Elementary Numerical Analysis

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Lecture No. # 35

Eigenvalues and Eigenvectors

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Today we are going to start a new topic and that is the eigenvalue problems. So, far we considered real vectors real matrices ,now even if matrix is a real matrix its eigenvalues they can be complex. So, that is why now our underlying field is going to be field of complex numbers. So, we will be considering complex matrices then the vectors also will be complex eigenvalues they are defined for square matrices.

We will show that if A is n by n matrix either real or complex then its roots are the eigenvalues they are given by roots of a polynomial of degree n, now as a consequence of fundamental theorem of algebra.

We know that if a polynomial has degree n then it has got exactly n 0 or n roots counted according to their multiplicity,that means, we will count if a 0 is repeated twice ,it will be considered as two zeroes.

Now, when we consider polynomial of degree bigger than or equal to 5, then we cannot have a formula for finding its roots like, if you have got a quadratic polynomial then we can write its two 0 in terms of the coefficients of our polynomial.

If you have got a x square plus b x plus c is equal to 0 ,then the roots can be written in terms of the coefficients a b c .This will not be possible, when your polynomial is of degree bigger than or equal to 5.

So, that is why for calculating the eigen values our methods ,they are going to give us only approximation. This was not the case with solution of system of linear equations

When we considered gauss elimination method or its variants, then the error came because of the finite precision whereas, the method was exact method in contrast for eigen values our method will be giving only an approximation. So, 1 tries to find as much information possible as of eigen values by say looking at a matrix.

So, there are some special matrices for which we will study what are their their eigenvalues; that means, we can if the matrix is a real symmetric matrix then its eigenvalues they are going to be all real and similar results then we will have some localization results, that means, we will find a region in the complex plane which is going to contain all our eigen values.

We are going to consider power method for finding the dominant eigen value of a matrix and then there are some variants of this method ,I am going to describe what is known as q r method for finding eigen values.

At present that is the most popular and the best possible method available for calculating eigen values or rather calculating approximations to eigen values of our matrix a. Now it is beyond this course, to prove convergence of q r method the description of q r method can be given easily and that is what I will do.

So, now we are going to start with complex vectors .When we consider the real vectors and complex vectors for real vectors ,what we had done was you can add 2 vectors. So, that is component wise addition you multiply a vector by a scalar. So, you multiply each component of your vector by that number. So, these things remain same for complex vector

It will be the real numbers they are replaced by complex numbers. So, again addition of 2 vectors will be component wise multiplication by a scalar will be same as before then matrix into vector multiplication will be exactly same as before.

There will be a change in the definition of inner product because we have to take into consideration the complex numbers then we had defined one norm infinity norm for real vectors that definition remains exactly the same the corresponding induced matrix norm the proof will have slight modifications,,,, but ,, but ,,, but ,, but let me not get into those details.

It they are the formula which you obtain is exactly the same as before. So, . So, now, let us quickly consider complex vectors then the inner product the vector norm and matrix norm. So, let us look at the complex vectors and the corresponding operations.

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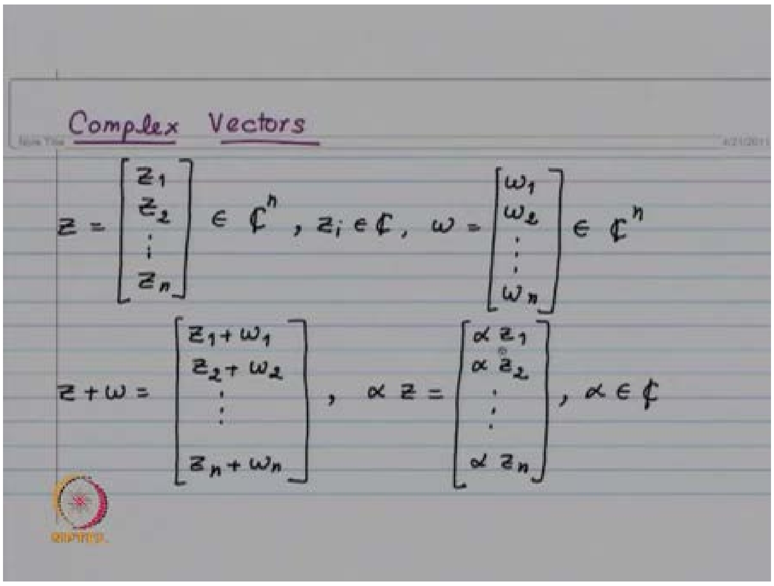

So, we have got z to be a complex vector z 1 z 2 z n. So, each z is going to be a complex number w is another n by 1 vector as I said before z plus w will be component wise addition. So, it is z 1 plus w 1 z 2 plus w 2 plus z n plus w n alpha times' z will be each component ,will get multiplied by alpha then inner product.

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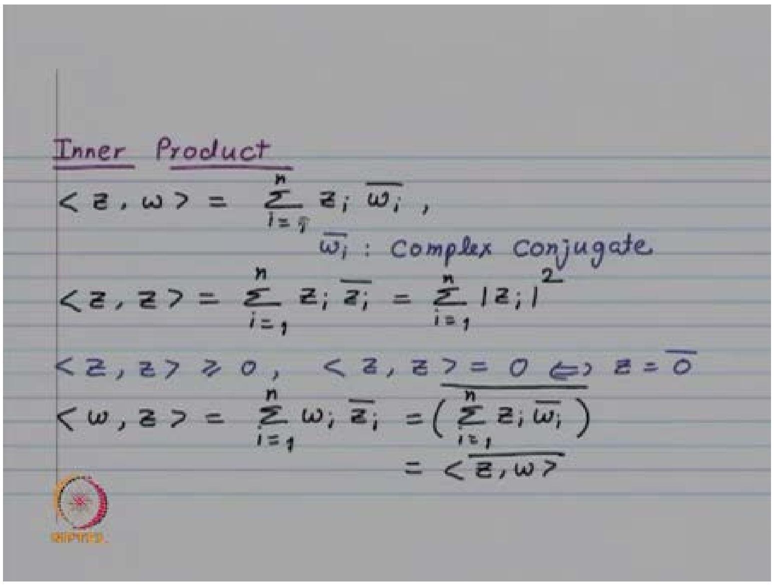

So, here when we had real vectors then the inner product was x comma y was summation x i y i ,now here change is you'll consider z i w i bar w i bar is the complex conjugate.

Now, when you consider inner product of z with itself it will be summation i goes from 1 to n z i z i bar. So, you have complex number you are multiplying by complex conjugate. So, it will be summation i goes from 1 to n mod z i square. So, thus inner product of z with itself will be bigger than or equal to 0 and it will be equal to 0 if and only if z is a 0 vector.

When you consider inner product of w with z it will be summation w i z i bar by our definition, which will be same as summation i goes from 1 to n z i w i bar and then complex conjugate. So that means, it is z comma w bar.

So, we have got conjugate symmetry inner product of w with z is complex conjugate of inner product of z with w

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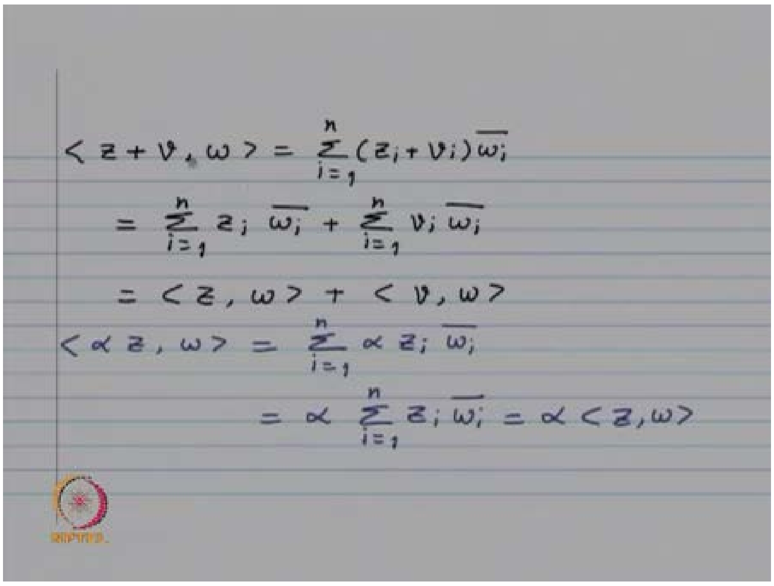

This is linearity in the first variable z plus v w will be summation i goes from 1 to n z i plus v i into w i bar split the summation into two summations .The first summation will be nothing,,,, but ,, but ,,, but ,, but inner product of z with w and the second summation is inner product of v with w.

Similarly, if you consider alpha z comma w this will be summation i goes from 1 to n alpha z i w i bar ,now alpha is independent of i. So, it will come out of the summation sign what remains in the summation that is inner product of z with w. So, our inner product will be linear in the 1 variable.

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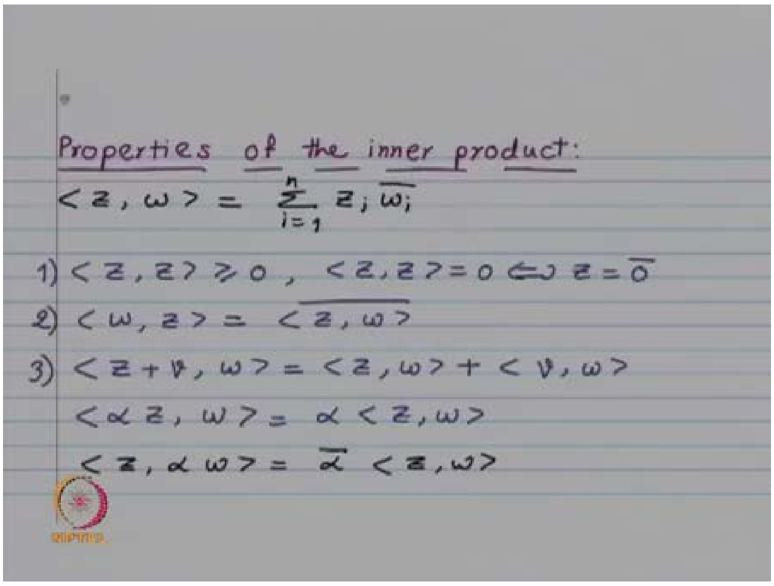

So, these are the properties of the inner product the 1 is positive definiteness 2 is conjugate symmetry and 3 property is linearity in the 1 variable ,when you consider z comma alpha w ,then alpha will come out as alpha bar because of the conjugate symmetry.

So, inner product is conjugate linear in the 2 variable. So, this the difference between real inner product and complex inner product that real inner product was symmetric now this is conjugate symmetric and we had linearity in both the variables for real inner product whereas, now complex inner product is going to be linear in the 1 variable whereas, conjugate linear in the 2 variable otherwise it is exactly similar.

Now, we had cauchy-schwarz inequality for real inner product. So, there is cauchy-schwarz inequality for complex inner product also and using this cauchy-schwarz inequality one considers the induced norm. So, that is induced norm by the inner product.

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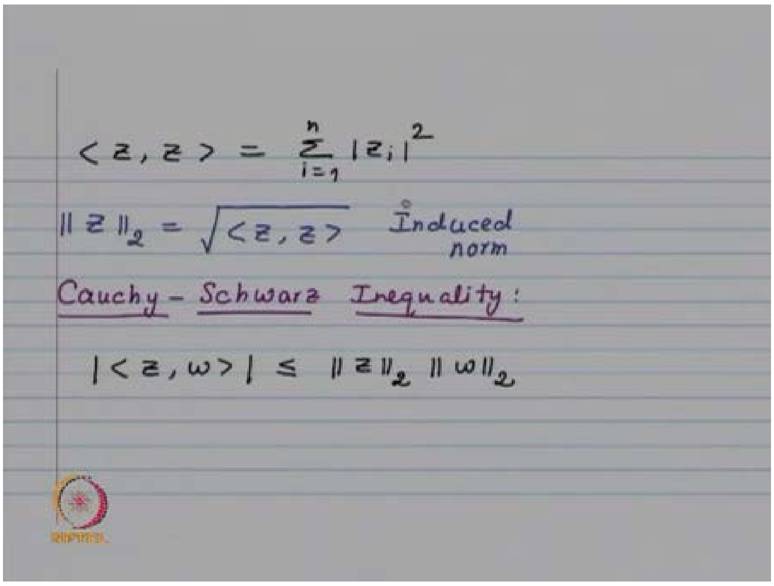

So, one show that it satisfies various properties of norm So, here is inner product of z with z is summation i goes from 1 to n mod z i square we define norm z 2 to be positive square root of z comma z and the cauchy-schwarz inequality is modulus of z comma w is less than or equal to 2 norm of z into 2 norm of w

I want you to notice that our complex inner product it is a map from c n cross c n to c. So, in general our complex inner product is a complex number,,,, but ,, but ,,, but ,, but when you consider inner product of a vector z with itself, then it is going to be a positive real number and that is why you can take its positive square root and then obtain a real number. In fact, the number is going to be bigger than or equal to 0 and that is our euclidian norm.

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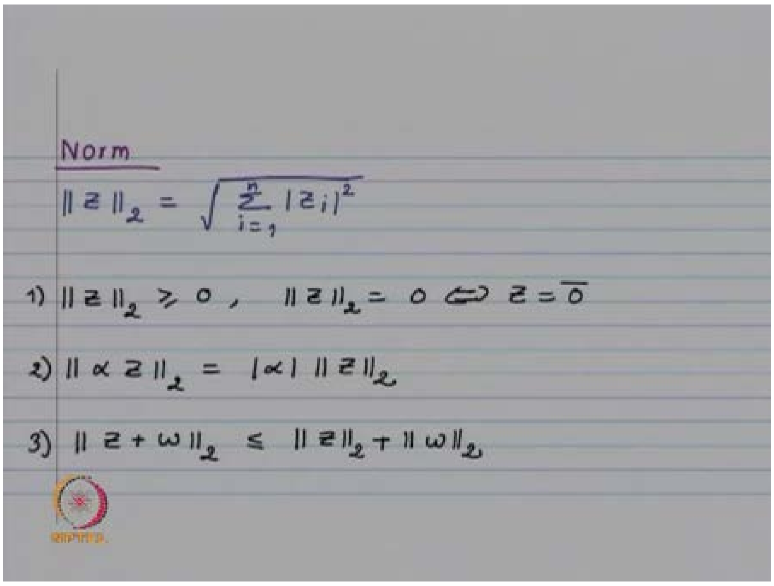

So, norm z 2 is positive square root summation goes from 1 to n mod z i square norm z 2 will be bigger than or equal to 0 it will be equal to 0 .If and only if z is equal to 0 vector that will follow from positive definiteness of inner product norm ,alpha z will be equal to mod alpha times norm z ,it will follows from the definition and the triangle inequality norm of z plus w is less than or equal to norm z plus norm w. So, it is for the triangle inequality that we need the cauchy-schwarz inequality. So, this is about the 2 norm

Now, analogously one can define 1 norm and the infinity norm. So, norm z 1 is going to be summation i goes from 1 to n mod z i and norm z infinity to be maximum of modulus of z i 1 less than or equal to i less than or equal to n. So, in the definition there is no difference instead of real numbers we have got complex numbers,,,, but ,, but ,,, but ,, but you are taking its modulus.

For 2 norms we are taking summation mod z i square. So, this modulus is important for real inner product space or for if the vector is real ,whether I write x i square or whether I write mod x i square ,the answer is the same whereas, for the complex number it is important that you should take modulus of z i square.

Now, we are going to look at the induced matrix norm. So, if you are given any vector norm then you define norm of the matrix to be maximum of norm a x by norm x x not equal to 0 and then for 1 norm and infinity norm; that means, if you are taking or if you are fixing vector norm to be 1 norm ,then look at the corresponding induced matrix norm for that we obtained an expression in terms of the elements of the matrix.

Similar thing was possible for norm a infinity whereas, for the 2 norm we have to be satisfied only with an upper bound. So, here the expressions for norm a 1 and norm a infinity they are going to remain to be exactly the same.

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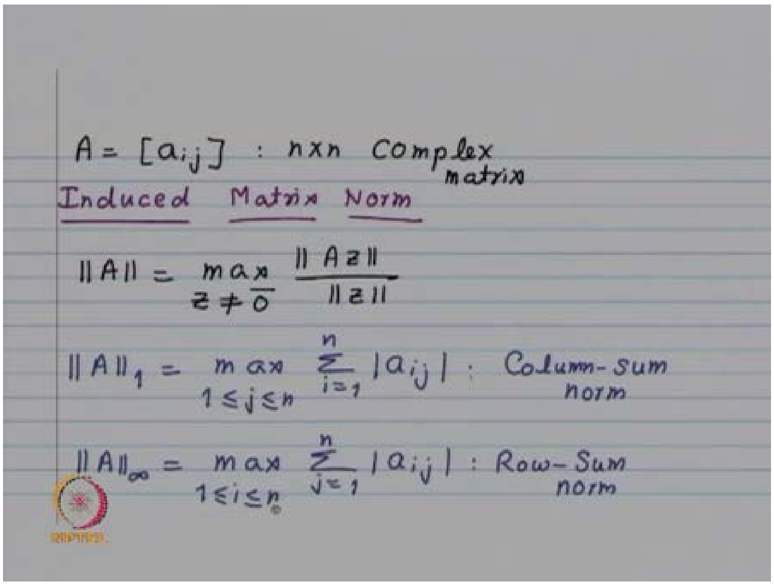

So, we are looking at the induced matrix norm. So, we have norm A 1 to be column sum norm. So, summation i goes from 1 to n modulus of a i j. So, look at the first column take the modulus ,add it up do it for all the columns whatever is the maximum that is norm A1 norm A infinity the expression is obtained by interchanging j and i. So, column sum norm becomes row sum norm. So, we have got norm A infinity to be summation j goes from 1 to n modulus of a i j 1 less than or equal to i less than or equal to n.

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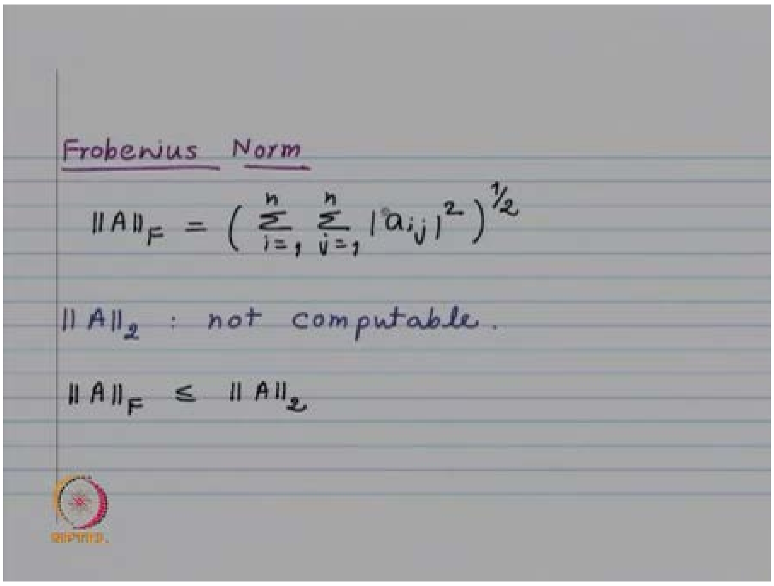

And then this is the frobenius norm. So, it summation over i summation over j mod a i j square raise to half norm A 2 is not computable ,,,, but ,, but ,,, but ,, but norm a frobenius here it is norm A 2 less than or equal to norm A F, here this less than or equal to should be bigger than or equal to.

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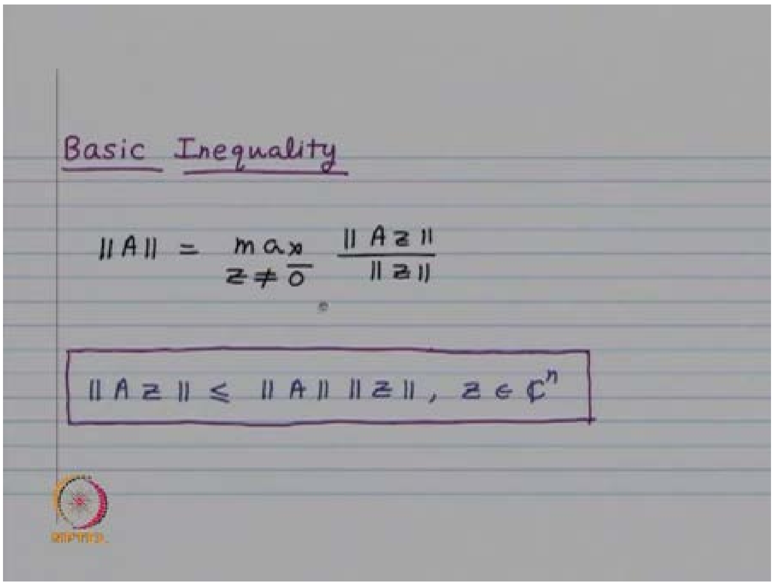
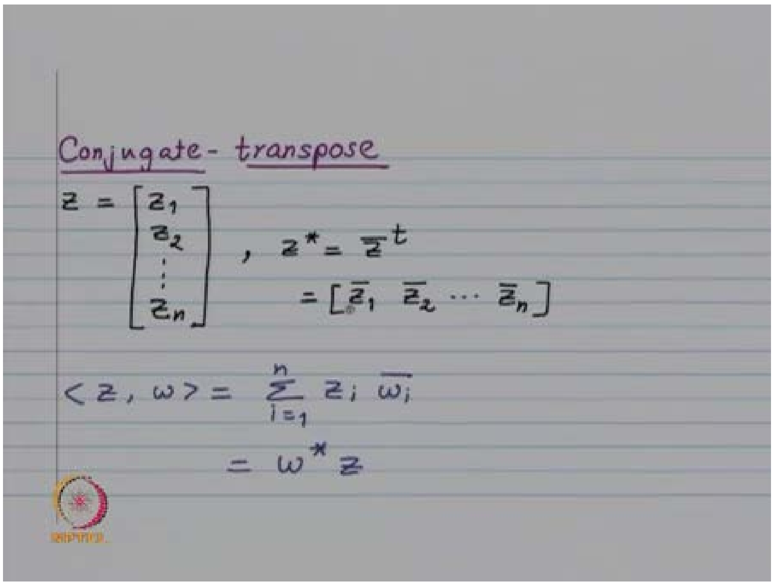


Then we have got this basic inequality norm A is maximum of norm A z by norm Z. So, from here we get norm A z to be less than or equal to norm A into norm z for z belonging to C n next we define conjugate- conjugate transpose. So, we defined the conjugate transpose for a vector as well as for a matrix

So, you take complex conjugate of each entry and then you take transpose. So, if you are taking conjugate transpose of a vector column vector then its conjugate transpose will be a row vector if the matrix is square matrix then conjugate transpose is again going to be equal to the matrix of size n.

So, this conjugate transpose we know that matrix multiplication is not commutative. So, if the conjugate transpose commutes with the matrix then it deserves a a special name it is a special class of matrices and those are known as normal matrices.

So, we are going to define normal matrix and then self-adjoint matrix q self-adjoint matrix these matrices their eigen values they have got some special property. (Refer Slide Time: 16:49)



So, here is definition z is vector z 1 z 2 z n z star is z bar transpose. So, it becomes a row vector z 1 bar z 2 bar z n bar.

Now, inner product of z with w this is our definition summation z i w i bar. So, in this notation we can write it as w star z w star, is going to be a 1 by n vector z i is n by 1 vector. So, when you take 1 by n vector multiplied by n by 1 vector you are going to get 1 by 1 matrix or you are going to get scalar. So, inner product of z with w will be same as w star Z.

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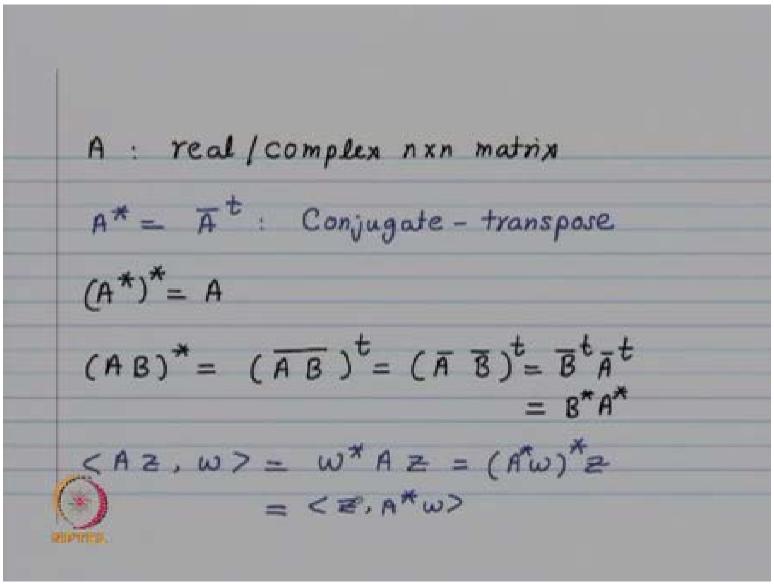

Next for a matrix A we define A star to be equal to A bar transpose conjugate transpose if you repeat the operation A star star is going to give you back matrix A then when you consider A B star this will be A B bar and then transpose A B bar will be same as A bar into B bar and then its transpose when you take A bar B bar transpose the order gets reversed. So, you get B bar transpose A bar transpose.

So, this will be equal to B star A star. So, A B star is B star A star and inner product of A z with w will be we have seen that this is the w star A z then w star A I write as A star w star because when you take the complex conjugate it will become w star A star star; that means, w star A and this is nothing,,,, but ,, but ,,, but ,, but z comma A star w.

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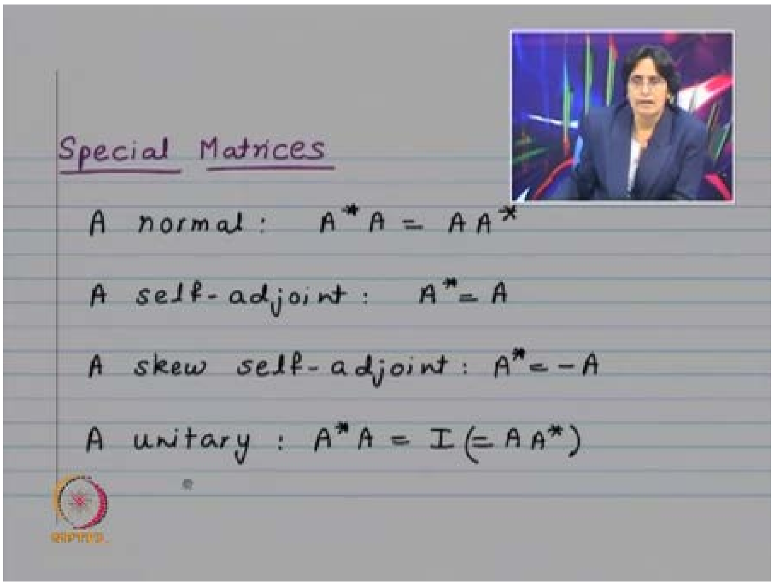

So, important property A z comma w a will go to the second variable as A star and here are the special matrices A star A is equal to A A star. So, that is class of normal matrices then A star is equal to A that is class of self-adjoint matrices if you consider A star is equal to minus A that is skew self-adjoint and lastly unitary matrix. So, we have got A star A is equal to identity and now for matrix we know that the left identity is same as the right identity left inverse is same as the right inverse. So, that is why you will have if A star A is equal to identity then automatically A A star is equal to identity.

Now, if you take 2 self-adjoint matrix ,if you add it up then again you are going to get a self-adjoint matrix .This result will not be true for product of matrices, because when you will consider A B star then you are you are going to have B star A star. So, if A star is equal to A B star is equal to B does not mean A B star is equal to A B because A B star will be equal to B A. So, these are some of the special matrices and they are going to their eigen values they are going to be something special or we can say something more about their eigen values

So, now we want to show we want to define eigen value eigenvector ,and then we want to show that they are roots of a characteristic polynomial. So, here is eigen value problem our notation is going to be a will be either a real matrix or a complex matrix,,,, but ,, but ,,, but ,, but it has to be a square matrix one defines eigen value and eigenvector only for square matrix

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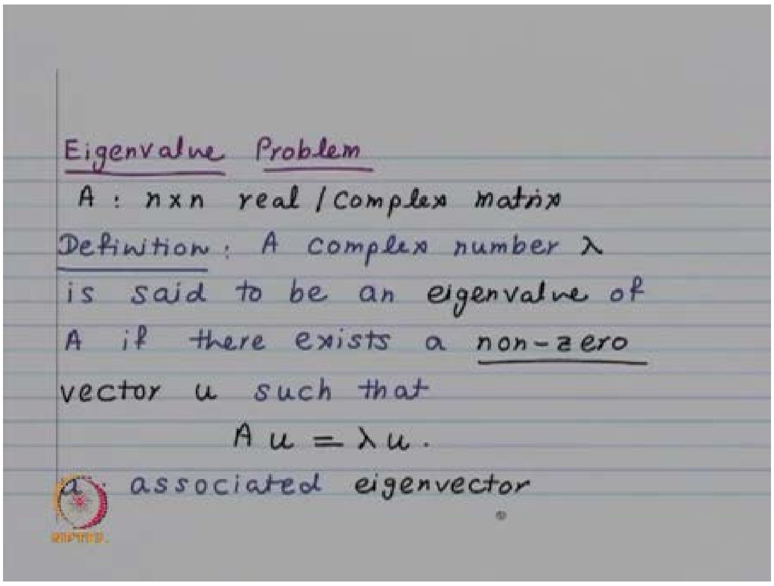

So, definition is a complex number lambda is said to be an eigenvalue of A. If there exists a non-zero vector u such that A u is equal to lambda u, and in that case u is called an associated eigenvector .This non-zero part is important ,because if you take a 0 vector then when you apply matrix A to it you are going to get a 0 vector ,then A u will be equal to lambda u for any lambda. So, lambda will be eigenvalue provided you have got a non-zero vector u such that A u is equal to lambda u.

Now, how to find a lambda like you cannot find,,,, but ,, but ,,, but ,, but at least we want some characterization. So, that characterization we are going to show that the lambda is nothing,,,, but ,, but ,,, but ,, but look at determinant of A minus lambda I A is matrix which is given to us then you look at matrix A minus lambda times identity

Look at its the determinant is something which we can calculate. So, you will get a polynomial in lambda of degree n and our eigen value is going to be 0 of this polynomial. So, we start with the definition that lambda is eigen value, provided we have got a non-zero vector u such that A u is equal to lambda u

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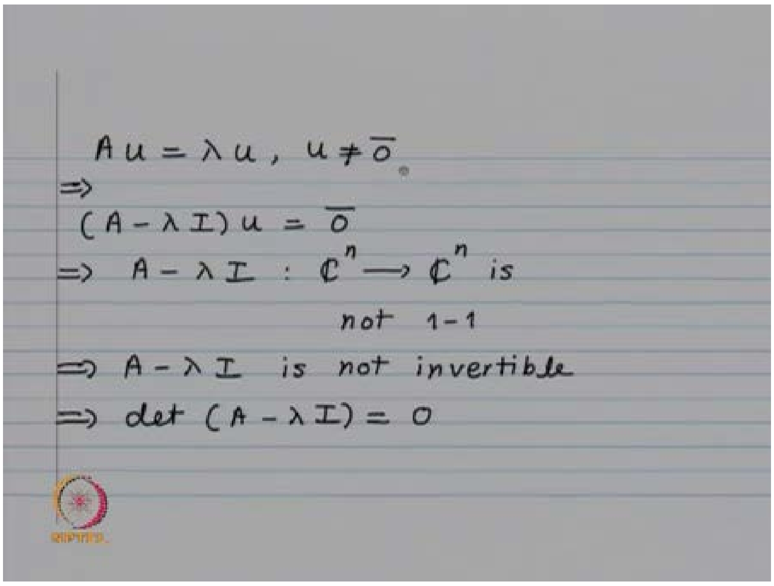

So, we have A u is equal to lambda u u not equal to 0 .This will imply that A minus lambda I u is equal to 0 vector which will mean that A minus lambda it is A n by 1 matrix. So, we can consider it has a map from C n to C n any vector in C n ,you apply A minus lambda I to it you again get A n by 1 vector. So, A minus lambda I from C n to C n it is a map this map is not 1 to 1 because we have got A minus lambda I ,u is equal to 0 vector where u is a non-zero vector and A minus lambda I into 0 vector is also equal to 0 vector.

So, we have got 2 vectors u bar and 0 vectors which have the same image, and that is the 0 vector. So, that is why A minus lambda I will not be 1 to 1 if A minus lambda I is not 1 to 1 it cannot be invertible, because for inevitability what we need is our map should be 1 to 1 and on 1 and in our case infinite dimensional spaces it is sufficient,,,, but ,, but ,,, but ,, but if A minus lambda I is 1 to 1 then A minus lambda will be invertible or if A minus lambda I on 2 it will be invertible. So, our we are starting with lambda is an eigenvalue u is eigenvector. So, map A minus lambda I will not be 1 to 1; that means, A minus lambda I will not be invertible. So, you have got A minus lambda I to be a singular matrix now if it is singular; that means, its determinant has to be equal to 0.

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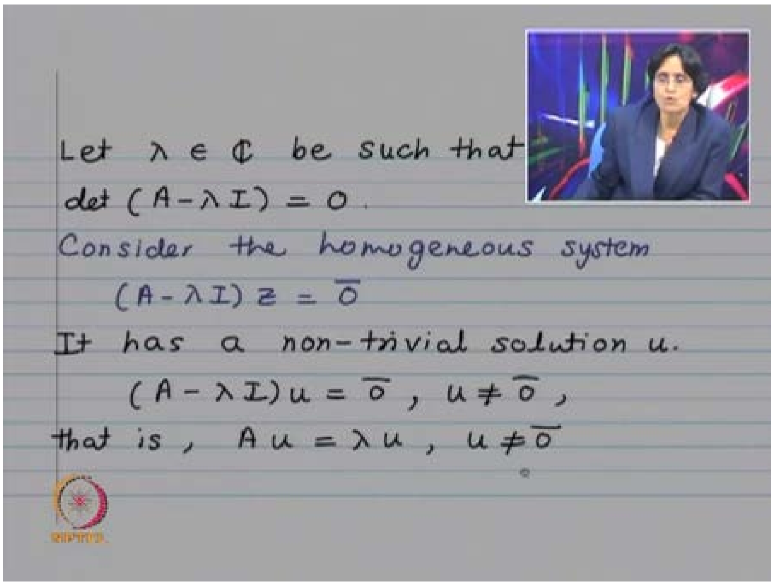

So, you get determinant of A minus lambda I to be equal to 0. Now conversely suppose lambda I is a complex number such that determinant of A, minus lambda I is equal to 0. So, you look at homogeneous system A minus lambda I z is equal to 0 vector. Now this homogeneous system it is going to have a non-trivial solution, because the coefficient matrix as determinant equal to 0. So, it has a non-trivial solution u such that A minus lambda I u is equal to 0 vector and that precisely means A u is equal to lambda u u not equal to 0 vector.

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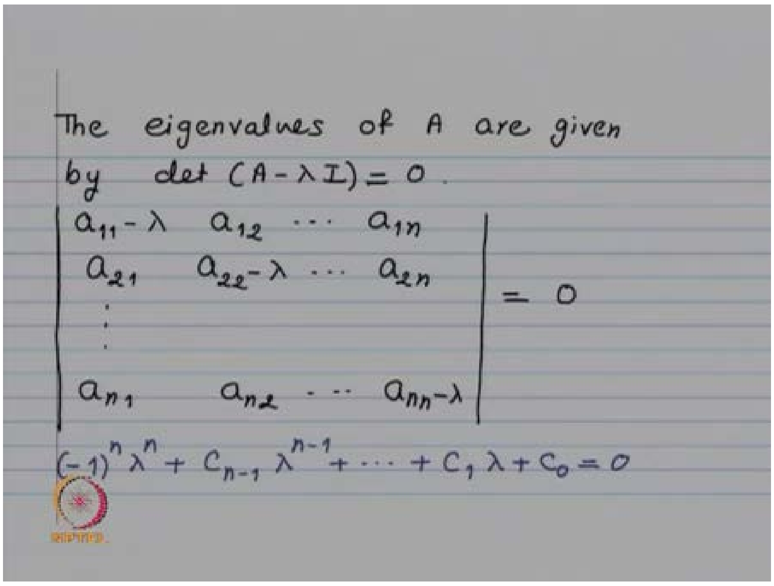

So, thus the eigen values of A they are given by determinant of A minus lambda I is equal to 0. So, this is the determinant of A minus lambda I when you will expand the determinant you are going to have minus 1 raise to n lambda raise to n plus c n minus 1 lambda raise to n minus 1 plus c 1 lambda plus c 0 is equal to 0.

So, you have a polynomial in lambda of exact degree n because the coefficient of lambda raise to n is non-zero it is minus 1 raise to n

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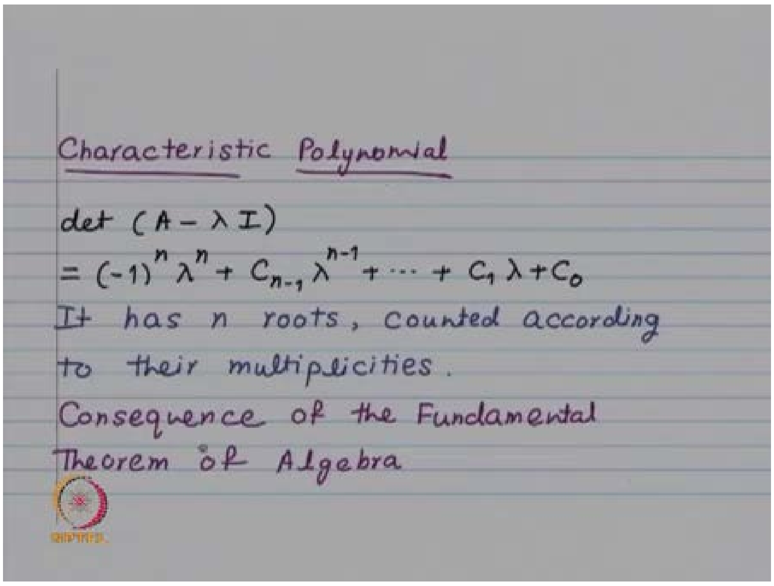

Now by consequence of the fundamental theorem of algebra ,this it is going to have n roots ,if you count them according to their multiplicities.

So, thus we know that the n by n matrix it is going to have at the most n eigen values and they are going to be roots of this polynomial. So, thus the problem of finding eigen values it gets reduced to finding roots of a polynomial

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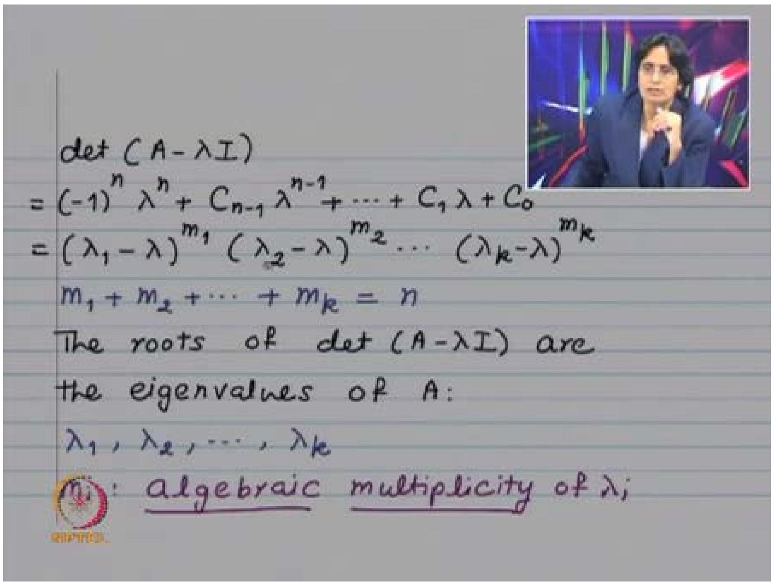

So, this determinant of A minus lambda I this polynomial now we factorize it. So, it will be lambda 1 minus lambda raise to m 1 lambda 2 minus lambda raise to m 2 into lambda k minus lambda raise to m k where the m 1 m 2 m k they add up to n.

So, you have got eigen values to be lambda 1 lambda 2 lambda k .These are distinct eigen values and the power m I that is known as the algebraic multiplicity of lambda i.

So, you count lambda 1 m 1 times lambda 2 m 2 times and lambda k m k times and that is how you have got exactly n eigenvalues counted according to their algebraic multiplicity

Now, there is another multiplicity associated with eigen value ,and that is known as geometric multiplicity. So, your geometric multiplicity is going to be number of linearly independent eigenvectors associated with a particular eigen value.

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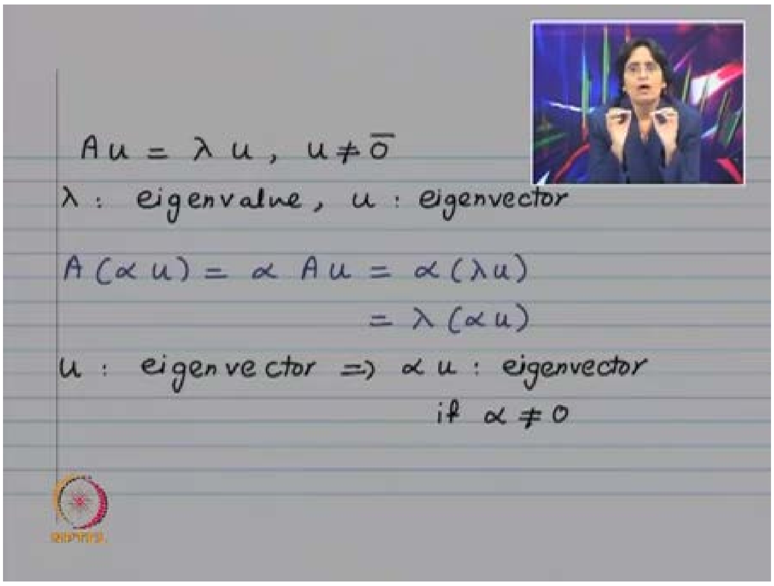

So, we have A u is equal to lambda u u not equal to 0 vector if I consider A of alpha u this will be alpha times A u A u is lambda u. So, it is alpha time's lambda u now alpha and lambda they are scalars those are complex numbers. So, they commute and then you can have lambda time's alpha u. So, if u is an eigenvector alpha u will also be an eigenvector provided alpha is not equal to 0. So, eigenvector is not unique

You have got infinitely many eigenvectors as soon as you find one eigenvector any non-zero multiple of it is also going to be an eigenvector.

Now, one defines what is known as eigen space. So, see what you have got is suppose , I have got a eigenvector then I take a multiple.

So, if you are in say r two you are going to have a straight line ,except what you do not want is multiply by 0. So, eigen space by definition is going to be all multiples and you add 0 to it. So, all non-zero vectors in your eigen space they are going to be eigenvectors associated with eigenvalue lambda and. So, there are infinitely many eigenvectors ,,,, but ,, but ,,, but ,, but when you consider number of linearly independent eigenvectors they are going to be finite and. In fact, the that number is going to be less than or equal to algebraic multiplicity.

So, if you have got lambda 1 to be an eigen value with algebraic multiplicity to be m 1. In that case you can have at the most m 1 linearly independent eigenvectors, the number can be less .We will consider an example where your number of linearly independent eigenvectors can be strictly less than algebraic multiplicity.

Your algebraic multiplicity is you consider factorization of characteristic polynomial and in that you have lambda 1 minus lambda term whatever its power that is our algebraic multiplicity and geometric multiplicity is number of linearly independent eigenvectors associated with it.

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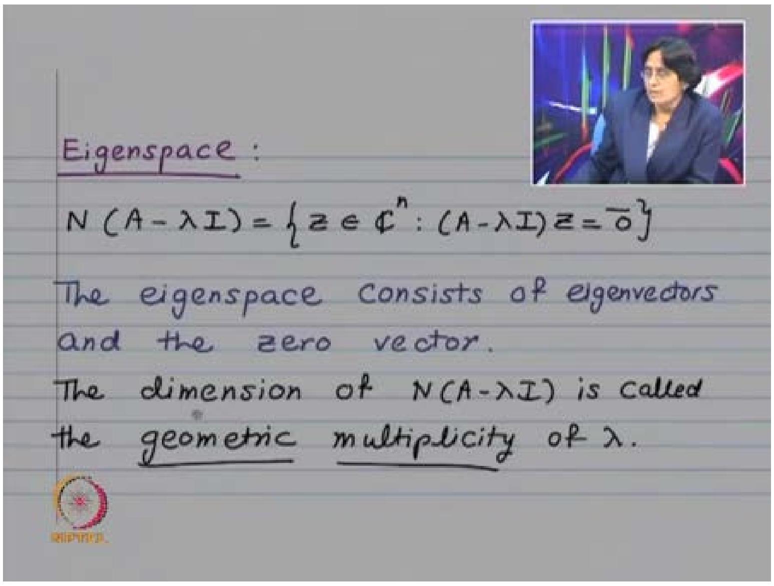

So, here is definition of eigen space null space of a minus lambda I is set of all z such that a minus lambda I z is equal to 0 vector, it is a subspace it consists of eigenvectors and a 0 vector the dimension of this sub space is called geometric multiplicity of our eigen value lambda then

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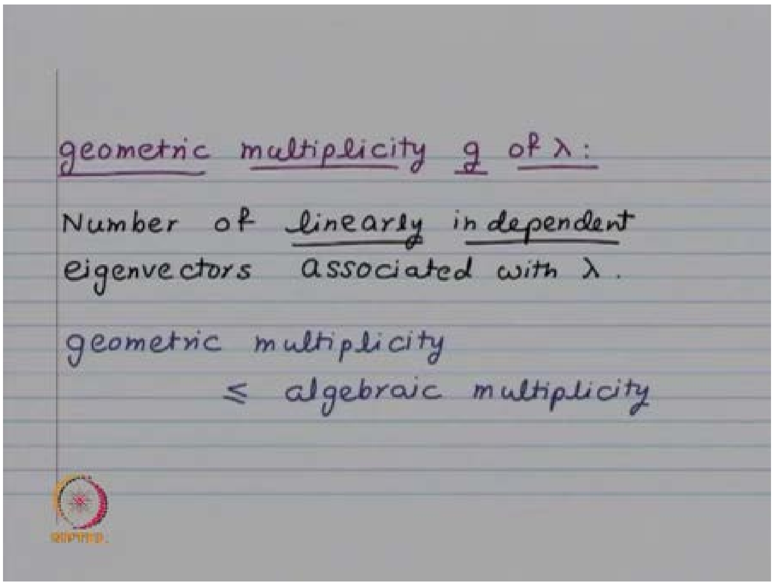

As I said it is same as number of linearly independent eigenvectors associated with eigenvalue lambda and geometric multiplicity ,will always be less than or equal to algebraic multiplicity.

So, now let me give you an example of 2 by 2 matrix a simple matrix for which in one case geometric multiplicity is strictly less than algebraic multiplicity and in another case they are equal. If your matrix is upper triangular matrix ,then your eigen values are going to be diagonal entries. So, for upper triangular matrices you do not have to do any computation just look at the diagonal entries those are your eigen values.

Now, when you considered gauss elimination method we reduced matrix a to upper triangular form,,,, but,, but ,,, but ,, but these elementary row transformations they do not preserve the eigenvalues .You have matrix a it has got certain eigen values you do elementary row transformations obtain to an upper triangular matrix,,,, but ,, but ,,, but ,, but the eigen values of upper triangular matrix which you have obtained will be completely different than your original eigen values.

This elementary row transformations they do not change the solution of system a x is equal to b, that is why it was useful there whereas, here it is not useful. So, now, let us consider a example.

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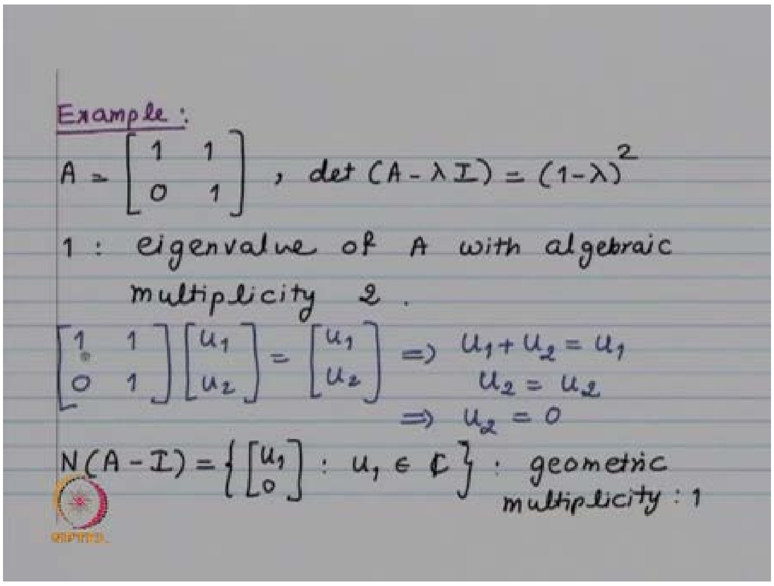

So, here is upper triangular matrix 1 1 0 1 the determinant of A minus lambda I is 1 minus lambda square. So, A has eigenvalue 1 with algebraic multiplicity 2. So, it is a repeated eigenvalue.

I look at its eigenvector. So, 1 1 0 1 u 1 u 2 is equal to u 1 u 2. So, you get u 1 plus u 2 is equal to u 1 and u 2 is equal to u 2 this second equation gives us no information the first equation tells us that u 2 has to be 0; that means, null space of a minus I is going to be vector u 1 0 u 1 belonging to c. So,your null space of A minus I which is all u 1 0 u 1 belonging to c. So,; that means, we have got multiples of vector 1 0.

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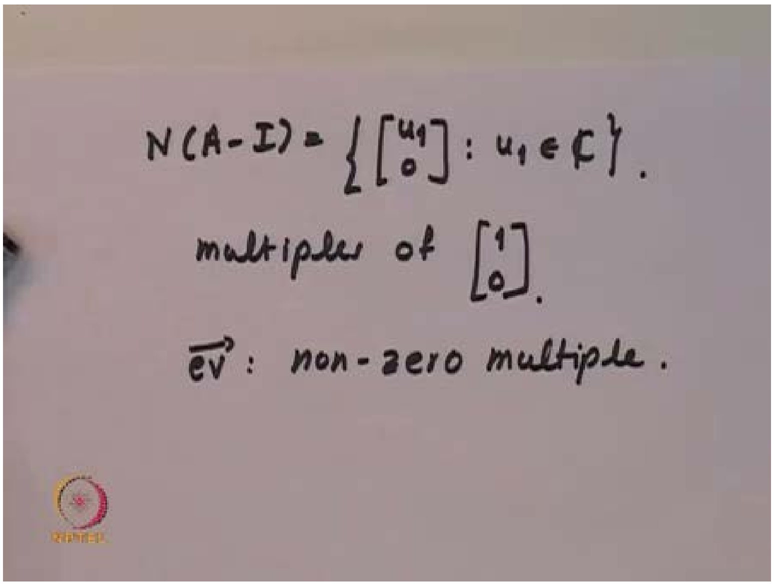

If you want eigenvector then it should be a non-zero multiple. So, for this example you have got 1 is eigenvalue with algebraic multiplicity 2 and geometric multiplicity to be 1. So, geometric multiplicity is strictly less than algebraic multiplicity now let me change this examples slightly let me make this 1 as 2.

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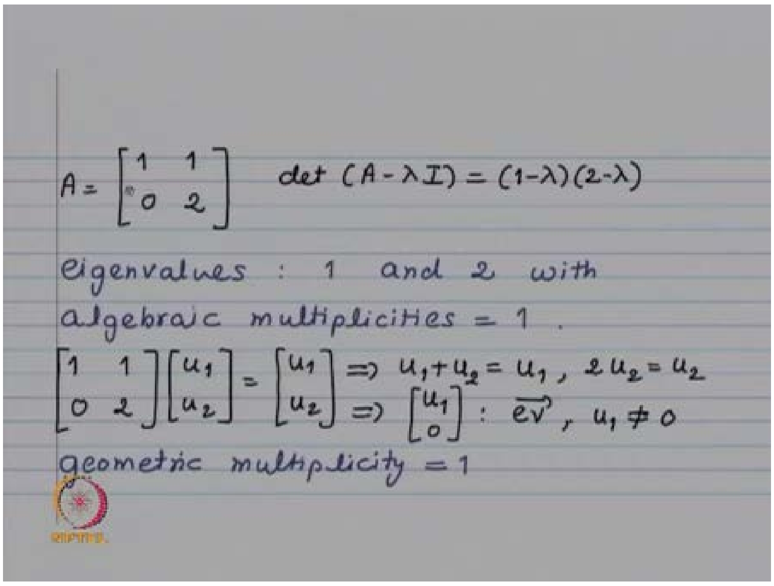

So, when you look at matrix 1 1 0 2 its characteristic polynomial will be 1 minus lambda 2 minus lambda. So, you have eigen values to be 1 and 2 with algebraic multiplicities in both the cases to be equal to 1.

When we try to consider the eigenvector then you are going to have u 1 plus u 2 to be equal to u 1 and 2 u 2 is equal to u 2. So, that means, u 2 has to be 0 and eigen vector will be of the form u 1 0 with u 1 not equal to 0. So, one will be eigenvector with geometric multiplicity to be equal to1.

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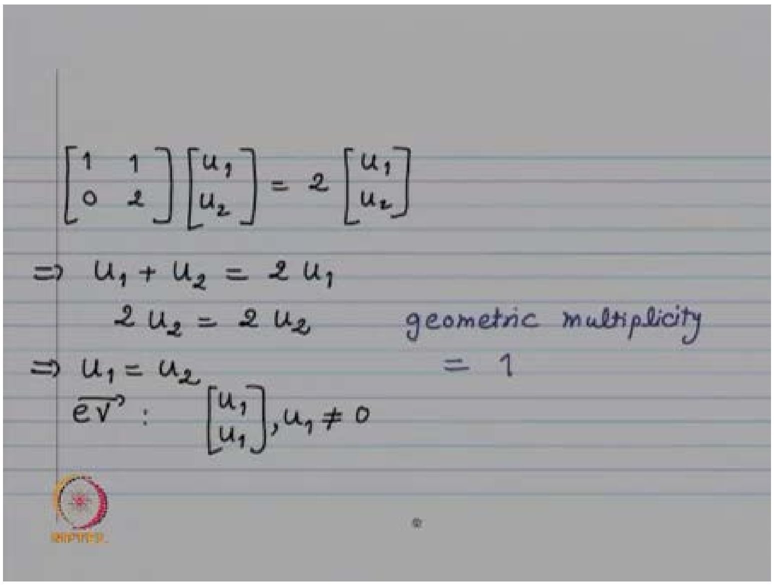

Next look at 1 1 0 2 u 1 u 2 into is equal to 2 times u 1 u 2. So, what will be the first equation it will be u 1 plus u 2 is equal to 2 u 1 ,second equation will be 2 u 2 is equal to 2 u 2. So, again the second equation does not give us any information from the first equation you will get u 1 is equal to u 2. So, any eigenvector associated with 2 will be of the form u 1 u 1 u 1 is not equal to 0 or equivalently it is going to be a non-zero multiple of vector u 1 u 1.

So, eigenvector of 1 will be 1 0 or any multiple eigenvector of 2 will be vector 1 1 or any non-zero multiple. So, . So, now, what we are going to do is we are going to consider eigenvalues of our special matrices. If the matrix is self-adjoint A star is equal to A then we will show that eigenvalues they have to be real if A star is equal to minus A then eigenvalues have to be purely imaginary or 0

For normal matrix we do not have any such structure your eigenvalues can be complex,,,, but ,, but ,,, but ,, but still for eigenvalues of normal matrix it has got some special property if you look at two distinct eigenvalues and corresponding eigenvectors then they are linearly independent for normal matrix something more is true.

Eigenvectors corresponding to distinct eigenvalues, they are going to be perpendicular to each other; that means, their inner product is going to be 0. If you consider eigenvectors of unitary matrix; that means, the matrix which satisfies A star A is equal to A. A star is equal to identity then the eigen values they are going to have modulus to be equal to 1. So, they will lie on unit circle, now what does these eigen values tell us.

So, these are going to be precisely the points where a minus lambda 1 will not be invertible at all other complex numbers our matrix A minus lambda 1 will be invertible. So, when you have got n by n matrix there are going to be at the most n complex numbers for which A minus lambda I will not be invertible for all other complex numbers A minus lambda I will be invertible.

So, let us show the properties of eigen values of special matrices the proofs are simple and straight forward.

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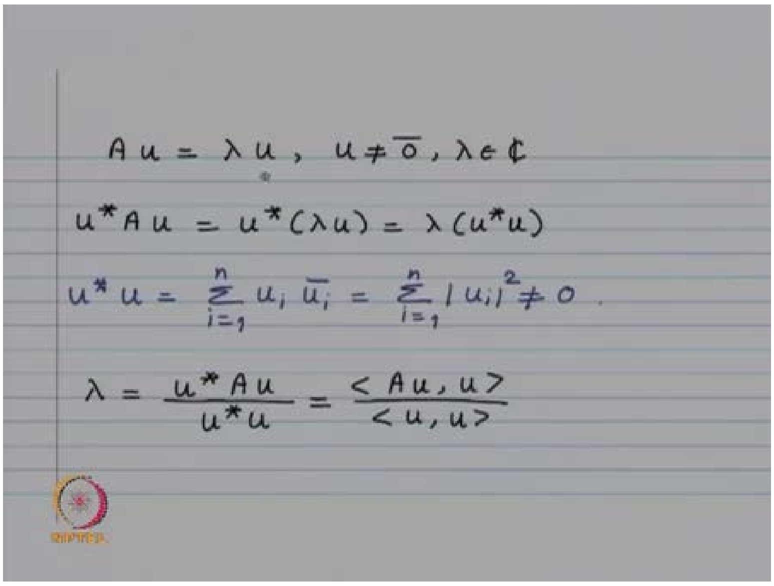

So, look at A u is equal to lambda u u not equal to 0 vector lambda complex number pre multiply by u star. So, you have got u star A u is equal to u star lambda u. So, which is same as lambda times u star u.

u star u will be summation I goes from one to n u i u i bar. So, that is summation i goes from 1 to n mod u i square u is not a 0 vector. So, that means, at least 1 u i will be non-zero and hence this summation will not be equal to 0. So, I get lambda to be equal to u star A u divided by u star u which is equal to in the notation of inner product it is A u comma u divided by u comma u.

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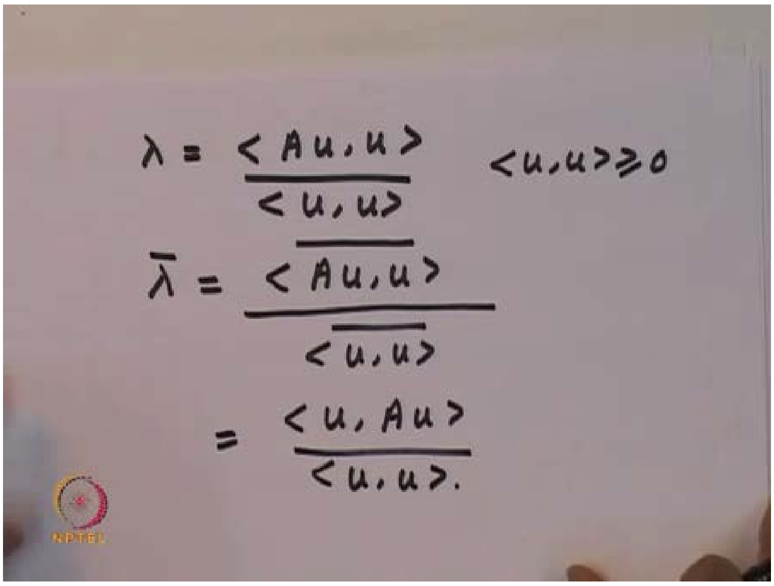

So, we have lambda to be equal to inner product of A u with u divided by inner product of u with u let me consider complex conjugate of lambda this is going to be complex conjugate of A u with u divided by complex conjugate of u with u now since inner product of u comma u is bigger than or equal to 0 here this u comma u bar will be same as u comma u and by conjugate symmetry the numerator is going to be inner product of u with A u divided by u comma u.

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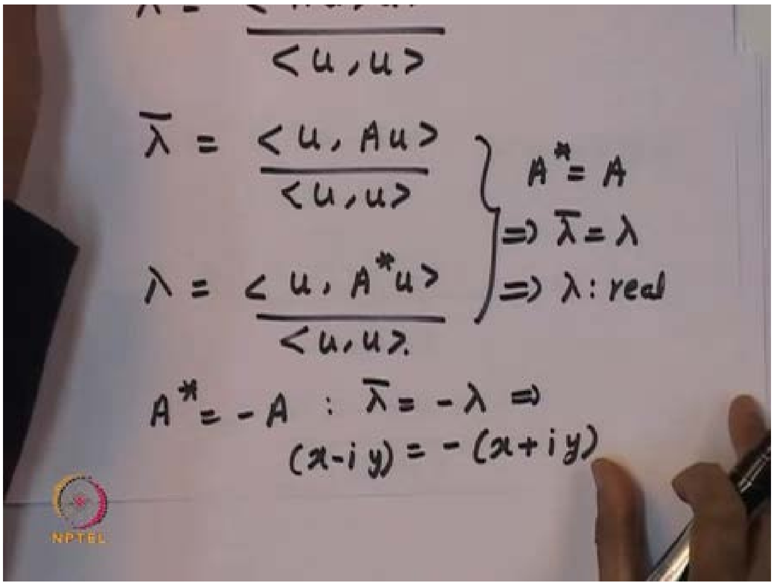

So, thus lambda is equal to A u comma u divided by u comma u and lambda bar is u comma A u divided by u comma u ,now lambda is also equal to this a when it goes to the second variable it goes as A star. So, it is going to be u A star u upon u comma u, now from here I can conclude that A star is equal to A, will imply that lambda bar is equal to lambda and which will mean that lambda is going to be real because lambda is a complex number its complex conjugate is equal to itself, that means, lambda has to be real.

Similarly, if A star is equal to minus A then your lambda bar is minus lambda. So, if lambda is equal to x plus y. So, it is say minus x plus y and lambda bar is going to be x minus y and hence in this case you are going to have if you have got A star is equal to minus A then lambda bar is equal to minus lambda and then this means that lambda is purely imaginary or zero.

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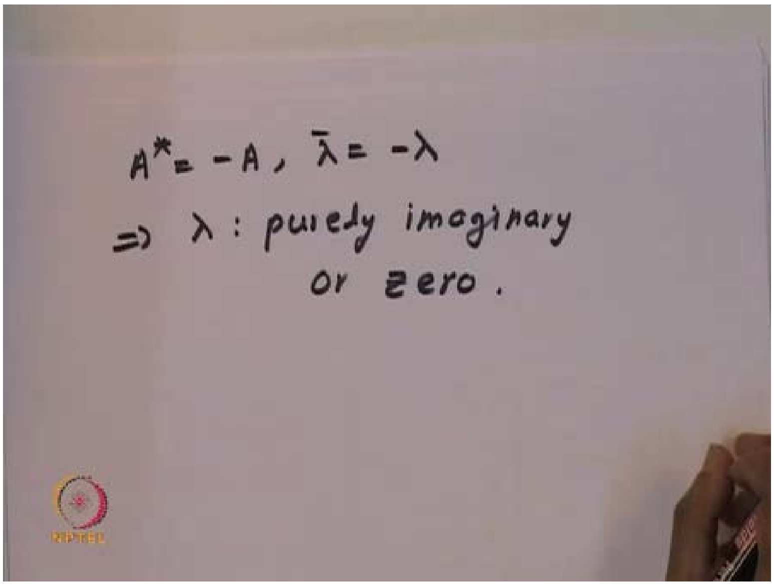

So, this is for self-adjoint and skew self-adjoint matrices now, for the normal matrix.

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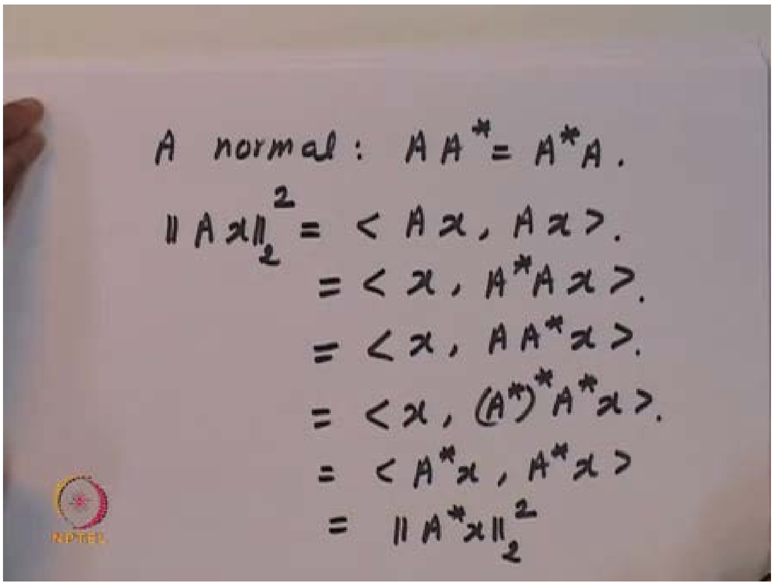

So, suppose A is normal. So, you have got A A star is equal to A star A consider norm A x its ecludian norm and its square this will be nothing,,,, but ,, but ,,, but ,, but inner product of A x with itself this a will go here as A start. So, it is x A star A x now use the property that A star A is same as A A star. So, it will be x A A star x which will be x now this A I can write as A star its star A star x. So, this is same as A star x A star x because this A star will go to the second variable as its star. So, this is nothing,,,, but ,, but ,,, but ,, but norm A star x 2 norm square.

So, an important relation that if A is normal then euclidian norm of A x is same as euclidian norm of A star x how does this property helps us for saying something about eigenvalues. So, what we have to proved is if A is normal then norm A x is same as norm of A star x then suppose lambda is eigenvalue of a then we have got A minus lambda I u is equal to 0.

So, norm of A minus lambda I u will be equal to 0 now a normal will mean that if I consider A minus lambda I its star that u into also will be 0. So, that will mean that lambda bar will be an eigenvalue of A star.

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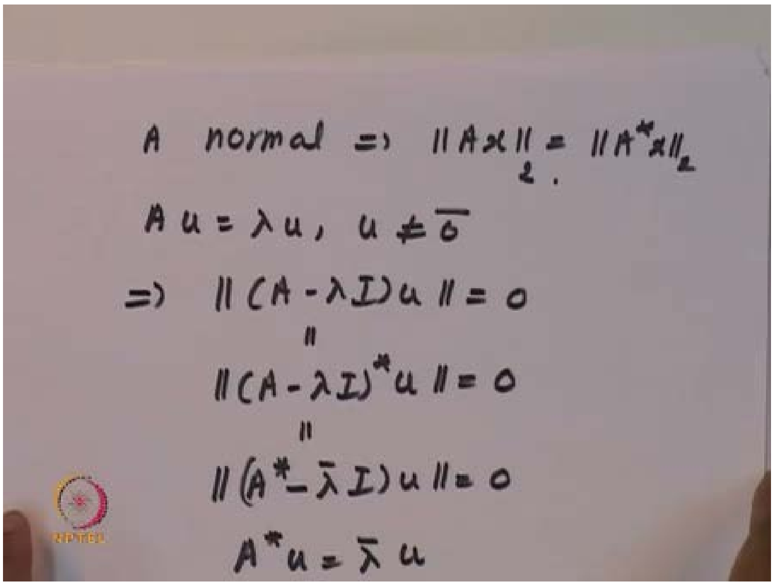

So, A normal implies norm A x is equal to norm of A star x its 2 norm then A u is equal to lambda u is not equal to 0 vector. So, norm of A minus lambda I u will be 0 this will be same as A minus lambda I star u is equal to 0 and this is equal to A star minus lambda bar I u is equal to 0 and thus A star u is equal to lambda bar u.

So, now for normal matrices the A and A star if lambda is eigenvalue of a lambda bar will be eigenvalue of A star and eigenvector is going to be the same. So, using this fact in our next lecture ,we will show that eigenvectors of a normal matrix associated with distinct eigenvalues, they are perpendicular ,then I am going to state scherus theorem spectral theorem and then we will go to localization of eigenvalues. So, thank you.