Elementary Numerical Analysis
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Lecture No. # 36
Spectral Theorem

In our last lecture, we have considered Eigen values of some special matrices like, if matrix is self adjoint, then we saw that Eigen values are real. If A is Q self adjoint, that means, the conjugate transpose is equal to minus of the matrix A, then the Eigen values are purely imaginary or they are 0. Then, for normal matrix; that means, if A star A is equal to AA star, we saw that if lambda is an Eigen value of A, then lambda bar, the complex conjugate is Eigen value of a star, whereas Eigen vector, it remains the same.

Now, using this result, we are going to show that for a normal matrix, Eigen vectors corresponding to distinct Eigen values; they are going to be perpendicular to each other. If we are looking at a general matrix, then Eigen values corresponding to distinct Eigen values, they are linearly independent. For normal matrices, we have got something more. So, we are going to look at this and then, we will consider Eigen values of unitary matrices.

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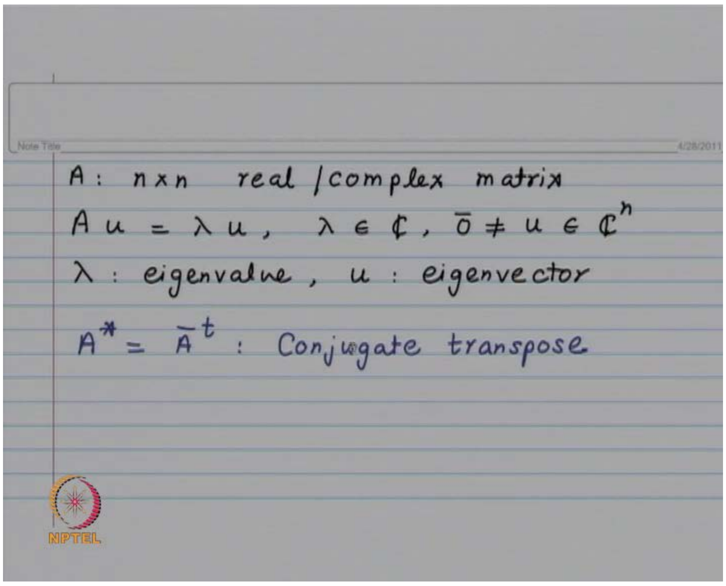

So, our notation is, A is n by n, either real or complex matrix. Au is equal to lambda u, where lambda is a complex number and u is a non 0 vector in c n. A star is equal to A bar transpose, that is the conjugate transpose.

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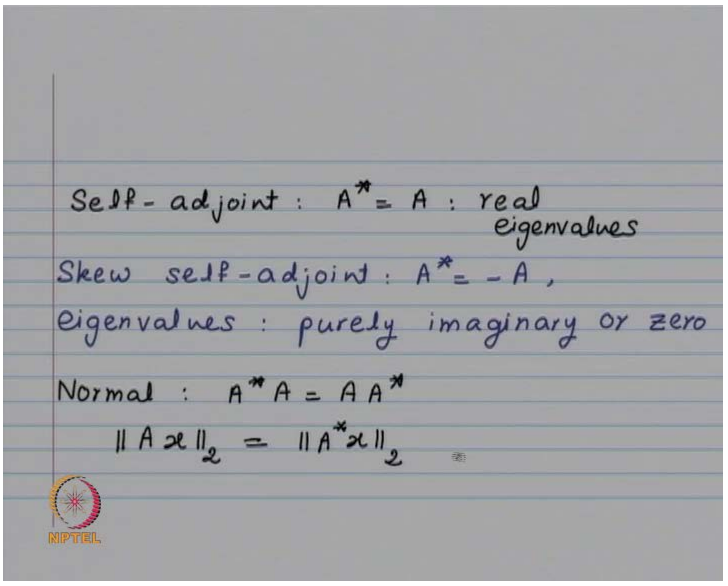

Self-adjoint matrix, that means, A star is equal to A, they have real Eigen values. Skew self-adjoint A star is equal to minus A, then the Eigen values are either purely imaginary or 0. For a normal matrix, A star A is equal to A star and using this, one shows that norm of Ax 2 norm, Euclidian norm is going to be equal to norm of A star x 2 norm and as a consequence of this result, Au is equal to lambda u if and only if A star u is equal to lambda bar u. Now, let us look at a unitary matrix, that means, A star A is equal to AA star is equal to identity.

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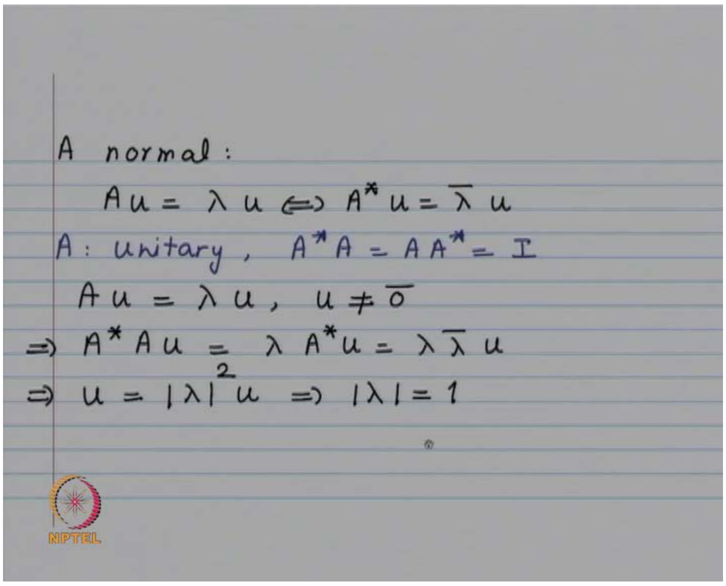

So, we have in particular, unitary matrix is a normal matrix. So, Au is equal to lambda u apply A star. So, we will have A star Au is equal to lambda times A star u, but A star u will be lambda, lambda bar u. So, it is going to be equal to lambda, lambda bar u and thus, we get u is equal to modulus of lambda square u. Since, u is a non 0 vector; it implies that mod lambda is equal to 1. So, thus for a unitary matrix, all the Eigen values, they are going to lie on unit circle.

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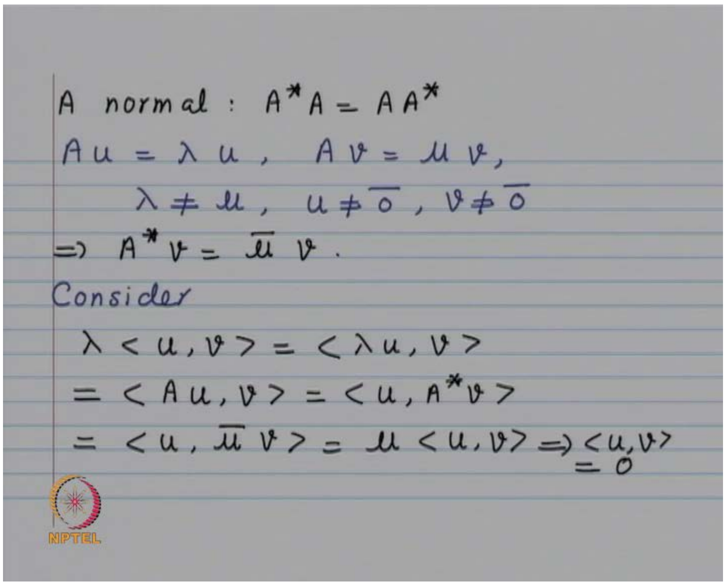

So, now we look at the case of distinct Eigen values for normal matrices. So, A star A is equal to AA star. Let Au v equal to lambda u. Av is equal to mu times v. So, we look at 2 distinct Eigen values, lambda and mu and u and v are associated Eigen vectors. So, we want to show that, inner product of u with v is going to be equal to 0. So, u is going to be perpendicular to v.

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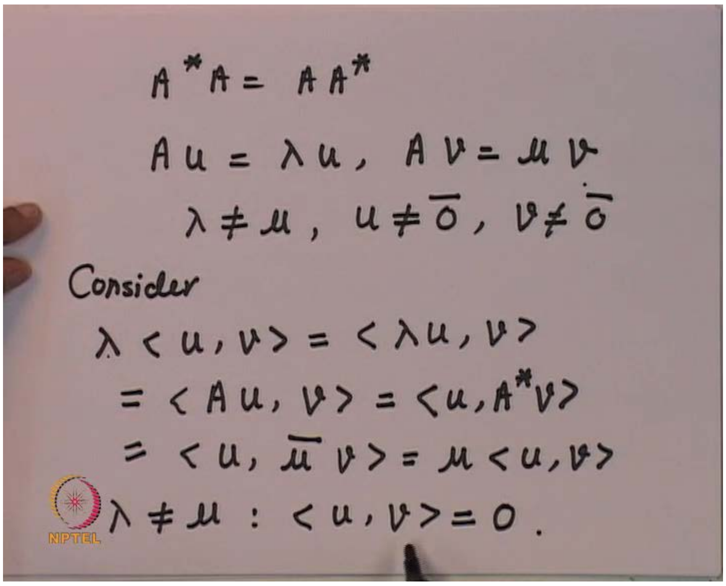

So, we have A star A is equal to AA star, Au is equal to lambda u, Av is equal to mu times v, where lambda is not equal to mu, u being a Eigen vector, it is not a 0 vector, v also is a non 0 vector. So, let us look at, so consider lambda times inner product of u with v. This will be lambda u, v using linearity of inner product in the first variable. So, this lambda goes inside as lambda u. Now, Au is equal to lambda u, so this will be inner product of Au with v. We have seen that A will go to the second variable as A star. So, it will be u A star v. Now, since Av is equal to mu times v, A star v will be equal to mu bar v.

So, this will be equal to u mu bar v and now, the inner product is conjugate linear in the second variable, so this mu bar will come out as mu. So, this will be mu time inner product of u with v. Since, lambda is not equal to mu; we get inner product of u with v to be equal to 0. So, if u and v are Eigen vectors corresponding to distinct Eigen values, then we get them to be perpendicular. So, this is property of normal matrices. In general, it will not be true.

Now, what we want to do is, we want to consider similar matrices. We want to show that similar matrices, they have the same set of Eigen values and algebraic multiplicity as well as geometric multiplicity that is going to be preserved.

So, let us first define what a similar matrix is. So, we have A and B, 2 matrices. They will be similar, if there exists an invertible matrix P, such that P inverse AP is equal to B. So, this is definition of similar matrices.

We are going to show that, they have they are going to have the same Eigen values. So, when you want to find Eigen values of A, if you want to simplify your matrix, then what is allowed, is similarity transformation elementary row transformations, which we used in Gauss Elimination method. They will change the Eigen values. So, they are not allowed, but similarity transformations, they will preserve Eigen values with algebraic multiplicity as well as geometric multiplicity.

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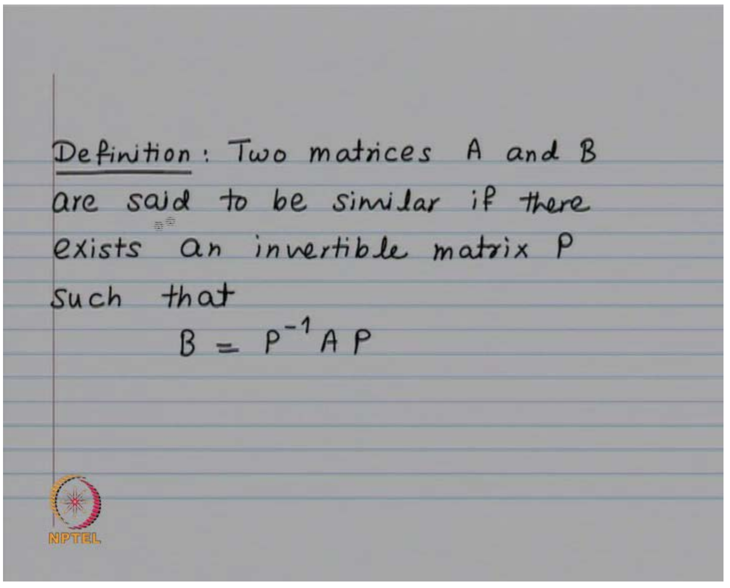

So, this is 2 matrices A and B of the same size are said to be similar, if there exists an invertible matrix P, such that B is equal to P inverse AP.

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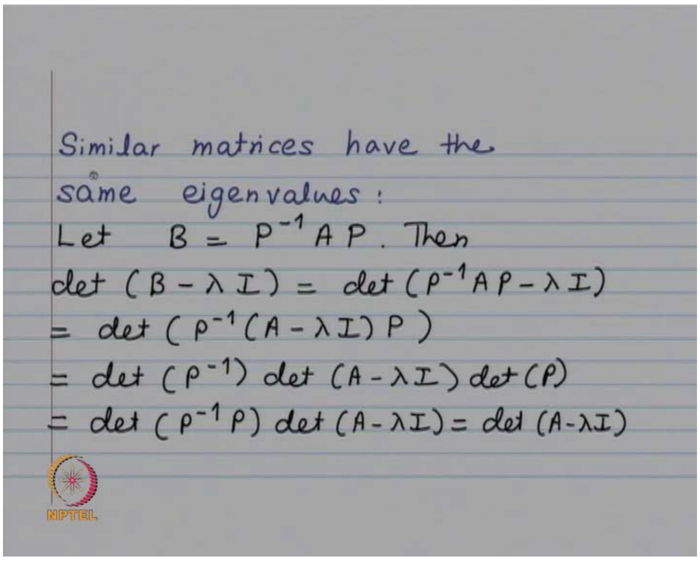

Now, this is our claim that similar matrices, they have the same set of Eigen values. So, what we are going to do is, we are going to look at the characteristic polynomial. We will show that the characteristic polynomial of matrix B is same as the characteristic polynomial for matrix A.

The Eigen values are nothing, but roots of the characteristic polynomial. So, if we show that they have the same characteristic polynomial, it will mean that they will have the same Eigen values. Then, the algebraic multiplicity is defined as you factorize. So, you have got characteristic polynomial in that, suppose lambda 1 is one of the Eigen value, then you look at lambda 1 minus lambda raise to m 1. You factorize. So, whatever is the power that is the algebraic multiplicity.

So, once we show that matrix B and matrix A, they have the same characteristic polynomial. It will also follow that the algebraic multiplicities, they are preserved and showing the characteristic polynomial. They are the same is by using properties of determinant.

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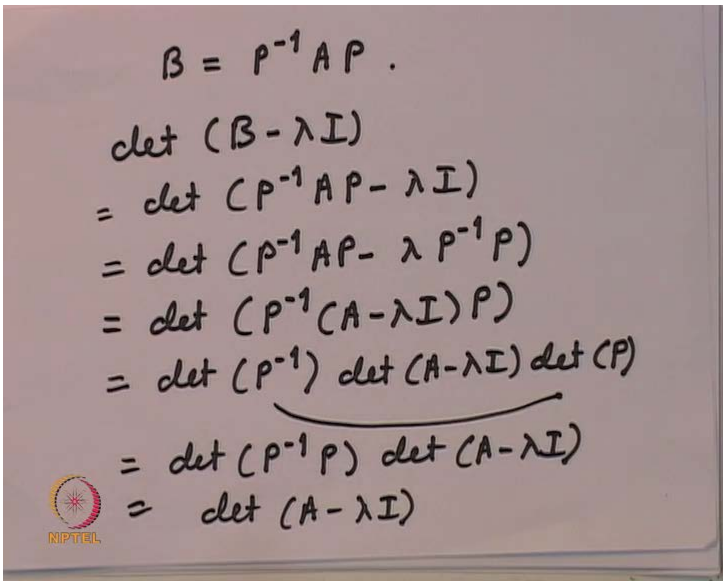

So, we have to look at say, we have got our B is equal to P inverse AP, then we look at determinant of B minus lambda I. That is the characteristic polynomial. This is going to be equal to determinant of P inverse AP minus lambda I substituting for B. This will be determinant P inverse AP minus lambda times P inverse P. For the identity I write P inverse P. This is determinant P inverse A minus lambda I P.

Now, using the property of determinant, this is going to be equal to determinant P inverse determinant A minus lambda I and determinant P. Now, let me combine these two. So, that is going to be determinant P inverse P, determinant A minus lambda I, and determinant of identity matrix is 1. So, we get determinant of A minus lambda I. So, thus both B and A, they have the same characteristic polynomial and hence, the same set of Eigen values with preservation of algebraic multiplicity.

Now, let us look at geometric multiplicity. The definition of geometric multiplicity is number of linearly independent Eigen vectors associated with the Eigen value. So, now we want to show that, matrix B, which is P inverse AP and matrix A, they have Eigen values with the same geometric multiplicities. B is equal to P inverse A P. So, Bu will be equal to P inverse AP u.

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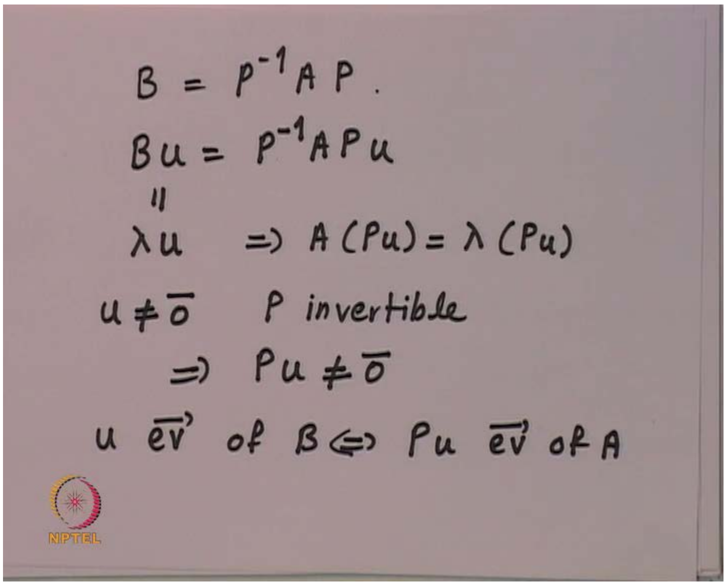

Suppose, Bu is equal to lambda u, then what we get is, so we have got P inverse APu is equal to lambda u. So, that implies A of Pu is equal to lambda times Pu. I am pre multiplying by P to get this. Now, u naught equal to 0 vector, it is an Eigen vector. P invertible, it implies that P of u also naught equal to 0 vector because if it were equal to 0, we can multiply by P inverse and get u to be equal to 0. So, that means, u Eigen vector of B implies Pu Eigen vector of A and converse also is true. So, this is going to be if and only if, let us look at the number of linearly independent Eigen vectors associated with B and lambda.

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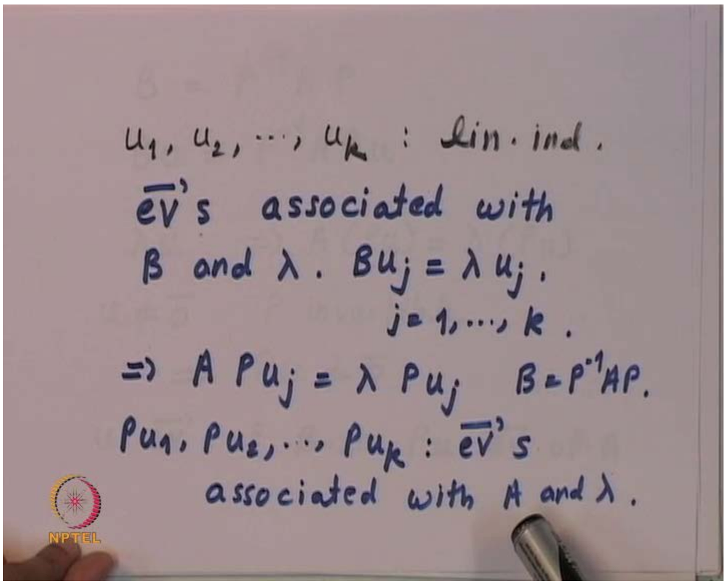

Suppose, you have got u 1, u 2 up to u k, these are linearly independent associated with B and lambda. That means, B of u j is equal to lambda times u j, j is equal to 1 to up to k. Now, just now we have seen that this implies that A Pu j is equal to lambda times Pu j. Our B was P inverse AP. So, this will mean that Pu 1, Pu 2 up to Pu k, these will be Eigen vectors associated with A and lambda. So, I am assuming that lambda is Eigen value of B with geometric multiplicity k. Then, I got k Eigen vectors associated with A and lambda. If I can show that these are linearly independent, then that will mean that geometric multiplicity of lambda as an Eigen value of A also is going to be k.

So, see there is a one to one correspondence between Eigen vectors of B and Eigen vectors of Au is Eigen vector of B. That will mean that P into u is Eigen vector associated with A and conversely. So, now, we are assuming that lambda is Eigen value of B with geometric multiplicity, say k. Then, you look at k linearly independent Eigen vectors associated with B and lambda. Then, Pu 1, Pu 2, Pu k, these will be Eigen vectors of A. So, only thing which remains to show is that Pu 1, Pu 2, Pu k, they are linearly independent.

Now, they will be linearly independent because P is invertible matrix. If P is not invertible, then such a result is not true. So, let me show quickly the linear independence.

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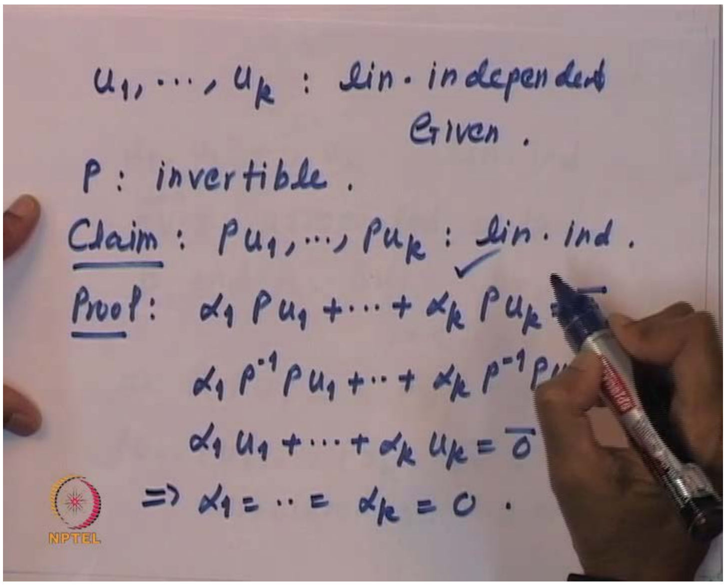

So, we have got our assumption is u 1, u 2, u k, these are linearly independent. This is given. Then, P is invertible. Claim Pu 1, Pu k are linearly independent and for the proof of this claim, we start with alpha 1, Pu 1 plus alpha k, Pu k is equal to 0 vector. If this implies that alpha 1, alpha 2 and alpha k, they all have to be 0, then that will mean that these are linearly independent. Now, in order to show that I am going to make use of the fact, that u 1, u 2, u k, are linearly independent and P is invertible. So, multiply by P inverse. So, we will have alpha 1 P inverse Pu 1 plus alpha k P inverse Pu k is equal to 0 vector. You are multiplying by P inverse P inverse into 0 vector is 0 vector. Now, this is nothing, but alpha 1 u 1 plus alpha k u k to be 0 vector and by using the fact, that this is linearly independent, it follows that alpha 1 is equal to alpha k is equal to 0. So, this proves our claim.

Similar matrices, they have got the same Eigen values with preservation of algebraic multiplicity and geometric multiplicity. So, if we have got matrices which are upper triangular matrices or diagonal matrices, we can calculate their Eigen values. In these two cases, the Eigen values, they are nothing but the diagonal entries. So, it will be good if I can find an invertible matrix P, such that P inverse AP is either a diagonal matrix or an upper triangular matrix. Now, P inverse AP is equal to D. Such matrices, they are known as diagonalizable matrices. That given a matrix A, if you can find the invertible matrix P, such that P inverse AP is equal to D, then the matrix is called diagonalizable. All matrices, they are not diagonalizable.

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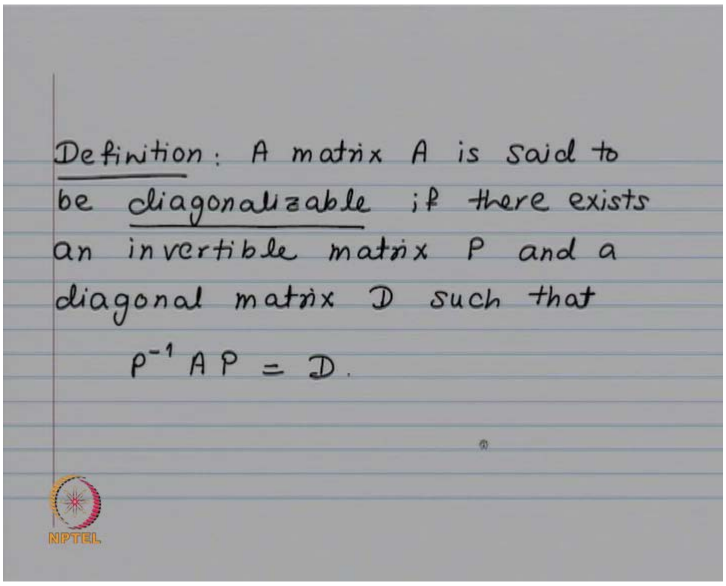

So, what we are going to do is, we are going to prove a characterization for the diagonalizable matrices and then, using that the characterization, one can give a counter example to show that not all matrices are diagonalizable. So, a matrix A is said to be diagonalizable, if there exist an invertible matrix A and a diagonal matrix D, such that P inverse AP is equal to D. So, this is the definition.

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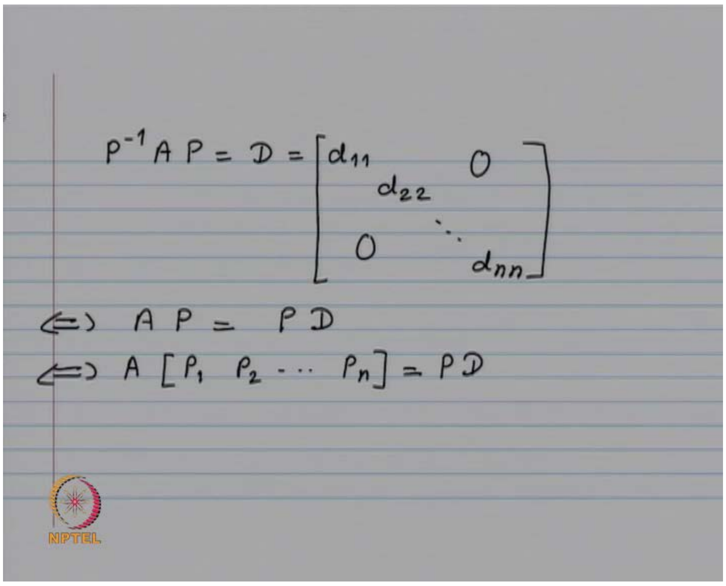

Now, what is P inverse AP is equal to D diagonal matrix d 11, d 22, d nn? So, this will mean that A into P is equal to P into D. Let me write columns of P as P 1, P 2, P n. So, we have got A times P 1, P 2, P n is equal to P times D.

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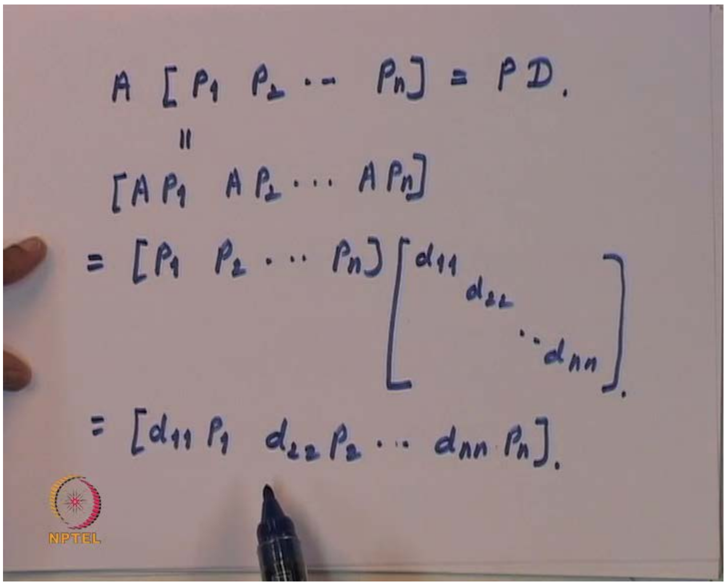

So, when you consider A and then, P 1, P 2 up to P n is equal to P into D. Now, this is nothing, but AP 1, AP 2 up to AP n. That is the property of matrix multiplication. What will be P into D? It will be P 1, P 2 up to P n multiplied by diagonal matrix d 11, d 22, d nn.

So, when you post multiply by a diagonal matrix, the first column will be d 11 P 1. The first column will get multiplied by d 11, second one will be d 22 P 2 and d nn P n. So, this is first column, second column and nth column. Here, you have got first column, second column, nth column and hence, what we have is AP j is equal to d jj P j. P is invertible matrix. P j is its jth column because it is invertible; P j will not be a 0 vector. If you have got one column to be 0, then that matrix is not invertible. So, this will mean that d jj, they are Eigen values of A and P j, these are Eigen vectors. P j is jth column of P and again, since P is invertible, P 1 P 2 up to P n, this set is linearly independent.

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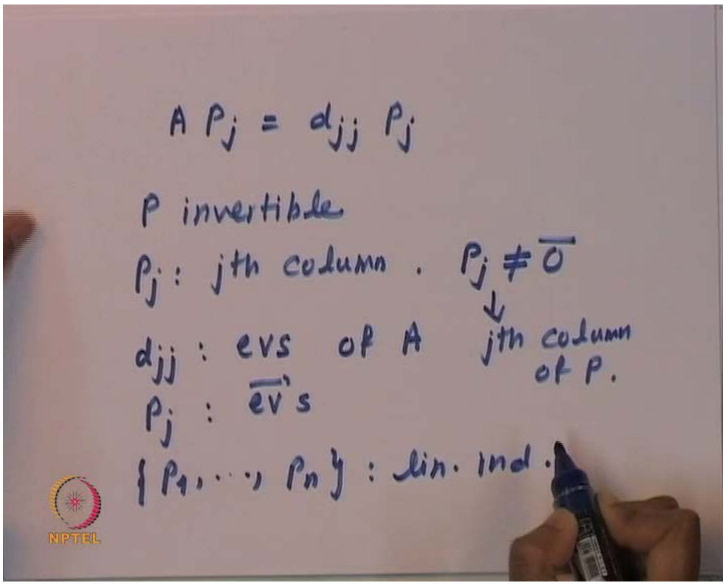

So, thus our matrix A will be diagonalizable if and only if, you have got n Eigen vectors linearly independent associated with A. We have seen example of a 2 by 2 matrix. For that matrix, it was upper triangular matrix with all entries to be equal to 1. So, the only Eigen value is 1 and there was only one Eigen vector associated. So, this one Eigen value 1 had algebraic multiplicity to geometric multiplicity 1, so only 1 linearly independent Eigen vector. So, it is a 2 by 2 matrix and you have got only 1 linearly independent Eigen vector.

So, now we have seen that a matrix is diagonalizable if and only if, you are looking at a matrix of size n, then it should have n linearly independent Eigen vectors. So, thus not all matrices, they are going to be diagonalizable, but we will see there is a big class of matrices which is going to be diagonalizable.

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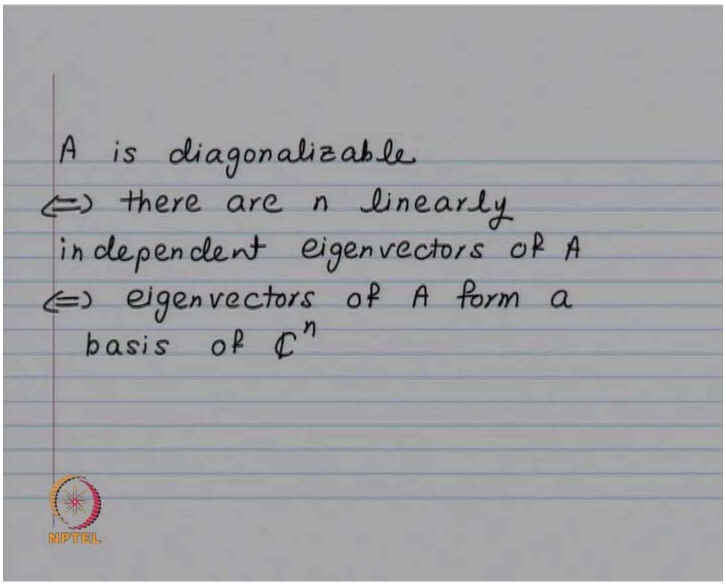

So, we have A is diagonalizable if and only if, there are n linearly independent Eigen vectors of A and that will mean that Eigen vectors of A, they will form a basis of C n because the dimension of C n is n. It is a finite dimensional space. So, if there are n independent Eigen vectors, they will also form a basis.

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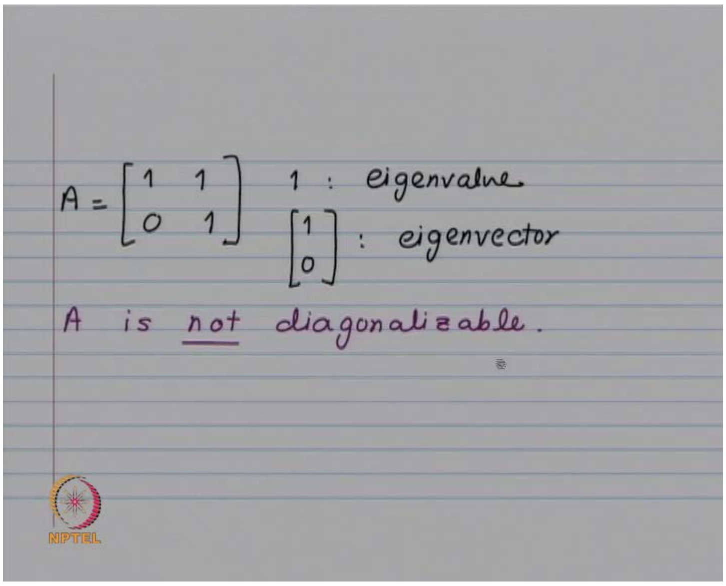

So, here is that example, A is equal to 1 1 0 1 is the only Eigen value. 1 0 or any non-0 multiple of it is going to be an Eigen vector. So, such a matrix is not diagonalizable. So, now not all matrices are diagonalizable, but when we are looking, we are interested in

Eigen values of matrix. So, even if I cannot reduce it to a diagonal form, if by using elementary, not elementary low transformations, by using similarity transformations, if i can reduce it to upper triangular form, then that will suffice because then I have got P inverse AP is equal to upper triangular matrix.

The Eigen values of A, they are same as Eigen values of upper triangular matrix and Eigen values of upper triangular matrix, they are the diagonal entries. So, whether all matrices, they can be equivalent similarly or whether, they can be similar to an upper triangular matrix. So, the answer is yes and that is Schur's theorem.

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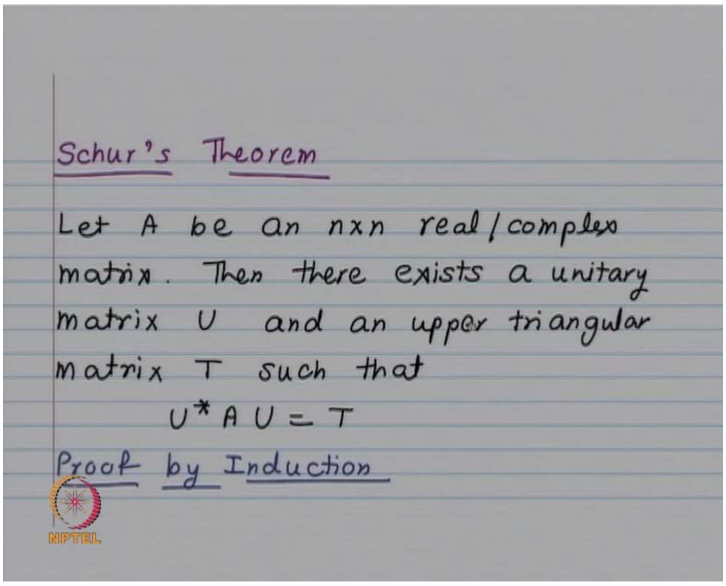

So, we have got, suppose A is n by n real or complex matrix. Then, there exist a unitary matrix u. So, not only invertible, but we have got something better. A unitary matrix U and an upper triangular matrix T, such that U star AU is equal to T.

Now, look at the statement of Schur's theorem. It is very important that it says, then there exists. So, the statement which we are making is existential, that we are saying that there exist some U, which is unitary, such that U star AU is equal to T. A and T, they are going to have the same Eigen values. There cannot be a constrictive proof for this theorem because if you have got constrictive proof, it will mean that you can find its Eigen values. Given any matrix A, you can find its Eigen values. Eigen values, they are related to finding the roots of the polynomial. Now, as soon as your polynomial is of degree bigger than or equal to 5, there cannot exist a formula. Like, if you have got a quadratic polynomial, then we know how to write down its roots. So, such a thing is not possible, but still it is an important theorem and the proof is by induction, but I am going to skip the proof.

What we are going to do is we are going to look at the special cases. Like now we know that for any matrix U, you can write U star AU is equal to T. So, what happens if your matrix is self-adjoint or if it is Q self-adjoint? So, let us see what one can deduce for these special matrices.

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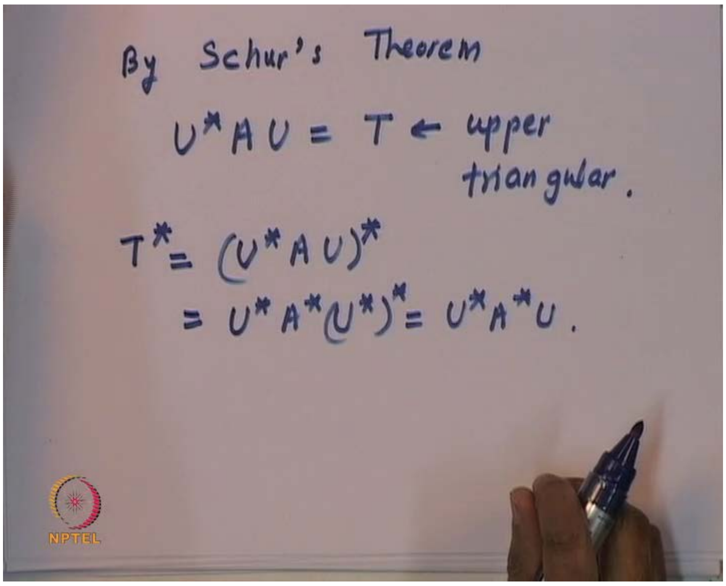

So, we have got by Schur's theorem. U star AU is equal to T, where T is upper triangular. So, what will be T star? T star will be U star AU its star. Now, this will be AB star is B star A star. So, it will be U star A star and then, U star its star. So, it is going to be equal to U star A star U. So, thus we have got T is equal to U star AU, then T star will be equal to U star A star U. So, if a star is equal to A, this will imply that T star is equal to T, T upper triangular. Then, T star is going to be lower triangular.

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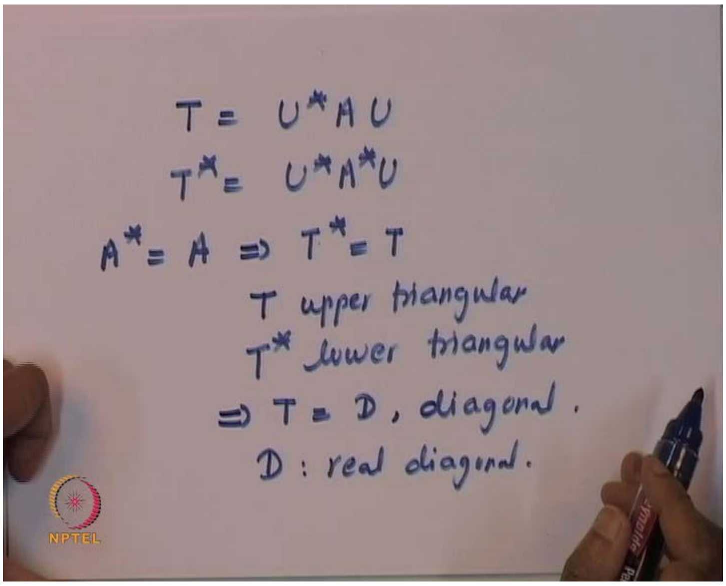

So, you have got a lower triangular matrix is equal to upper triangular matrix, and that will imply that T is going to be a diagonal matrix D because left hand side is lower triangular, right hand side is upper triangular. So, it has to be diagonal and because you have got D star is equal to D, you are taking here conjugate transpose. So, that will mean that D is going to be real diagonal.

Now, it fits in our, whatever we have been saying. The Eigen values of A are same as the Eigen values of T. Now, we showed that T is going to be a diagonal matrix and the Eigen values of A are the diagonal entries. It is going to be a real diagonal matrix. So, for self-adjoint matrix, the Eigen values which are going to be diagonal entries, they are going to be real. This part we have seen earlier, that self-ad joint matrices, they have got real Eigen values.

Now, the same idea or the same proof, it tells us that if A is Skew self-adjoined, then your again T will be a diagonal matrix, but now the entries, diagonal entries, they will be either 0 or purely imaginary.

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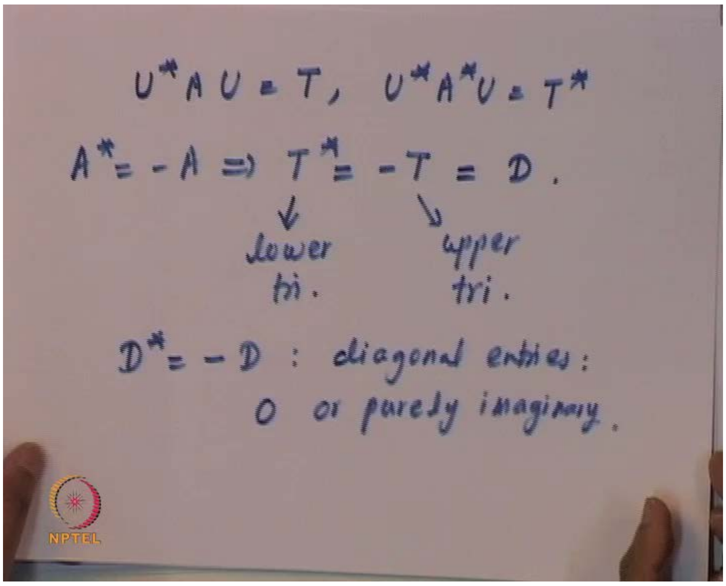

So, we have got U star AU is equal to T, U star A star U is equal to T star, A star is equal to minus A implies T star is equal to minus T. This is lower triangular, this is upper triangular. So, it has to be equal to a diagonal matrix D and D star will be equal to minus D. So, diagonal entries will be 0 or purely imaginary.

Now, let us look at normal matrix. So, for the normal matrix, it is not (()) to see that in this case also T is, in fact a diagonal matrix. So, that we are going to do as a tutorial problem. So, that means, the self-adjoint matrices, then skew self-adjointer matrices and more generally, normal matrices, they are going to be diagonalizable.

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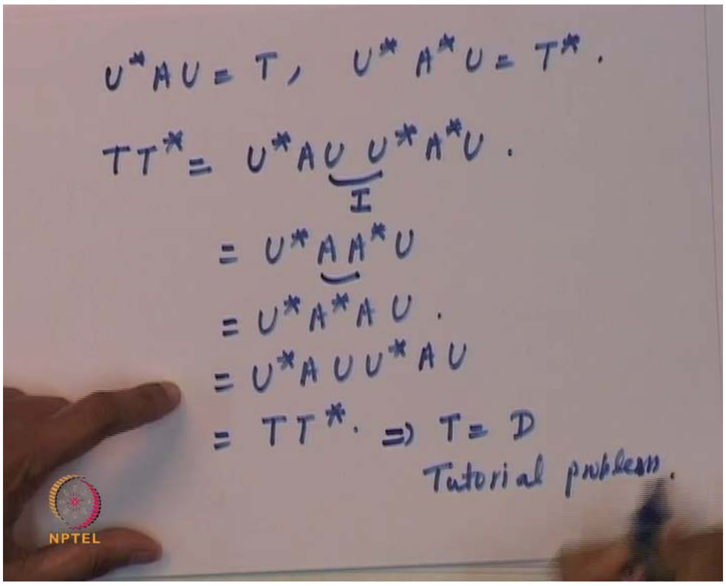

For diagonalizability, what we wanted was existence of an invertible matrix, such that P inverse AP is equal to D. Now, we have got something more. We have got a unitary matrix, where U star is equal to U inverse. So, we have got U star AU is equal to D for normal matrices and this is known as Spectral theorem. So, we have for the normal matrix, let me write down U star AU is T, U star A star u is equal to T star. So, consider TT star, that is going to be equal to U star AU and U star A star U. This is identity. So, this will be U star AA star U.

Now, normal matrix, so this will be U star A star AU and now, let me introduce UU star here, which is identity, AU and this is nothing, but TT star and this implies T to be a diagonal matrix, that is going to be a tutorial problem.

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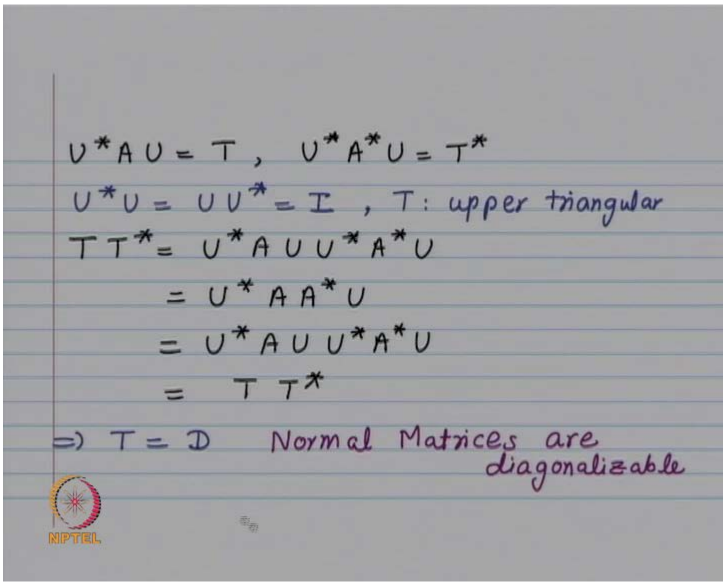

So, normal matrices, they are going to be diagonalizable. So, not all matrices are diagonalizable, but at least we have got a big class of matrices, which is the class of normal matrices, they are going to be all diagonalizable. Now, there is another class of matrix, that is, if your matrix has n distinct Eigen values. In that case, the corresponding Eigen vectors, they are going to be linearly independent. So, in C n, we will have n linearly independent vectors. They will form a basis and we saw that diagonalizability, it means existence of a basis of Eigen vectors.

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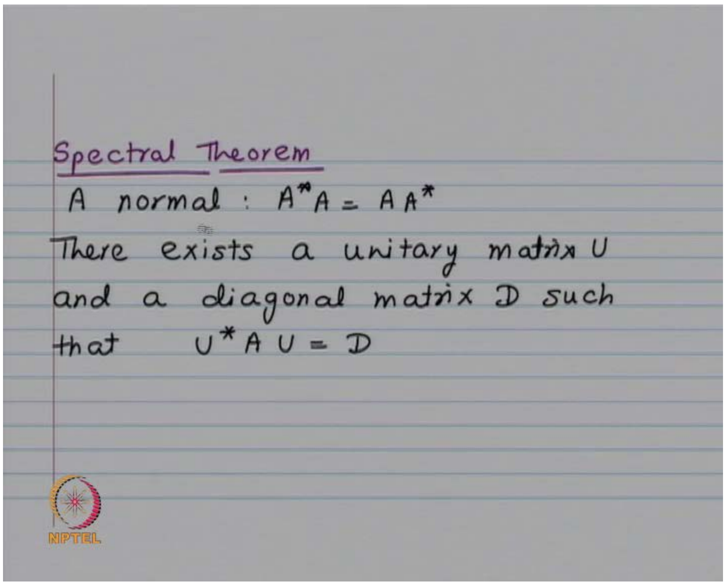

So, if you have got a matrix with distinct Eigen values, then you are going to have P inverse AP is equal to D. In this case, matrix P will be only invertible, it need not be unitary, the unitary matrix, that is for normal matrices. So, we have, this is the Spectral theorem. If A is normal, that means, A star A is equal to AA star, then there exists a unitary matrix U and a diagonal matrix D, such that U star AU is equal to D. So, once again, I want you to notice that they exist. We are not giving a recipe. If we could have done, that would have been ideal, but that is just not possible. So, we are going to have U star AU is equal to D.

Now, using Spectral theorem, I want to calculate or I want to get an expression for Euclidian norm of matrix A. We had defined matrix norms, induced matrix norm. So, we had norm A 1, norm A infinity and norm A 2. Norm A 1 is nothing, but column sum norm. Norm A infinity is the row sum norm. So, these 2 norms, you can compute. We have got a formula in terms of the elements of the matrix, whereas for norm A 2, we had only an upper bound. Upper bound is that Frobenius norm. We are going to show that if A is a normal matrix, then norm A 2 will be modulus of the biggest Eigen value.

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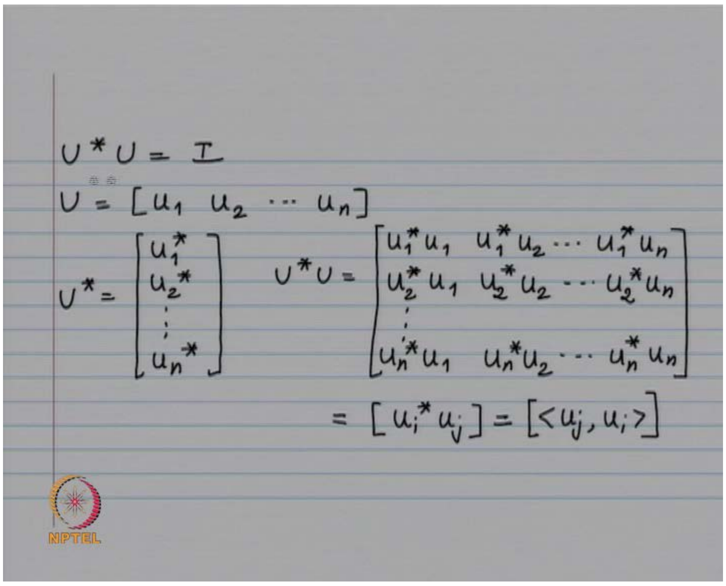

Now, before we do that, I want you to notice that U star U is equal to identity. It means the columns of U, they are Orthonormal. So, we have got U is a matrix. I denote its columns by U 1 U 2 U n. U star will be U 1 star U 2 star and U n star. So, when I look at U star U, this is going to be this multiplied by this. So, it is going to be U 1 star U 1, U 1 star U 2 and U 2 star U n. Then, U 2 star U 1, U 2 star U n and U n star U 1, U n star U n and this is equal to identity matrix. It will mean that U 1 star U 1 will be 1. So, that is nothing, but the Euclidian norm of U 1. U 1 star U 2 will be 0, which we mean that U 1 and U 2 are perpendicular. Similarly, U 1 and U n will be perpendicular. So, this U 1 U 2 U n, which are columns of u, so we are going to have columns of U are Orthonormal.

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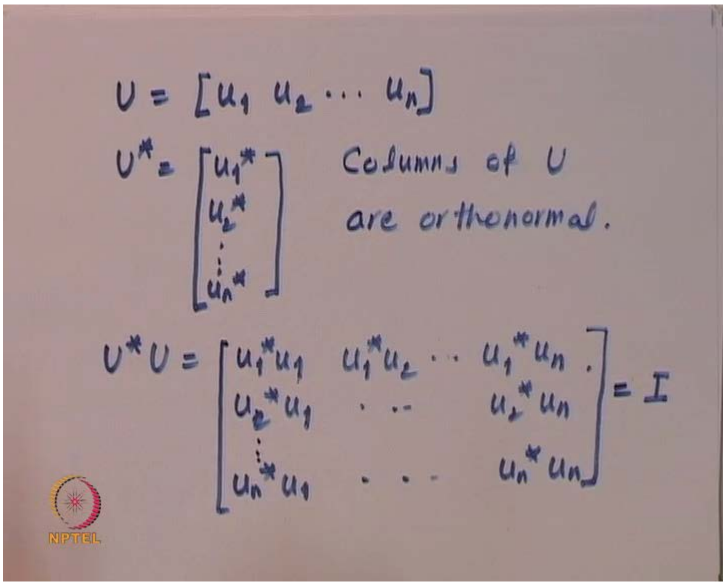

So, if you have got an invertible matrix, then its columns, they are linearly independent. If you have got a unitary matrix, that means, the inverse of U is nothing, but its conjugate transpose. Then, the columns of U, they are Orthonormal. That means, any 2 distinct columns, they will be mutually perpendicular and the Euclidian length of each column vector is going to be equal to 1. Similar result is true for row vectors.

So, for the row vectors, we have to use the fact that UU star is equal to identity. Now, we have got by spectral theorem for normal matrix u star Au is equal to D. So, the entries on the diagonal of D, those are our Eigen values and columns of u, those are our Eigen vector. So, that means, for normal matrix, we have got Eigen vectors to be orthonormal. We already saw this for distinct Eigen values.

If you have got A to be a normal matrix, lambda and mu to be distinct Eigen values corresponding Eigen vectors, they are mutually perpendicular. What we are saying now is, if A is a normal matrix, there is a basis of, there is an orthonormal basis which consists of Eigen vectors.

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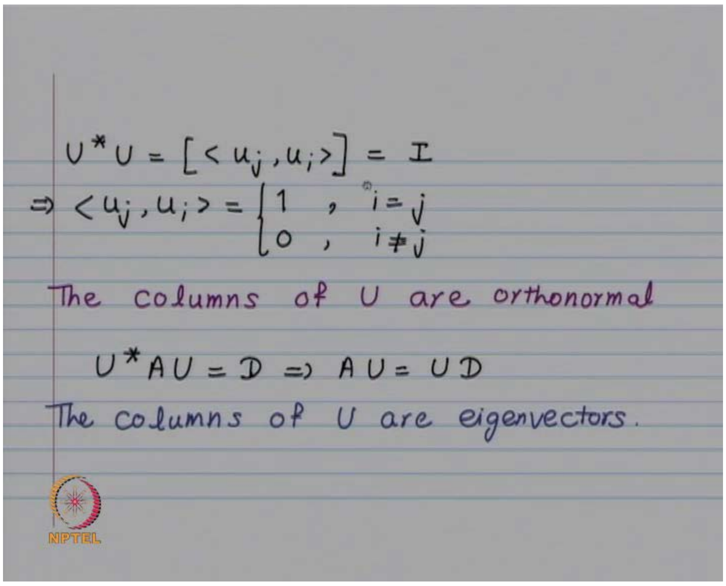

So, we have u star u is equal to identity. That means, inner product of u j with u i is 1 if i is equal to j and 0, if i not equal to j. The columns, that means, the columns of u are orthonormal. u star Au is equal to D. That means, Au is equal to u into D. So, the columns of u are nothing, but Eigen vectors.

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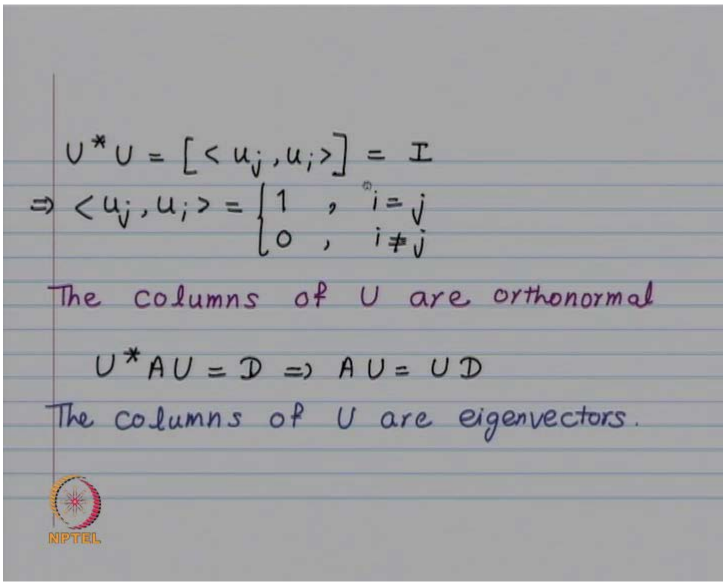

So, we can state Spectral theorem as if A is normal, then A has n orthonormal Eigen vectors, not just vectors, but Eigen vector. So, you have got Au j is equal to lambda j u j, j is equal to 1, 2 up to n. These lambda j's, they need not be distinct. Those are the Eigen values which may be repeated.

Now, consider any z belonging to C n a vector in C n. U 1, u 2, u n is going to form a basis. So, I can express z as a linear combination of u 1, u 2, u n. So, z is summation j goes from 1 to n alpha j u j. If you take inner product of z with u k, this is going to be equal to summation j goes from 1 to n alpha j u j u k. Using linearity of inner product in the first variable, this will be summation j goes from 1 to n alpha j, u j, u k. This is going to be one only when j is equal to k. So, this is equal to alpha k. So, any vector z can be written as summation j goes from 1 to n, z, u j u j.

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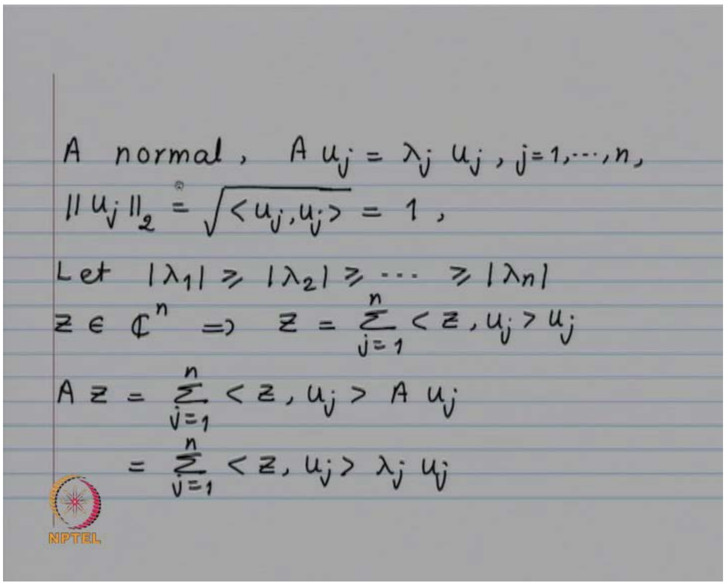

So, we have for normal matrix Au j is equal to lambda j u j norm u j to be equal to 1. Let me arrange Eigen values lambda 1 in the descending order of modulus, mod lambda 1 bigger than or equal to mod lambda 2 bigger than or equal to mod lambda n. Then, just now we saw that z can be written like this, this combination. So, A z will be take A inside A u j is equal to lambda j u j. So, this is going to be for A z.

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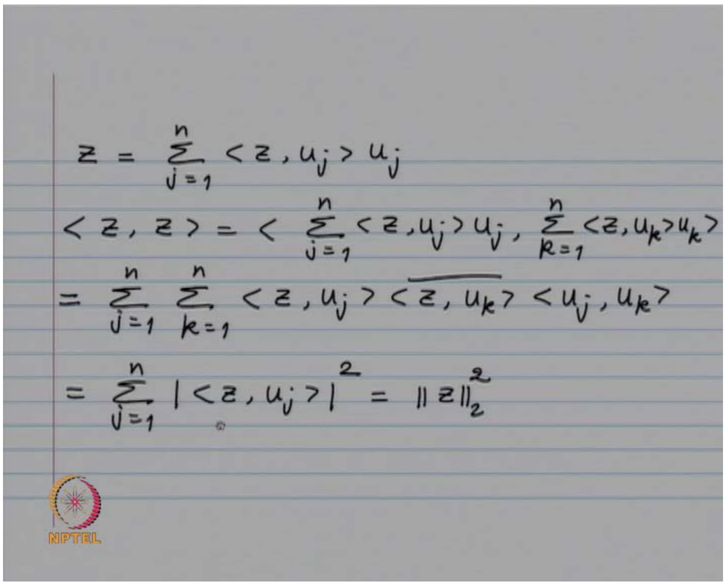

Now, norm z square is going to be nothing, but summation j goes from 1 to n modulus of z, u j square. So, we are going to look at what I am trying to show is Euclidian norm of A or norm A 2 is going to be equal to modulus of lambda 1. So, for that, we make use of fact that if A is normal, there are n orthonormal Eigen vectors, write any vector z as a linear combination, consider A z because for norm A 2, we have to look at maximum of norm A z by norm z. So, this proof, I will complete in the next lecture and then, we are going to look at some localization results for Eigen values and then approximate methods for calculating Eigen values. So, thank you