Elementary Numerical Analysis

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Lecture No. # 21
Vector and Matrix Norms

We have considered some methods for solving a system of linear equations. So, the methods which we have considered are choas decomposition, then gauss elimination method, with and without partial pivoting.

Now, these methods they are meant for solving big systems of linear equations using a computer. If you are doing hand computations, then you will be solving small system, say, n is equal to 3, n is equal to 4 and then, it does not matter which method you use, but the methods which we have considered and their relative merits, it is meant for a big system of linear equations.

So, you are necessarily going to use computer, now in computer no matter how powerful your computer is, you have finite precision; that means, after the decimal point you are going to have a fixed word length, so that is going to introduce a error. Also, the linear system which you are trying to solve, it may be coming from some experimental data. In that case, there is going to be experimental error also, we want to solve an exact equation A x is equal to b, but because of the finite precision of the computer, you are going to solve a perturb system; that means, there is going to be error in the elements of your matrices, matrix - coefficient matrix. Similarly, there will be error in the right hand side, so you get a computed solution.

Now, one wants to know, how near your computed solution is to the exact solution. So, for that we need to have a measure to decide, how near your computed solution is to the exact solution, so that is why we are going to consider norms. Now, the norm will give us a way to calculate distance between two vectors, you are familiar with Euclidean norm. So, in r 3 the Euclidean length is going to be x 1 square plus x 2 square plus x 3 square whole thing raise to half.

Then, we are going to define some other vector norm and then, we will be considering the matrix norm. So, today's topic is going to be vector and matrix norms, and as has been the case so far, our numbers, they are going to be all real numbers. So, matrices involved are real, then vectors involved they are real, so let us define a norm. So, x is going to be a vector in R n and we will define norm.

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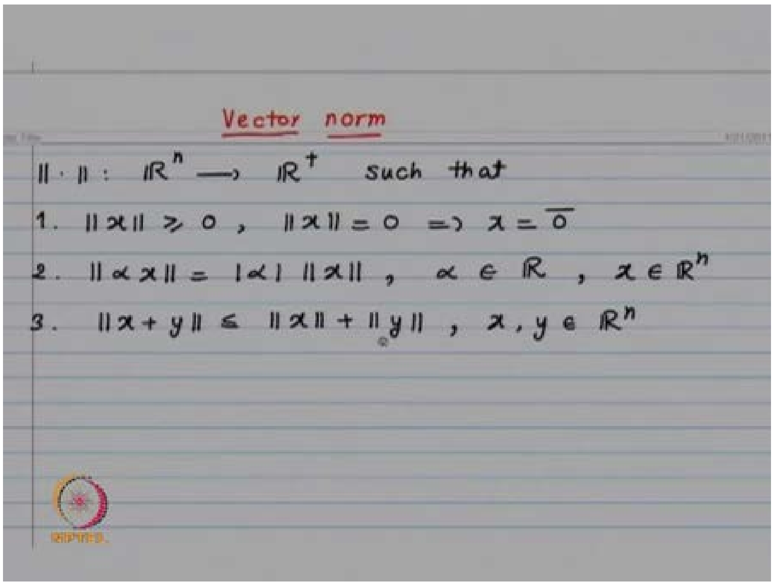

So, norm is a function from R n to R plus, so R plus is all non-negative real numbers, which should satisfy the property that norm x is bigger than or equal to 0. Actually, that is included, when I say that norm is from R n to R plus; that means, norm x is bigger than or equal to 0, it should be equal to 0 should imply x is equal to 0 vector, and the converse also should be true that x is equal to 0 vector, that should imply that norm x is equal to 0.

The second condition is alpha is a real number; alpha times x; that means, you multiply each component of x by alpha. So, norm of this new vector should be equal to modulus of alpha times norm x, this should be valid for all alpha belonging to R and for all vectors x belonging to R n. And the third condition is known as triangle inequality that norm of x plus y is less than or equal to norm x plus norm y, two vectors in R n you add component y , so if you have a vector x - x 1, x 2, x n; y is vector y 1, y 2, y n. When you want to add x plus y, you add the corresponding components. So, it will be x 1 plus y 1, x 2 plus y 2, x n plus y n, and scalar multiplication is alpha times x will be vector with the components alpha x 1, alpha x 2, alpha x n.

Now, let us look at some of the examples of norms, you can define norm in various manners, we are going to consider three norms. So, one is the Euclidean norm and then second is known as a one norm, and the third norm is the maximum norm or infinity norm. And the reason we consider 1-norm and infinity norm will become clear when we look at corresponding matrix norm.

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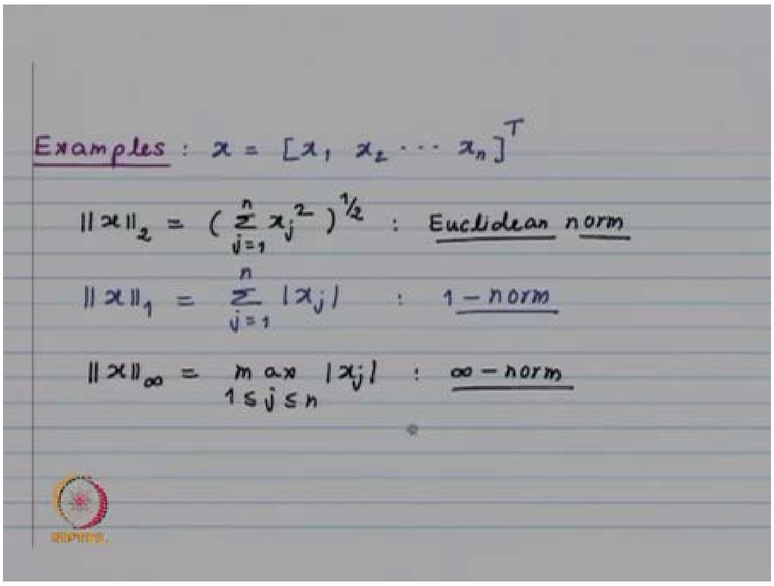

So, here Euclidean norm is summation j goes from 1 to n x j square whole thing raise to half, then norm x 1 is equal to summation j goes from 1 to n modulus of x j and norm x infinity is maximum mod x j 1 less than or equal to j less than or equal to n. Now, it is easy to verify that all these three definitions, they will satisfy the three properties of the norm, so that norm x should be bigger than or equal to 0, so look at Euclidean norm. You are considering summation x j square and then, you are taking positive root - positive square root, so it is going to be bigger than or equal to 0. If it is equal to 0 then, your sum of the squares is 0, so it has to be the each component has to be 0; that means, x is a 0 vector and if x is a 0 vector norm x will be equal to 0.

Similarly, when you consider 1-norm and infinity-norm, it is going to satisfy that it is bigger than or equal to 0 and is equal to 0, if and only if, x is equal to 0 vector, it is simple to verify.

Next is, norm alpha x should be equal to mod alpha times norm x, so this also for all the three definitions you substitute, look at the formula and then you can verify. The third property is the triangle inequality. So, this triangle inequality, it is easy to verify for 1-norm and infinity-norm, its straight forward. Like, look at the 1-norm, so norm of x plus y, it will be summation j goes from 1 to n modulus of x j plus y j, modulus of x j plus y j, x j and y j these are real numbers, so that will be less than or equal to mod x j plus mod y j and then separate the summation.

Similarly, for the infinity-norm it is straight forward, the triangle inequality for the 2-norm one needs to use what is known as Cauchy Schwarz inequality. So, I am not going to prove Cauchy Schwarz inequality, but I will state Cauchy Schwarz inequality and show you how using Cauchy Schwarz inequality, we get the triangle inequality for the 2-norm.

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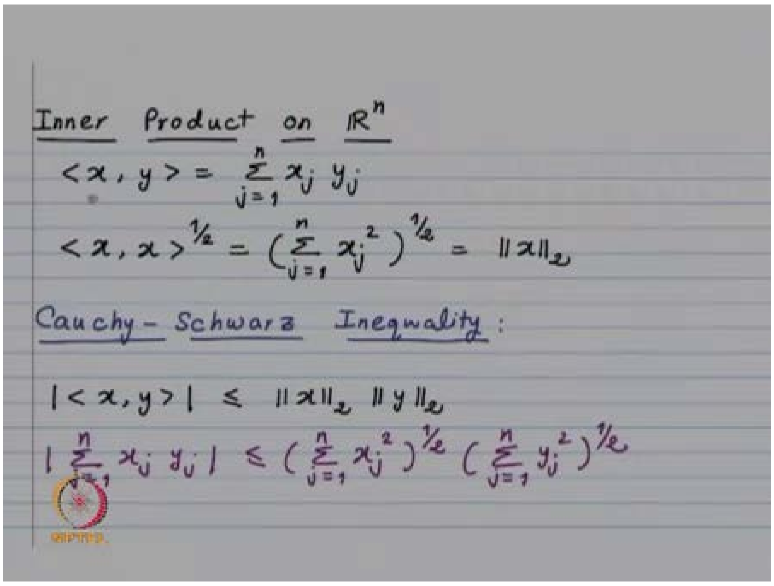

So, now this inner product on R n, so its inner product of x comma y, x and y are vectors in R n, it is defined as summation j goes from 1 to n x j y j. We will come across inner product little later and then, that time we will study the properties of the inner product. In fact, we did use inner product, when? We had talked about the gauss points. The gauss points they were 0's of the Legendre polynomials, that was need they were needed in gaussian quadrature and that is where we had talked about the inner product, so, you are familiar with inner product. So now, look at the inner product on R n and then, you consider positive square root of x comma x, so that is nothing, but our 2-norm of x.

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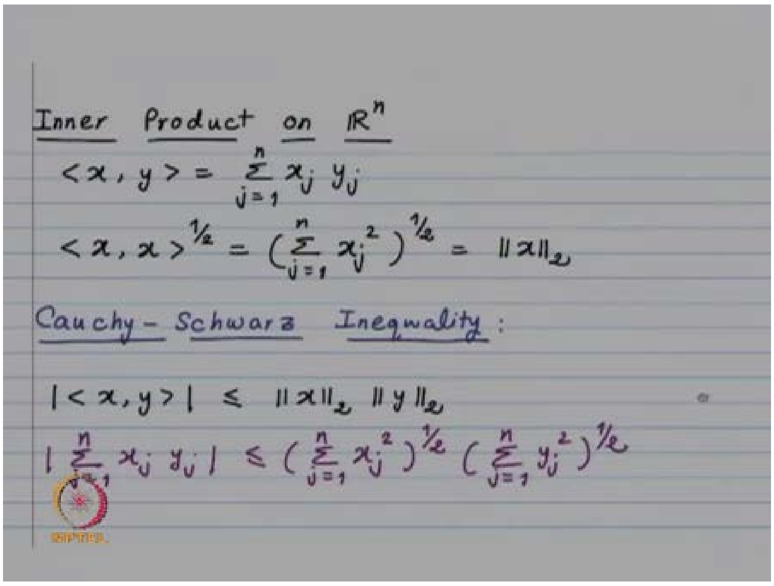

So, x comma x raise to half is equal to norm x 2, so the Cauchy Schwarz inequality is modulus of inner product of x with y is less than or equal to norm x 2 into norm y 2. So, that is modulus of summation j goes from 1 to n x j y j, because that is our definition of inner product of x with y, is less than or equal to summation j goes from 1 to n x j square raise to half which is norm x 2 and this quantity is norm y 2, so this is the Cauchy Schwarz inequality.

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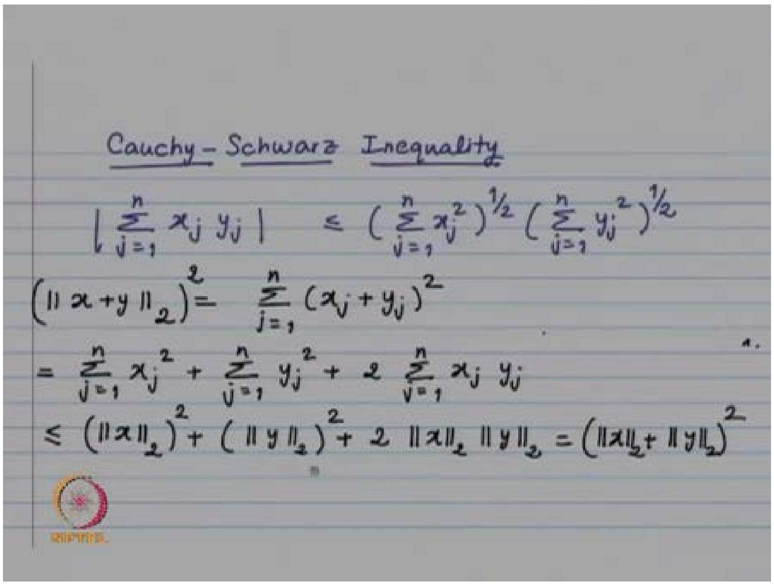

And now, look at norm of x plus y, it is 2 norm and then square of it, so that is summation j goes from 1 to n x j plus y j square; x j plus y j is going to be j th component of x plus y. So, I expand a square and I will get summation over j x j square plus summation over j y j square plus two times summation x j y j. This is nothing but 2 norm of x square summation over j y j square is, norm y its 2 norm square plus, now I use the Cauchy Schwarz inequality to dominate this by two times norm x 2 norm y 2, so this is nothing but norm x plus norm y square. So, now you take positive square root and then that will prove the triangle inequality for 2 norm, so this is about the vector norms.

Now, I want to consider matrix norm, so I have got a to be n by n matrix, so I want to define the norm. So, it should satisfy our three properties, the positive definiteness, triangle inequality, and the third one was that, alpha times a should be equal to mod alpha times norm a. So, why not treat our a as a vector of length n square because, what is matrix? It is an arrangement of n square number, we write them as rows and then columns. So, as such the matrix A, it has got n square elements, so I can treat it as a n square a vector of length n square, I know how to define vector norm.

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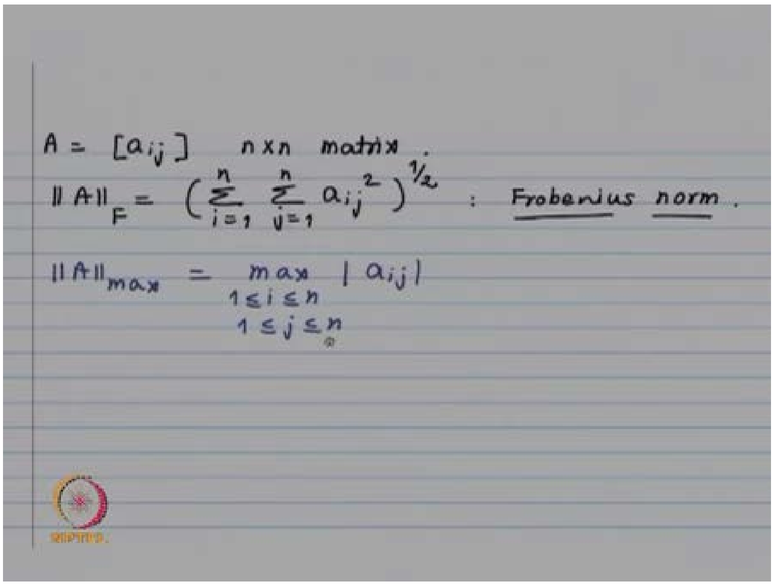

So, then I will consider the corresponding norm, norm which corresponds to the Euclidean norm will be a i j square. Now, instead of single summation you will have summation i goes from 1 to n, summation j goes from 1 to n and then, whole thing raise to half, so this is known as frobanius norm.

And then, norm A max, now I am not writing infinity, because norm A infinity I want to reserve that symbol for something else, so norm A max will be maximum of modulus of a i j, 1 less than or equal to i, 1 less than or equal to j less than or equal to n. And then, one can verify that these definitions, they will satisfy our three properties of norms. Now, what we want to do is, we want to relate our vector norm and the matrix norm, because look at our system of linear equation, it is A x is equal to b, so we are considering matrix into vector. So, I will like to have some relation between A - the matrix norm and a vector norm and another thing is I can multiply two matrices A and b, if they are I am considering square matrices, you cannot multiply two vectors, but you can multiply two matrices.

So, then again norm of A b and norm A norm b, I will like to have some relation between them. So, in order to do that what we are going to do is, we are going to consider induced matrix norm.

So, we will have we are going to define a general way of defining a matrix norm. So, we are going to start with a vector norm and from that we will give a general definition of induced matrix norm. And then, since we are interested in our 1-norm, 2-norm, infinity-norm, we will see what are going to be the corresponding induced matrix norm, so now, first definition of Induced matrix norm.

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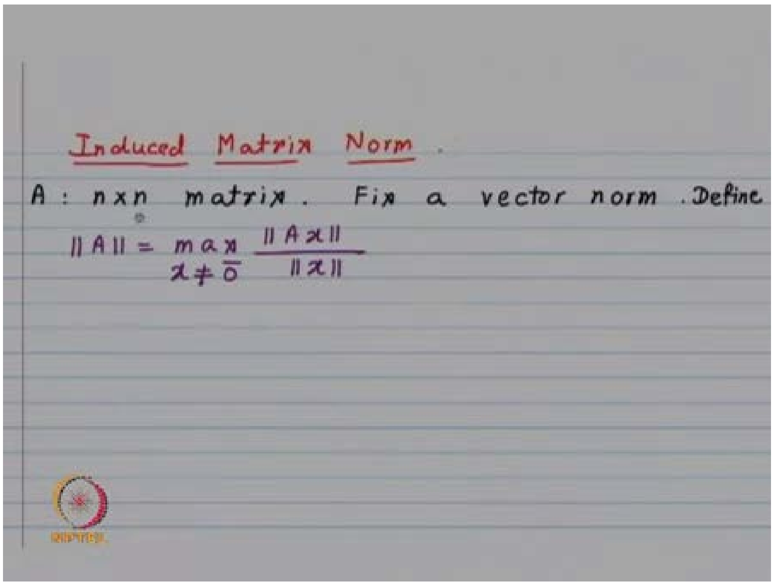

So, A is n by n matrix, you fix a vector norm, any norm which you want; that means, it should satisfy the three properties of norm. And now, I am going to look at maximum of norm A x by norm x, x not equal to 0 vector that is my definition of norm of A. So, now, the first thing we want to show is that if I define norm A in this manner, then it satisfies those three properties of norm.

So, the first property is positive definiteness that norm A is bigger than or equal to 0, and it is equal to 0, if and only if, A is a now 0 matrix. The second property will be, norm alpha A should be equal to mod alpha times norm A, and the third property is the triangle inequality norm of A plus B is less than or equal to norm A plus norm B. So, three properties we are going to deduce using the fact that our vector norm satisfies these properties, you will see that the proofs are straight forward.

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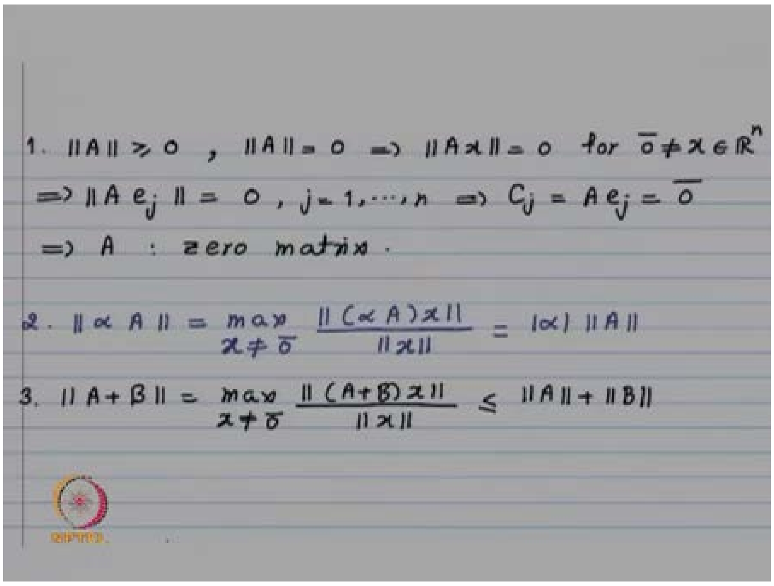

So, norm A, A is bigger than or equal to 0 that is clear, norm A is equal to 0, it will imply that norm A x is equal to 0. Remember, our norm A is maximum - I have written here - so norm of A will be maximum of norm A x divided by norm x, x not equal to 0 vector. So, if norm A is 0 then norm A x is going to be equal to 0 for any non-zero vector in R n. So, now, look at our canonical vectors, e j is going to be vector with 1 at j th place and 0 elsewhere.

So, I will have norm A e j to be equal to 0, but what is our A e j? A e j is nothing but the j th column. Now, using the property of vector norm, you will get C j to be a 0 vector and you get a to be a 0 matrix. So, we are using the property of vector norm, the second property is, look at norm alpha a, this will be by definition maximum x not equal to 0 vector norm of alpha A x by norm x, but then our alpha times A into x will be nothing but alpha times A x.

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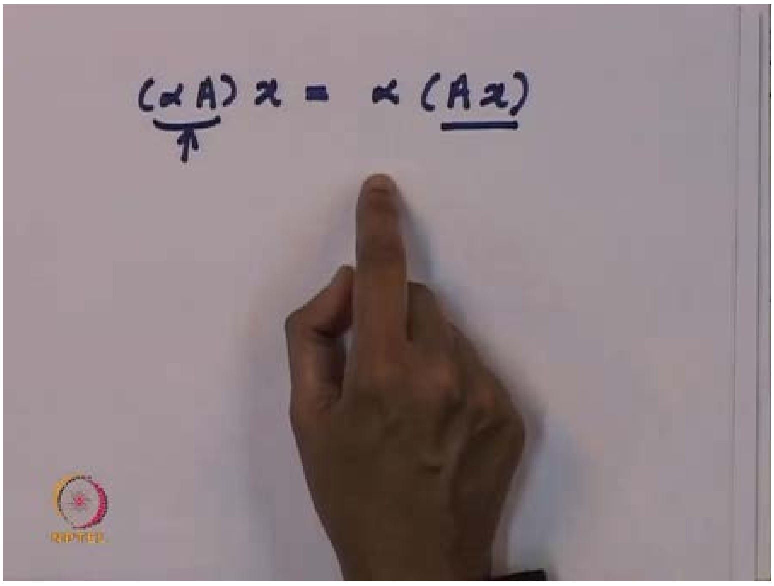

What is the difference? Here, you are multiplying matrix A by alpha, here you are first calculating A into x and then, multiplying by scalar alpha.

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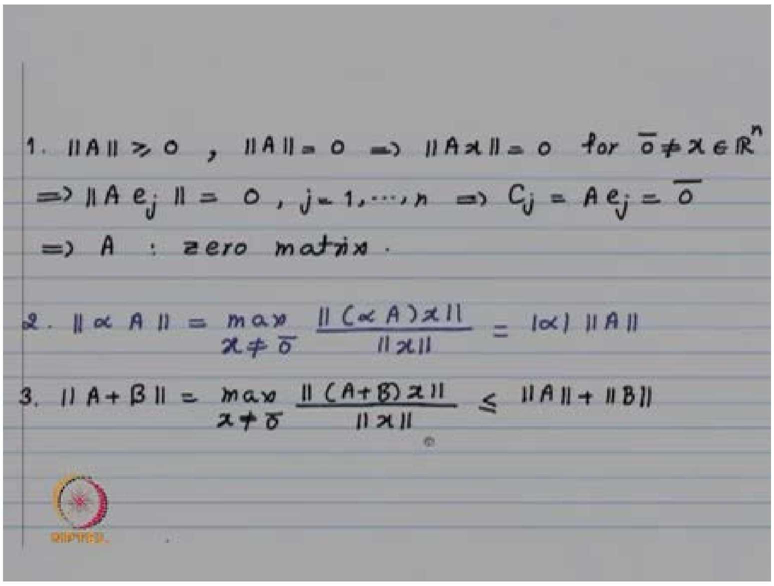

You have norm alpha A is maximum x not equal to 0 vector alpha times A x divided by norm x, and using the property of vector norm you get mod alpha times norm A because, mod alpha comes out it does not depend on maximum, so it will come out of the maximum, and maximum of norm A x by norm x, x not equal to 0 vector is norm A. And similarly, triangle inequality norm of A plus B is maximum x not equal to 0, norm of A plus B x divided by norm x use the fact that norm of A plus B x is less than or equal to norm A x plus norm B x.

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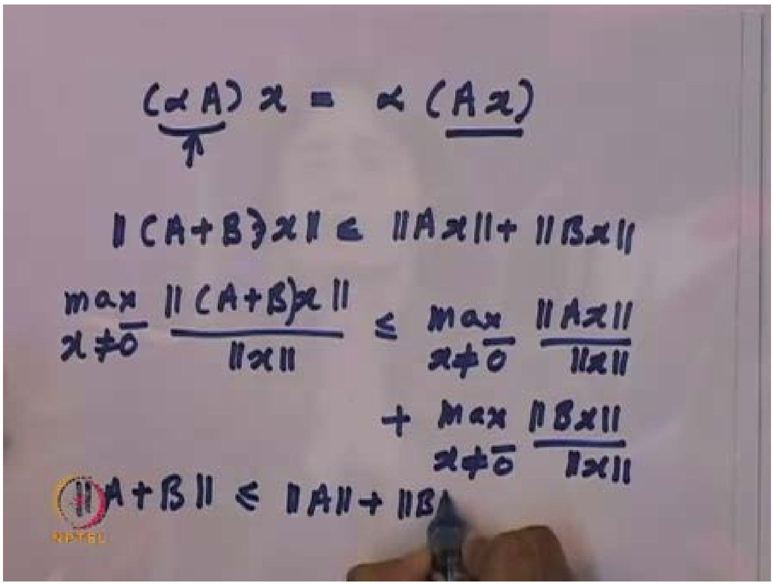

So, maximum of norm of A plus B x, x not equal to 0 vector divided by norm x will be less than or equal to maximum of norm A x by norm x, x not equal to 0 vector, plus maximum of norm B x, x not equal to 0 vector, divided by norm x. So, this proves that norm of A plus B is less than or equal to norm A plus norm B.

We defined induced matrix norm and then, we showed that it satisfies all the properties of our vector norm. Now, we are going to prove one of the basic inequality which we will be using frequently later on. So, that inequality is going to relate the matrix norm to vector norm and we are going to show that norm of A B is less than or equal to norm A into norm B. Now, these two properties they will follow from our definition of induced matrix norm.

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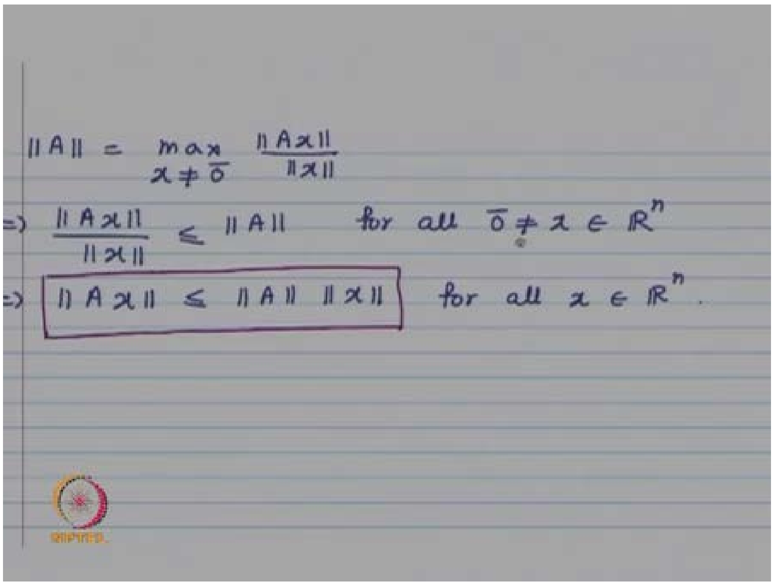

So, our norm A is maximum norm A x by norm x, x not equal to 0 vector; that means, for all non-zero vector norm A x by norm x is less than or equal to norm A, which will give us norm A x to be less than or equal to norm A into norm x for all non-zero vector, but if x is a 0 vector, norm x will be 0 A x being a 0 vector again this will be 0, so this fundamental inequality will be true for all x belonging to R A.

Now, I want you to note one thing that here norm A x - that means vector norm - norm A means, induced matrix norm and norm x; that means, again vector norm. So, it is clear from the context about which norm we are talking about, A being a matrix this will be a matrix norm; x being a vector, this will be a vector norm.

So, this is the first inequality and now, let us show the consistency condition; that means, if you have got A and B to be two matrices take their product, so A B is going to be again a n by n matrix, so norm A B is less than or equal to norm A into norm B. So, this fact will be proved using our basic inequality that norm A x is less than or equal to norm A into norm x.

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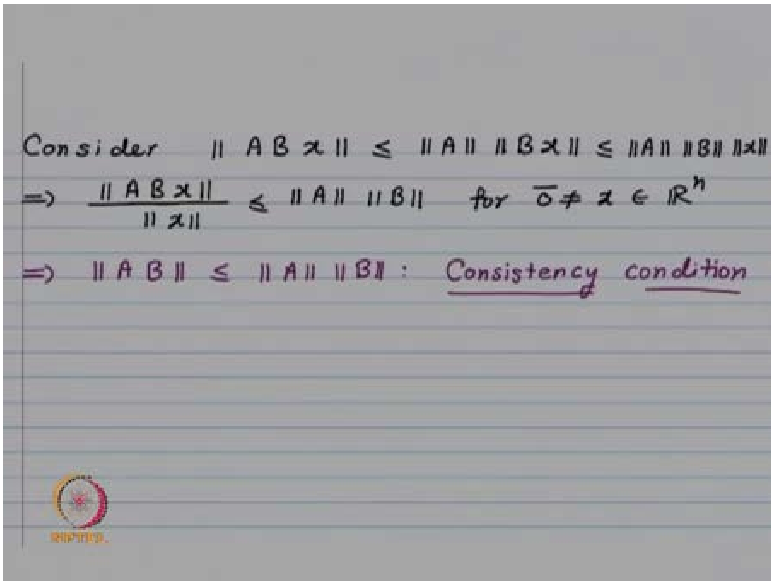

So, you have got norm A B x to be less than or equal to, by fundamental inequality, it will be norm A into norm B x, treat B x has one vector and A as our matrix. So, you have got norm A into norm B x, this is less than or equal to… Now, use fundamental inequality for B x, so norm B x will be less than or equal to norm B into norm A. Now, if x is not equal to 0 vector, norm x will not be 0, so I can divide and you will get norm A B x upon norm x to be less than or equal to norm A into norm B for all non-zero vectors in R n, so now take maximum of this over x not equal to 0.

The right hand side is independent of x, so you get norm A B to be less than or equal to norm A into norm B, so this is consistency condition. So, when one talks about vector norm, it should satisfy those three properties. When one talks about matrix norm, it should satisfy the three properties of vector norm and in addition norm A B to be less than or equal to norm A into norm B. This is a very elegant definition, start with any vector norm you define the induced matrix norm showing that it satisfies the three properties of vector norm and then the consistency condition it was straight forward, so very nice situation.

But then, given a matrix A, I will like to calculate its norm, like given a vector x, if I want to calculate its Euclidean norm - I know- that take the squares of their components, add them up, take the positive square root. If I want 1-norm then I will take the modulus of each component add it up, if I want infinity-norm look at the moduli of its components and look at the maximum. So now, suppose, I take one of these three norms and then, I want to know what the induced matrix norm? Now, in the definition of induced matrix norm, you have got maximum over all non-zero vector of certain quotient, that quotient is norm A x by norm A.

So, how am I going to calculate this maximum, I have got infinitely many vectors, so then I cannot calculate this induced matrix norm. So, now, we are going to derive a formula for calculating norm A 1; that means, when you fix vector norm to be 1, the corresponding induced matrix norm. So, we will have a formula in terms of the components of the matrix, so it is going to be something one can calculate.

Similarly, for infinity-norm of the matrix, we will have a formula whereas, the nice case of Euclidean norm, well, one has to be satisfied only with an upper bound. So, the Euclidean norm of matrix is something which you cannot calculate, so now, let us derive a formula for norm of A 1. We are fixing our vector norm to be 1-norm, and then, I am going to look at norm A x 1 divided by norm x 1 and look at their maximum.

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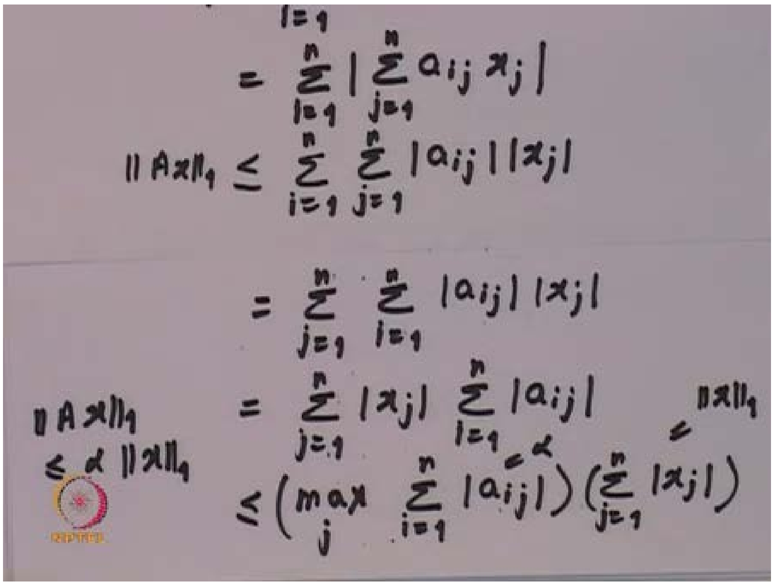

So, I have got norm A 1 is maximum of norm of A x 1 divided by norm x 1, x not equal to 0 vector, I want to find a formula for this. Now, the components of A x, let me write as A x i, this will be summation a ij x j - j going from 1 to n that is, the matrix into vector multiplication. Norm of A x1, this will be summation i goes from 1 to n modulus of A xi that is our definition of 1-norm. Now, let me substitute for A xi, so it is going to be summation i goes from 1 to n modulus summation j goes from 1 to n A ij x j.

Now, this will be less than or equal to summation i goes from 1 to n, summation j goes from 1 to n modulus of a i j modulus of x j, so we are using triangle inequality. Now, this is finite summation, so I can interchange the order of the summation, so, I am going to have. So, this is our norm A x1 is less than or equal to this, so this is same as summation j goes from 1 to n, summation i goes from 1 to n modulus of A ij and then, mod x j.

Now, summation is over i, here you have x j, so it will come out of the summation sign. So, it will be summation j goes from 1 to n mod x j summation i goes from 1 to n modulus of A ij. So here, what we are doing is, you fix j; that means you are fixing the column, look at the entries in that column take their modulus and then add it up. So, I can say that this is less than or equal to maximum over j of the quantity, summation i goes from 1 to n modulus of A i j into summation j goes from 1 to n mod x j and this is nothing but norm x 1.

Let me call this quantity to be alpha, so what we have is norm A x 1 is less than or equal to alpha times norm x 1. And hence, you will have norm A 1 to be less than or equal to this number.

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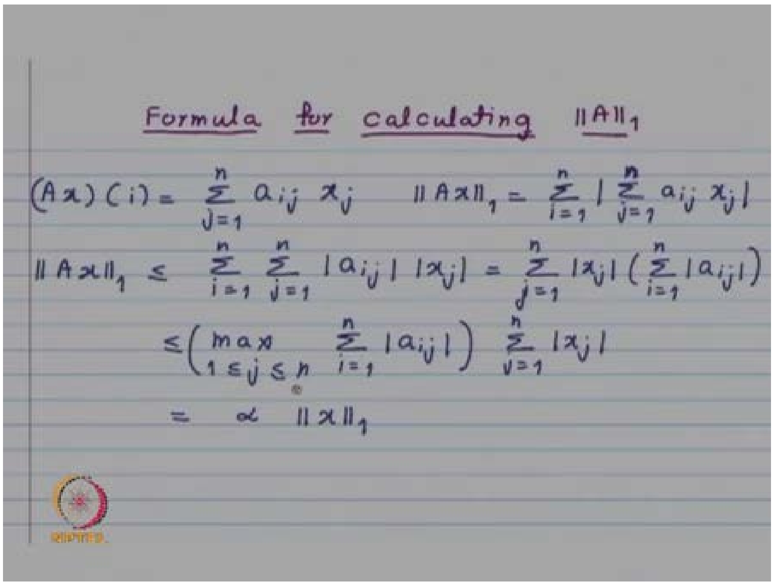

So, here, we have norm A x1 is less than or equal to alpha times norm x 1, where alpha is this number, summation i goes from 1 to n modulus of a ij 1 less than or equal to j less than or equal to n. So, I can very well calculate alpha, I will look at the columns of my matrix A, so look at the first column then take the modulus and add it up. So, that means I am considering 1-norm of the first column, then 1-norm of the second column and 1-norm of the nth column. So, I will get n numbers, the maximum among them that is going to be my alpha.

And my norm A 1 is going to be less than or equal to this alpha, but we will like to show that it is not only less than or equal to, but it is equal to. So, we have proved that norm A 1 is less than or equal to this number, and now, let us show that, in fact, it is equal to maximum.

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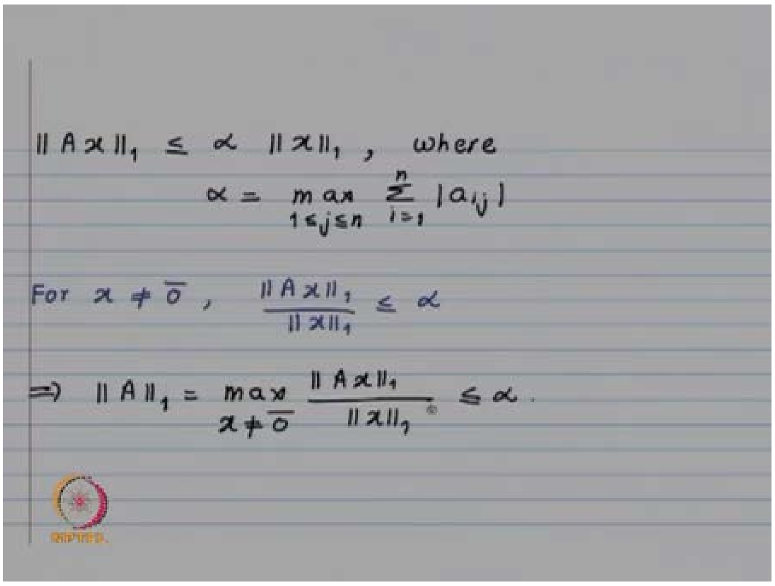

So, you have norm A x1 is less than or equal to alpha times norm x 1, where alpha is this quantity and hence you have got norm A 1 to be less than or equal to alpha.

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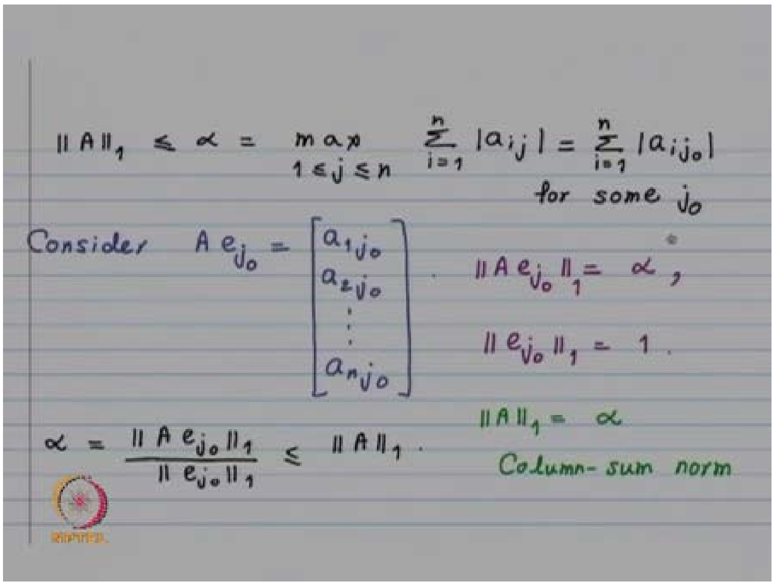

Now, look at this maximum, this will be equal to summation i goes from 1 to n modulus of a ij 0, for some j. Such a j 0 need not be unique, it depends on your matrix, may be all the columns they are going to have the same 1-norm, but you are looking at maximum of m numbers, so look at least one number which is equal to maximum and let that column b equal to j 0 th column.

Now, if you consider A times e j 0, e j 0 is a canonical vector with 1 at j 0 th place and 0 elsewhere. So, A is j 0 that gives you j 0 th column, so using this fact we will show that alpha is equal to norm AS 1.

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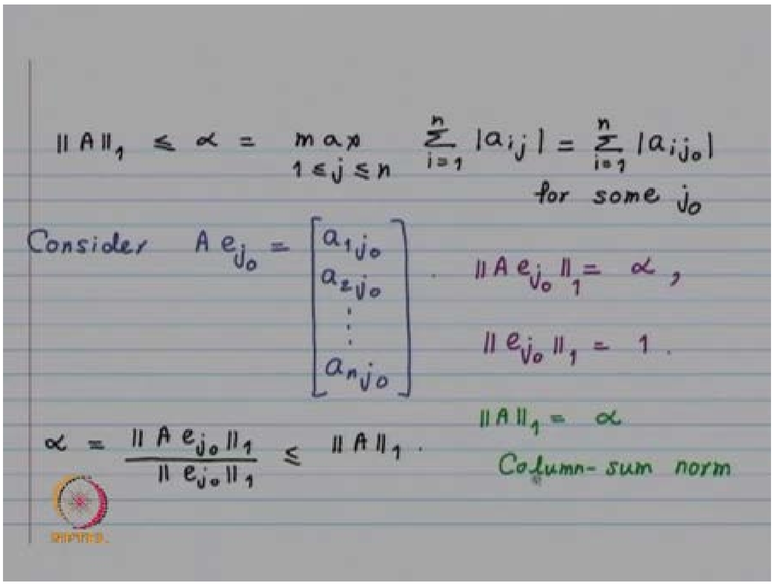

So, consider A ej0 that is the thing, but the j 0th column, so norm A e j 0 1 is equal to alpha and if you consider norm e j 0 its 1-norm that is going to be equal to 1. So, you have got alpha which is equal to norm A ej01 divided by norm ej01, this is going to be less than or equal to norm A 1. So, thus we have proved that norm a 1 is equal to alpha and that is known as column sum norm, because what you do is, in each column you are adding the moduli of the entries and looking at their maxima, so here is a formula for norm A 1.

Now, we are going to look at norm A infinity - now norm A infinity we will see that. So, here, what was the crucial result? It was that you had finite summation, so you interchange the order of the summation. Now, in case of the infinity-norm, you do not have to do interchange, because there is going to be only one summation and one maximum. Here, in order to show that as such norm A infinity, it is going to be row sum norm. That means, what you do is, look at each row and take the module of the entry and add it up, you will get n numbers, among this n numbers whichever is going be the bigger one that is going to be norm A infinity, so this fact now we are going to prove.

Now, in order to show that norm A infinity is equal to this showing less than or equal to will be simple, showing that it is equal to, that is, going to be a bit involved in case of 1-norm, it was easy you just looked at the corresponding canonical vector. Here, we will have to construct a vector where, the infinity-norm is going to be at A, but then the proof is going to be something similar.

So, you fix infinity-norm and then, our aim is to calculate norm A infinity which is maximum of norm A x infinity divided by norm x infinity, x not equal to 0 vector, so let us start with norm A x infinity.

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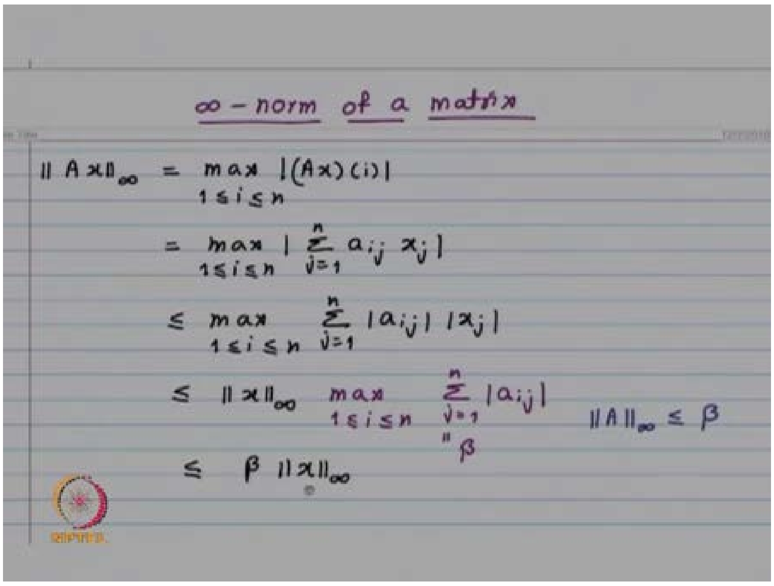

So, norm A x infinity will be maximum of modulus of A xi, 1 less than or equal to i less than or equal n, by definition of infinity-norm. Now, A xi as before will be given by summation j goes from 1 to n a ij x j then, for this modulus use triangle inequality. So, you will get less than or equal to, maximum one less than or equal to i less than or equal to n summation j goes from 1 to n modulus of a ij mod x j.

So, there are no two summations here now this mod x ji will dominate by norm x infinity. So, norm x infinity is going to be maximum of mod x jj going from 1 to n. So, if I dominate that it will come out of this summation and then maximum. So, you have norm x infinity maximum summation j goes from 1 to n mod a ij 1 less than or equal to i less than or equal to n. So, here you are fixing first i is equal to 1 then, you are looking at you are varying j; that means, you are looking at the elements in the first row taking their modulus summing it up.

Then put i is equal to 2, so you do it for the second row and put i is equal to n. So, like that you are going to get n numbers maximum among them that we are denoting by beta. So, we get norm A x infinity to be less than or equal to beta times norm x infinity, from this conclude that norm A x infinity divided by norm x infinity is less than or equal to beta for x not equal to 0 vector take their maximum that is norm A infinity, so you have got norm A infinity to be less than or equal to beta.

And now, we want to show the other way inequality, that now we want to show that we have proved that norm A infinity is less than or equal to beta. Now, let us show that beta is less than or equal to norm A infinity, so that combining these two inequalities we can show that norm A infinity is in fact equal to beta.

So, now again beta is maximum summation j goes from 1 to n modulus of a i j maximum over i. So, this maximum will be attended for some i is equal to i 0. Such a i 0 can be more than 1, so take one of them. So, get hold of i 0 such that beta is equal to summation j goes from 1 to n modulus of a i 0 j, and now let us construct a vector.

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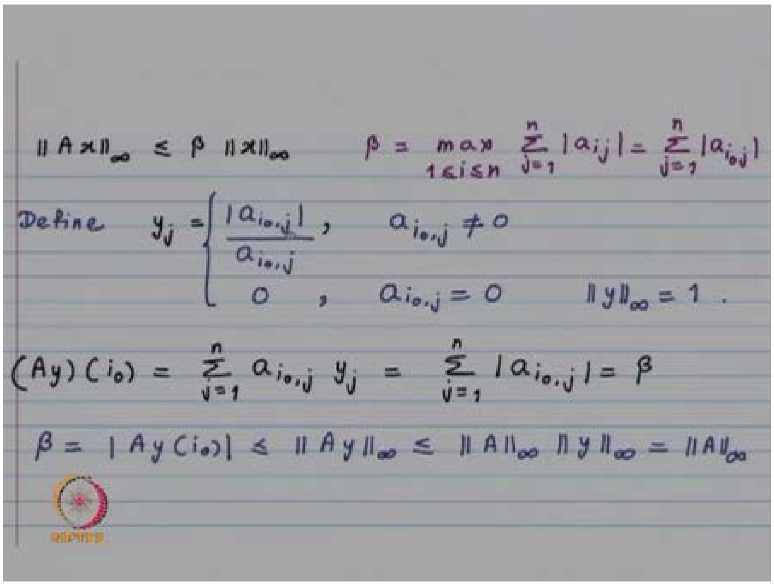

So, we are looking at elements of the i 0’th row, if a particular element is non-zero then you define y j to be mod a i0j divided by a i0j. If that number is equal to 0 then, you define it to be equal to 0. Now, if I define such a matrix modulus of y j for any j is going to be either 1 or 0. And that will mean that norm y infinity should be equal to 1 at least 1 of the y j should be not 0, because if y j is equal to 0; that means, each entry in the i 0’th row will be 0.

So, you will get beta to be 0 and that will be in that a is a 0 matrix. So, now, I have constructed a vector y j, now I look at A y of i 0, so i 0’th component of my vector A y that will be given by summation j goes from 1 to n a i0j y j. Now, look at the where we have constructed y j, if corresponding y j is not 0 then, a i0j multiplied by y j will be mod a i0j divided by a i0j, so a i0j will get canceled and you will get modulus of a i0j.

If the y j is equal to 0, it will not contribute to your summation, so you will get this is equal to summation j goes from 1 to n modulus of a i0j and this is going to be equal to beta. So, we have got beta is equal to i 0’th component of our vector a y, i 0’th component will be less than or equal to norm of A y infinity-norm; norm A y infinity, we know that it is less than or equal to norm A infinity into norm y infinity and then, you get our norm A y infinity will be less than or equal to norm A infinity into norm y infinity - norm y infinity will be 1, so you will get norm A infinity.

So, we have got beta to be less than or equal to norm A infinity and what was beta? It was this maximum, so this is going to be row sum norm, so now we have obtained a formula for norm A 1 and norm A infinity. Now, you know why I called norm A max for the term maximum over i n j of modulus of a i j, because I want to reserve norm A infinity to the definition of induced matrix norm when the vector norm is infinity-norm.

So, we have got a formula for norm A 1, we have got a formula for norm A infinity, but unfortunately for norm A 2 we have got only an upper bound. So, let us now calculate that upper bound, that upper bound we will be again using Cauchy Schwarz inequality. So, again the method is the same, we will look at norm of A x2 norm obtain it to be less than or equal to say a gamma times norm x 2, so that gamma will give us an upper bound.

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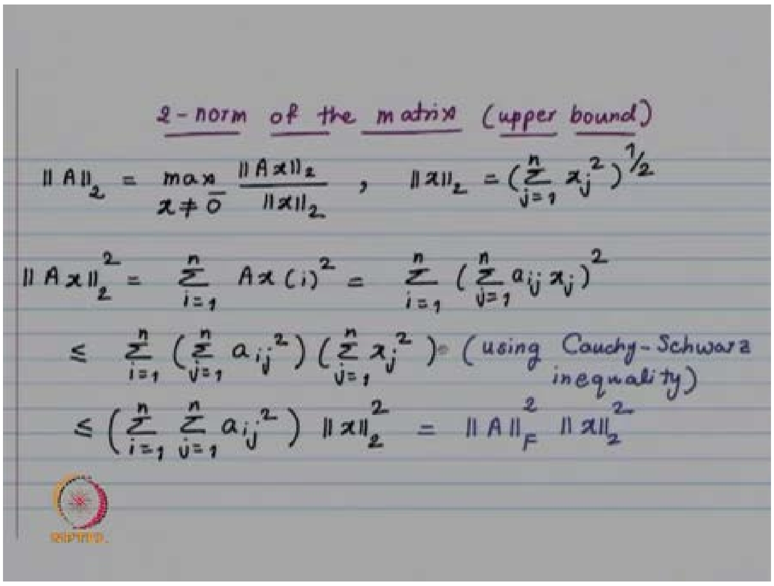

So, we have norm A 2 to be maximum of norm A x by norm x, x not equal to 0 vector, recall that Euclidean norm 2 of x is summation j goes from 1 to n, x j square or raise to half .

So, look at norm A x, it is 2 norm square, it will be summation i goes from 1 to n a xi square - by definition. Now i th component of A x is given by summation j goes from 1 to n a ij x j, so this is A xi and its square. Now, apply Cauchy Schwarz inequality here, so square of this will be less than or equal to summation j goes from 1 to n a i j square and summation x j square.

So, using the Cauchy Schwarz inequality you get this quantity to be less than or equal to summation i goes from 1 to n, summation j goes from 1 to n a ij square, summation j goes from 1 to n xj square. This is nothing but Euclidean norm of x, its square and then you have got this number. Now, if you recall this is what is known as frobenius norm, so you have got norm A f square into norm x square, so that gives you 2 norm of A to be less than or equal to norm A x. Now, let us recapitulate our norm, so we have got our norm A 1 that is the column sum norm, norm A infinity that is the row sum norm, and norm A 2 is less than or equal to norm a frobenius.

So, this is the best one could do for the 2 norm, but then for the 2 norm, we have got the basic inequality norm of A x, its 2 norm it is going to be less than or equal to norm A frobenius into norm of x 2 norm, so that inequality is available. If your matrix A is symmetric, then 1-norm and infinity-norm they are going to be the same because for column sum norm that gives us 1-norm, if you do corresponding thing for row you get infinity-norm.

So, for the symmetric matrix both 1-norm and infinity-norm they are going to be the same, for a general matrix they can be different. So, now, we have proved or we have defined vector norm and matrix norm, using these we are going to analyze the behavior of the perturbed system A x is equal to b is the original system or the system which we want to solve, because of the use of computers we are going to solve system A plus delta A x cap is equal to B plus delta B. So, x is the exact solution x cap is the computed solution and we will like to say something about norm of x minus x cap.

So, this perturbation of the linear system and sensitivity of the computed solution to the change in the right hand side or to the perturbation that is going to be topic of our next lecture, so thank you.