Elementary Numerical Analysis

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Lecture No. # 30
Tutorial-4

Today, we are going to consider some of the problems for the Newton's method. So, we look at the f x is equal to 0. We want to find a root. So, approximation to this root is obtained by Newton's method. In Newton's method, we have proved that there is quadratic convergence. In fact, that is the advantage of Newton's method.

So, we will look at specific example and in that, we will show that the quadratic convergence is achieved. Then, we will look at one more example for finding approximation to square root of a function using Newton's method. After that, we will consider some problems for solution of system of linear equations. So, we will calculate condition number in terms of maximum magnification and minimum magnification of the coefficient matrix a. Then, we will look at the residual obtained estimates for the residual and also, we will consider the condition number how it depends on the scaling.

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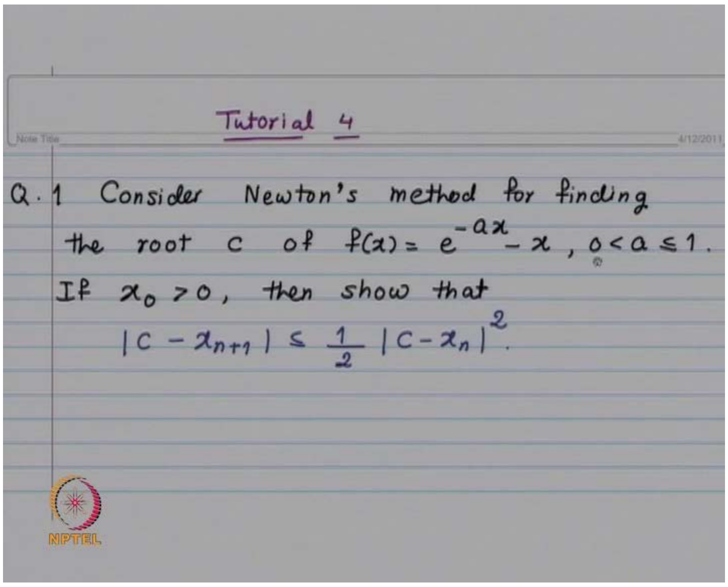

So, our first example is, we want to find root of this function f x is equal to e raise to minus a x minus x, where on our condition is that, 0 should be less than a, less than or equal to 1. We will show that this function is going to have a unique root in the interval 0 to infinity. Then, the x 0 is the initial approximation in the Newton's method. So, if this initial approximation you choose to be bigger than 0, then we will show that modulus of c minus x n plus 1 is less than or equal to 1 by 0 modulus of c minus x n square c is the exact root x n plus 1 is the n plus first iterate in the Newton's method. So, this is the error in n plus first iterate c minus x n square, that is going to be the error in the nth iterate and error in the n plus first iterate is less than or equal to 1 by 2 times error in the nth iterate it is square.

So, this will illustrate the quadratic convergence of Newton's method. So, the first thing we are going to show that this function has a unique root in interval 0 to infinity.

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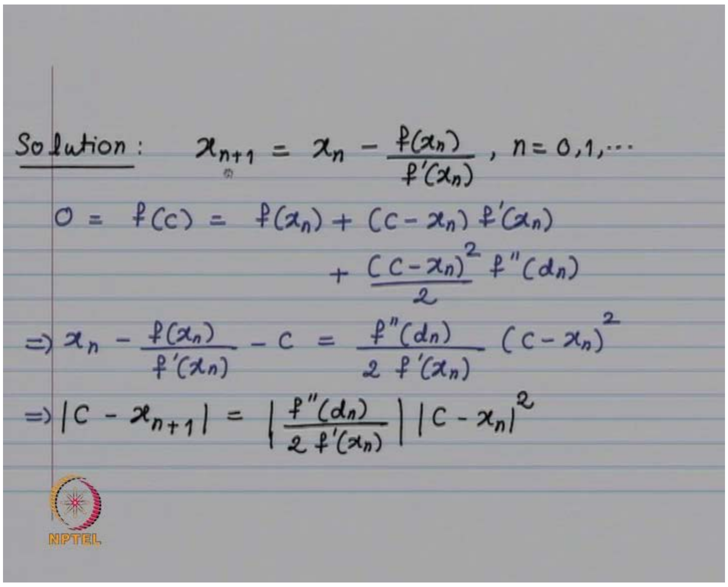

Look at the Newton's iterate, t is by definition x n plus 1 is equal to x n minus f x n upon f dash x n, n is equal to 0, 1, 2 and so on. So, I am recalling, how the error in the n plus first iterate and error in the nth iterate, they are related f of c is equals to 0 right down the Taylor expansion for f of c. So, that is going to be equal to f of x n plus c minus x n f dash x n plus c minus x n square by 2 f double dash of d n. This d n is going to lie between c and x n.

Then, what we do is we divide by f dash x n and take this term on the other side. So, that gives you x n minus f x n upon f dash x n minus c. This will be equal to f double dash d n. These 2 you are dividing by f dash x n and c minus x n square. So, now, this is nothing, but x n plus 1. So, the left hand side is the x n plus 1 minus c. I am taking modulus. So, it is modulus of c minus x n plus 1. This is equal to modulus of f double dash d n upon 2 times f dash x n and multiplied by c minus x n square.

In this particular example, we want to show that modulus of c minus x n plus 1 is less than or equal to 1 by 2 into modulus of c minus x n square. So, that means, I need to show that this coefficient of c minus x n square is going to be less than or equal to half.

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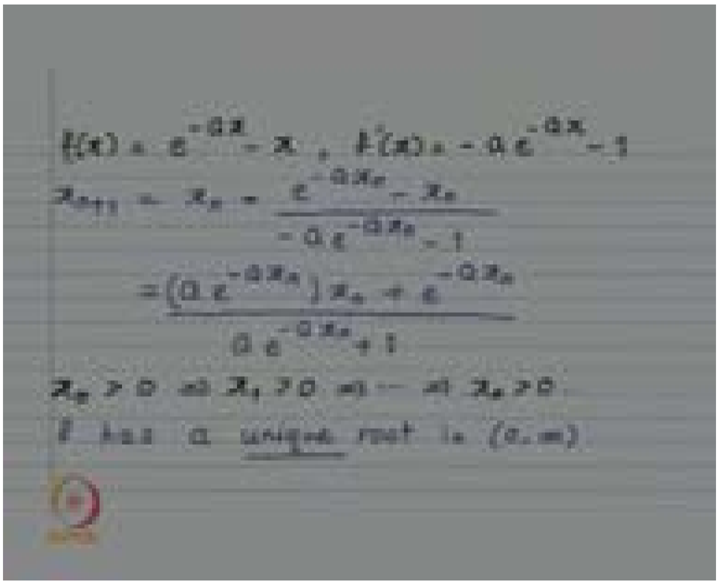

So, here is proof of unique root in interval 0 to infinity our function is f x is equal to e raise to minus ax minus x. So, f dash x is going to be e raise to minus a x minus 1 x n plus 1 is equal to x n minus f x n. So, that is e raise to minus a x n minus x n divided by f dash x n. So, it is minus a e raise to minus a x n minus 1. So, denominator you have got negative sign, here this is negative sign. So, this will become plus. Then, x n and this x n is going to get cancelled. So, you are left with a e raise to minus a x n multiplied by x n and then, from here, plus e raise to minus a x n divided by a e raise to minus a x n plus 1.

So, we are starting with x 0 to be bigger than 0. Now, look at x 1. Our a is lying between 0 and 1. So, this number is bigger than 0. Exponential is always bigger than 0, x 0 is greater than 0. Similarly, the denominator will be bigger than 0. So, that will mean that x 1 is bigger than 0. You continue and then, you get x n to be bigger than 0. In fact, I am not showing that f has a unique root. I will tell you how it follows. Look at f dash x. f dash x is going to be less than 0. If f dash x is less than 0, that will mean that f is going to be a strictly decreasing function.

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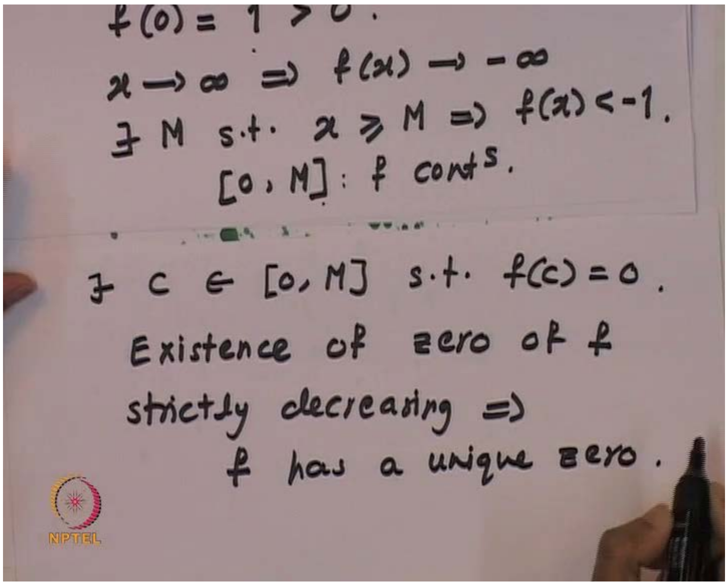

When you consider f of 0, f of 0 is going to be equal to 1. So, our f x is e raise to minus a x minus x, f dash x is, this is e, f dash x is minus a e raise to minus a x minus 1, a is bigger than 0. So, f dash x will be less than 0 on 0 to infinity. This implies f to be strictly decreasing. When you consider f of 0, it is going to be equal to 1. So, it is bigger than 0 and as x tends to infinity, f of x will tend to minus infinity. If f of x is tending to minus infinity, then there will exists some m, such that x bigger than or equal to m will imply that f of x is say, less then minus 1. F x is tending to minus infinity as extend to infinity. So, by definition this means that for some m, whenever x is bigger than or equal to m f x is going to be less than minus 1.

Now, you look at interval 0 to m. Your function f is continuous. At f of 0, it is 1; at m, it is going to be f of m will be less than minus 1. So, by intermediate value by theorem for the continuous function, there has to be some c in the interval 0 to m, such that we will have c, there will exist c in the interval 0 to m, such that f of c is equal to 0. So, that means we have to proved existence of 0 of f and now, strictly decreasing implies f has a unique 0. So, this was the proof for showing that our function f has a unique 0 in the interval 0 to infinity.

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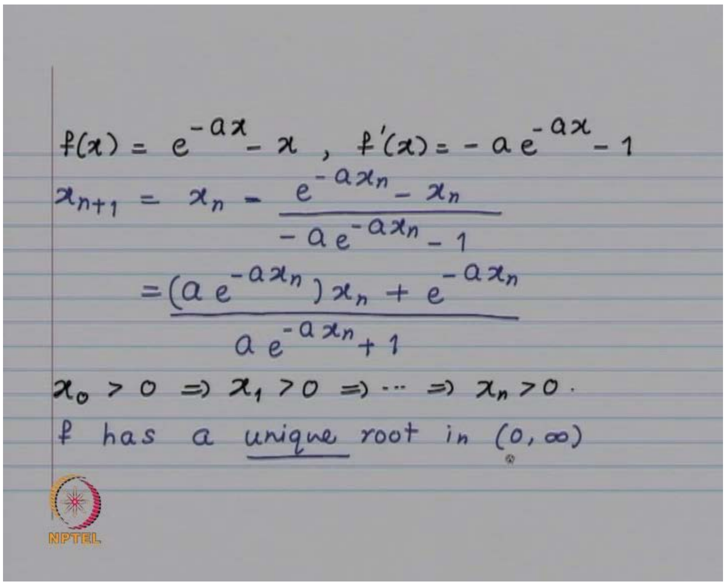

Now, we are looking at x 0 to be bigger than 0. If the initial approximation is bigger than 0, we calculated formula for x n plus 1 in terms of x n and using that formula, we saw the x 1 has to be bigger than 0 and then, use the same argument. X 1 bigger 0 will imply x 2 bigger than 0 and in general, your x n, they are going to be bigger than 0. Then, we look at c minus x n plus 1 and then, c minus x n. So, we obtain x n plus 1 to be this much. Then, f x is e raise to minus a x minus x and f dash x is minus a is e raise to minus a x minus 1. The second derivative is going to be a square e raise to minus a x.

So, when you consider modulus of f double dash of d n divided by 2 times f dash x n, this is going to be equal to a square e raise to minus a d n divided by 2 times a e raise to minus a x n plus 1 and now, this is going to be less than or equal to half. Why? Because our 0 is less than a, less than or equal to 1, so a square will be less than or equal to 1, e raise to minus a d n, a is bigger than 0, d n is going to lie between our point c and x n. So, that is why, e raise to minus a d n also will be less than 1 divided by 2 a, e raise to minus a x n plus 1. So, this number is going to be something bigger than 0. So, I can dominate 1 upon a e raise to minus a x n plus 1 by 1. So, numerator is less than or equal to 1, 1 upon this factor is going to be less than or equal to 1 and then, you have got 2. Then, you get this to be less than or equal to half and that gives you modulus of c minus x n plus 1 less than or equal to half c minus x n square. So, this illustrates the quadratic convergence.

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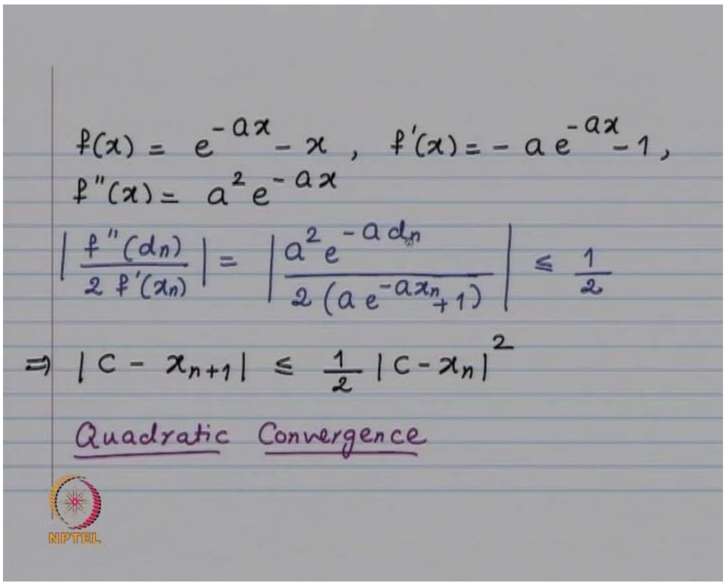

Now, why was it necessary to obtain this expression? Using this expression, we show that all x n's, they are bigger than 0. Our c is in the interval 0 to infinity. So, we have got c is in the interval 0 to infinity, x n is in the interval 0 to infinity, point d n will lie between c and x n. X n can be to the left of c or to the right of c. So, d n also will be in the interval 0 to infinity and then, using that you get e raise to minus a d n to be less than or equal to 1. So, in order to obtain this estimate, we need to look at what is x n plus 1 and then, we have proved the quadratic convergence.

So, now, in our next example what we are going to do is, when we want to find a square root of a positive real number, then we want to apply Newton's method to that to find an approximation to this square root. So, our function f x is going to be equal to x square minus a, where a is bigger than 0. So, we want to find the root of this. So, x square minus a is equal to 0 and let us decide that we want to find a positive square root. So, to this function, we will apply Newton's method, then that will give you a formula for finding x n's. Now, remember the Newton's method may not converge always. When it converges, it is going to converge quadratically. We have got our formula x n plus 1 is equal to x n minus f x n divided by f dash x n. If this sequence, you are defining a sequence, x 0 is your initial approximation and then, you are defining x n. If these x n's converge to a point, then that point or that number is definitely going to be 0 of a function.

So, let me repeat. The Newton's method defines a sequence of real numbers. If that sequence is convergent, then the limit is going to be 0 or root of a function. If there is convergence, then the convergence is going to be quadratic provided your 0 is a simple 0, but there may not be convergence. We have seen some sufficient conditions for which guarantee that there is the convergence in Newton's method. So, we are going to look at Newton's method for finding root of x square minus a is equal to 0. We will show that the sequence or the Newton's iterates they converge. This is our next problem.

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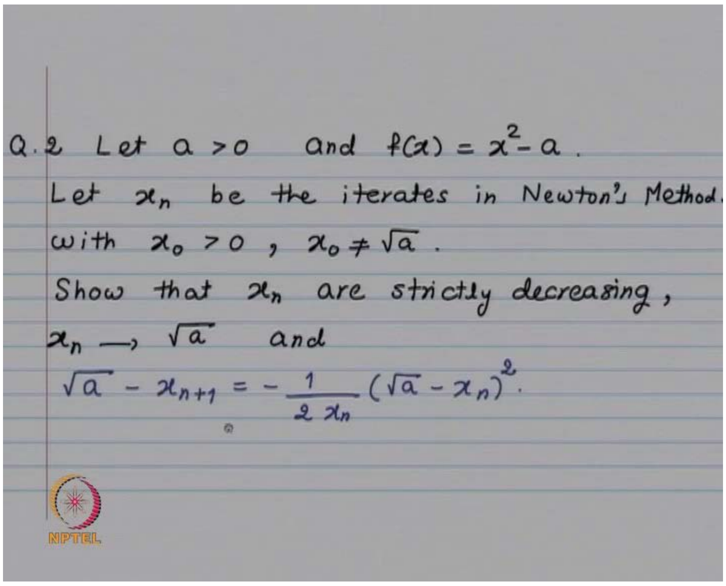

So, we start with a to be bigger than 0. Function f x is x square minus a, x n's are iterates in Newton's method. About the initial approximation, our condition is x 0 should be bigger than 0, and x 0 should not equal to root a. We want to show that x n's, they are strictly decreasing, x n's converge to root of a at root a minus x n plus 1. So, that is going to be the error in the n plus first iterate. This is equal to minus 1 upon 2 x n multiplied by root a minus x n square. So, here is error in n plus first iterate, here is error in nth iterate and then, it is square. So, once I show this, then that will mean that we have got quadratic convergence. We will keep a track as to where we need x 0 to be bigger than 0 and x 0 to be not equal to root a. If x 0 is equal to root a, then x 1 also will be equal to root a, and the sequence generated by Newton's method is going to be a constant sequence. So, here that means, you have found the root.

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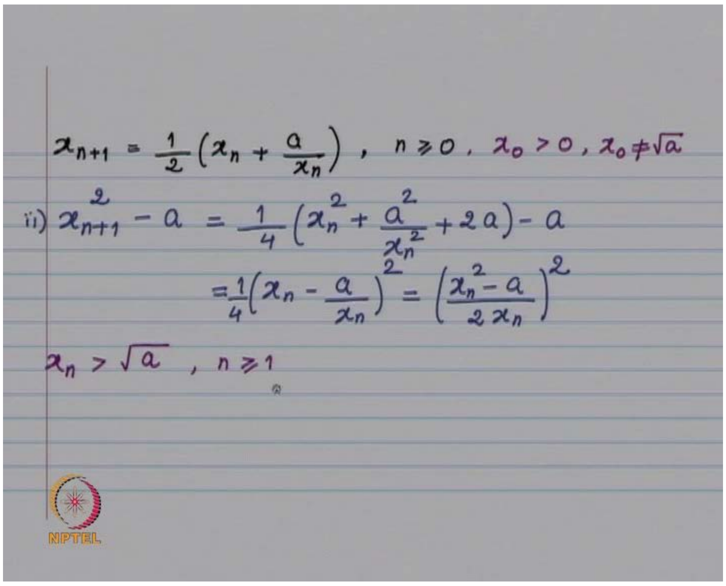

F x is equals to x square minus a. Its derivative will be given by 2 x, x n plus 1 is equal to x n minus f x n divided by f dash x n. This is equal to x n minus f x n will be x n square minus a divided by f dash x n which is 2 x n. So, to start with, when I look at x 1, I am going to have x 0 in the denominator. So, that is why I need x 0 to be bigger than 0, x 0 we say it should not be equal to be root a, because if it is equal to root a, then what will happen is x 1. So, x 0 square will be equal to a. So, this term will go away and you will have x 1 is equal to x 0. So, you get a constant sequence. So, this is equal to 1 by 2 x n plus a by x n. You have 2 x n square minus x n square divided by 2 x n. So, that gives you 1 upon 2 x n and then, this a upon 2 x n is written here.

So, x n plus 1 is equal to 1 by 2 into x n plus a by x n, a is bigger than 0. So, that will mean that x n's, they are going to be bigger than 0. X 0 bigger than 0 implies x 1 bigger than 0, that will imply x 2 bigger than 0 and so on. So, this is our first step, x n plus 1 is 1 by 2, x n plus a by x n. Then, let us look at x n plus 1 square minus a. So, square both the sides and subtract a. Square of this is 1 by 4 x n square plus a square by x n square plus 2 a minus a. So, this will be equal to 1 by 4.

Now, here you have got 2 a by 4. So, that is going to be a by 2 and then, this is minus a. So, you will have minus a by 2 and that gives you, x n minus a by x n square. So, this will be equal to nothing, but x n square minus a divided by 2 x n whole square and that gives you x n to be bigger than root a for n bigger than or equal to 1.

So, we start with x 0 to be bigger than 0. Then, your x 1 it is bigger than root a, and this condition will be satisfied by x 2, x 3 and so on. So, your first x 0, you have got 0, you have got root a. So, your x 0 which you choose, it may be to the left of root a, or to the right of root a, but after the first iterate x 1, x 2, x 3, they are all going to lie to the right of root a. Now, this becomes important because we are going to show, so we have got now x n to be bigger than root a. Now, we will show that they are strictly decreasing.

So, if you have got a monotonically decreasing sequence which is bounded below, then such sequence is always convergent. That is a property of real sequences that monotonically increasing sequence which is bounded above. That is convergent or monotonically decreasing sequence, which is bounded below, that is going to be convergent. So, we have got x n to be bigger than root a, for n is equal to 1, 2 and so on. So, we have got a sequence which is bounded below and now, let us show that it is a decreasing sequence.

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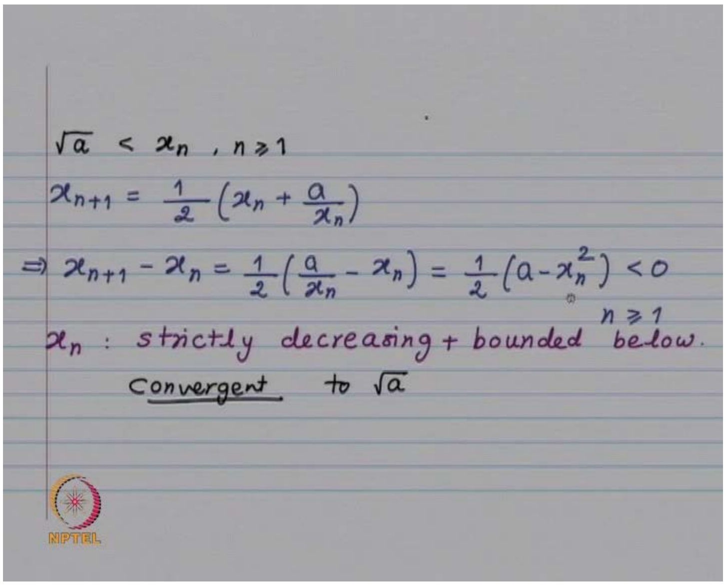

So, x n plus 1 is 1 by 2 x n plus a by x n. We obtain this expression. Look at x n plus 1 minus x n. So, we are subtracting minus x n from both the sides. So, what you will have will be 1 upon 2 a by x n, here it is 1 by 2 x n minus x n. So, that is going to be minus 1 by 2 x n. So, this is equal to 1 by 2 a minus x n square and then, they should be divided by x n. So, we have got x n plus 1 minus x n is equal to 1 upon 2 x n, a minus x n square. Just now we have proved that x n is bigger than root a, for n is equal to 1, 2 and so on. So, this will mean that x n square will be bigger than a, and that implies a minus x n square is going to be less than 0.

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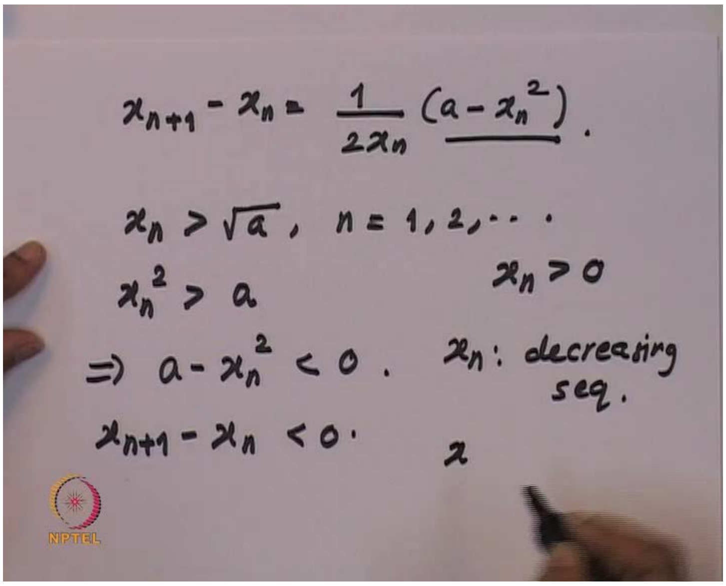

So, we have got x n plus 1 minus x n to be less than 0 because this a minus x n square is less than 0 and our x n, they are going to be bigger than root a and hence, bigger than 0. So, this implies that x n is a decreasing sequence. So, we have got a decreasing sequence which is bounded below and hence, it will converge.

Now, just strictly decreasing bounded below, it tells us that it is convergent. Why it should converge to root a? It is because if the sequence generated in the Newton's method if it is convergent, it has to converge to root of your function f. Roots of our function are plus root a, and minus root a. The sequence which is generated, it is going to be sequence which is bigger than 0. So, if it is convergent it has to converge to root a. It cannot converge to minus root of a. So, thus we have proved that the sequence generated in the Newton's method, it is converging to root of a, and now, let us show the quadratic convergence.

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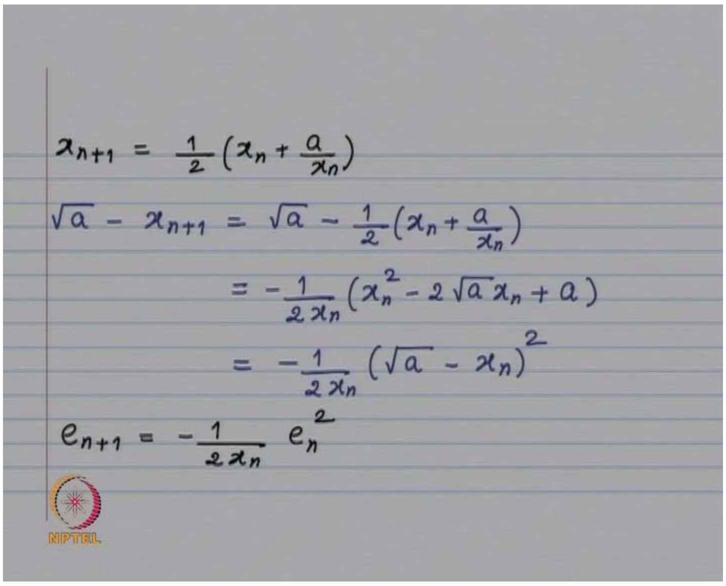

So, x n plus 1 is 1 by 2, x n plus a by x n, it is converging to root of a. So, let me look at root a minus x n plus 1. That will be root a minus 1 by 2 x n plus a by x n. Now, this is going to be equal to minus 1 upon 2 x n. So, I am taking x 2 x n as the denominator. So, I will have here, x n square, then you will have 2 root a x n because there is minus sign here, it will be minus 2 root a x n plus 2 x n is taken out. So, it will be a, which is equal to minus 1 upon 2 x n and root a minus x n whole square. This is e n plus 1 minus b 1 upon 2 x n e n square.

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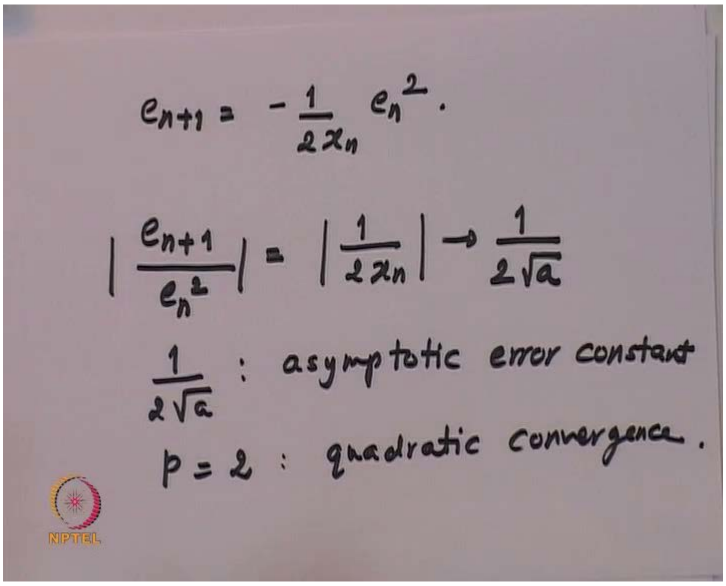

So, we have e n plus 1 is equal to minus 1 upon 2 x n, e n square. So, e n plus 1 upon e n square, this is going to be equal to, let me take the modulus. This will be 1 upon 2 x n. This will converge to 1 upon 2 root a, because x n is tending to root a. So, this will mean that 1 upon 2 root a, that is going to be our asymptotic error constant and if I recall the earlier notation, then p is equal to 2. So, that is quadratic convergence.

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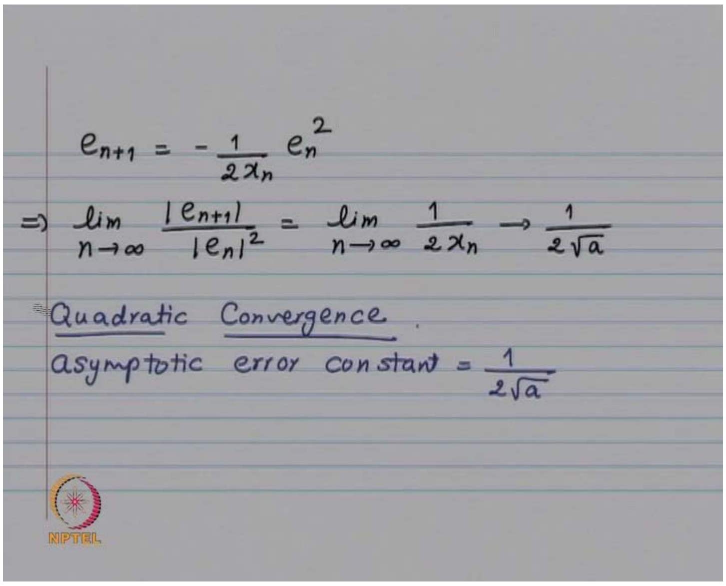

Limit as n tends to infinity, modulus of e n plus 1 by mod e n square, that is equal to 1 upon 2 root a and that is the asymptotic error constant and we have got quadratic convergence. We want to consider some of the problems, which are related to system of linear equations. So, when we talked about the condition number, we had obtained a lower bound for the condition number which was condition number of a, is bigger than or equal to norm c j by norm c i, where c j is jth column, c i is ith column.

So, using this estimate, we had said that if the columns are not balanced, if they are out of order, then your matrix is ill conditioned, but one says for the columns, it is true for the rows also. So, the result about that condition number of a, is bigger than or equal to norm c j by norm c i. I had left that as an exercise. So, that is what now we will do. We will define what is known as minimum magnification of a. Relate the minimum magnification of a to norm of inverse and then, we will obtain condition number as ratio of maximum magnification divided by minimum magnification and then, I will recall one of the example which we had considered of a 2 by 2 matrix. So, that matrix was ill conditioned. So, we showed that you change the right hand side slightly and perturbation in the solution is too big. So, here is the problem.

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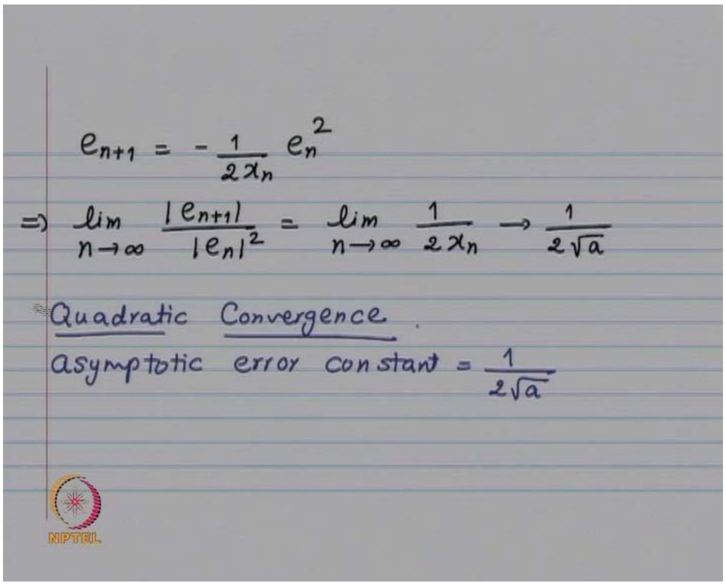

A is an invertible matrix, minimum magnification of a is minimum of norm a x by norm x, x not equal to 0. If instead of minimum, we have maximum here that is our definition of norm a. So, we want to show that minimum magnification of a is nothing, but 1 upon norm a inverse. The proof is straight forward we start with minimum magnification of a. This is our definition. So, this will be same as minimum x not equal to 0 vector 1 upon norm x divided by norm a x. So, this becomes equal to 1 upon maximum x not equal to 0, norm x divided by norm a x.

When we got a to be invertible matrix, then x not equal to 0 vector implies a x not equal to 0 vector. So, proof of this result is, where a x is equal to 0 vector. It will imply that a inverse a x is also 0 vector.

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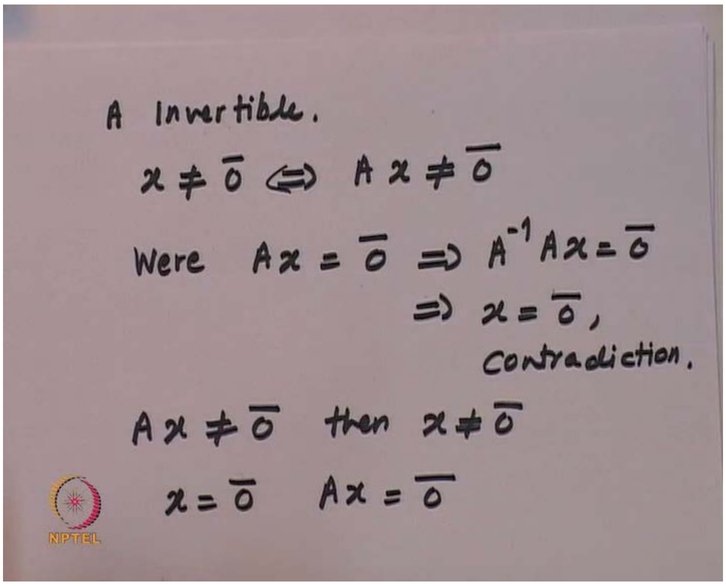

I am applying a inverse and this will mean that, x is equal to 0 vector contradiction, and if a x is not equal to 0 vector, then x cannot be 0. This is for any matrix, that if you have got x is equal to 0 vector, a x is always 0, whatever is the matrix whether it is invertible or not. So, for a invertible matrix, we have got this if only if condition.

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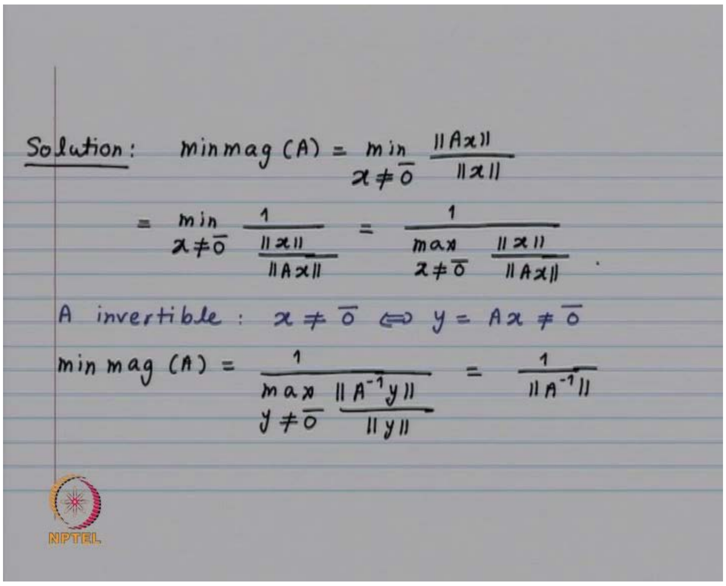

So, now look at the minimum magnification of a. So, here we are taking maximum over x norm x upon norm a. Let me put x is equal to y. So, y is equal to a x, then when I this will be same as 1 upon maximum y not equal to 0. For x, we have got a inverse y divided by norm y and that is nothing, but 1 upon norm a inverse. Now, once we show this the condition number is norm a into norm a inverse, norm a is maximum, norm a x by norm x x not equal to 0 vector and then, this norm a inverse which is in the numerator, I write as 1 upon norm a inverse in the denominator. So, this is maximum magnification of a just now we showed that 1 upon norm a inverse is minimum magnification of a.

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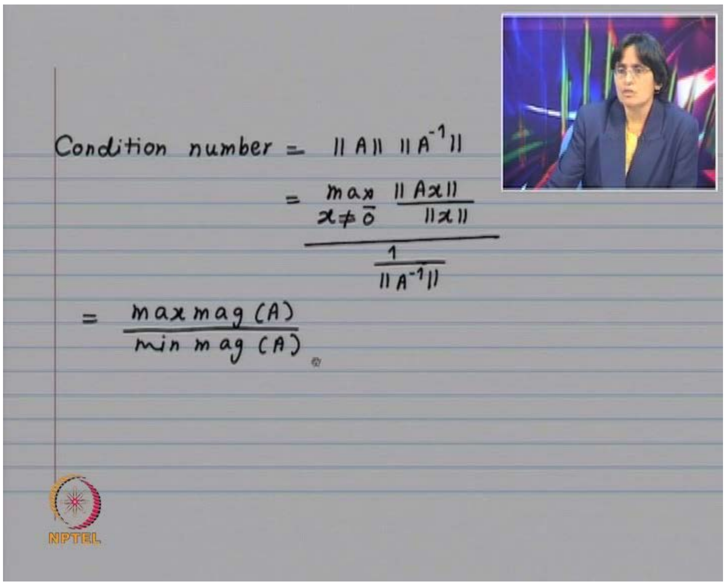

So, the condition number is going to be equal to ratio of maximum magnification of a divided by minimum magnification of a. So, now, let me recall the 2 by 2 examples which we had considered for this matrix. Its infinity norm is 1999 norm. That means, row some norm, take the absolute value of the entries, look at the first row, look at the second row and take the sum whichever is the maximum. A of vector 1 1 is going to be 1999 divided by this. So, you have norm x infinity is 1 1, norm a x infinity is going to be maximum of these numbers.

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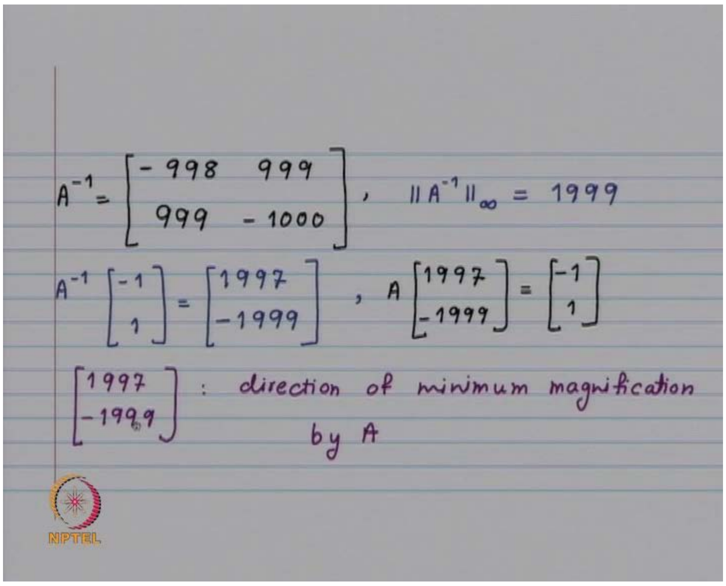

So, you have for this particular vector, norm a x is divided by norm x is equal to norm a infinity. So, this 1 1 is going to be the direction of maximum magnification for this matrix. Now, let us calculate or let us find out the direction of minimum magnification. So, the direction of the minimum magnification is going to be direction of maximum magnification for A inverse. So, A inverse is this matrix. Its infinity norm is again the same as before. A inverse of minus 1 1 is going to be this vector 1997 minus 1999. So, minus 1 1 is the direction of maximum magnification for A inverse. This relation you can write as A of this vector is equal to minus 1 1. So, this vector defines direction of minimum magnification by A. The direction maximum magnification was 1 1 and minimum magnification will be decided by this vector.

Let us go back to our system of linear equations. So, we have got a x is equal to b. We obtain an approximate solution x cap or it is our computed solution. There is some error because you are using computers. So, we have got finite precession or you may be using some indirect method, such as Jacobi method or Gauss-Seidel method. So, what we want do is, so residual is something which you are going to calculate, like I find a computed solution x cap. I will like to know whether this x cap is near to the exact solution. Exact solution we do not know. So, what we cannot calculate norm of x minus x cap or we cannot calculate the relative error. What we can calculate is a x cap minus b. If x cap were exact solution, then a x cap will be equal to b and the residual will be 0. Otherwise, it will be something non 0.

So, what one wants to know is, if the residual is small, then whether I can say that my computed solution is near to the exact solution. We have got computed solution x cap. I calculate a x cap minus b. So, suppose this number is small, its norm is small. In that case, whether one can say that x cap is near to x, so that is the question. Now, once again it is going to depend on the condition number. If your condition number is small, then the residual small will mean that x cap is near x. If the condition number is big, then it may not be case. So, let us prove the corresponding result.

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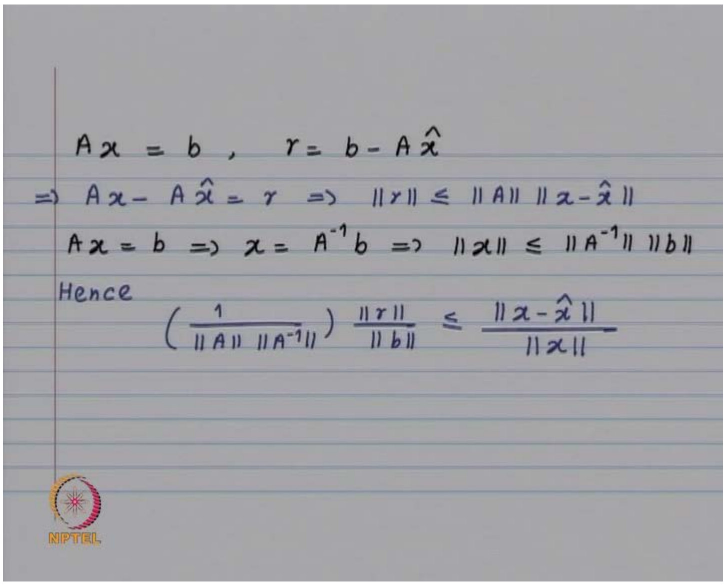

So, A is invertible matrix, x cap is approximate solution, r is the residual b minus a x cap and then, this is going to be the relative error in the computed solution. This will be less than or equal to norm A into norm A inverse, the condition number into norm r. So, what it means is, this number is a big number, then even though this is small. It can happen that the residual, it can happen the relative error in the computed solution can be big. If this is something reasonable, then the small residual implies the nearness of the computed solution to the exact solution. So, we have a x is equal to b r is equal to b minus a x cap. So, that will mean that a x minus a x cap is equal to r, just from these 2 equation. A is invertible matrix. So, x minus x cap is equal to A inverse r, norm of x minus x cap will be less than or equal to norm A inverse into norm r.

Now, norm b from here will be less than or equal to norm A into norm x. So, we get 1 upon norm x to be less than or equal to norm A into norm b. So, in the earlier estimate, what I had written there should be norm r by norm b. Now, that is something logical. Here, you are taking relative error. So, here you should not look at the absolute error. Absolute error will be norm r, norm of b minus a x cap. So, here also you should look at relative error. So, this is norm r by norm b. So, if this relative error is small and if the condition number is not too big, it will imply that this relative error will be small and then, other way inequality similar.

So, we have got A x minus A x cap is equal to r. So, do not apply A inverse. From here, you can say that norm r is less than or equal to norm A into norm of x minus x cap, A x is equal to b. So, you have got x is equal to A inverse b. So, norm x is less than or equal to nor A inverse into norm b. So, from combining these 2, you will get 1 upon norm A into norm A inverse, norm r into norm b to be less than or equal to this relative it.

Now, we want to consider an example about scaling. When you multiply a matrix by a non 0 number, it does not change the condition number. The condition number remains the same if I multiply the matrix by a non 0 number, but if I multiply, say one of the row by a non-0 number, then the condition number is going to be effected. So, we are going to look at a 2 by 2 example and in that, we will try to determine the non 0 scalar alpha, by which if I multiply the first row, then the condition number is minimized.

So, that is the scaling that the matrix is given to you. So, you look at its rows and columns and try to see that they are not, say out of proportion. If you multiply a row by a non 0 constant, you do not change the system. It is the same system as the original system, but new system may have a better condition number. If you multiply a column of the coefficient matrix by a non 0 scalar, then your solution gets changed, it is a new system, but there is a relation between the earlier solution and solution, which you will obtain in the new system and that is if you multiply the jth column by a non-0 constant, the corresponding solution will change only in the jth component. So, that we have seen. So, we are going to look at 2 by 2 matrix or a 2 by 2 system.

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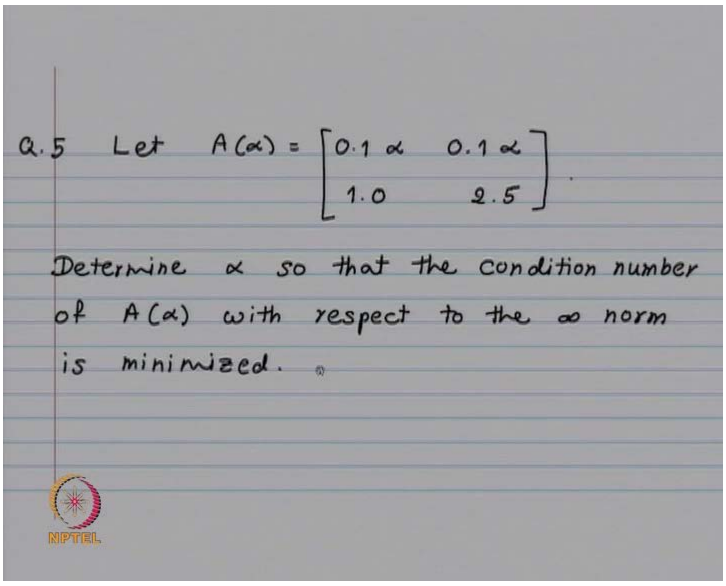

So, we have this is a matrix. What I want do is determine alpha, so that the condition number of A alpha with respect to the infinity norm is minimized now here is a 2 by 2 matrix. So, I can calculate its inverse, infinity norm is specified. So, what we are going to do is, look at A alpha, look at its norm A alpha infinity, calculate A alpha inverse, calculate its infinity norm and then, we will have to consider 2 cases and in both the cases, we will see that we get the same alpha. So, we are trying to find alpha which minimizes the condition number of our coefficient matrix.

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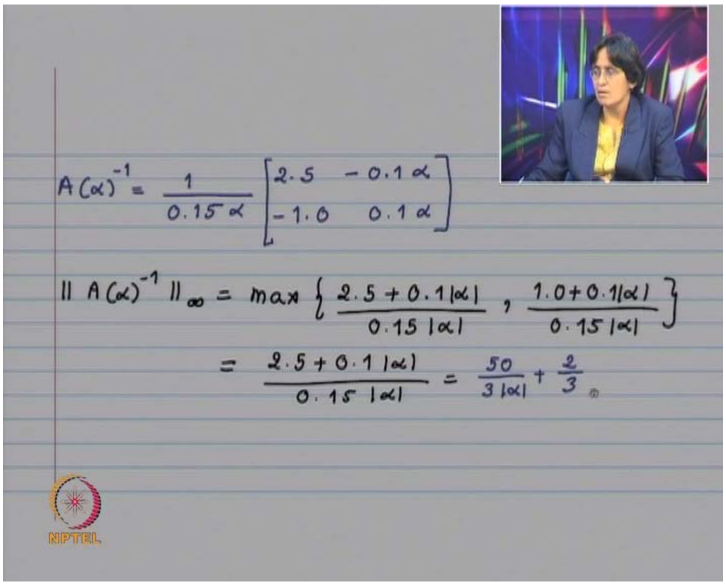

So, we have, this is our matrix a alpha when you look at the infinity norm of the matrix, that is row sum norm. You have to take the modulus. So, first row, its sum will be 0.2 into mod alpha. This second row, its sum is 3.5 and norm A alpha infinity will be maximum of these 2 numbers. You look at A alpha inverse. The inverse of this matrix A alpha is given by this. Our alpha is going to be not 0. So, this is the inverse.

Now, we will calculate its infinity norm. So, norm A alpha inverse will be the first row gives us 2.5 plus 0.1 mod alpha divided by 0.15 mod alpha. The second row gives you 1.0 plus 0.1 mod alpha divided by 0.15 alpha. So, the maximum of the 2 number is going to be the first number, which you can simplify and get 50 upon 3 mod alpha plus 2 by 3. So, we have got norm A alpha to be maximum of these 2 numbers. Norm A alpha inverse is given by this quantity.

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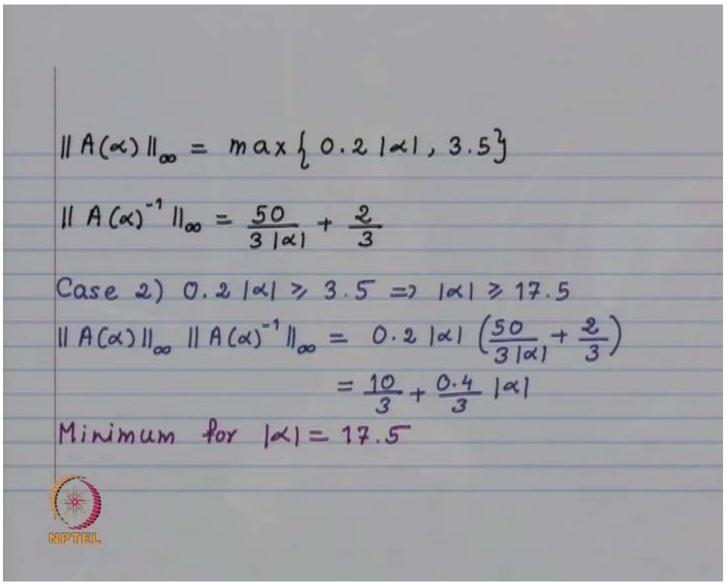

Case 1 will be 0.2 mod alpha will less than or equal to 3.5. That means, mod alpha will less than or equal to 17.5, when I take the maximum of the 2. Now, the maximum is 3.5, say it will be 3.5 times this. So, this is going to have minimum to be equal to 17.5. So, you will have minimum of norm A alpha, norm A alpha inverse to be 17.5 and in the second case also, you are going to get exactly the same value.

So, now in the next lecture, we are going to start new topic and that is the approximate solution of initial value problem. So, thank you.