mod lambda by mod mu, where lambda and mu are any eigen values of matrix A.

Here we have A is invertible matrix, lambda and mu are eigen values of A, mod lambda to be less than or equal to norm A, 1 upon mu - mod mu - will be less than or equal to norm of A inverse.

Take the quotient; this is the condition number of matrix A it is bigger than or equal to this and if your eigen values are arranged in descending order of modulus, then we have mod lambda 1 mod lambda n less than or equal to norm A into norm A inverse.

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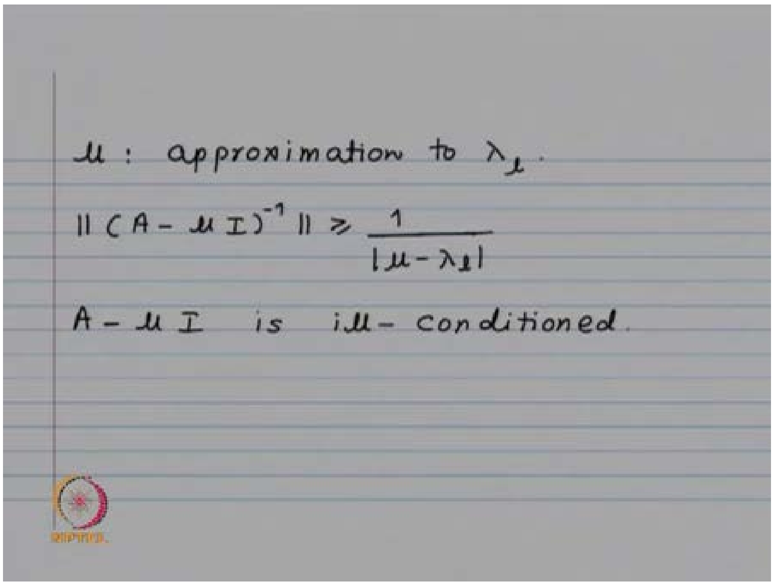

Let us go back to our A minus mu i inverse; mu is approximation to lambda l. So, norm of A minus mu i inverse is going to be bigger than or equal to 1 upon modulus of mu minus lambda l; if mu is approximation to lambda - a good approximation - good approximation will mean that denominator is small; that means, 1 upon mod mu minus lambda l will be big and then your matrix A minus mu i is going to be ill conditioned.

Now, we have considered the perturbation theory; in the perturbation theory, if your matrix a is ill conditioned - that means, if norm a into norm a inverse is big, then the solution is sensitive to the perturbation in the right hand side.

We had looked at a x is equal to b, perturb b is slightly; consider the nearby system a of x plus delta x is equal to b plus delta b, then even though norm delta b by norm b is small norm delta x by norm x can be big.

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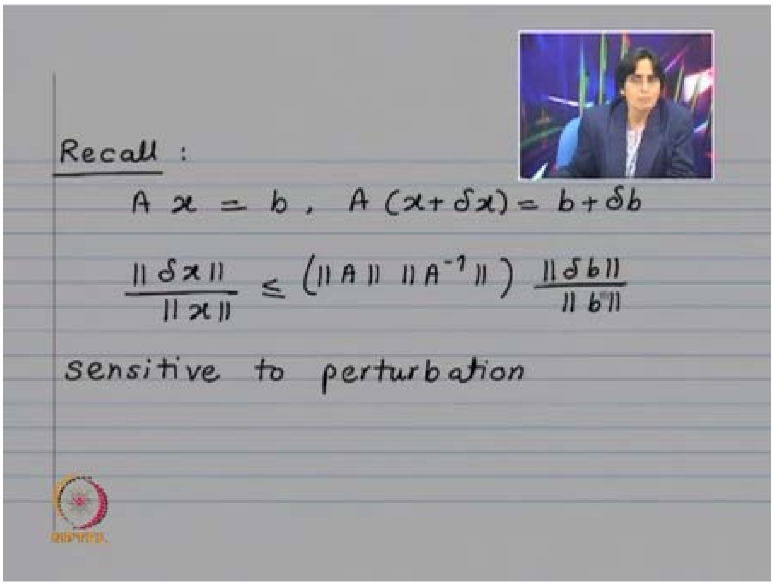

In fact, we had proved this inequality - norm delta x by norm x is less than or equal to norm A into norm A inverse into norm delta b by norm b.

This can be small, but if you are multiplying by a big number then norm delta x by norm x which is the relative error in the computed solution; x is the exact solution, because of finite precision instead of b it is going to be b plus delta b; so, x plus delta x will be computed solution.

The relative error in the computed solution will be norm delta x by norm x; this can be big even though norm delta b by norm b is small.

Now, this situation is going occur in our case we are calculating our z k; the kth iterate was given by a minus mu i inverse z k minus 1 - the earlier iterate divided by its norm. So, we need to calculate a minus mu i inverse z k minus 1, which we denoted it by r k minus 1.

That means, we need to solve a minus mu i r k minus 1 is equal to z k minus 1; now, this a minus mu i will be ill conditioned and hence our solution, r k minus 1, will be sensitive to the perturbation. This is what in general - at each stage we will be solving a minus mu i r is equal to q.

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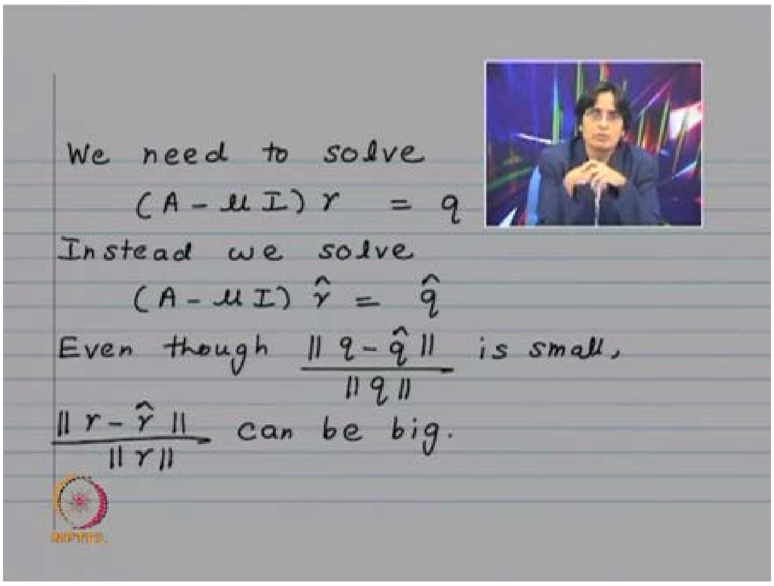

Now, in practice instead of this you are going to solve A minus mu i r cap is equal to q cap, where q cap is the near by vector.

Even though norm of q minus q cap divided by norm q - even though this is small, because A minus mu i inverse has a big norm, norm of r minus r cap by norm r can be big. This is something we need to worry about, because as such when we talk about the approximation mu should be available to lambda l.

Then the better the approximation our task should be simplified; it should not face such difficulty that when you have better and better approximation of mu to lambda l available then your problem becomes more and more ill conditioned.; then, the relative error in the computed solution becomes bigger and bigger.

In this particular application, it does not matter because we are not interested in the computed solution, but we are interested in direction; we are trying to find an eigen vector; what is important is the eigen direction.

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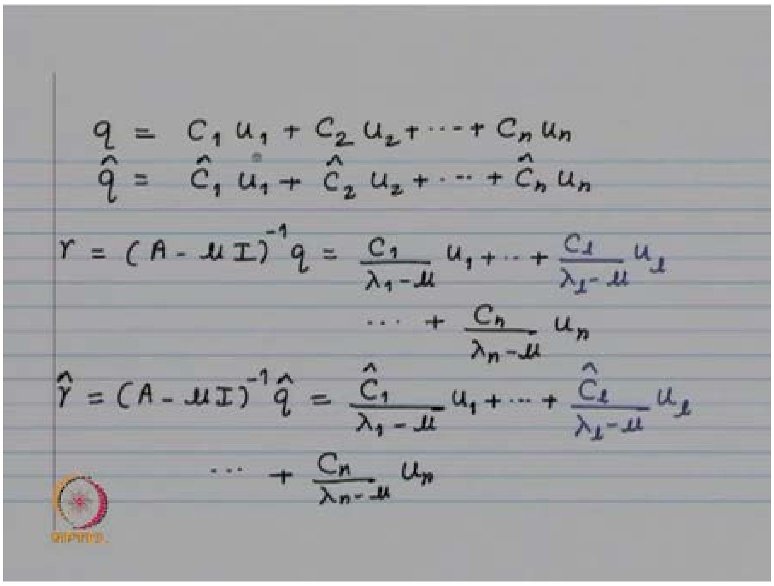

Let me make it more specific - suppose our q the - right hand side q - we are looking at q to be a minus mu i r is equal to q and this is the… Instead, we are going to solve this - our A has a basis of eigen vectors u 1, u 2 and u n; hence, our q will be c 1 u 1 plus c two u two plus c n u n - a linear combination.

Q cap, which is nearby vector will also have a linear combination c 1 cap u 1 plus c two cap u two plus c n cap u n; r is A minus mu i inverse q; A minus mu i inverse u 1 is going to be c 1 upon lambda 1 minus mu u 1 plus c l upon lambda l minus mu u l plus c n upon lambda n minus mu u n; here, for r cap you are going to have c 1 cap upon lambda 1 minus mu u 1 etcetera.

q n q cap are near. The distance between c 1 and c 1 cap c 2 and c 2 cap that is going to be small, when you look at r minus r cap look at this component, here you have c l upon lambda l minus mu into u l; here you have c l cap upon lambda l minus mu u l. So, c l minus c l cap is small, but you are multiplying by big number - 1 upon lambda l minus mu.

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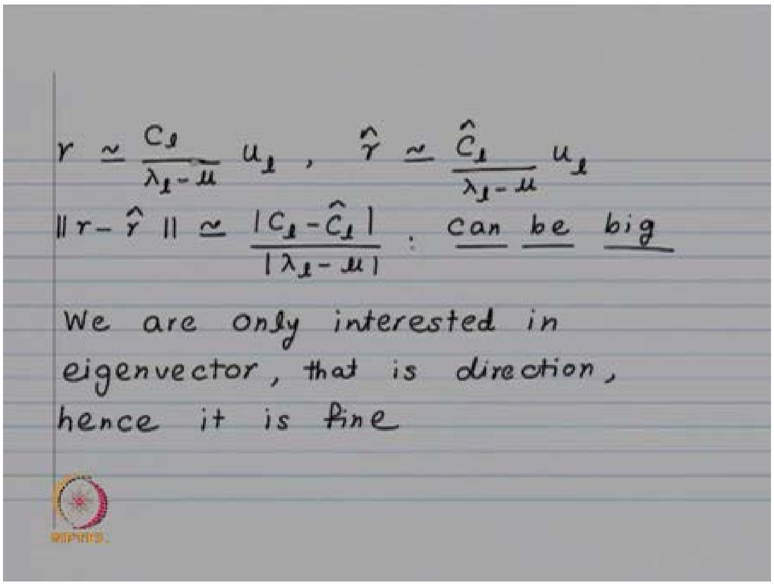

So, our r is approximately equal to c l upon lambda l minus mu u l, because that is going to be the significant part - r cap will be c l cap upon lambda l minus mu u l. Norm of r minus r cap will be approximately equal to modulus of c l minus c l cap upon mod lambda l minus mu.

Now, this can be big, but we are not interested in what r cap is we are interested only in the eigen vector that is the direction. So, even when you calculate r cap then what we do is you normalize it.

The exact solution is r; then, you consider r upon norm r, because when you are applying inverse power method you calculate a minus mu i inverse z k minus 1 and divide by its norm; so, a minus mu i inverse z k minus 1 is r k minus one - you are dividing by its norm; instead of r k minus 1 you are going to have some r cap k minus one, where if you consider the values they may differ, but for both r k minus 1 and r cap k minus 1 significant part is going to be in the direction of u l.

We are dividing by norm then the numerical values - they do not really matter; what is happening is the direction for both the exact solution and the computed solution.

The component in the direction of u l becomes significant and that is why there is no contradiction in our method; that when you have better and better approximation to lambda l that will give us faster convergence to eigen vector corresponding to eigen value lambda l.

So far we have been talking about approximation to eigen vector; in the power method we had approximation to the eigen vector associated with lambda 1; when you applied it to an inverse it is eigen vector associated with lambda n and in the inverse power method it is eigen vector associated with eigen value lambda l.

I have obtained an approximation to eigen vector; now, what about eigen value? When we talked about exact eigen value and exact eigen vector we said that if I know eigen vector then finding eigen value is easy - you have to just find constant of proportionality; if v is eigen vector of matrix a then look at two vectors a into v and v, these two they are proportional; I find what is the constant of proportionality between the two. Here, our eigen vector is only approximate. So, what best eigen value approximation can I choose?

That means - suppose I give you an eigen vector approximation and I want to know this is eigen vector corresponding to which eigen value? The best way it can be done is by considering the Rayleigh quotient. So, the Rayleigh quotient has got some minimization property, which we now consider.

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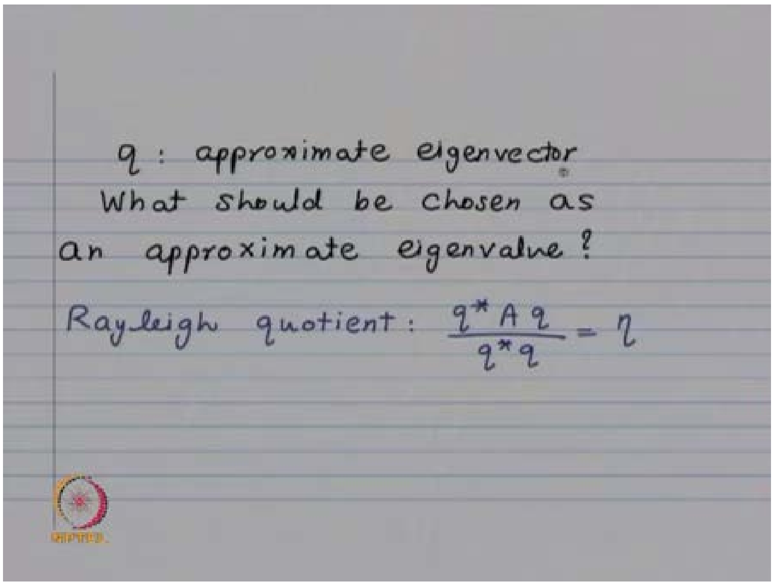

You have q to be an approximation eigen vector; the question is what should be chosen as an approximate eigen value? The answer is choose Rayleigh quotient which is q star a q divided by q star q which is equal to eta.

Now, let me tell you what is the minimization property of Rayleigh quotient. So, we consider eta to be equal to q star A q; so, our q is approximate eigen vector.

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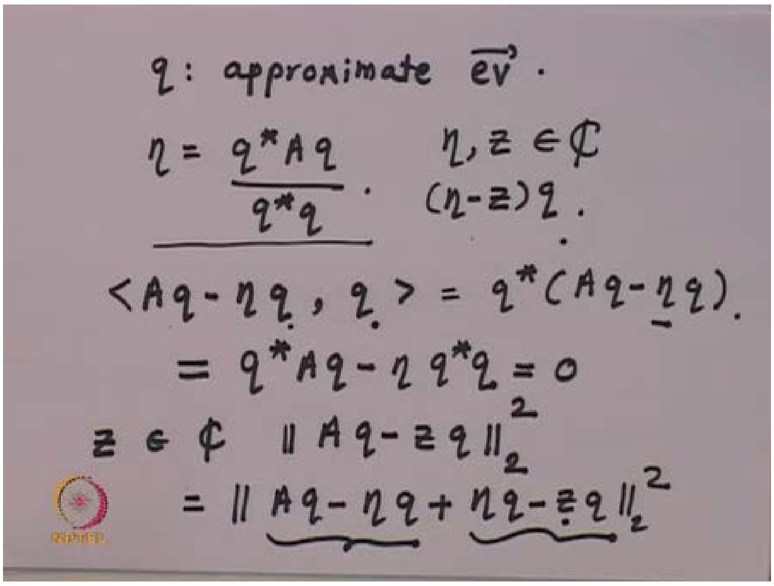

Then, eta is q star A q divided by q star q; if I consider A q minus eta q - its inner product with q, this will be nothing but q star A q minus eta q which is going to be equal to q star A q minus eta - is a complex number - so, it is minus eta q star q; using this result, this is going to be equal to zero.

That means A q minus eta q is going to be perpendicular to vector q; now, let z be any complex number and look at norm of A q minus z q - its two norm, let me take its square; this will be equal to norm of A q minus eta q plus eta q minus z q - add and subtract; here is one vector here is another vector, our eta and z these are complex numbers. So, we have got vector eta minus z into q; A q minus eta q is perpendicular to q - it is perpendicular to any multiple, which will mean that these two vectors they are perpendicular; we can use Pythagoras theorem.

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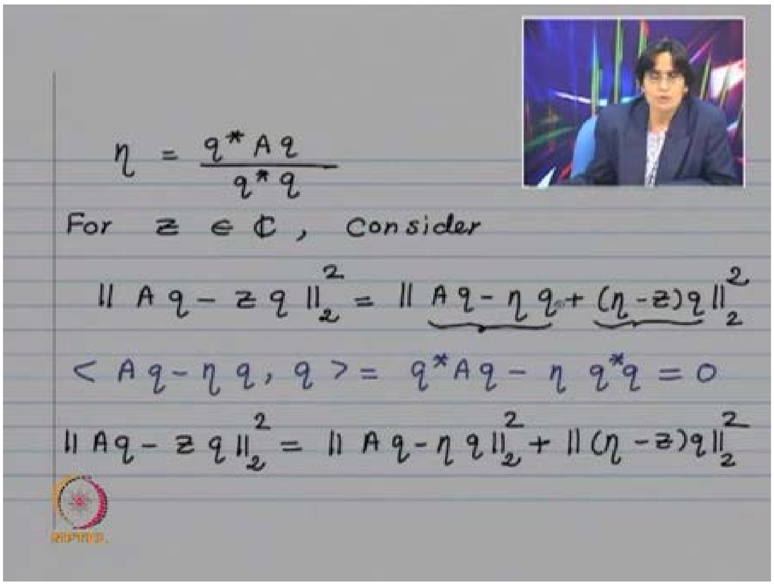

So, we have got norm of A q minus z q square to be equal to A q minus eta q plus eta minus z q for any z belonging to c.

Using orthogonality property, this vector is perpendicular to this vector - eta minus z into q; this two norms square will be nothing but square of norm of this Aq minus eta q square plus norm of eta minus z q square.

What does this relation tell us? It tells us that two norm of A q minus z q is going to be bigger than or equal to two norm of A minus eta into q.

Our vector q is approximate eigen vector; we cannot hope to find a complex number lambda such that A q is equal to lambda q; but, what we are saying is - look at the Rayleigh quotient q star A q divided by q star q that we denote by eta.

Now, consider two norm of a q minus eta q; this norm will be less than or equal to norm of a q minus z q, where z is any complex number. The Rayleigh quotient eta minimizes two norm of a minus z i into q where z varies over the complex plane and that is why if you are given q to be an approximate eigen vector the best you can do is choose eigen value approximation to be the Rayleigh quotient q star a q divided by q star q.

If your q happens to be exact eigen vector, then your eta will be - suppose your e q is equal to lambda q, then q star a q will be q star lambda q; that means, lambda times q star q because lambda is a complex number.

You have eta to be equal to - in the numerator lambda times q star q divided by denominator is q star q; so, you get eta is equal to lambda. So, the Rayleigh quotient associated with the exact eigen vector is nothing but the eigen value itself; if it is not an eigen vector then it minimizes two norm of vector a minus z i into q.

Next, I want to consider q r decomposition. The reason I am going to consider q r decomposition is I want to describe what is a q r method for calculating eigen values of a matrix A or finding approximations to eigen values of matrix A.

Now, the description of q r method is easy; you will see that we can quickly describe what is q r method. What is difficult is to show its convergence and also why it works; unfortunately these things are involved, so we are not going to do these things in detail; but, I want to tell you what is a q r method, which is the currently used method for - or it is the best method available for calculating eigen values of matrix A.

There is a relation between power method and q r method. What we have done is we have considered power method to find approximation to one eigen vector; now, instead of one eigen vector you can consider several eigen vectors together.

That gives rise to what is known as simultaneous iteration, and the best implementation of simultaneous iteration is done by q r method. Anyway, let us first look at what is q r decomposition of a matrix A; for simplicity, let me assume a to be an invertible matrix and also let us look at the matrix to be real matrix; q r decomposition is available for complex matrices also, but just for the sake of simplicity let us restrict ourselves to real matrices. So, our assumption is A is real n by n matrix and we want to write it as q into r where q is an orthogonal matrix; that means q transpose q is equal to identity and r is an upper triangular matrix.

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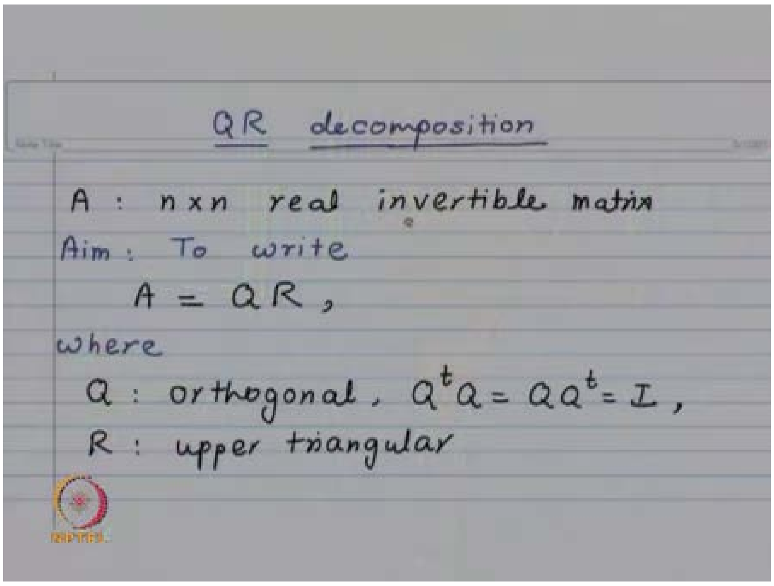

Here is A, n by n real invertible matrix; aim is to write A is equal to q into R where q is orthogonal; that means, q transpose q is equal to q, q transpose is equal to identity and R is upper triangular.

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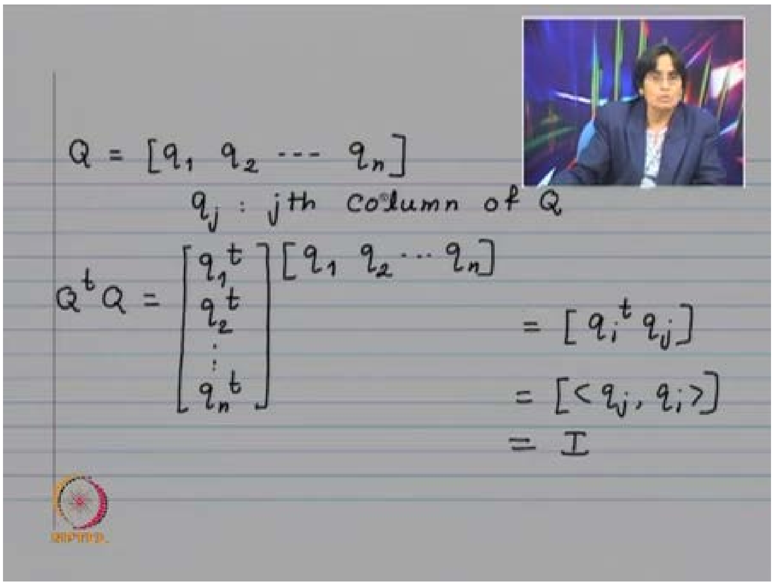

Let me write it as q 1, q 2 and q n - these are the columns of Q. Q is going to be a n by n matrix and these are - q 1 is the first column, q 2 is the second column and q n is nth column.

Now, what will be Q transpose? Q transpose will be given by q 1 transpose, q 2 transpose and q n transpose; q n is n by 1 vector, when I take its transpose then it becomes a row vector; here, q 1 is the first column of Q; q 1 transpose is going to be first row of Q transpose; then, q 2 transpose and q n transpose multiplied by q 1, q 2, q n. So, when I consider Q transpose Q it will be first row into first column, first row into second column and so on - the usual matrix multiplication .

The i jth element of Q transpose Q will be given by ith row here - that will be q i transpose multiplied by jth column here, so it is q i transpose q j; our notation is inner product of q j with q i. Q transpose Q is equal to identity. So, identity matrix means 1 along the diagonal and 0 elsewhere; q j q i is the i jth entry of q transpose q.

So, q transpose q is equal to identity is equivalent to saying that inner product of q j with q i will be 1 if i is equal to j 0 if i not equal to j, which will mean that the columns of q are ortho normal.

We had seen similar relation for unitary matrices, when we considered q star q - q is equal to identity; q star means conjugate transpose, q unitary means the columns of q are orthonormal - it is exactly - it is a special case.

Actually, that q transpose q is equal to identity; that means, the columns of q are orthonormal; now, orthogonal matrix also satisfies q q transpose is equal to identity.

From that we can deduce that the rows of q; they are going to be orthonormal. Orthogonal matrix is that matrix which has - if you look at any column its Euclidean norm is going to be equal to 1 and that column will be perpendicular to any other column and similar property holds for rows of matrix q.

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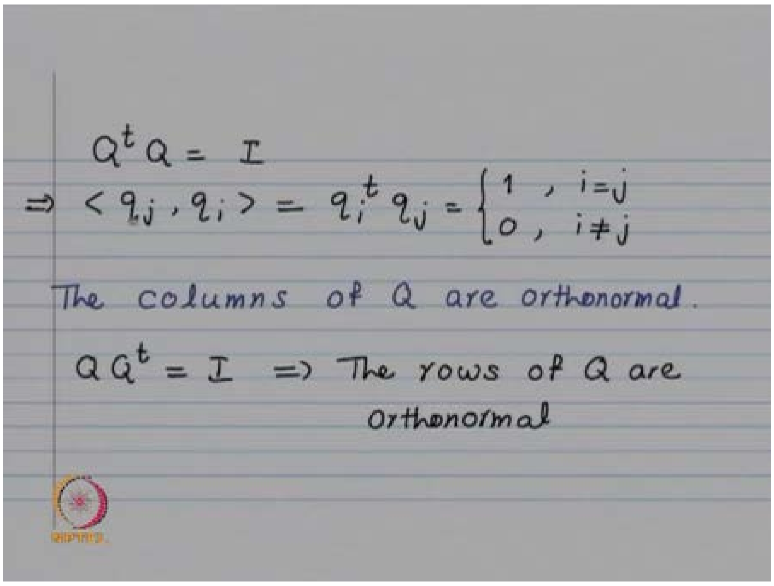

We have got Q transpose Q is equal to identity; that means, inner product of q j with q i is one if i is equal to j, 0 if i not equal to j.

So, the columns of q are orthonormal and q q transpose is equal to identity; that means, the rows of q are orthonormal.

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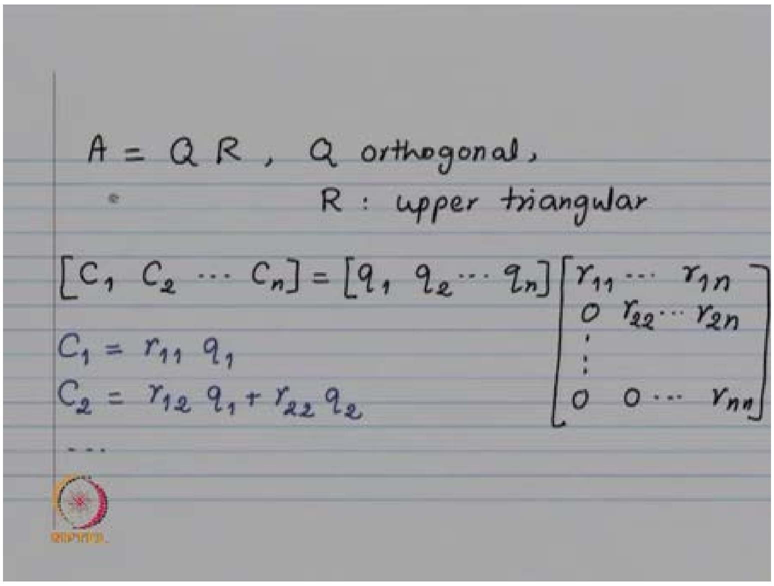

Now, we are trying to write A as Q into r, where Q is orthogonal and

R is upper triangular. So, let me write columns of A as c 1, c 2 and c n. So, c 1, c 2 and c n is equal to q 1, q 2 and q n and then here is upper triangular matrix. So, below diagonal call the entries are going to be 0.

Now, let me equate the columns; the first column c 1 will be nothing but r 1 1 times q 1 second column c 2 will be given by r 1 2 into q 1 plus r 2 2 into q 2 etcetera.

This is just property of matrix multiplication; we have got c 1, c 2 and c n to be columns of our original matrix A; suppose, we write it as q into r, then first column of a which is c 1 will be r 1 1 times q one.

So, c 1 is r 1 1 times q 1 the second column c 2 will be r 1 2 times q 1 plus r 2 2 times q 2; now, look at the first relation - c 1 is equal to r 1 one times q one.

Now, q 1 - just now we have seen that the columns of q they form an orthonormal say; that means, Euclidean norm of q 1 is going to be equal to one.

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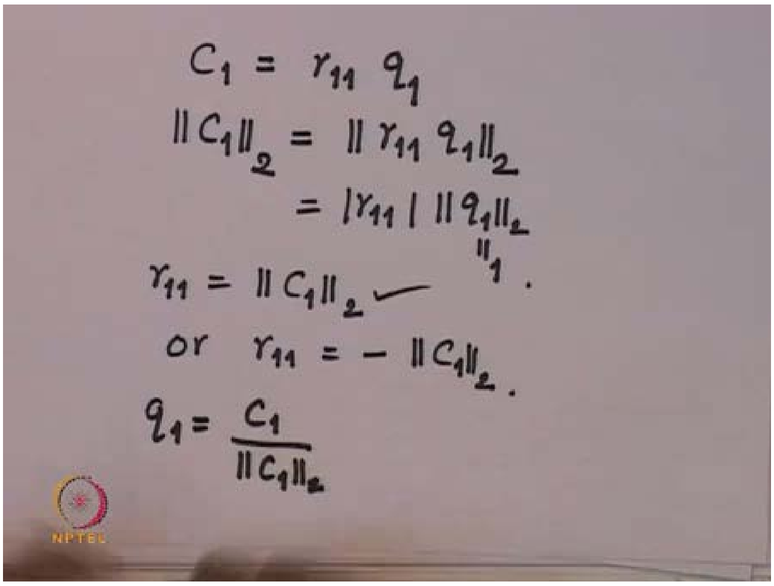

We have got c 1, which is the first column of A; this is r 1 1 times q 1, so norm of c 1 - its two norm is equal to norm of r 1 1 q 1; this will be modulus of r 1 1 and then norm of q 1 2 norm and this is equal to one.

For modulus of r 1 1 we have got a choice - you can either choose r 1 1 to be equal to norm of c 1 or r 1 1 is equal to minus norm of c 1 2; you get q 1 to be equal to c 1 divided by norm of c 1 2, if you choose this. So, the matrix a is given to us; that means, I know what its column c 1 is; I am trying to write it as q into r where q is orthonormal and r is upper triangular .

If you want to consider q 1 - the first column, it will be nothing but the c 1 normalized like the first column of a which we are denoting by c 1 it need not have Euclidean norm to be equal to one, so you divide by its norm.

Looking at c 1 is equal to r 1 1 into q 1, we get q 1 will be nothing - first of all r 1 1, we have to make a choice either you choose it to be positive or you choose it to be negative.

For the sake of definiteness, let us choose r 1 1 to be bigger than 0, then our r 1 1 will be nothing but c 1 divided by its Euclidean norm - that is for the first. That means, we have determined what should be the first column of q and what should be the entry r 1 1 in the upper triangular matrix.

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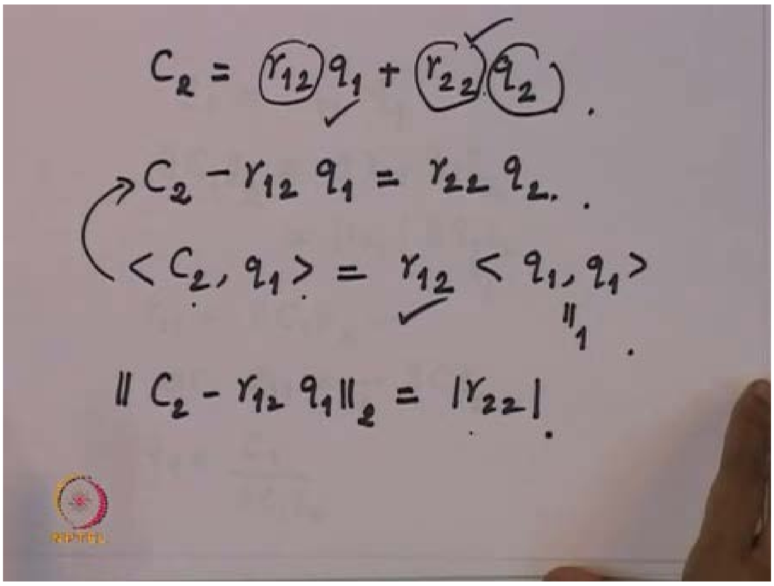

Now, in the second one, you have the relation c 2 is equal to r 12 q 1 plus r 22 q 2; we have already determined q 1, so, we have c 2 minus r 12 q 1 is equal to r 22 into q 2 .

Now, I need to determine what is r 12 what is r 22 and what is q 2; q 1 is determined; these are the three things to be determined. I make use of the fact that q 1 and q 2 are perpendicular to each otherl; inner product of c 2 with q 1 will be r 12 and inner product of q 1 with q 1 - this is going to be equal to 1.

This determines r 12, because c 2 is known q 1 is known take their inner product and that will be r 12; now, after this go to this relation and then take norm of c 2 minus r 1 to q 1 - its two norm - will be modulus of r 22 because norm q 2 is 1. So, once again choose r 22 to be bigger than 0 so that you would have determined r 22 and then q 2 will be nothing but - we have determined r 22 - q 2 will be c 2 minus r 12 divided by r 2 2.

In this manner we can determine the matrix q and matrix r. So, any invertible matrix can be written as product of q into r.

In our next lecture we will consider the relation between a is equal to q into r and Gram-Schmidt Orthonormalization process, then I will describe what is a q r method and then we will consider an efficient way of finding q r decomposition matrix - q r decomposition of a matrix by using what are known as reflector.

So, this we are going to do in the next lecture. Thank you.