**Elementary Numerical Analysis**

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**Lecture No. # 19**

**Cholesky Decomposition**

We are considering solution of linear system of equation. So, we have got n equations in n unknowns. We describe the method, which is known as gauss elimination method. So, the system of linear equation, we write as A x is equal to b and then using elementary row transformations, we reduce the system A x is equal to B to an upper triangular system U x is equal to y. So, the coefficient matrix U is upper triangular it will mean that the last question has got only one unknown x n last but what but one equation has got only two unknowns x n x n minus 1 and So, on.

So, the solution of upper triangular system can be obtained by back substitution. The gauss elimination method which we described, we needed an additional assumption to the assumption that A is invertible; that look at the matrix A it is A n by n matrix. Consider, sub matrix formed by first k rows and first k columns of matrix a, we denote that matrix by A k it is A k by k matrix, it is known as principle leading sub matrix. And then, we assume that determinant of Ak is not equal to 0 for k is equal to 1 to up to n. Under this assumption, we showed that gauss elimination method is equivalent to writing coefficient matrix A as product of two matrices l and U, l is unit lower triangular matrix; that means, on the diagonal you have got the entries to be equal to 1 and it is A lower triangular matrix.

U is upper triangular matrix, such decomposition we proved that it is unique. And then the system x is equal to b it is decomposed into two systems. So, you have got A x is equal to b then that is equal to l into u x is equal to b. So, we look at u x is equal to y and l y is equal to b. So, the system l y is equal to b b is given to us. So, we calculate y by forward substitution. Now, y which we have obtain that becomes A right hand side and then we have got u x is equal to y. So, you apply back substitution. So, thus Ax is equal to b can be obtained as solution of two systems, one lower triangular another upper triangular.

Next, yesterday, from the L U D composition we deduced, what is known as L D V decomposition? In the L U D composition l was unit lower triangular and u was upper triangular. So, the upper triangular the diagonal entries of U need not be equal to 1. Now, this U we decompose or we write U is equal to D into V; where D is going to be a diagonal matrix and V is going to be unit upper triangular.

Then, using the uniqueness of L U D composition, one can show that - this L D V decomposition is also unique. And now, today, we are going to look at the case, when a is positive definite and for the positive definite matrix, we are going to prove Cholesky decomposition; where we are writing A as g into g transpose, where g is a lower triangular matrix. So, our starting point is going to be L D V decomposition, which is available for matrices satisfying the condition that determinant of A k is not equal to 0, for k is equal to 1 to up to n, the notation is A is equal to A j n by n matrix. Assumption, determinant A k not equal to 0, for k is equal to 1 to up to n, A is equal to l u l unit lower triangular u upper triangular.

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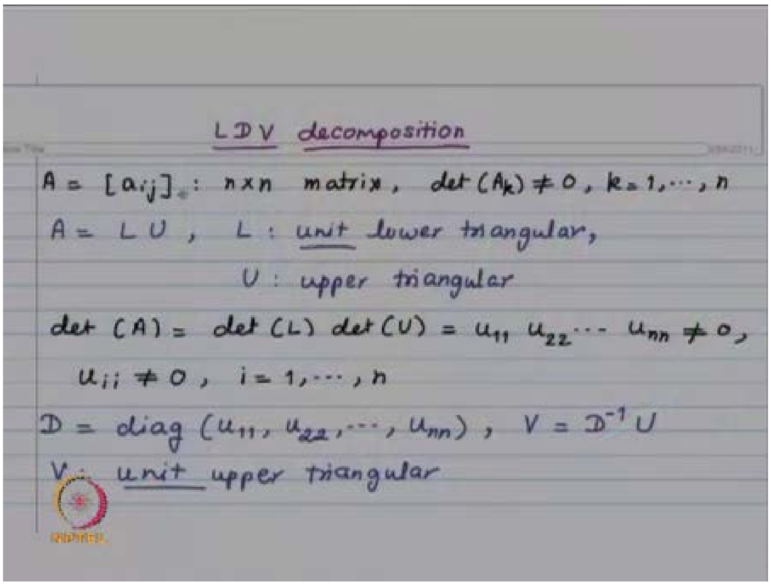


Now, determinant of A is determinant of L into determinant of U, determinant of l is equal to 1, determinant of U is product of the diagonal entries. So, you have got u 1, 1 u 2 2 u n n not equal to 0. That will imply that u i not equal to 0 for i is equal to 1 to up to

n. Now, write d as diagonal u 1 1 u 2 2 u n n and define v is equal to d inverse u, then this v becomes unit upper triangular and when you substitute in l u, then you are going to get A is equal to l d v. Now, this decomposition is unique l unit lower triangular d diagonal and u unit upper triangular. So, now, let us look at the case, when A is symmetric matrix.

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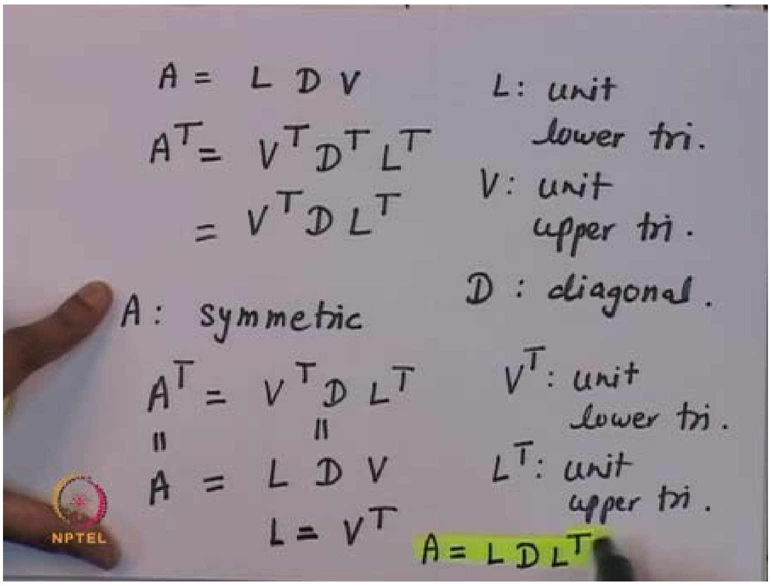


So, we have got A is equal to L D V that will give us A transpose to be equal to V transpose, D transpose, L transpose. Our L is unit lower triangular, V is unit upper triangular. And D is diagonal, since D is diagonal it becomes V transpose D and then L transpose, if A is A symmetric matrix, then we have got A transpose is equal to A, A transpose is V transpose, D L transpose, A is L D V. Now, V transpose is going to be unit lower triangular L transpose is going to be unit upper triangular.

These two are equal. So, by uniqueness of the L D V decomposition, what we get is L has to be equal to V transpose and does for A symmetric matrix, A is going to be equal to L D L transpose. So, we have this, provided A is symmetric. So, now, this is for symmetric matrix. Now, we will consider A positive definite matrix, A positive definite matrix has got 2 properties it is A symmetric matrix and if you look at A non-zero vector x.

Then x transpose A x is going to be bigger than 0. So, now, for A positive definite matrix, we are going to have A is equal to L D L transpose, where L is A unit lower triangular matrix. For positive definite matrix, we will show that the diagonal entries of D, they are going to be all strictly bigger than 0. Once we prove that we define A new diagonal matrix, which we denote by D raise to half consisting of diagonal entries as square root of D 1 1, square root of D 2 2 square root of D n n.

So, we have got A is equal to L D L transpose, this D we write as D raise to half into D raise to half and then associate 1 D raise to half with L and another d raise to half with v and that will give us Cholesky decomposition G, G transpose. So, let us do that.

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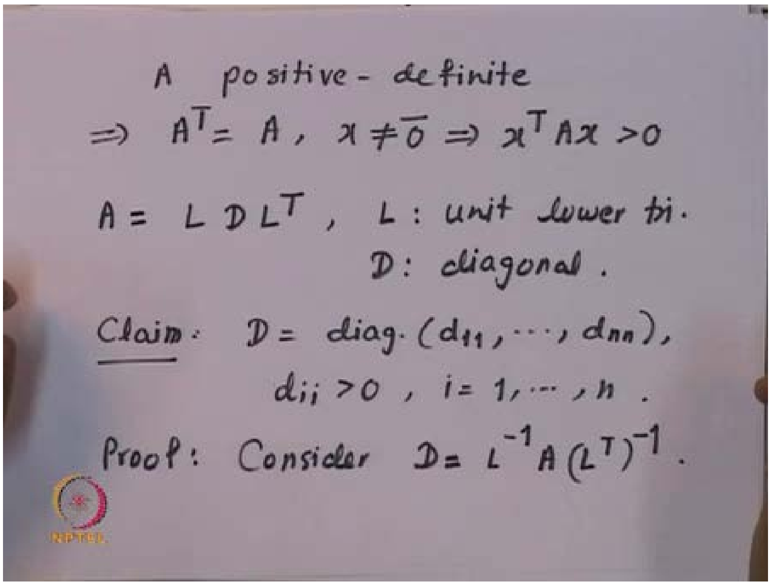


We have a positive definite; that means, A transpose is equal to A and x not equal to 0 vector, implies x transpose A x is going to be bigger than 0. This is definition of positive definite matrix. Now, A is written as L D L transpose, where L is unit lower triangular and D is diagonal. So, our claim is that if i write D as diagonal d 1 1 up to d n n, then each d i i is greater than 0. This is our claim; so, in order to prove the claim. So, proof: you consider, matrix D to be equal to L inverse A L transpose inverse L is A unit lower triangular matrix. So, that means, determinant of L is going to be equal to1 and hence it will be invertible.

And if L is invertible, L transpose is also invertible. In fact, for A invertible matrix A transpose inverse is nothing but A inverse transpose. So, we have written D as L inverse A L transpose inverse. We will show that the D is positive definite or in fact, that we do not need to show that D to be positive definite what we want is d i i, they should be bigger than 0.

So, these entries d i i, they are going to be given by e i transpose d e i, where e i is our canonical vector and using positive definiteness of our matrix A, we will show that all these entries they are bigger than 0.

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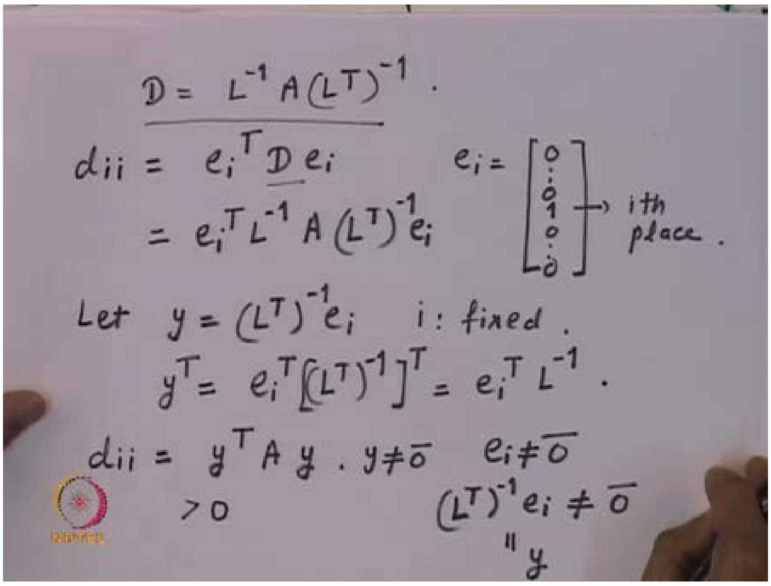


So, now, D is L inverse A L transpose inverse. di i is going to be e i transpose D e i, where e i is canonical vector with 1 at ith place and 0 else where, you can verify that d i is going to be e i transpose d e i, that is going to be e i transpose L inverse A L transpose inverse e I substitute for d from here. So,i get this, let y be equal to L transpose inverse e i, i is fixed in that case y transpose is going to be e i transpose L transpose inverse and then transpose of whole thing. Now, the transpose and inverse they compute L transpose inverse will be L inverse transpose and transpose of transpose gives us original matrix. So, this is nothing but e i transpose L inverse.

And hence, d i i will be y transpose A y. Our vector e i is not A 0 vector, because you have got one entry which is equal to 1, when you consider L transpose inverse e i, this is also not equal to 0, because if y is equal to 0, then i can multiply throughout by L transpose and then i will get e i, which is equal to L transpose y it is equal to 0. So, this y is A non-zero vector. And now recall the definition of positive definite matrix that for positive definite matrix whenever vector x is not equal to 0 x transpose A x is greater than 0. And hence, y is not equal to 0 vector and you will get y transpose A y is bigger than 0.

So, let us recapitulate what we have done, we are assuming our matrix A to be positive definite; using symmetry we know that A can be written as L D L transpose, where L is A unit lower triangular matrix and D is a diagonal matrix and then using the other property of positive definiteness of matrix A we proved the diagonal entries they are going to be all bigger than 0. So, now, we are ready for proving the Cholesky decomposition.

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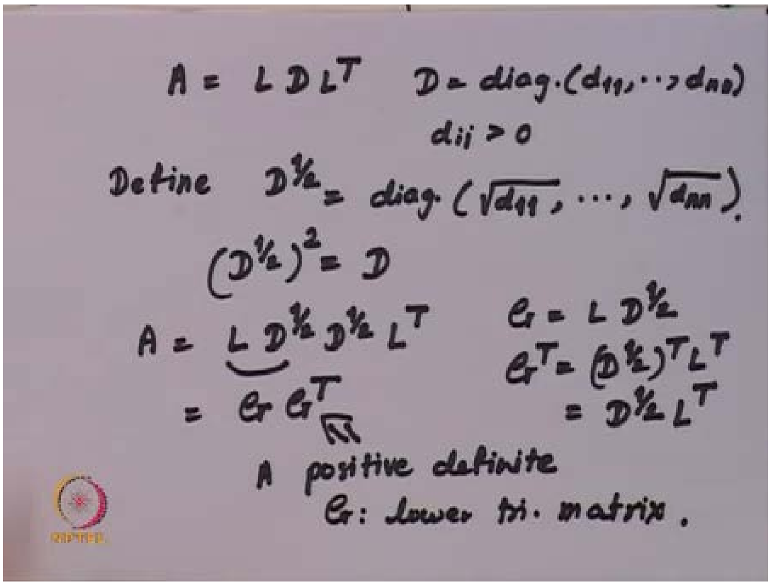


So, we have A is equal to L D L transpose D being a diagonal matrix and d i they are bigger than 0, you define D raise to half its a notation to be a diagonal matrix with diagonal entries as square root of d 1 1 square root of d n n. This matrix will have property that D raise to half its square is going to be your matrix D.

And hence, A is going to be equal to A L D raise to half D raise to half L transpose. Now, let me call this matrix to be G. So, if g is equal to L D raise to half, g transpose will be D raise to half transpose L transpose, but D is a diagonal matrix. So, it is D raise to half L transpose. So, that means, A is equal to G G transpose. So, A positive definite will imply that A can be written as a G G transpose. And G is going to be a lower triangular matrix. Now, the diagonal entries of G need not be equal to1. So, compare with L U D composition. In case of L U D composition, you had A is equal to L into U L unit lower triangular U upper triangular, if the matrix is positive definite A is equal to G G transpose where G is a lower triangular matrix and then G lower triangular it will mean that G transpose is upper triangular.

So, what have we achieved, when we want to determine L U D composition of our matrix A, we need to determine L and we need to determine U. Now, we have got A is equal to G G transpose. So, one’s I determine G, then G transpose is determine.

So, that means, our work is going to get reduced by half. Now, what about uniqueness like you have A is equal to G G transpose, whether such a decomposition will be unique. If you do not impose any condition, if you just say that G should be a lower triangular matrix, then such a decomposition will not be unique, because remember how we have done this G G transpose, it was we had our diagonal matrix D. We proved that diagonal matrix the entries are bigger than 0 and then we looked at positive square root now for a each of the entry, if I take negative square root, then also my D raise to half will have property that D raise to half square is equal to D. So, if you want uniqueness, then you have to put some extra condition for example, you can say that G should be a lower triangular matrix with diagonal entries to be bigger than 0. The normalization could be that it should be said that the diagonal entries are less than 0 or some other, but then for the sake of the definiteness, we will consider the matrix G to be such that it is lower triangular and its diagonal entries they are all bigger than 0. So, this is Cholesky decomposition.

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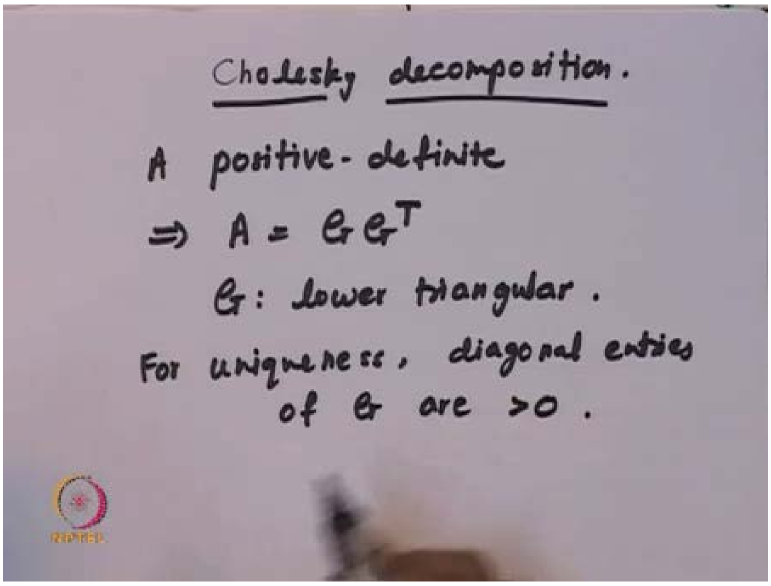


So, we have A positive definite implies A can be written as G G transpose ,where G is lower triangular matrix and for uniqueness impose the condition that diagonal entries of G are strictly bigger than 0.

Now, this it has a converse, if you are given that A is equal to say, M M transpose; where M is invertible matrix, then A is going to be positive definite. So, this part we will prove but before that let us look at how to determine G and G transpose like we have done in case of L U D composition. One way of finding L U D composition is reduce the coefficient matrix A to an upper triangular form using gauss elimination method. The final form which you get upper triangular form that is our u and when you do gauss elimination method you have got those multipliers.

Using those multipliers, you construct your lower triangular matrix, other way is start with A is equal to L into U. So, entries of L and entries of U they are determined by taking matrix multiplication of L into U and equating to the corresponding element of A.

So, now in order to find the Cholesky decomposition that is what we are going to do, we will not go through like start with - gauss elimination method that get L U D composition then get L D V decomposition and then write D raise to half. So, instead of that we will just start with a is equal to G G transpose the entries of G are unknown and they will be obtained by taking the matrix multiplication of G and G transpose and equating with the corresponding entry of your matrix A.

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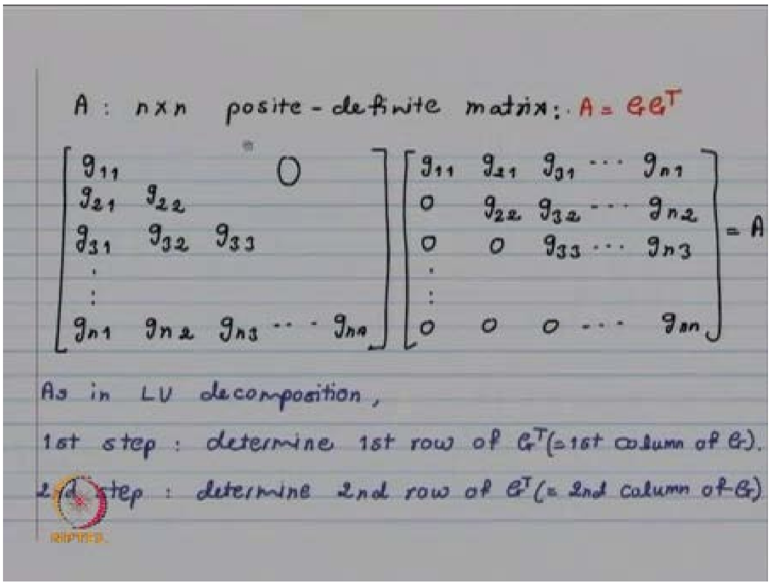


So, here is n by n positive matrix given. Now, you want to write it as G G transpose, where G is a lower transpose matrix. So, all the entries here they are 0, when you look at its g transpose all the entries below diagonal they will be 0. So, as in case of L U D composition our first step is going to be determine the first row of G transpose and that will also in first column of G.

So, consider the first row here and multiplied by first column. So, that will give us the entry a 1 1, then first row into second column, first row into third column and first row into nth column, exactly same as we have done in case of L U decomposition with a difference that in L U decomposition; you determine the first row of U and you need to determine then first column of L. Now, here once you determine, this row this column will be automatically determine. So, that is what makes our work to be reduced by half.

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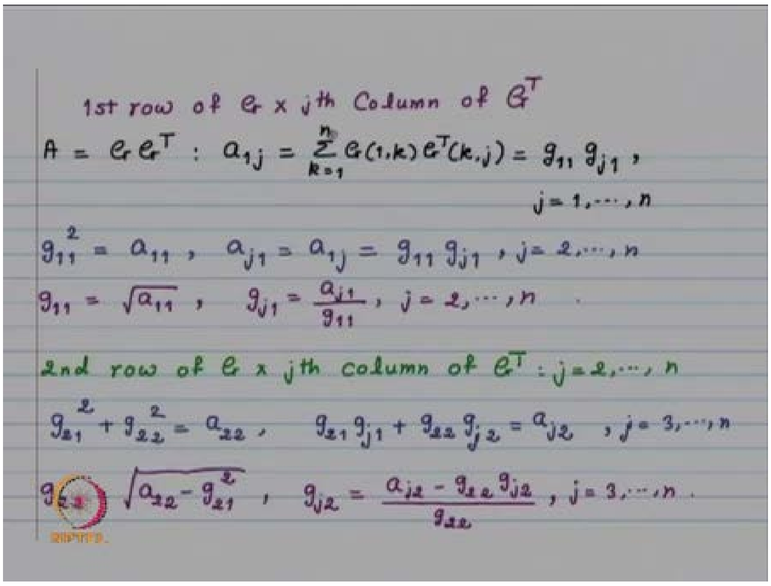


So, now let us look at first row into various columns of G transpose. So, first row of g into jth column of G transpose, when you do that you are going to get the entry a1 jth is is equal to summation k goes from 1 to n. First row of g the entries will be g 1, k then g transpose the entries of jth column. So, they will be G transpose k j, the notation is slightly different. So, here 1 denotes the row index and k denotes the column index. So, first row that is y this one is fixed with jth row. So, that is why this j is fixed

Now, remember that our G is a lower triangular matrix. So, in the first row only one entry is non-zero and that is g 1 1. So, the summation will get reduce to g 1 1 and then G transpose 1,j. So, that is going to be g j 1 j going from 1 to up to 1. So, put j is equal to 1 that will give you g 1 1 square to be equal to a 1 1 and; that means, g 1 1, we take positive square root. So, you have got root of a 1 1 then for the other values a j one is same as a 1 j because our matrix is symmetric. And then you have got g 1 1 g j 1 for j is equal to 2 up to n g 1 1 is determined. So, g j one will be a j 1 divided by g 1 1 j is equal to 2 up to 1.

So, we have determined the first row of G transpose, which is same as first column of G. Now, second row of G into jth column of G transpose. So, in the second row of G, you have got only two entries in the first row there was only one non-zero entry. Now, there are possibly two non-zero entries. So, the summation will be g 2 1 square plus g 2 2 square is equal to a 2 2 and then you will have g 2 1 g j one plus g 2 2 g j 2 a j 2 for j is equal to 3 up to n. So, you determine g 2 2 by square root of a 2 2 minus g 2 1 square and g j two will be equal to a j 2 minus g 2 2 g j 2 divided by g 2 2, you are considering. So, g j 2 g 2 1 g j 1 g 2 2g j 2 that is correct and g j 2 . So, it will be a j 2 minus actually this term should be g 2 1 g j 1, this should be different it should be our g j 2 is going to be equal to a j 2 minus g 2 1 g j 1 divided by g 2 2.

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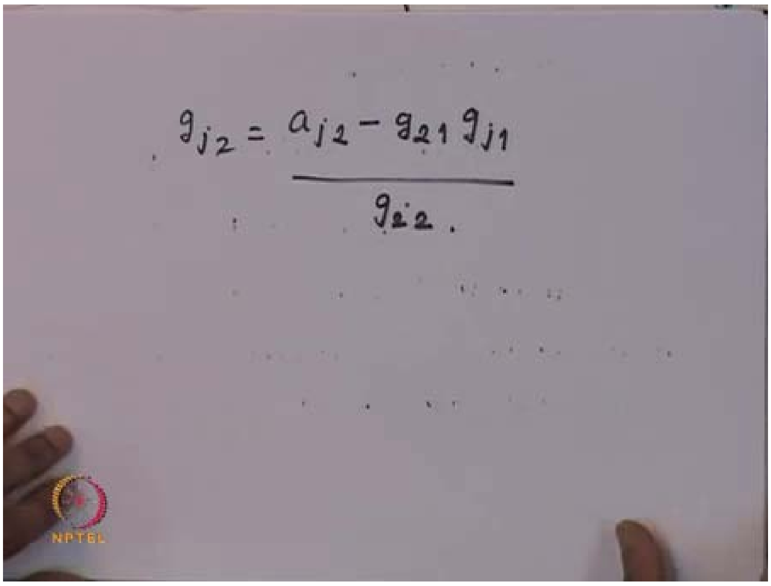


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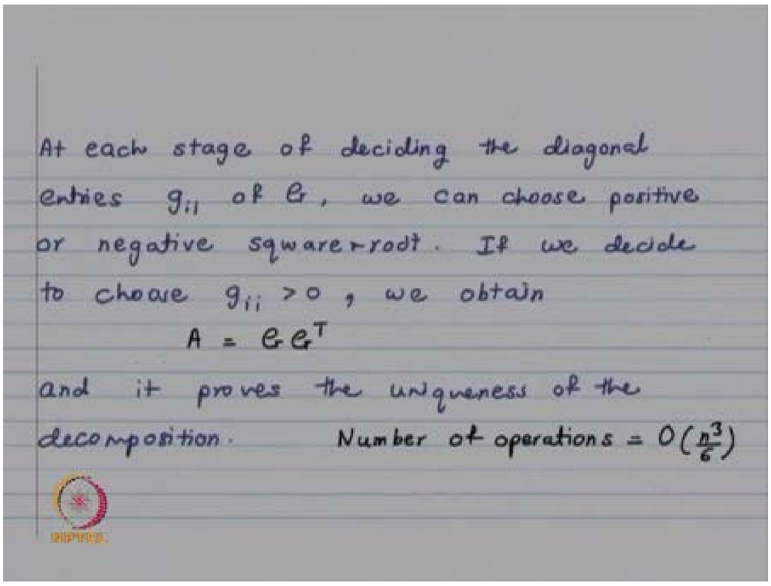


So, we continue like this and then we determine G G transpose, we have calculated G G transpose. And this will be the number of operations; they are going to be of the order of n cube by 6. In case of L U D composition, they were of the order of the n cube b 3. And now, because of the symmetry you have got A is equal to G G transpose it is going to be of the order of the n cube by six half the number. So, now, more than this reducing the number of operations by half that is an important factor, but this decomposition is a stable way. So, now, what to stability means we are going to discuss it later on. So, if matrix A given to be a positive definite then you can write this G G transpose.

Now, let us prove converse that if i have got matrix A to be equal to M M transpose M need not be even lower triangular, if M is a invertible matrix then A is equal to M M transpose is going to be positive definite.

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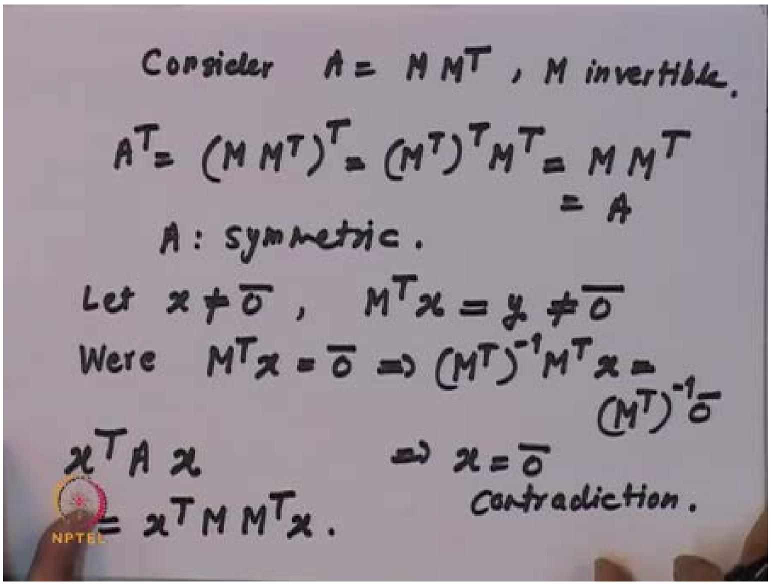


So, consider A is equal to M M transpose, where M is invertible; look at A transpose. A transpose will be M M transpose transpose of that then when you take the A B transpose that is B transpose A transpose. So, you get M transpose transpose into M transpose, but the M transpose transpose gives us M and then M transpose which is equal to A.

So, A is symmetric. Now, let x be not equal to 0 vector, m invertible means M transpose also is invertible. So, you consider, M transpose x. This will be let me call it y, this is not a 0 vector, because where M transpose x equal to 0 vector; this will imply that M transpose inverse M transpose x will be equal to M transpose inverse into 0 vector. This will imply that x to be equal to 0 vectors. So, that will be contradiction, because you are starting with x is equal to 0 vectors. So, now, M transpose x is equal to y which is not a 0 vector.

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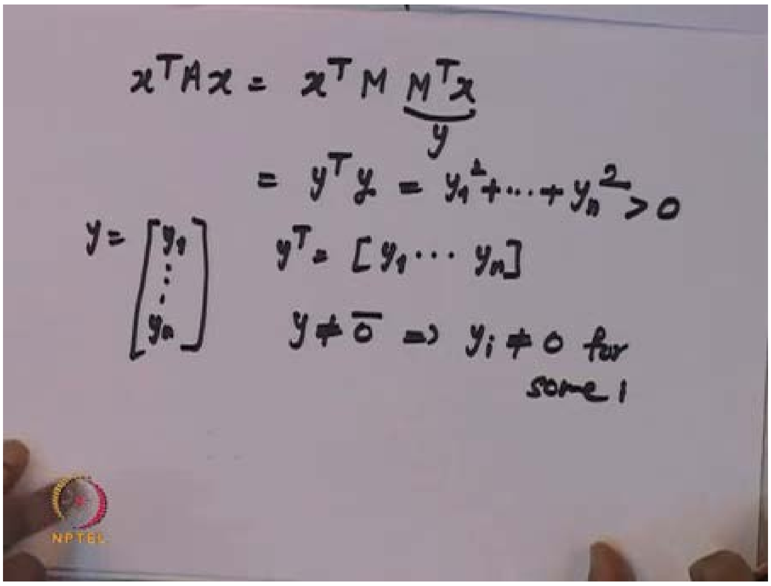


So, look at x transpose A x I am starting with x to be a non-zero vector and I am looking at x transpose a x. This will be x transpose, M M transpose x substituting for A then we have m transpose x is equal to y. So, x transpose A x will be equal to x transpose M M transpose x, this is y. So, this will become y transpose y. If y is our vector y 1, y 2, y n. Then y transpose is row vector y 1, y 2, y n. So, y transpose y becomes you consider this y is this and you have got y transpose y, you have to consider y transpose into y. So, that will give you y 1 square plus y n square. Now, y not equal to 0 vector implies y i not equal to 0 for some i, at least 1 y i has to be non-zero. Now, y 1 square plus y 2 square plus y n square, this will be strictly bigger than 0, because each entry is bigger than or equal to 0 and 1 entry is strictly bigger than 0. So, we have proved that if A is equal to M M transpose with m to be a invertible matrix, then it is a symmetric matrix and x transpose a x is bigger than 0. So, it is a positive definite matrix. So, now, we have got, if A is a positive definite matrix then you can write A as G G transpose, conversely if A can be written as G G transpose with G to be invertible then A is going to be positive definite. So, this gives us a practical way to determine whether your A is going to be positive definite or not.

Look at the definition of positive definiteness, we say that it should be symmetric, fine I can verify that it is a symmetric. The other part is for every non-zero vector x, we want x transpose A x is bigger than zero. So, now, how am I going verify this, we have got infinitely many vectors. So, for all non-zero vectors we want x transpose x to be bigger than 0. So, now, we have got, if and only if condition, A is positive definite if and only if A can be written as G G transpose with g to be a invertible matrix in the Cholesky decomposition for the normalization we had said that all the diagonal entries they should be bigger than 0. So,G being a lower triangular matrix its determinant will be product of the diagonal entries and it will be bigger than 0's it will be invertible.

So, given a matrix A, you first verify whether a transpose is equal to A, you write a computer program for that next you try to write a G G transpose. So, look at a matrix A is given to you try to do like this. So, in the first stage you are trying to determine the first row, if g 1 1 your taking positive square root, if this number is less than 0, then your matrix is not positive definite, if this number is equal to 0, you are dividing. So, then your matrix is not positive definite. So, at any stage when matrix A is given to you, you have verified that it is a symmetric matrix, now you try to write A at G G transpose for this also you can write a program, if at any stage your algorithm breaks down, now how can the algorithm break down? you are taking square roots. So, if at any stage the number of which you need to take the square root, if it turns out to be less than 0 your matrix is not positive definite; if your diagonal entry becomes 0 then again your procedure is going to break down. And then your matrix is not positive definite. So, here is a practical test for determining whether a matrix A is positive definite or not and such case they are prayer. So, it is one of the occasions where practically you can determine whether matrix A is positive definite or not.

So, now, So, far we had considered system, where the coefficient matrix A was invertible and in addition determinant of a k is not equal to 0 for k is equal to 1 to up to n where k is the principle leading sub matrix. And we have a class of matrices which is the positive definite matrices where this is applicable, but then if you have given only that a is A invertible matrix you are going to come across the systems that A x is equal to y A is invertible matrix. So,i know that A x is equal to bit is going to have a unique solution.

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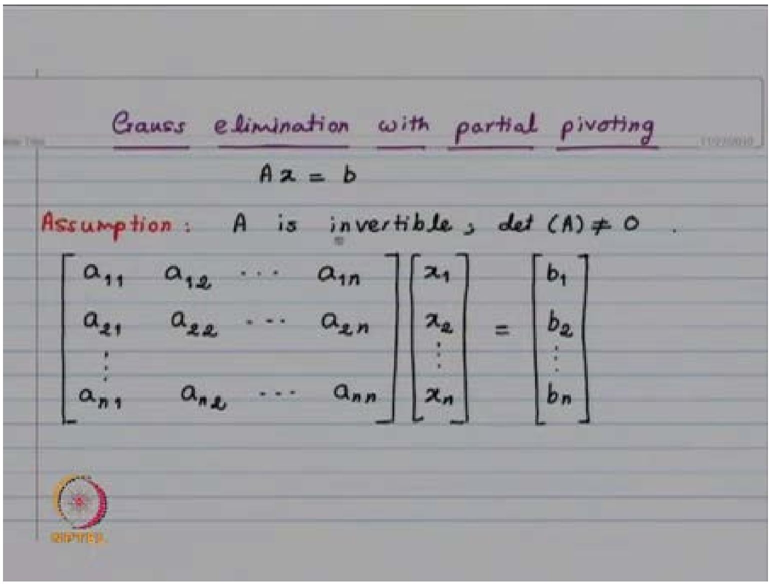


Now, how to find that solution that is where the modification of gauss elimination comes into picture and that is known as gauss elimination with partial pivoting. So, now let us discuss that gauss elimination with partial pivoting. So, now, our assumption is A is invertible; that means, determinant of A is not equal to 0, this is only given to us and we look at the system in the gauss elimination what we were doing where we were introducing 0's in the first column below the diagonal. So, a 1 1 was our pivot and we looked at multiplier when we want to introduce 0 here, we multiplied the first row by a 2 1 by a 1 1 and subtracted from the second row. And So, on now if A is invertible it can very well happen that this a 1 1 is equal to 0.

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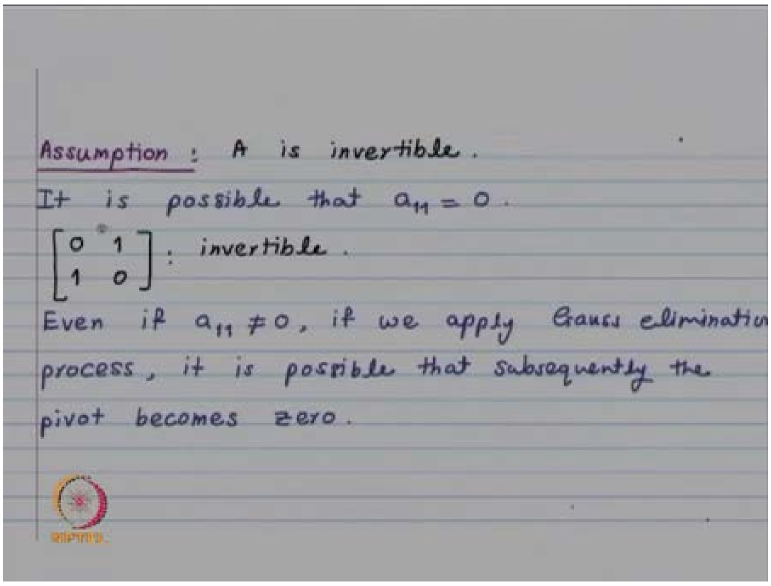


we had already considered this example a 2 by 2 matrix it is a invertible matrix because determinant is equal to minus one and the pivot or the first entry a 1 1 is equal to 0 now even if a 1 1 is not equal to 0. You should avoid to divide by a small number, when you are doing exact computations are or when we do is a mathematics we say that you cannot divide by 0.

When your computations are going to be with the help of a computer you should not divide by a small number, now what happens when you divide by a small number that discussion we will do later on. So, now what I can do is I have been given that matrix A is invertible e, I look at the first column it can very well happen that the first entry a 1 1 is equal to 0, but at least 1 of the entry in the first column it has to be non-zero because if all entries are equal to 0 I have go to zero column; that means, determinant of A is 0. So, a will not be invertible.

So, i will look at the entries in the first column and look the entry, which has got maximum modulus. So, suppose that is in the kth row. So, I will interchange kth row and the first row. So, by this interchange I have done my best I am going to divide by the entry which is going to occupy the position 1 1 in the matrix. So, in the first column among the entries I take the entry with the maximum modulus.

After interchanging these two rows I proceed as before and then this will take care of first column, then one do similar thing for the second column. So, on and this is known as gauss elimination with partial pivoting.

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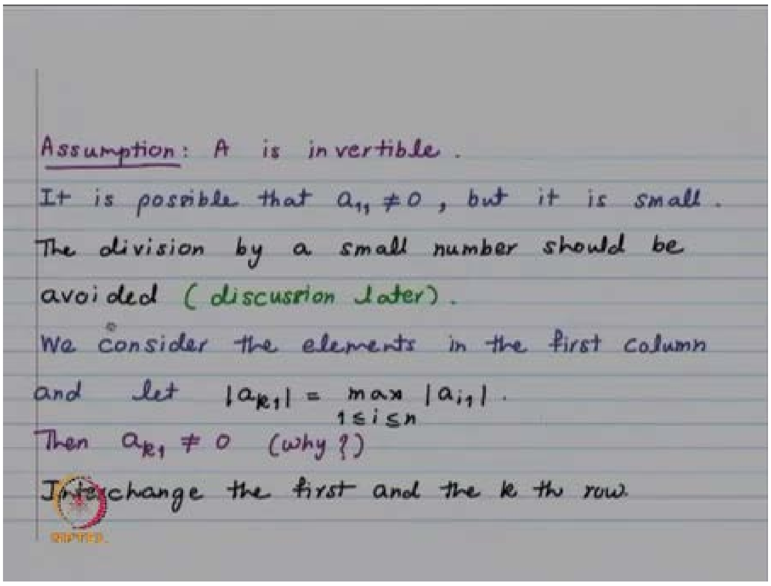


So, here suppose modulus of a k one is maximum of modulus of a i 1 1 less than or equal to i less than or equal to n. So I am considering the entries in the first column and looking at the entry which has got maximum modulus and then interchange the first and the kth row.

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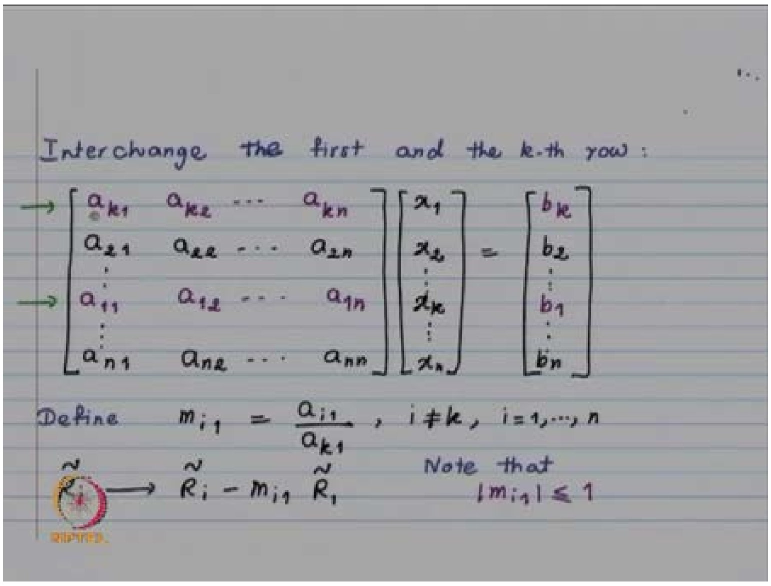

So, what was kth row before that has become the first row, what was first row that has become the kth row. Now, notice that by interchanging the 2 rows; that means, I am interchanging the order of my system of equations, I do not change the system because we want to find solution of A x is equal to b. So, whatever changes we make it should give us an equivalent system.

Now, n questions are given to you you are just changing the order of the equation. So, you do not change the system. So, we have interchange and now our multipliers are going to be m i 1 is equal to a i 1 by a k 1 the modified entry and and then R i tilde are my new roots. So, this is my r 2 tilde r 3 tilde r n tilde. So, all the rows are going to be same as before except for one row and then I introduce 0 here. So, as I said is going to be exactly same as gauss elimination method, which we had discussed earlier. Now, I will like to you to notice one thing that if do this modulus of m i 1 is going to be less than or equal to 1. So, this factor becomes important, when we are going to discuss the stability. So, when I do this my matrix will get modified to this matrix.

So, because we are interchanging rows, the new entries I am denoting by a 1 1 tilde a 2 1 tilde a n one tilde i have introduced all 0 here and then this is the modified matrix, sub matrix and this now i want to do similar thing for the second column. So, in the second column look at the entries diagonal onwards and look at the maximum entry, this maximum entry again should not be 0, because what will happen is if all the entries here they are 0 then, what our matrix looks like a 1 1 tilde 0 0 0 a 2 1 tilde 0 0 0. So, you have got two columns which are linearly dependent or what I can do is multiple or a subtract a multiple of the first column from the 2 column and then get a 0 column.

So, think about it. So, all these entries among these entries at least one has to be not equal to zero. So, we will interchange second and kth equation and continue as before.

So, the difference between gauss elimination, which we discuss before and gauss elimination with partial pivoting is at each stage we are interchanging roots. So, that is the gauss elimination with partial pivoting and tomorrow we will show that the gauss elimination was equivalent to writing A is equal to l into u. So, this gauss elimination with partial pivoting is equivalent to writing p into a is equal to l into u, where p is a permutation matrix; that means, it is a obtained from the identity matrix by interchange of roots, this again will be useful when we want to consider the backward error analysis. So, we will continue tomorrow, thank you.

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