Elementary Numerical Analysis

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Lecture No. # 27

Quadratic Convergence of Newton's Method

We are considering solution of non-linear equations. Last time we defined newton's method and secant method and also bisection method; so, for the bisection method we have seen that the convergence is very slow. Now, today, we are going to show that newton's method, it converges quadratically or the order of convergence is two; and for secant method it is going to be better than linear convergence, but less than quadratic convergence.

So, let me recall the definition of order of convergence which we defined last time. So, we look at a sequence xn of real numbers converging to c. Then let en plus 1 be the difference between c and xn plus 1, so c minus xn plus 1; so, if modulus of en plus 1 divided by modulus of en raise to p, so en is going to be error at the nth stage, en plus 1 is the error at the n plus first stage.

So, look at the quotient, modulus of en plus 1 divided by modulus of en raise to p; if limit of this is equal to m where m is not 0, so limit as n tends to infinity modulus of en plus one divided by mod en raise to p; if it is equal to m not equal to zero, then we say that m is the asymptotic error constant, and p is order of convergence; so, this p we are going to show that, in case of fix point iteration it is going to be equal to 1, in case of newton's method it is going to be equal to 2, and in case of secant method it will be about 1.6; so, better than linear, but less than the newton's method, which is quadratic convergence.

So, first let us show the linear convergence in the fix point method. Now, in all these, like for newton's method, for fix point iteration, in order to show the order of convergence we are going to use mean value theorem or extended mean value theorem; and for the secant method we will use the error in the polynomial approximation.

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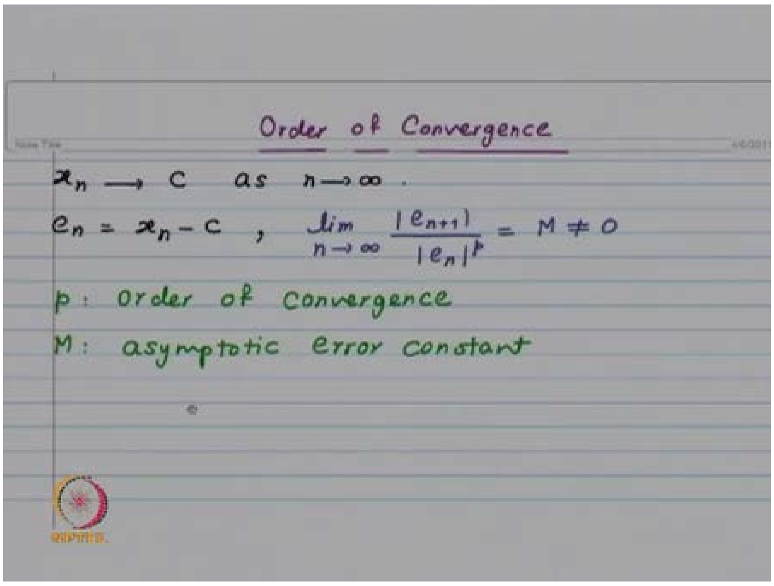

So, here is the order of convergence, xn is converging to c as n tends to infinity; by en we denote the error xn minus c, and limit as n tends to infinity modulus of en plus 1 by mod en raise to p, if it is equal to m not 0, then p is the order of convergence, and m is the asymptotic error constant.

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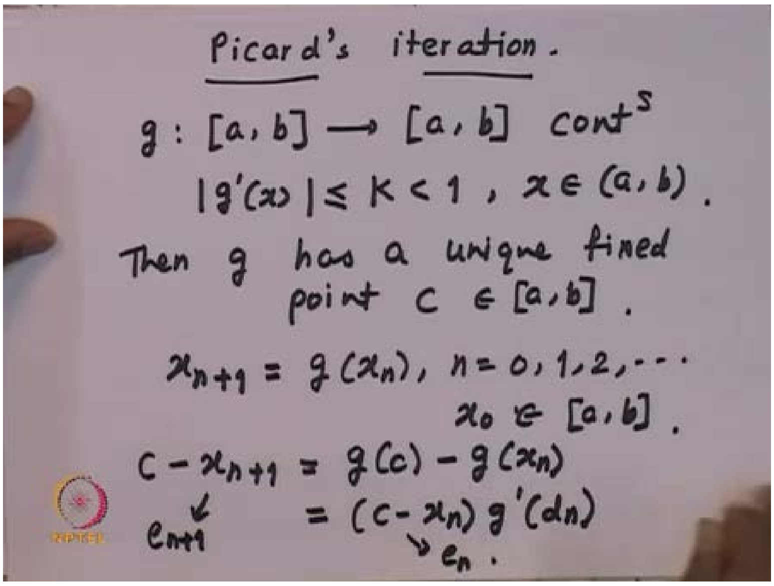

So, let us first look at the fix point iteration or Picard’s iteration. So, we have got g to be a map from interval a b to interval a b, it is a continuous map, and modulus of g dash x is less than or equal to K less than 1 for x belong to open interval a b.

Then g has a unique fixed point c in interval a b; and our iteration is xn plus 1 is equal to g xn, n is equal to 0, 1, 2, and so on, and x 0 is starting point which is any point in the interval a b. So, when I consider c minus xn plus 1, this is equal to g of c - c being a fix point - minus g of xn; and now, I use mean value theorem to write the right hand side as c minus xn g dash of dn.

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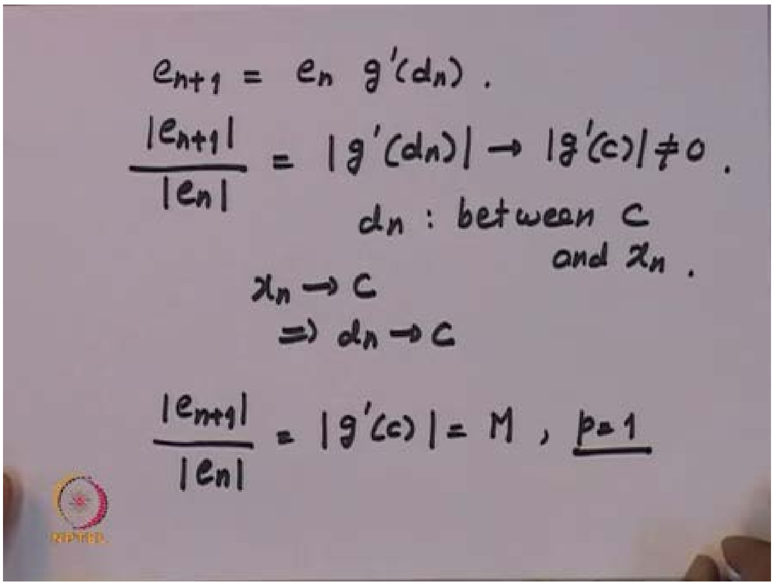

So, this is our en plus 1, and this is our en; look at our en plus 1 is en multiplied by g dash d n. So, modulus of en plus 1 divided by modulus of en is equal to mod g dash d n, d n is between c and xn; our xn is going to converge to c, so that will imply that d n also will converges to c then assuming continuity of the derivative, this will tend to modulus of g dash c.

So, if this is not equal to 0 then we have got mod en plus 1 divided by mod en is equal to mod g dash c, so this will be our M, and our p is going to be equal to 1; and does for the fix point iteration the order of convergence is going to be equal to 1. So, this is under the condition that g dash c is not equal to 0; if our fix point c is such that g of c is equal to c and g dash is equal to 0, in that case we are going to get order of convergence to be equal to 2. So, this part we will see little later.

So, now from the fix point iteration let us go to newton's method. Now, the newton's method it need not always converge. So, first we are going to show that if the iterates in the newton's method, if they converge then they have to converge to a 0, if the convergence is there then it is to the zero of the function; then we will look at an example where the iterates in the newton's method they domain not converge, but they will oscillate.

We will then consider sufficient conditions for the convergence of newton's method. Now, what we are going to do is, this newton's method we will write it as fix point iteration and we have got sufficient condition for convergence of fix point iteration, so we will just translate those, so that gives us a sufficient condition for convergence of newton's method; then we will look at the some other set of conditions which also gives us convergence in the newton's method and then we will look at the order of convergence of newton's method.

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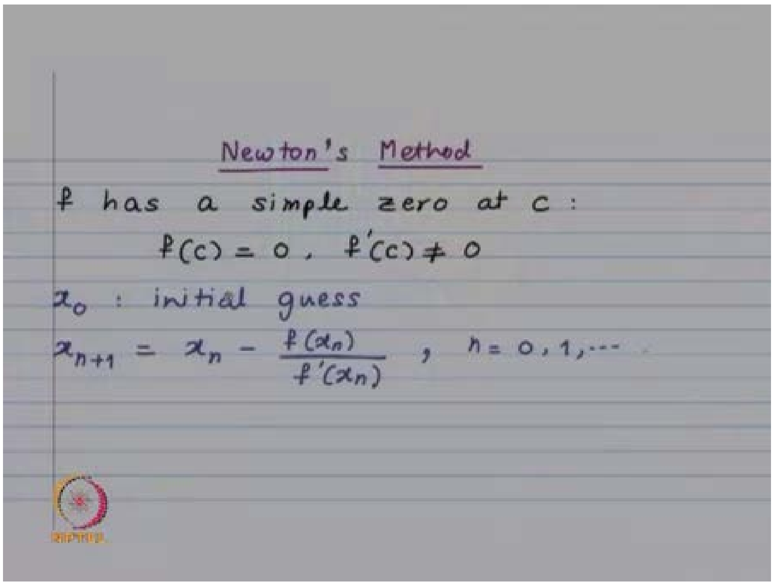

So, here is newton's method, f has a simple zero at c, so this is our assumption, that means, f of c is equal to 0, f dash c is not equal to 0, x 0 is our initial guess, xn plus 1 is equal to xn minus f xn divided by f dash xn, n is equal to 0 1 2 and so on. And we had seen yesterday interpretation of newton's method as you look at the tangent to the curve at point xn f xn, see where the tangent cuts x axis, so the intersection of the tangent to the curve at xn f xn and the x axis.

That is going to give us our next iterate xn plus 1. So, we need the condition that at no point the tangent should become horizontal, that will be the case if at some point f dash xn is equal to 0; so, that is why we our starting assumption is f of c is equal to 0 f dash c not equal to 0. So, if you are in neighborhood of c f dash xn will not be 0 in any case at present we are assuming that the iterates are defined.

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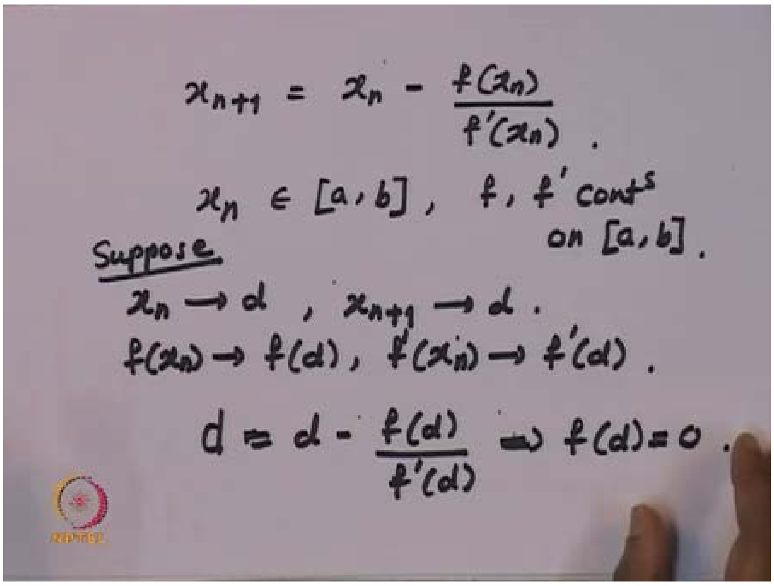

Now, suppose, iterates converge, so if we have got, we have got iterates xn plus 1 is equal to xn minus f xn divided by f dash xn. Suppose, that all these iterates xn they lie in the interval a b and function f and its derivate, they are continuous on interval a b.

Now, suppose xn is converging to d, so suppose then xn plus 1 also will tend to d, because it is the same sequence by continuity of f f xn will tend to f of d f of xn or other f dash of xn will tend to f dash of d. So, we get c or rather d is equal to d minus f of d divided by f dash d; so, that gives us f of d to be equal to 0. So, if iterates xn in the newton's method, if they are converging then they have to converge to a zero of our function. Now, let us look at an example where the iterates they may not converge .

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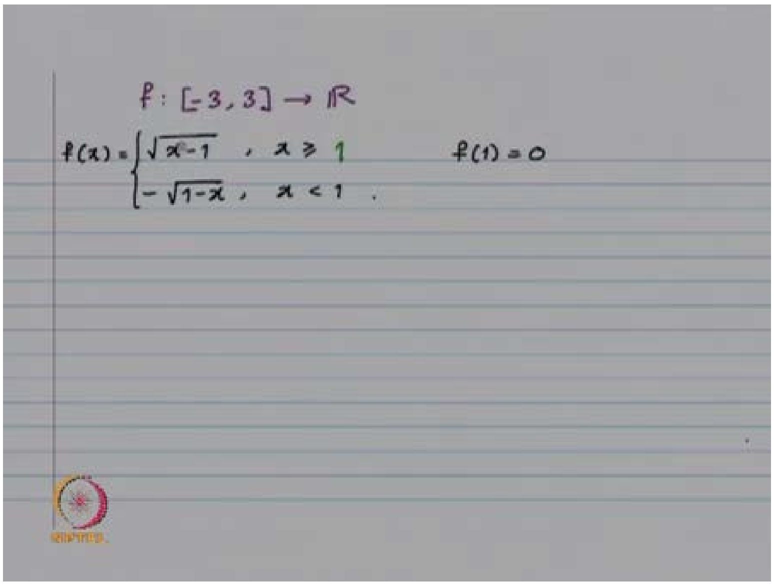

So, look at this example, f is defined on interval minus 3 to 3 taking real values; and the definition is fx is equal to root of x minus 1 x bigger than or equal to 1 and for x less than 1, the definition is minus root of 1 minus x 1 can see that f of 1 is equal to 0, and that is going to be unique 0. In the interval minus 3 to 3 the function has only 1 0 and that is one now we need to calculate the derivative.

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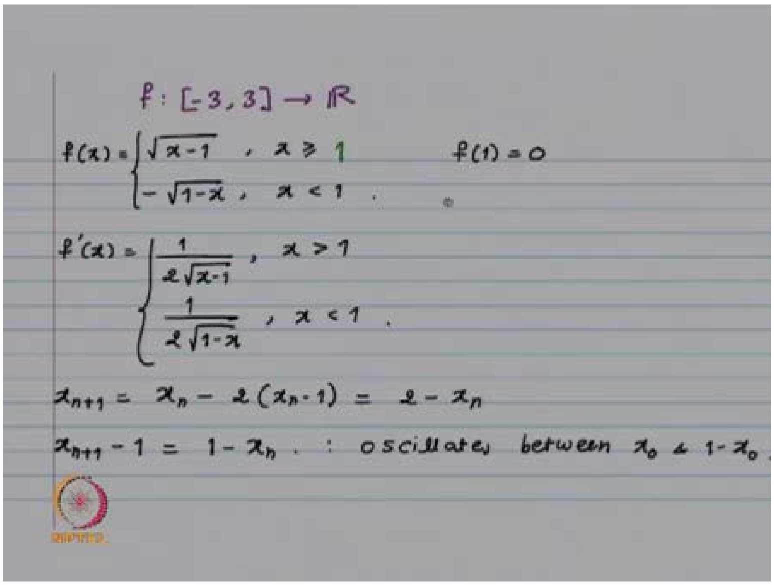

So, f dash x is going to be 1 upon two times root of x minus 1 if x is bigger than 1; and 1 upon two root of 1 minus x if x is less than 1.

So, this function will be differentiable in the interval minus 3 to 3 except at point 1. So, this is our f dash x; now xn plus 1 is going to be equal to xn minus f xn divided by f dash xn. So, you have got xn plus 1 is equal to xn f xn divided by f dash xn, so f dash xn is in the denominator, so it will go in the numerator and you will get minus 2 times xn minus 1 provided your xn is bigger than 1.

If xn is less than 1, then it is going to be minus root of 1 minus xn and then there is this 2 into root of 1 minus xn again going in the numerator; so, again whether xn is bigger than 1 or xn is less than 1, f xn divided by f dash xn is going to be 2 times xn minus 1, so this is equal to 2 minus xn.

From this relation I conclude that xn plus 1 minus 1 is equal to 1 minus xn. So, if I start with x 0, if I take x 0 is equal to 1, and then of course it is we have already found the 0. So, if x 0 is not equal to 1, then our x 1 is going to be equal to 2 minus x 0, it is going to oscillate between x 0 and two minus x 0.

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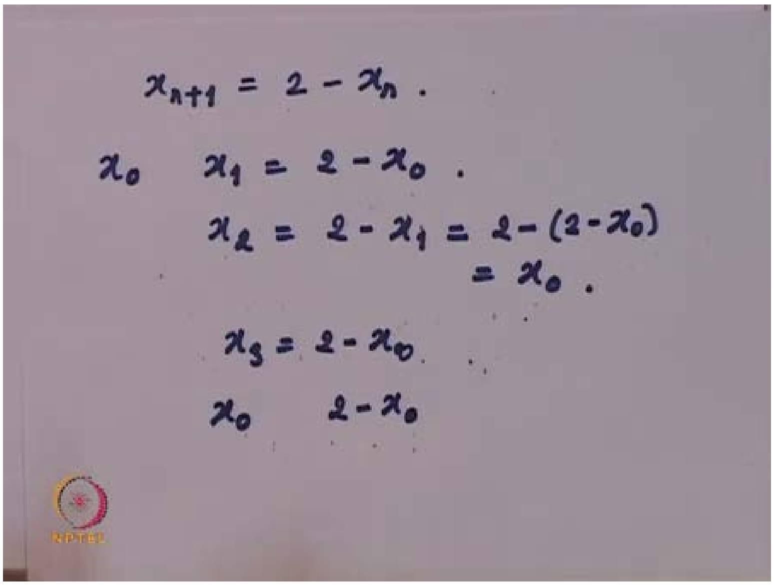

So, we have got xn plus 1 is equal to minus xn. So, you start with x 0, x 1 is going to be 2 minus x 0, x 2 will be 2 minus x 1, so it will be 2 minus 2 minus x 0 which is equal to x0. So, x 2 is equal to x 0 that will give you x 3 to be 2 minus x 3 will be equal to 2 minus x 0 and so on.

So, our sequence xn that is going to oscillate between x 0 and 2 minus x 0; so, in this example we had our function is defined on interval minus 3 to 3, there is a single 0 and that is 0 is equal to 1. Now, no matter how near you choose your x zero to one no matter how you are starting point is near to the 0, you are the sequence which you get it is a oscillatory sequence, this is a pathological example.

In general there are more chances of the convergence of iterates provided your starting point is near your 0. Now, in this example if you happened to choose your starting point to be the 0 itself, then you will you have convergence, you are going to get the constant sequence, but otherwise it remains oscillatory and we have no convergence. Now, look at the function, it is continuous on the interval minus 3 to 3, but it lacks differentiability at an interior point.

So, now, let us look at some of the sufficient conditions for convergence of newton's iterate. So, we have as I said we are going to try to write newton's method as an fix point iteration and then for the fix point iteration we have got sufficient condition.

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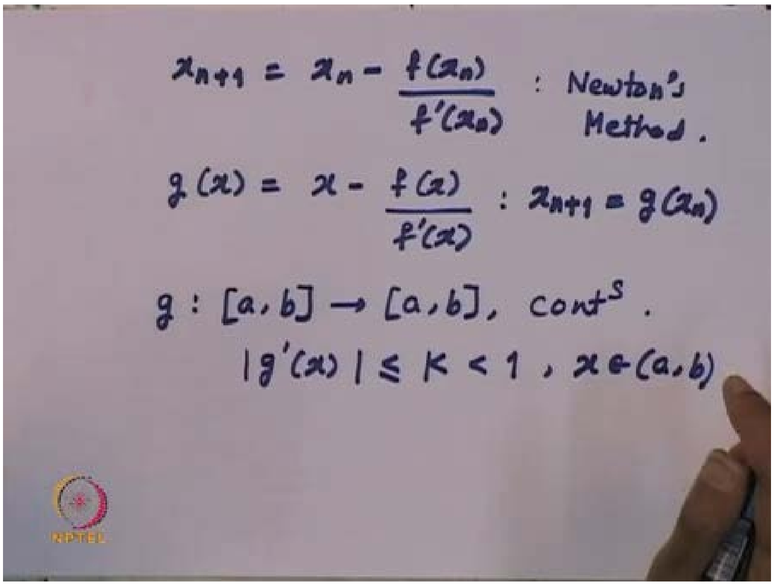

So, our newton's iterates are xn plus 1 is equal to xn minus f xn divided by f dash xn. So, if I define my function g to be g x is equal to x minus fx upon f dash x, then I can write the newton's iterates as xn plus 1 is equal to g xn. So, this i can look at…, so this is newton's method; and the iterates can be written as a fix point iteration for this function g.

Now, for the convergence of Picard's iteration what we had was g should map interval a b to interval a b, it should be continuous; and modulus of g dash x should be less than or equal to k less than 1 for x belonging to interval a b. So, we needed continuity of our function g, and we needed differentiability over open interval a b and the derivative should be less than or equal to k, where k is less than 1 for each x belonging to a b. So, under these conditions we showed that g has a unique fix point and no matter what starting point x 0 you choose in the interval a b.

The Picard's iterations xn plus 1 is equal to g xn, they are going to converge to the fix point our g now is g x is equal to x minus fx upon f dash x. So, first thing we need to assume is that, f dash x should not vanish, so f dash x should not be equal to 0, so that our function g is defined on the interval a b. When we will look at the derivative of g, the second derivative of function f will come into picture, so our function f should be twice differentiable; and then let us calculate g dash x with g x is equal to x minus fx upon f dash x, whatever condition we get we will say that that should be less than 1; so, let us write down this condition

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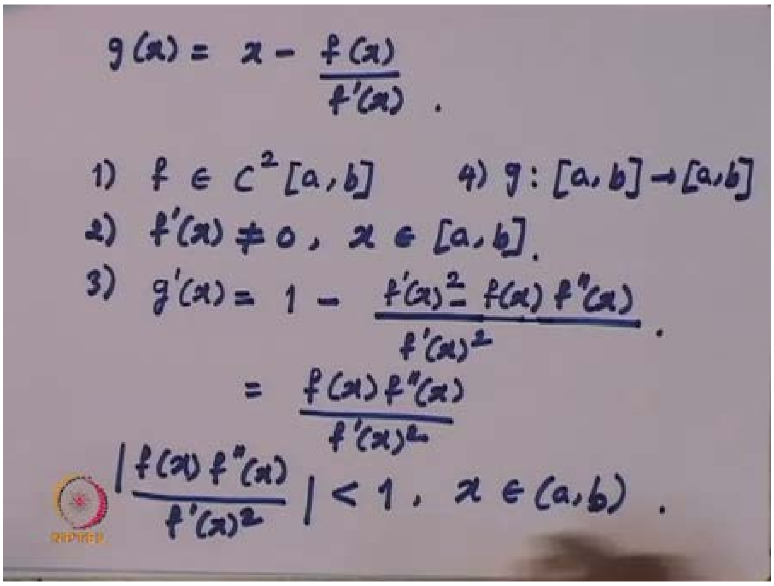

g x is fx x minus fx upon f dash x; so, the first condition is f should be 2 times continuously differentiable on interval a b.

Second, f dash x should not be equal to 0 for x belonging to a b; third condition will be look at g dash x, so g dash x will be derivative of x is 1 and for fx upon f dash x let me use the quotient rule, so it will be f dash x square, then denominator into derivative of the numerator, so it will be f dash x square minus fx into derivative of the denominator, so it is going to be f double dash x; so, this is going to be equal to fx f double dash x upon f dash x square; so, we want modulus of fx f double dash x by f dash x square, this should be less than 2 for x belonging to a b; and an important condition is that, g should map interval a b to interval a b. So, if these conditions are satisfied, then our newton's method it is going to converge; and we have seen that when it converges it is going to converge to 0.

Also these conditions they imply that g has a unique fix point in the interval a b and fix point of g is nothing but 0 of f. So, we are going to have a unique 0 of function f and the newton's method or the newton's iterates they are going to converge; now, this is one set of conditions; so, here we had just translated, so let us see more say geometric conditions for convergence of newton's method.

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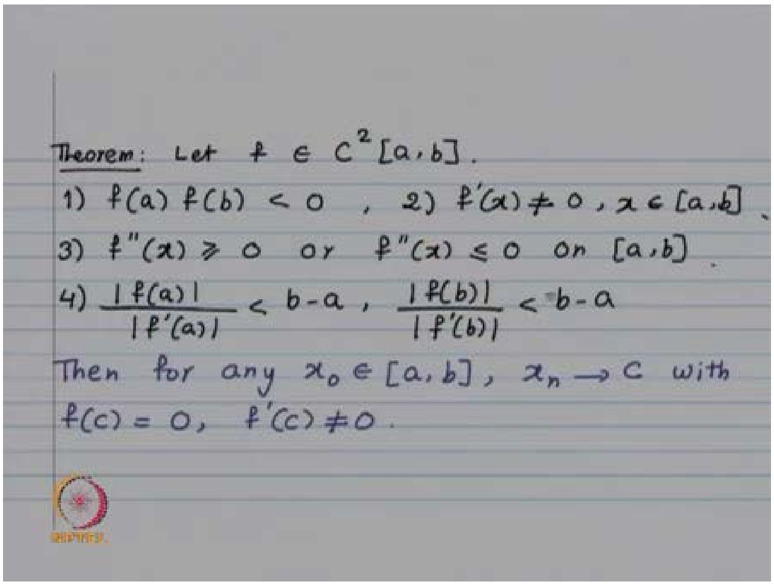

And these methods are…, so this is as before that f should be 2 times continuously differentiable, then f a into f b should be less than 0; that means, f a and f b they are of opposite signs; f dash x not equal to 0 x belonging to a b, this also was assumed in the earlier set of conditions; f double dash x bigger than or equal to 0 or f double dash x less than or equal 0 on close interval a b; and modulus of f a upon modulus of f dash a should be less than b minus a.

And mod f b by mod f dash b should be less than b minus a. Then for any x0 in a b the Newton's iterates xn will converge to c with f of c is equal to 0, f dash c not equal to 0. So, look at the first condition, f a into f b is less than 0; so, by the intermediate value theorem f has at least one 0 in the interval a b; if you assume that f dash x is not equal to 0 in the interval a to b, along with this condition consider f double dash x bigger than or equal to 0.

The second derivative tells you something about concavity and convexity of the function. Now, if f double dash x is strictly bigger than 0, that will mean that f dash has to be strictly increasing. If f double dash x is strictly less than 0, that will mean that f dash is strictly decreasing; so, this condition f dash x is not equal to 0, it tells us that f dash is going to be of the same sign, it will be either bigger than 0 or it will be less than 0; the fact that f a into f b is going to be less than 0, that tell us that there is at least one 0, then f dash x it will be either bigger than 0 or less than 0; if f dash x is bigger than 0, f will be strictly increasing; if f dash x is less than 0, it will be strictly decreasing; that means, there is going to be unique 0 in our interval a b.

Then the third is something about the convexity and concavity; and the last condition that those conditions are imposed to guarantee that if you choose the starting point say x 0 is equal to a or x 0 is equal to right hand point b, then the next iterate they will lie in the interval a b. Because what we want is all our iterates they should be in the domain of our f, f is defined on interval a b; so, I am not going to prove this theorem, but I am just going to show you that the last condition implies that if I choose x 0 is equal to a, then x one is going to be in the interval a b.

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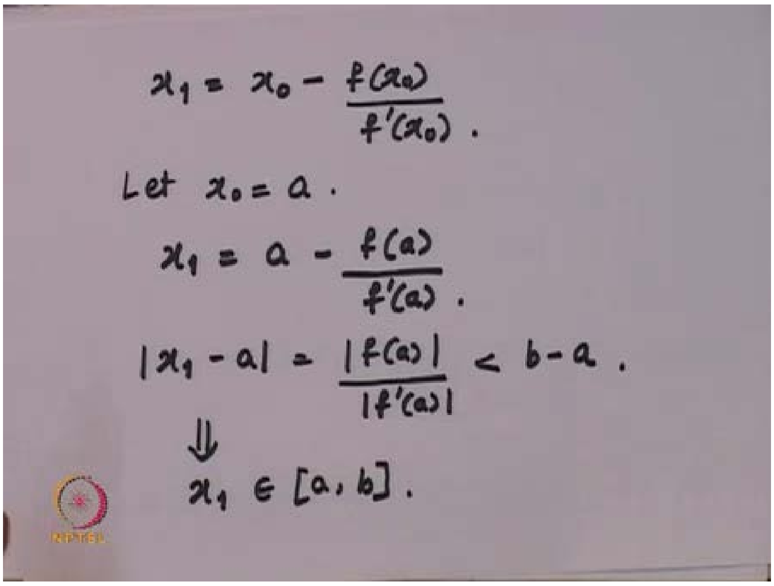

And the proof is simple. So, our x 1 is going to be equal to x 0 minus fx 0 upon f dash x 0; let x 0 be equal to a, then we have x 1 is equal to a minus f a divided by f dash a. So, modulus of x 1 minus a will be equal to modulus of f a divided by mod f dash a and this we are assuming to be less than b minus a, so this will imply that our x 1 is in the interval a b.

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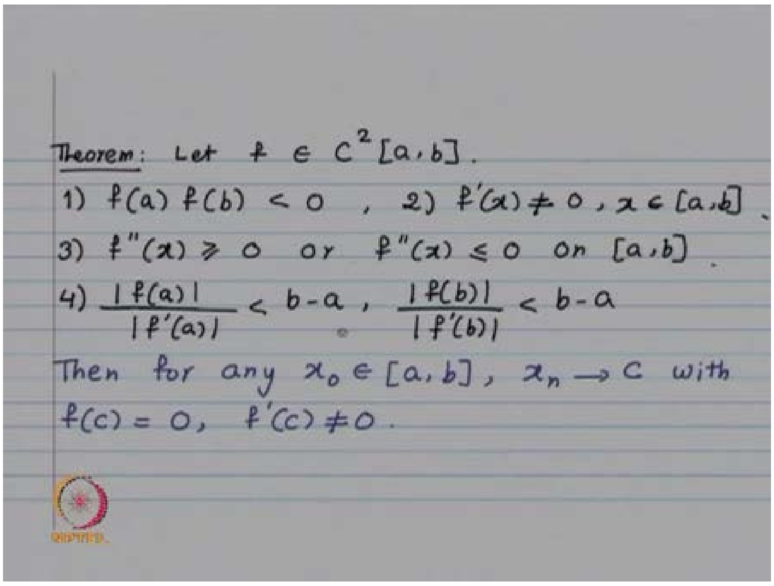

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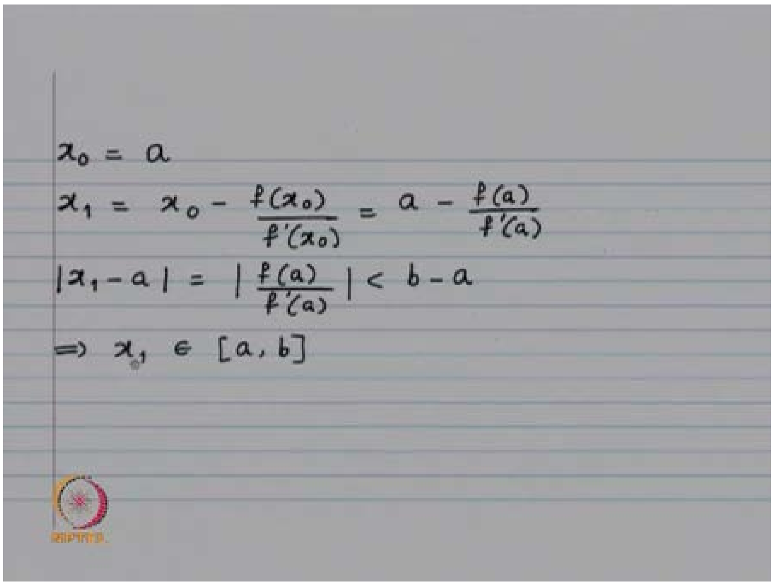

So, this last condition mod f a by mod f dash a less than b minus a, that implied that the first iterate x 1 is going to be in the interval a b. And a if you choose a x 0 is equal to b then the other condition will guarantee that x 1 belongs to interval a b. Now, we want to show that the iterates in the newton's method they converge quadratically.

This is the advantage of newton's method, like this is one of the plus point that is why newton's method is so popular; that if it converges, it is going to converge quadratically; Picard's iteration it converges only linearly. Now, let me show you that the newton's method, it is going to converge quadratically; that means, when I consider the error at the n plus first stage, modulus of en plus 1 divided by mod en square that will converge to a non-zero constant as n tends to infinity, so because mod en square, so that too is the order of convergence.

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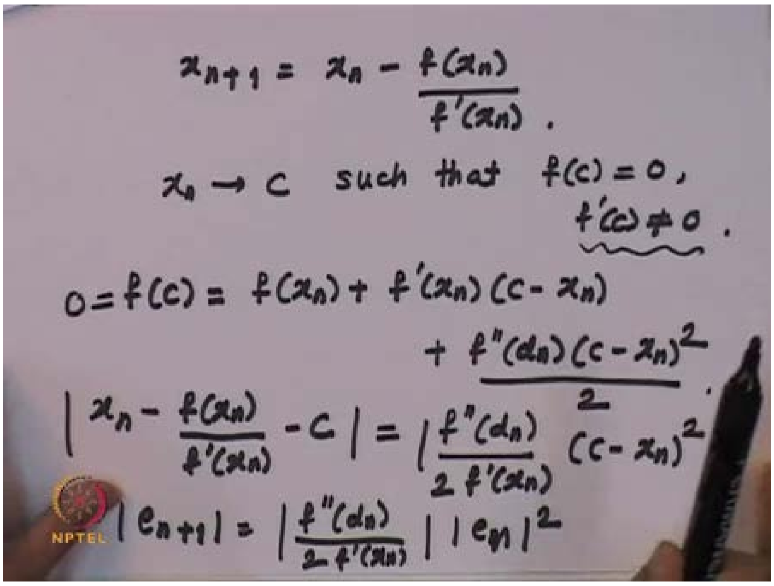

So, our iterates are xn plus 1 is equal to xn minus f xn upon f dash xn, xn's are converging to c such that f of c is equal to 0, and f dash c is not equal to 0. So, let me look at f of c and write Taylor's series expansion, so it is going to be f of xn plus f dash xn into c minus xn plus f double dash at some d n into c minus xn square divided by 2; so, this is the extended mean value theorem or truncated Taylor series for expansion. Our f of c is equal to 0, so what I do is, I divide throughout by f dash f xn and I take this first two terms on the other side; so, when I do that I will have xn minus f xn divided by f dash xn minus c, so what I have did is I am dividing throughout by f dash xn and taking the two terms on the other side, so i have xn minus f xn by f dash xn minus c is equal to f double dash d n divided by 2 times f dash xn into c minus xn square; take mod of both the sides, here xn minus f xn upon f dash xn that is our xn plus 1.

So, this is modulus of xn plus 1 minus c, so that is our modulus of en plus 1 is equal to modulus of f double dash dn divided by 2 times f dash xn into c minus xn is en, so it will be modulus of en square. Look at this condition f dash of c not equal to 0 xn is tending to c, so for n large enough f dash xn is not equal to 0, so we have modulus of en plus 1 upon mod en square is equal to….

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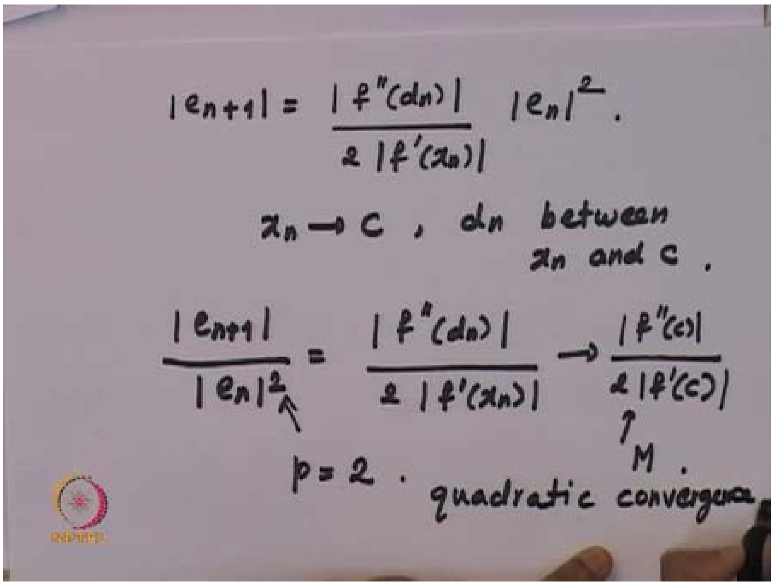

So, you have mod en plus 1 is equal to mod f double dash dn by 2 times mod of f dash xn into mod en square, xn is tending to c, d n lies between xn and c; so, assuming second derivative to be continuous you get modulus of en plus 1 divided by mod en square is equal to mod f double dash d n divided by 2 times mod of f dash xn, which converges to mod f double dash c divided by 2 times f dash c. So, this will be our asymptotic error constant and our p will be equal to 2; so, we have got quadratic convergence; so, this was for the newton's method.

Now, we are going to look at secant method. So, in the secant method what we do is, we start with two points x 0 and x 1; in newton's method we have got only one x 0, and then we looked at the tangent to the curve at x 0 fx 0; for the secant method as the name suggest we are going to look at two points on the curve x 0 fx 0 x 1 fx 0, look at the straight line joining them c where it cuts x axis, that is going to be our x 2.

And then you consider x 2 and x 1, look at the secant which passes through x 1 fx 1, x 2 fx 2 c, where it cuts x axis that is going to give us x 3 and so on, so this is the secant method. So, as we showed that when the iterates in the newton's method converge, they have to converge to a zero of our function, same thing we will show for the secant method.

Then we are going to show that our formula is going to be symmetric in xn and xn minus 1; xn plus 1 in the secant method, the formula is in terms of xn and xn minus 1 the values of function.

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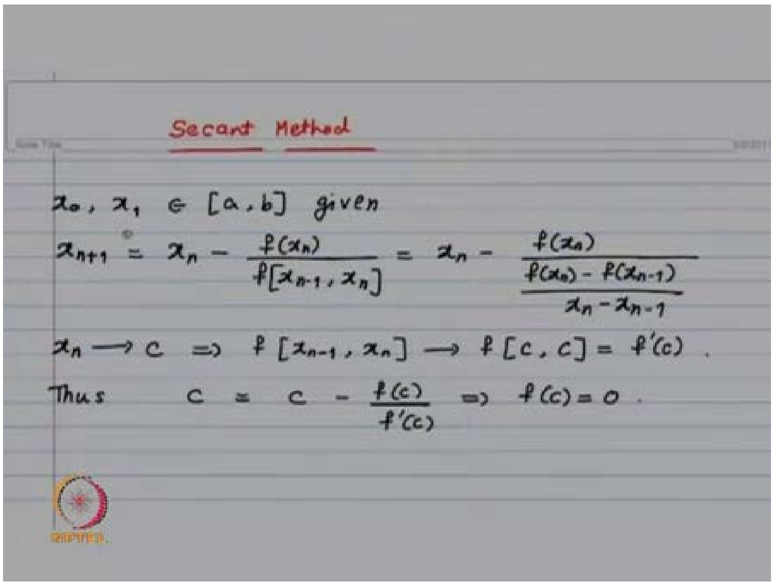

So, we will show that it is symmetric and then we will consider the order of convergence in secant method; x 0 and x 1 they are in the interval a b, what we are doing is f dash xn in the newton’s method is replaced by the divided difference based on xn minus 1 and xn. So, xn plus 1 is equal to xn minus f xn divided by divided difference based on xn minus 1 xn, so I substitute, it is going to be xn minus f xn divided by f xn minus f xn minus 1 divided by xn minus xn minus 1.

Suppose, xn converges to c, then xn minus 1 also converge to c, so it is the same sequence, and continuity of the divided difference gives us that this will converge to f of c c, but our definition of divided difference when the arguments are repeated it is f dash c; so, you get c is equal to c minus f c divided by f dash c; so, that will give you f of c is equal to 0. So, whenever the iterates converge, they are going to converge to 0 of our function.

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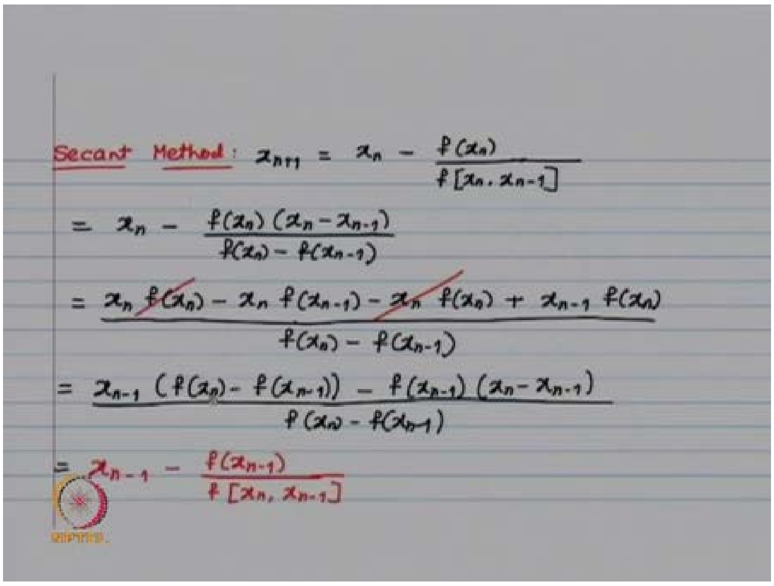

Now, here is the symmetric like xn plus 1 is equal to xn minus f xn upon f xn xn minus 1; so, if I interchange xn and xn minus 1, that means, if I consider xn minus 1 minus f of xn minus 1 divided by divided difference based on xn xn minus 1 I am going to get the same result.

And this result is something expected, because what we are doing is we are looking at point's xn minus 1 and xn, these we have obtained by the iteration process so far. Now these two points I look at the corresponding points on the curve, I join them by straight line, so then whether I the order should not matter, what matters is the two points xn minus 1 and xn; whereas, if you look at the formula xn plus 1 is equal to xn minus f xn divided by f of xn minus 1 xn the divided difference, it is not evident, how I can instead of xn I can write xn minus 1 and instead of xn minus 1 write xn.

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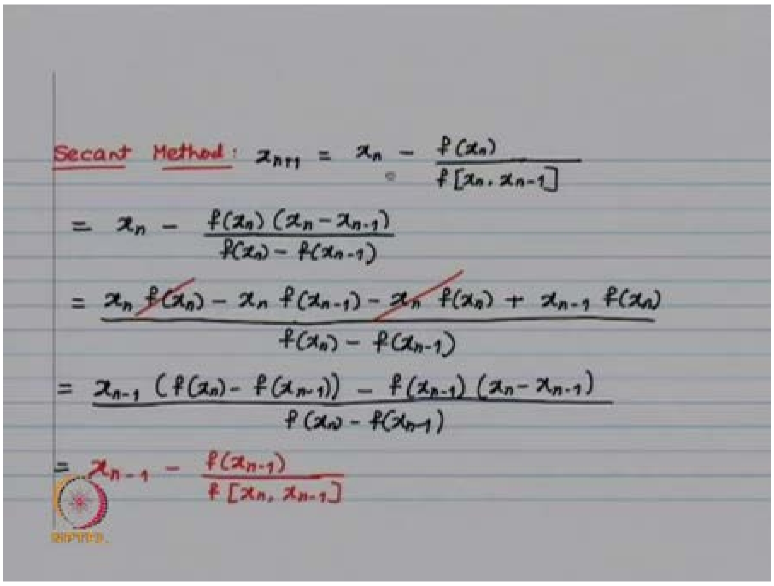

So, one has to do a bit of calculation. So, let us work out the details. So, we have got xn plus 1 is equal to xn minus f xn divided by the divided difference, I substitute for the divided difference, so I have xn minus f xn multiplied by xn minus xn minus 1 and then divided by f xn minus f xn minus 1.

So, now, multiply this xn by f xn minus f xn minus 1, so I will get xn f xn minus xn f xn minus 1, then minus xn f n and then plus xn minus 1 f xn; so, this xn f xn will get cancelled, so you have xn minus 1 f xn and minus f xn f of xn minus 1; so, we are writing, what we are doing is, we are adding and subtracting xn minus 1 f xn minus 1 from here, what I have is xn minus 1 f xn, so it is this term, then minus xn f xn minus 1 it is this term; so, I am subtracting xn minus 1 f xn minus 1 and I am adding it.

When I do that I will get xn minus 1 minus f xn minus 1 and this is nothing but the divided difference based on xn xn minus 1. So, this formula and this formula it is the same, its symmetric in xn minus 1 and xn. So, now, we want to look at the order of convergence in the secant method; earlier what we did was we looked at f of c is equal to zero for newton’s method, then for f of c we wrote the Taylor’s formula. Now, here what you will have to do is, you will have to consider the error in the interpolating polynomial and then the remaining proof will be similar.

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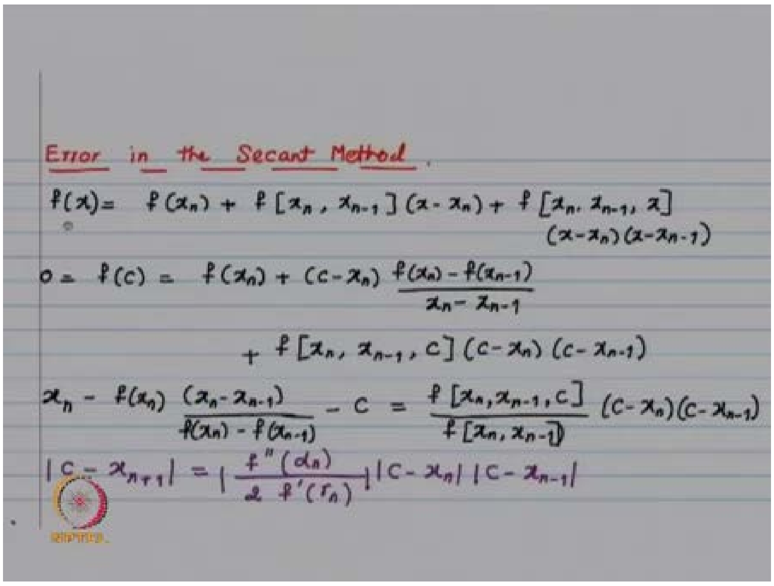

So, we have got fx is equal to f xn plus divided difference based on xn xn minus 1 into x minus xn plus this is the error term; so, this is linear approximation; this is a polynomial of degree less than or equal to 1 which interpolates the given function at xn and xn minus 1; this is the error term f xn xn minus 1 x x minus xn x minus xn minus 1, so this is from our polynomial interpolation.

So, the results from polynomial interpolation we keep on needing them often like all our numerical integration it was based on the polynomial interpolation; numerical differentiation also the polynomial interpolation it came into picture. Now, for this solution of non-linear equations, linear approximation when you consider the tangent line approximation that means, your interpolation point is repeated twice; you get Newton's method when you take the points xn and xn minus 1 and fit a polynomial of degree less than or equal to 1 you get secant method.

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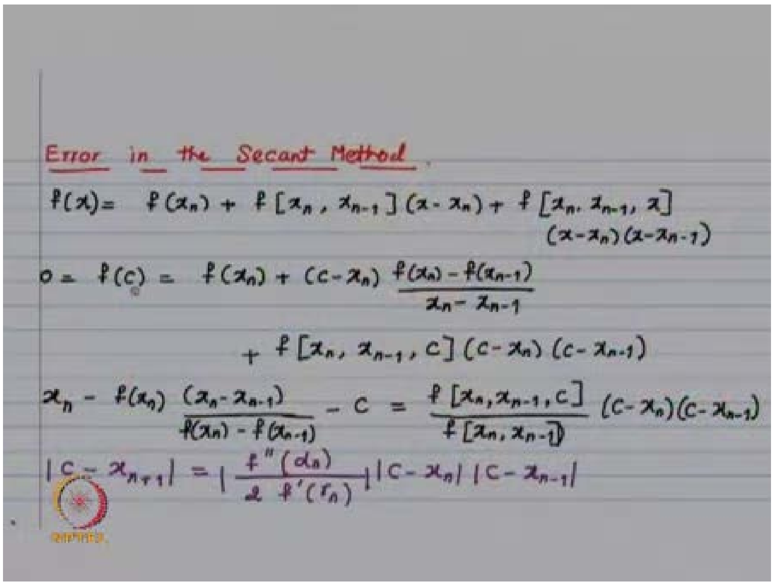

So, now, you have got this f x, so write 0 is equal to f of c, so I am substituting x is equal to c, so it will be f xn plus c minus xn, this is the divided difference plus f of xn xn minus 1 c and then c minus xn into c minus xn minus 1. As we did in case of newton's method, let us divide by this divided difference, so you are going to have it to be equal to xn, I am going to take this term on the other side and I will be multiplying by xn minus xn minus 1.

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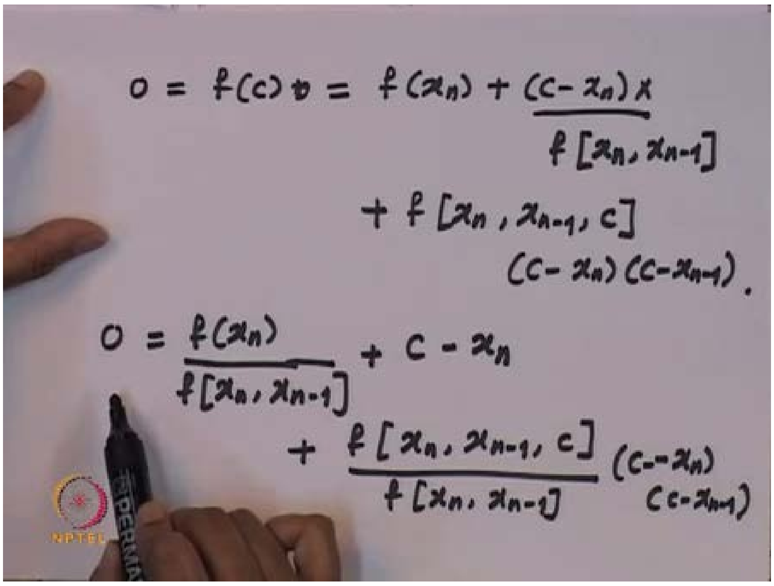

So, as such what we have is, we have got 0 is equal to f of c plus or is equal to f of xn plus c minus xn divided by c minus xn into multiplied by divided difference xn xn minus 1 plus divided difference based on xn xn minus 1 c multiplied by c minus xn c minus xn minus 1, so I will divide by this divided difference. So, I am going to have 0 is equal to f xn divided by divided difference based on xn xn minus 1 plus c minus xn plus divided difference based on xn xn minus 1 c divided by f of xn xn minus 1 and multiplied by c minus xn c minus xn minus 1. So, from here I have got this, this divided difference.

Now, if you take this on the other side, then what I am going to get is, so I am going to take this on the another side, so it will be xn minus f xn divided by f of xn xn minus 1 minus c is equal to the right hand side f of xn xn minus 1 c divided by f of xn xn minus 1 into c minus xn c minus xn minus 1, this is our xn plus 1.

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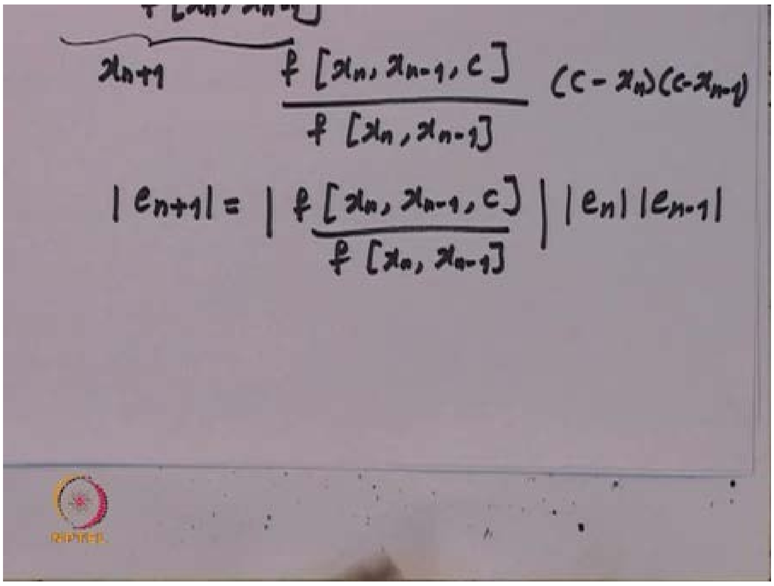

So, we have got xn plus 1 minus c, take the modulus, so you will have modulus of en plus 1 to be equal to modulus of f of xn xn minus 1 c divided by f of xn xn minus 1 and mod en mod en minus 1.

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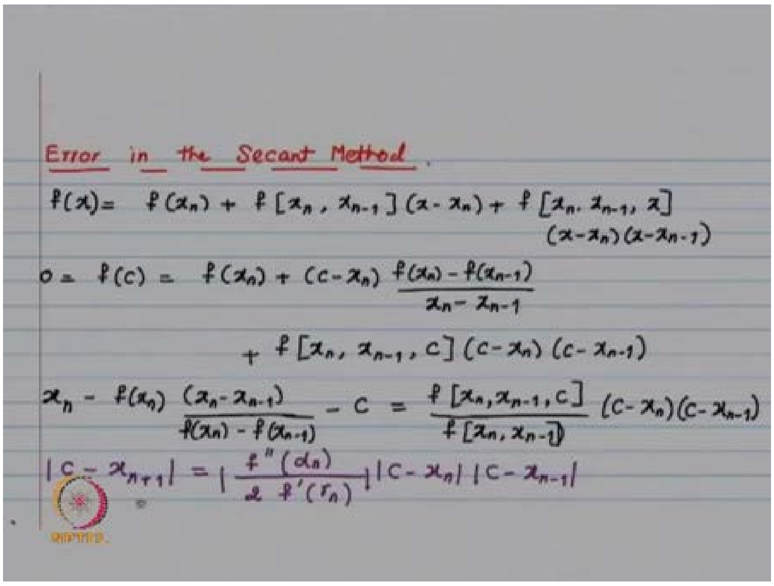

So, here now we have got, if you compare with the newton’s method we had modulus of en plus 1 is equal to something into mod en square, but now we have got this mod en and mod en minus 1.

So, we will see next time that, this is going to make the order of convergence to be less than 2, it will be about one point six. Next time we are going to consider one more method which is known as regula falsi method; and then we will compare these methods, what are the advantages? What are the drawbacks? And then we are going to consider iterative methods for solution of system of linear equation. So, thank you.