Elementary Numerical Analysis

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Lecture No # 15

Tutorial –2

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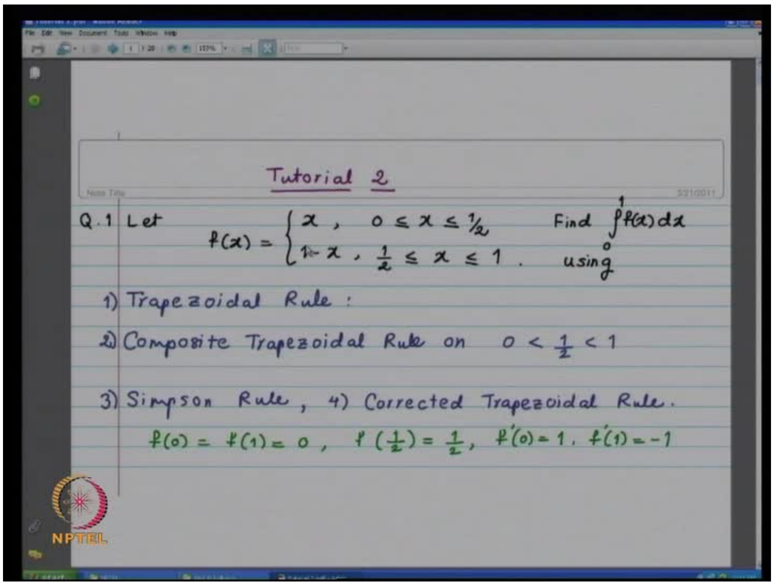

Today we are going to solve some problems based on numerical quadrature and after that we will consider numerical integration some of simple problems first problem is we are given a function f x which is piecewise linear so f x is defined as equal to x for 0 less than or equal to x less than or equal to half and 1 minus x if 1 by 2 less than or equal to x less than or equal to 1.

We want to find an approximation to integral 0to 1 f x d x by using trapezoidal rule then composite rule based on partition 0 less than half less than one; that means, we will be applying trapezoidal rule to interval 0 to half and trapezoidal rule to interval half to and adding it up then Simpson rule. So, basic Simpson's rule and corrected trapezoidal rule in all these rules what will come into picture will be value of function f at 2 end points 0 and 1.

And at the midpoint half in addition for corrected trapezoidal rule we will need f dash at 0 and f dash at 1 our function f is such that it vanishes at the 2 end points. So, f of 0 is 0f of 1 is 0 and f at half is going to be equal to half f dash at 0 is going to be equal to 1 f dash of 1 is going to be equal to minus 1 because on the interval half to 1 our function is defined as 1 minus x. So, when take its derivative. So, at 1 we are taking the left handed derivative.

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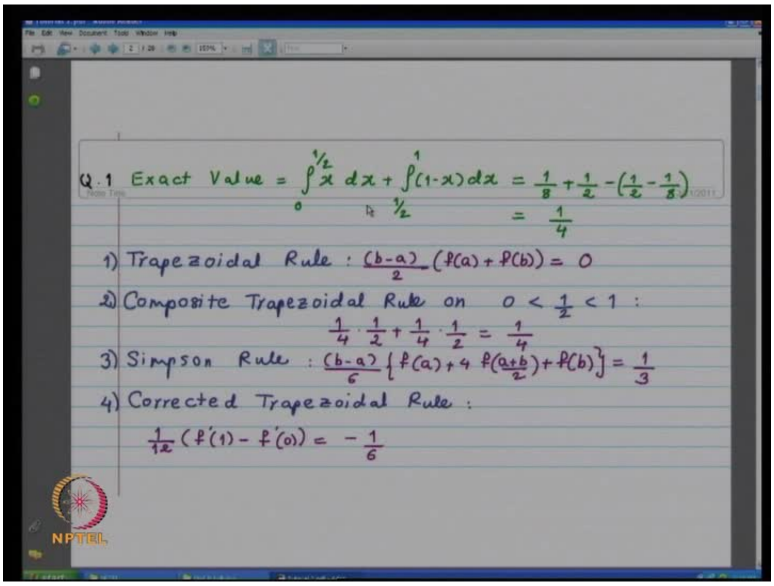

So, now we are going to substitute in these values in various formulae to obtain approximation to integral 0 to 1 f x d x. So, let us look at first the exact value the integral 0 to half x d x plus half to 11 minus x d x because that is how our function f is defined when you integrate you get the value to be equal to 1 by 4 the first rule is trapezoidal rule.

Our formula is b minus a f a plus f b by 2 in our case a is 0 b is 1 f at 0 is 0 f at 1 is 0. So, the approximation using trapezoidal rule which we get is equal to 0 now we want to apply composite trapezoidal rule we are going to apply it on the partition 0 half and 1.

That means on the interval 0 to half we apply trapezoidal rule on the interval 0 to half our function is f x is equal to x; that means, it is a linear function and the error is 0 for linear functions for trapezoidal rule. So, using our trapezoidal rule on the interval 0 to half we are going to get exact value of the integral similarly on the interval half to 1 our function is 1 minus x. So, it is linear we approximate our integral over half to 1 by trapezoidal rule.

So, there again the error is going to be 0. So, composite trapezoidal rule is going to give us exact value. So, it is the using composite trapezoidal rule you get 1 by 4 into half 1 by 4 because b minus a on the interval 0 to half it is half. So, 1 by 2 and then this 2. So, it is 1 by 4 then we will be looking at value of f of 0 plus value at f of half at 0 the value is 0 at half the value is half. So, that is why 1 by 2 plus 1 by 2 to 1. So, again the length of the interval is half divided by 2 that gives us 1 by 4.

Now, we will be considering f at half plus f at 1. So, that is going to be equal to 1 by 2 and f at 1 is 0. So, it is 1 by 4. So, for the composite trapezoidal rule you get the value to be equal to 1 by 4 next for Simpson's rule this is the formula f a is 0 f b is 0 f at a plus b by 2at the midpoint it is value is half. So, when you simplify you get it to be 1 by 3 and in the corrected trapezoidal rule you have trapezoidal rule plus this term trapezoidal rule gives us 0. So, you have only this term and the value which you obtain by corrected trapezoidal rule is 1 by 12 f dash 1 minus f dash 0so it is going t o be equal to minus 1 by 6.

Thus the best result which we obtain in this example is when you are using composite trapezoidal rule and that is because our function is piece wise linear and in the composite trapezoidal rule we have used our partition such that on each sub interval our function is linear if instead of applying composite trapezoidal rule to interval 0 to half and half to 1 if I would have used say interval 0 to 1 by 3 then 1 by 3 to 2 by 3 and 2 by 3 to 1then our error would not have been equal to 0 because on the interval 0 to 1 by 3 our function will be linear.

On 2 by 3 to 1 also it will be linear, but on 1 by 3 to 2 by 3 it will be piece wise linear and if you compare the result which you obtained using trapezoidal and Simpson then Simpson's approximation is a better approximation than the trapezoidal and the corrected trapezoidal rule that is going to be the worst approximation.

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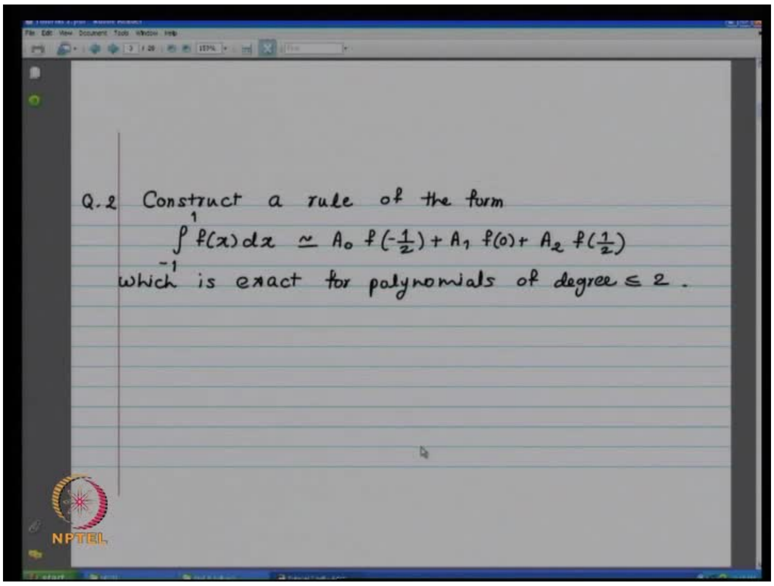

The value of the integral is positive and the value which you obtain is negative. So, it is a simple example to compare various our numerical quadrature rules now the next example we want to construct a rule of the type a 0 f of minus half plus a 1 f of 0 plus a 2 f of half which is exact for polynomials of degree less than or equal to 2 so; that means, we want to determine a 0 a 1 a 2 in such a manner that there is no error when you use this formula for f to be a polynomial of degree less than or equal to 21 way of doing this problem is we have got our points minus half 0 and half. So, you fit parabola; that means, a polynomial of degree less than or equal to 2.

Then you integrate and then you are going to get values of a 0 a 1 a 2 now our interpolating polynomial p 2 itself will be equal to function f if f is a polynomial of degree less than or equal to 2 and that is why our numerical integration also will be exact if f is a polynomial of degree less than or equal to 2.

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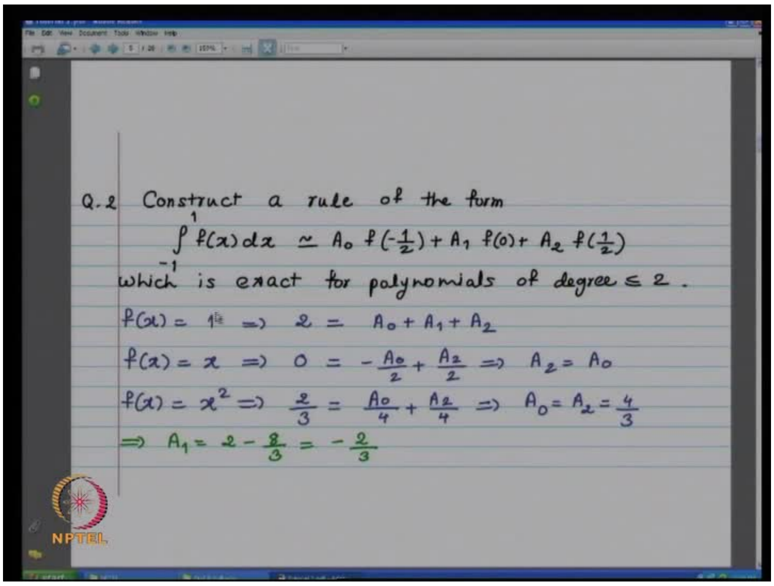

We will do it slightly differently we want that there should not be any error for polynomials of degree less than or equal to 2. So, if we guarantee that there is no error for 3 functions f x is equal to 1 f x is equal to x and f x is equal to x square then there will not be any error for a general quadratic polynomial of the form a 0 plus a 1 x plus a 2 x square now when we take these 3 functions f x is equal to 1 f x is equal to x and f x is equal to x square we will get 3 equations in 3 unknowns a 0 a 1 a 2 and then we determine a 0 a 1 a 2. So, f x is equal to1 then the left hand side is going to be equal to 2 f x is equal to 1 integral minus 1 to 11 d x. So, that will be 2 is equal to f x is a constant function. So, f of minus half is 1 f of 0 is 1 f of half is 1 and hence we get a 0 plus a 1 plus a 2.

So, this is our first equation next look at f x is equal to x when you integrate you are going to get x square by 2 evaluation between 1 and minus 1 that will give us to be 0 and this will be equal to now f x is equal to x so f of minus half will be minus half f of 0 will be 0 f of half will be plus half. So, you get minus a 0 by 2 plus a 2 by 2

So, this equation gives us a 2 has to be equal to a 0now this is for function f x is equal to x if f x is equal to x square then the integration is going to be equal to 2 by 3 and here f of minus half will be 1 by 4 f of 0 will be 0 f of half also will be equal to 1 by 4 and hence we will get a 0 plus a 2 divided by 4.

From this equation we have got a 2 is equal to a 0. So, using that fact we get a 0 is equal to a 2 is equal to 4 by 3 then we have not use this equation. So, use this equation to deduce that a 1 is equal to minus 2 by 3

And thus the co-efficiencies a 0 a 1 a 2 they are uniquely determined by the condition that the formula should be exact for polynomials of degree less than or equal to 2 now in the next example.

We are given a rule; that means, we are given the weights we are given the interpolation points and we are ask to determine the degree of precision of this formula that means we want to find the highest degree of the polynomial for which the rule is exact; that means, there is no error.

Now, when we want to determine degree of precision what we have to do is we have to proceed in an orderly manner; that means, check whether there is no error for f x is equal to 1 then look at f x is equal to x if there is no error for f x is equal to x then go to f x is equal to x square.

But if there is a error for f x is equal to x then; that means, the degree of precision is going to be only constant polynomials it can happen that there is no error for the constant functions there is no error for function f x is equal to x square.

But in that case if there is a error for f x is equal to x then we say that our degree of precision is going to be only 0; that means, it is going to be exact for only constant polynomial. So, here in this example or in this problem we are given the interpolation points we are given the nodes we are given the weight sand we want to determine the degree of precision.

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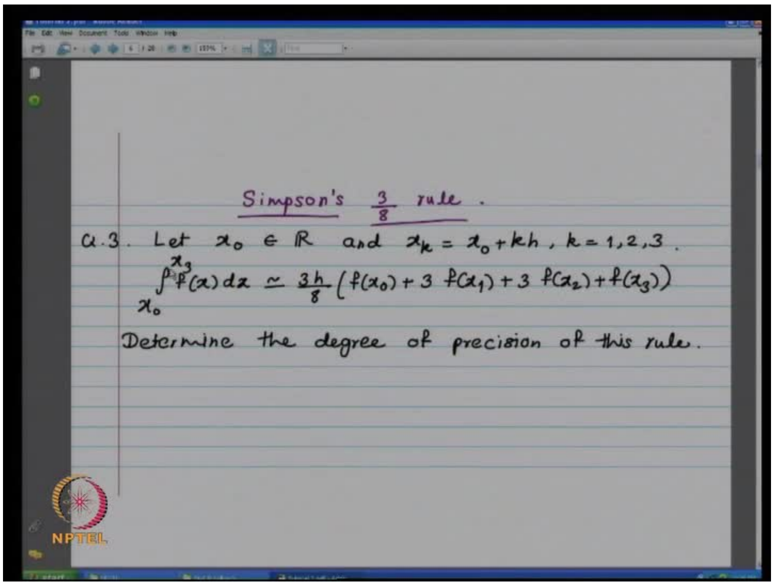

So, it is integral x 0 to x 3 f x d x is approximately equal to 3 h by eight f x 0 plus 3 f x 1 plus 3 f x 2 plus f x 3 where x 0 is any real number and x k are given by x 0 plus k h k is equal to 123; that means, x 0 x 1 x 2 x 3

They are going to be they form a equidistant partition which means x 1 minus x 0 is h x 2 minus x 1 is h x 3 minus x 2 is equal to h. So, we want to know the degree of precision of this rule; that means, the highest degree polynomial for which there is no error.

What we are going to do is we are going to assume that without loss of generality let x 0 be equal to 0. So, we are just changing the position of our point to start with x 0 is any real number, but we do not lose any generality of the problem if I can if i assume x 0 to be equal to 0. So, if i assume x 0 is equal to 0 my x 1 is going to be equal to h x 2 is equal to 2 h x 3 is equal to 3 h.

This is just for the sake of convenience and then we will write the formula now it will become integral 0 to 3 h f x d x is approximately equal to 3 h by 8 and then f at 0 plus 3 times f at h plus 3 times f at 2 h plus f of 3 and then we will try a calculate for f x is equal to 1 integral 2 to 3 h f x d x and formula using formula we are going to get the value. So, check whether these 2 are equal if they are equal then we will go to f x is equal to x.

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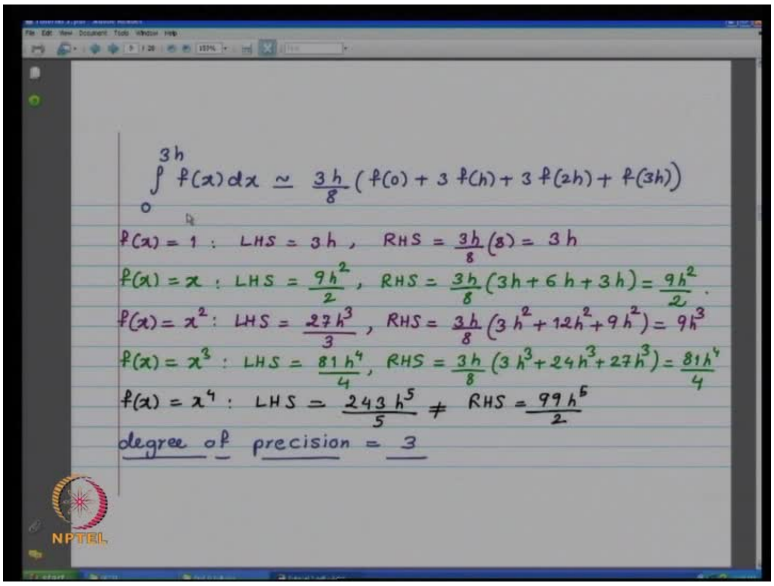

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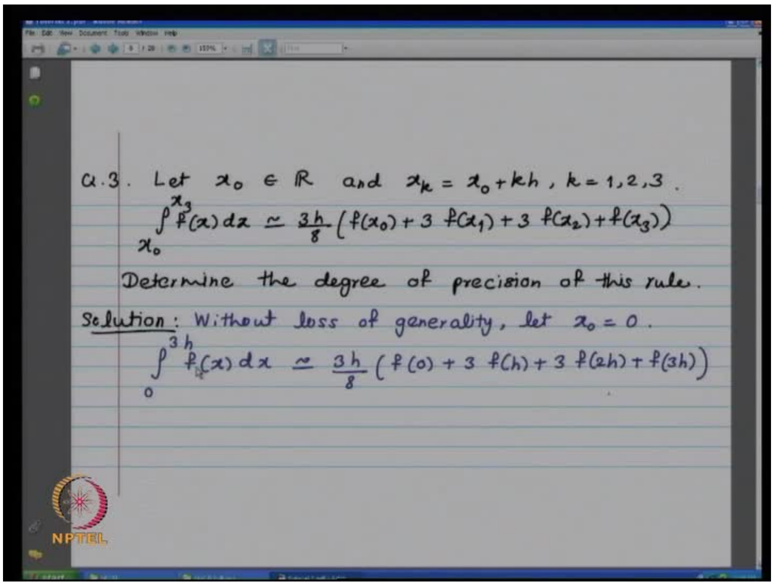

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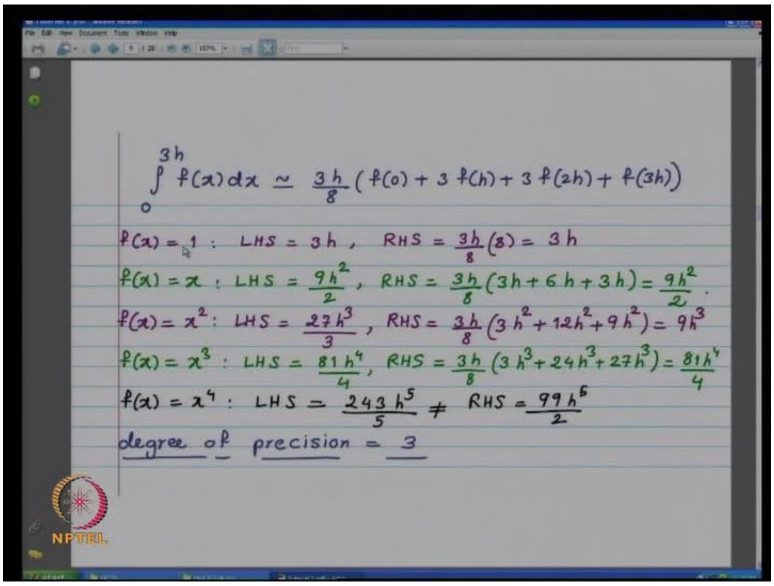

And we will continue till we get a function or we get a power of x for which the 2 values are not equal or not equal that will determine the degree of precision of our rule so this is by assuming x 0 is equal to 0 if I assume x 0 is equal to 0 integral x 0 to x 3 f x d x becomes integral 0 to 3 h f x d x which is approximately equal to 3 h by 8 f of 0 plus 3 f h plus 3 f of 2 h plus f of 3when you put f x is equal to 1 this is our left hand side LHS. So, LHS is 3 h RHS will be 3 h by 8 f x is constant 1. So, it will be 1 plus 3 plus 3 plus 1. So, that is 8. So, you get 3 h.

So, there is no error for f x is equal to 1 next for f x is equal to x the integral is going to be x square by 2 evaluated between 0 and 3 h. So, that gives us 9 h square by 2RHS will be 3 h by 8 f of 0 now it is going to be 0 then it will be 3 h plus 6 h plus 3 h. So, that again is going to be 9 h square by 2.

Consider f x is equal to x square the exact integral is going to be 27 h cube by 3; that means, it is going to be 9 h cube the right hand side will be 3 h by 8 no contribution from this term here it will be 3 h square this will be4h square multiplied by 3. So, that will be 12 h square plus here it is going to be 3 h square. So, that is 9 h square. So, when you simplify you get it to be nine h cube. So, again no error and f x is equal to x cube

So, the exact integral will be 81 h raise to4 by 4 and right hand side also is the same next f x is equal to x raise to 4. So, this is the first time when the left hand side is not equal to right hand side and hence the degree of precision is going to be equal to 3. So, here this is an example of a newton cotes formula.

We have considered newton cotes formula for like trapezoidal rule when we had considered 2 equidistant points in the interval a to b then we had Simpson's rule. So, we had the 2 end points and a midpoint and then we fitted a polynomial of degree less than or equal to 2.

Now, if you consider interval a b and look at 4 equidistant points including the 2 end points. So, you have 4 points you fit a cubic polynomial and then you integrate. So, because you are fitting a cubic polynomial if the function itself is a cubic polynomial then there is no error and hence no error in the integration formula.

And if you now here what was given to us it was given to us the formula and then we decided what is the degree of precision. So, this is Simpson's 3 eighth rule a special case of a newton cotes formula if we were given Simpson's rule the direct formula and ask to determine the degree of precision then we would have again obtained no error for 1x x square x cube and as we have noticed before that this is something unexpected.

You are fitting a polynomial of degree 2. So, there should not be any error for quadratic polynomial, but we get a result that no error for cubic polynomial as well whereas, here for the we are fitting cubic polynomial and no error for cubic polynomial.

Now, we are going to look at composite trapezoidal rule. So, we will take a special example where we will apply composite trapezoidal rule to find an approximate value of our integral a to b f x d x we have got a formula for the error in the composite trapezoidal rule it involves second derivative of the function now the function which we are going to integrate is f x is equal to 1 by x on the interval 1 to 7. So, we choose our interval 1 to 7 because our function f x is equal to 1 by x it has got a singularity at 0. So, whatever

Interval you are choosing for integrating that should not include 0 for this function f x is equal to 1 by x we can calculate it second derivative and then in the error formula we have got second derivative and then we have got power of h and some constant.

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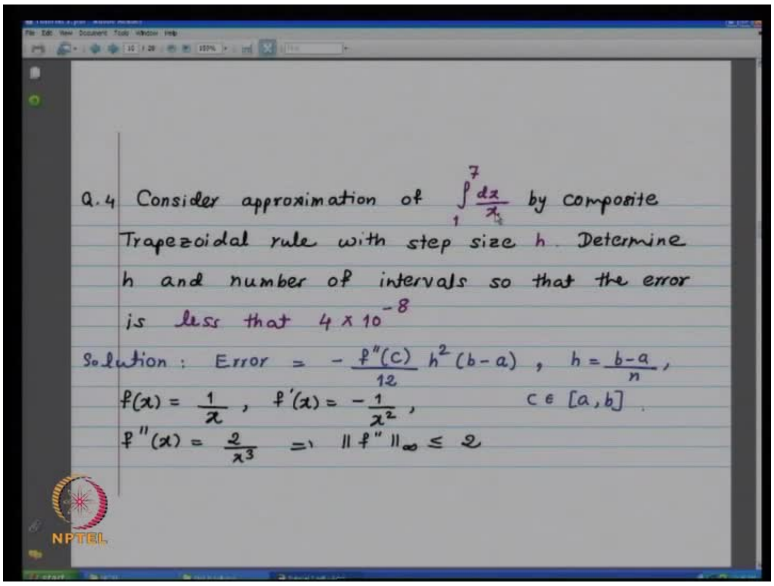

So, we will try to determine the length of the sub interval. So, that we achieve the desired accuracy now once we determine h, h is going to be equal to b minus a by n. So, it will also tell us equivalently the number of sub intervals next we will consider the same example for composite Simpson’s rule and then find the step size h and the number of sub intervals to achieve the same degree of precision. So, our function is d x by x integral 1 to 7 and we want to determine the step size h such that the error is less than 4 into 10 raise to minus 8.

In the case of composite trapezoidal rule in the case of composite trapezoidal rule error is given by minus f double dash C divided by 12 h square into b minus a where h is b minus a divided by n.

Point c is going to be some point in the interval a to b in this example b minus a is going to be seven minus 1. So, it is 6 f x is 1 by x f dash x is minus 1 by x square and f double dash x is 2 by x cube point C that it is it exists, but we do not know what that point C is. So, one dominates f double dash C by norm f double dash infinity 1 upon x cube is going to be a decreasing function on the interval1 to 7. So, the maximum will be attained at the left hand point and that gives us norm f double dash infinity to be less than or equal to 2

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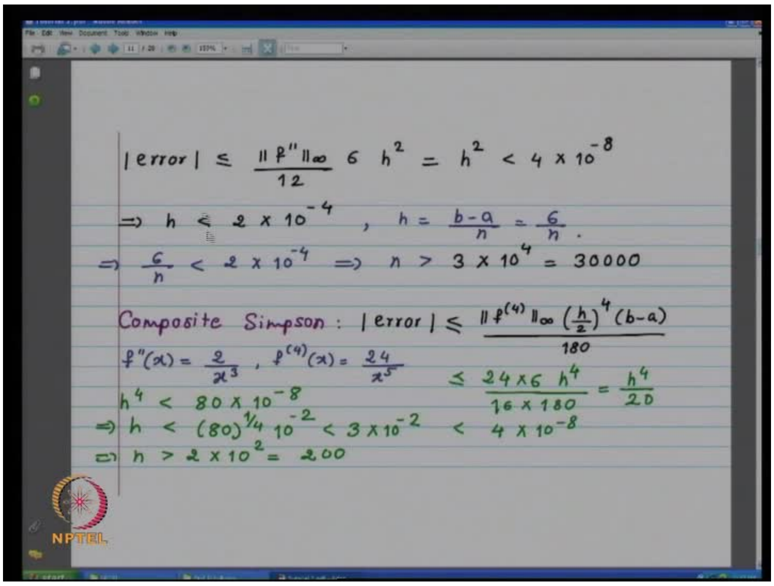

So, our error take the modules. So, modules of the error will be less than or equal to 2 by 12 b minus a is going to be 6. So, you are going to have 6 into 212. So, that will get cancelled. So, you will get h square should be less than 4 into 10 raise to minus 8 and h is going to be equal to nothing, but 6 by n. So, modules of error is less than or equal to h square which is less than which we want to be less than 4 into 10 raise to minus 8.

And hence h should be less than 2 times 2 into 10 raise to minus 4h is equal to 6 by n. So, that gives you 6 by n should be less than 2 into 10 raise to minus 4 or n should be bigger than 3 into 10 raise to 4. So, that is 3000. So, if you choose n to be bigger than 3000 then the error will be less than 4 into 10 raise to minus 8 in the composite trapezoidal rule.

In the case of composite Simpson's rule you have fourth derivative of the function h by 2 raise to4 b minus a by 180. So, calculate the fourth derivative that is 24 by x raise to 5 again the maximum will be attained at the left end point.

So, no rmf 4 infinity will be 24 h by 2 raise to4. So, I write it as h raise to 4 and 2 raise to 4 is 16180 from here and b minus a is 6. So, that 6 is here. So, this going to be equal to h raise to 4 divided by 20.

Suppose we want this to be less than again the same number 4 into 10 raise to minus 8 then modules of the error will be less than 4 into 10 raise to minus 8 provided h raise to4is less than 80 into 10 raise to minus 8 which gives us h to be less than 80raise to 1 by 410 raise to minus 2. So, we have got 3 raise to 4 is 81.

So, that is why 80 raise to 1 by 4 will be less than 3 into 10 raise to minus 2 and that we mean that h is 6 by n. So, our n should be bigger than 2 into 10 raise to 2; that means, 200. So, here we have like for the composite trapezoidal rule you will be evaluating your function if there are n intervals then you will be evaluating it at n p plus 1 points in case of composite Simpson we evaluate the function at the n plus 1 patrician points and in addition at the midpoint; that means, 2 n plus 1.

Now, in order to achieve the desired accuracy in case of trapezoidal rule we need number of sub intervals to be 3000; that means, we will be evaluating our function 3000 plus 1 times.

For the Simpson's rule we have got 200. So, you we will be evaluating it 200 into 2 plus 1. So, that means it is going to be401 that is much less than in case of the composite trapezoidal rule.

This is about the function evaluation and then you evaluate the function you multiply by weight and then you are going to add all these numbers. So, this illustrates that the higher the order of h the formula it should be preferred that we had composite trapezoidal composite Simpson if your number of intervals is the same then in the case of composite Simpson we had double the computation, but if you fix a desired accuracy that will be achieved with much less effort in case of composite Simpson's rule as compare to composite trapezoidal rule provided your function f is sufficiently differentiable.

You need your function to be4times differentiable if your function is only twice differentiable then we will not get h raise to 4 term in the composite Simpsons rule then the order of convergence gets reduced to h square. So, for smooth function the higher the power of h will be available and that formula will be more efficient as compare to formula with lower power of h.

So, these were some of the simple examples to illustrate our theory now we are going to consider what is known as Romberg integration if we remember the corrected composite trapezoidal rule.

So, corrected trapezoidal rule was obtained by considering cubic Hermite polynomial; that means, we looked at a cubic polynomial which interpolates our function f at point a and point b and also the derivatives at point a and point b. So, the formula which one gets it involvesf a f b f dash a f dash b now the term which contains the derivative it is of the form constant times f dash a minus f dash b. So, if we consider composite corrected trapezoidal rule; that means, our interval a b it is divided into n equal parts on each sub interval you apply corrected trapezoidal rule because of this f dash a minus f dash b when we apply 2 sub intervals and add it up all the derivative terms gets canceled and then what remains is only 2 end derivatives f dash a and f dash b.

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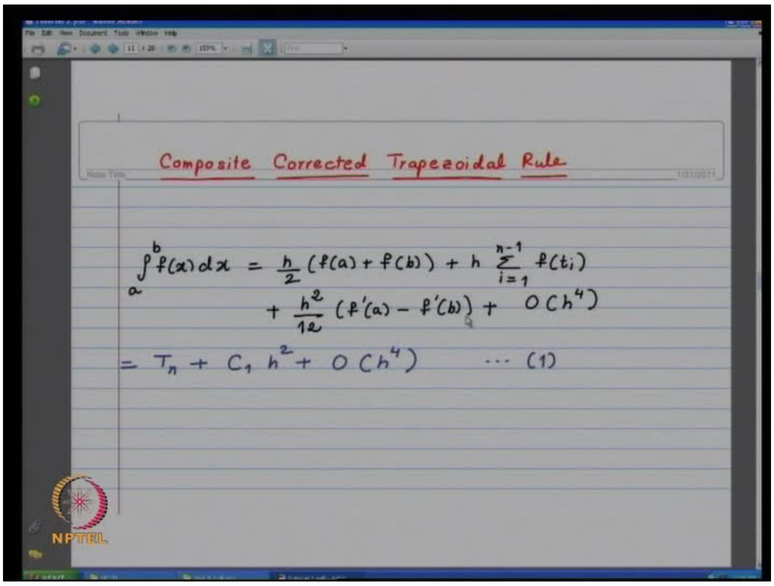

So, corrected composite trapezoidal rule is trapezoidal rule plus a term which contains the derivatives at the 2 end points so it is of the form integral a to b f x d x is equal to the trapezoidal rule plus h square by 12 f dash a minus f dash b plus term of the order of h raise4.

Here the error is of the order of h raise to 4 but one needs to know what is f dash a and what is f dash b. So, you have order of convergence h square in the trapezoidal rule in the corrected trapezoidal rule you have got order of convergence h raise to4with the rider that you should know what is f dash a and what is f dash b now 1 wants to know whether one can do something and get a formula which will have order of convergence or which will have error to be of the order of h raise to4but what should involve only function values because the derivative values they are not available. So, we look at this corrected trapezoidal rule more carefully.

You have integral a to b f x d x is equal to T n plus C1 h square plus term of the order of h raise to4what is C1C1is f dash a minus f dash b divided by 12; that means, our C1 is independent of the partition; that means, it does not matter the what patrician I am looking at what is c 1 is it is only f dash a minus f dash b upon 12 if I change my partition the term will remain the same C1 will remain the same. So, why do not I do like this that I look at a partition with n intervals. So, I will have such integral a to b f x d x d x to be equal to trapezoidal rule plus a term plus a term of the order of h raise to4now instead of n intervals I will consider a partition with n by 2 intervals.

That means earlier our sub intervals they had length to be equal to h now I will look at length to be 2 h. So, then I will have integral a to b f x d x is equal tot2 n so; that means, that is the trapezoidal rule based on partition with n by 2 interval plus I will have C1 the same constant and instead of h square I will have 2 h square because now the length of the partition is 2 h.

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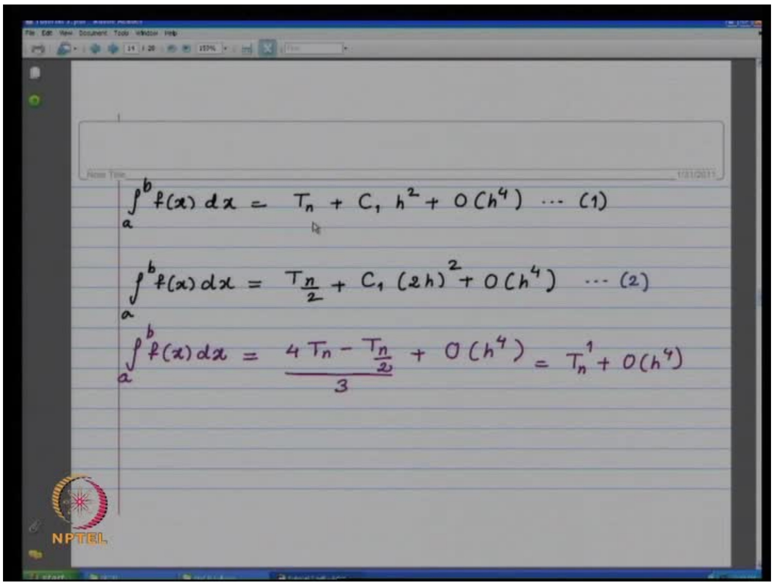

Plus term of the order of h raise to4so I have got T n plus C1 h square T n by 2 plus C12 h square nowI will take a combination of t n and T n by 2. So, as to get rid of the term which contains h square. So, let me explain so we have integral a to b f x d x is T n plus C1 h square plus term of the order of h raise to4integral a to b f x d x is T n by 2 n denotes the number of intervals in the sub partition plus C12 h square plus term of the order of h raise to4.

Now, what one can do is multiply the first equation by4and subtract the 2 equations. So, left hand side will be4times integral a to b f x d x minus integral a to b f x d x. So, that is going to be 3 times integral a to b f x d x here it will be4 T n minus T n by 2 the numerator here look at this term it is going to be C1 times4 h square minusC1 times4 h square. So, this will get cancelled and then term of the order of h raise to4big o of h raise to4that means it is less than or equal to constant times h raise to4and then I divide by 3 throughout. So, I get integral a to b f x d x is equal to4t n minus T n by 2 divided by 3 plus term of the order of h raise to4.

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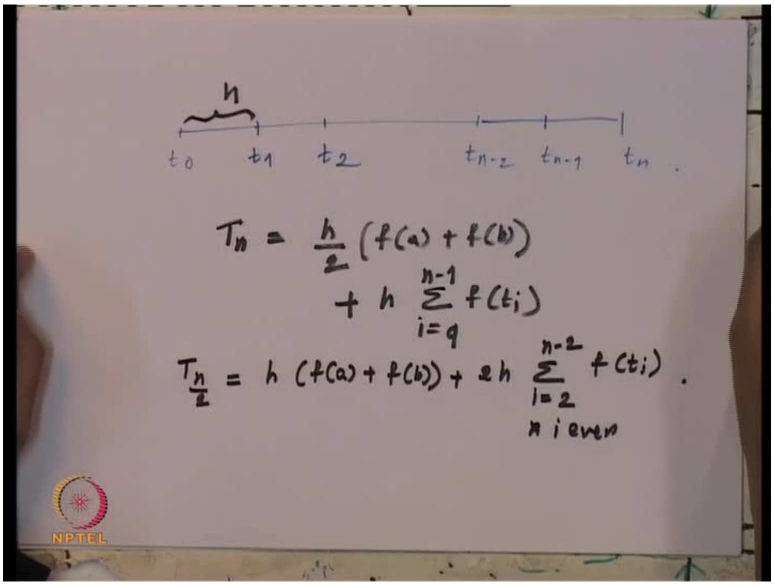

So, I have got T n 1. So, integral a to b f x d x is equal to T n 1 plus a term of the order of h raise to4now what we are going to do is we are considering this partition say t 0 then t 1 t 2 t n minus 2 t n minus 1 t n and this is going to be of length h. So, t n is going to be h by 2 f a plus f b plus h times summation f t ii goes from 1 to n minus 1.

When we look at T n by 2 then it is going to be h times f of a f a plus f b because wherever there is h it is going to be 2 h plus2 h times summation i goes from 2 to n minus 2 i even f of t iso here t n by 2 and t n they have some points they are in common like for the t n by 2 what comes into picture is a b and all even order t i's when you want to calculate t n then to these points we add points t1 t 3 up to t n minus 1 so; that means, whatever was the works done for t n by 2 calculation of the function values at t i's for i even that work one uses and then one looks at the combination.

4t n minus t n by 2 divided by 3 whatever formula you get that is going to give you the error to be of the order of h raise to4so this is the first step of Romberg integration. So, if your function f is going to be4times differentiable then we obtain a formula which involves only function values no derivative values which gives us the error to be less than or equal to constant times h raise to4now whether suppose my function f is6times differentiable is it possible to obtain error obtain a formula in which case error is less than or equal to constant times h raise to6so such a thing is possible.

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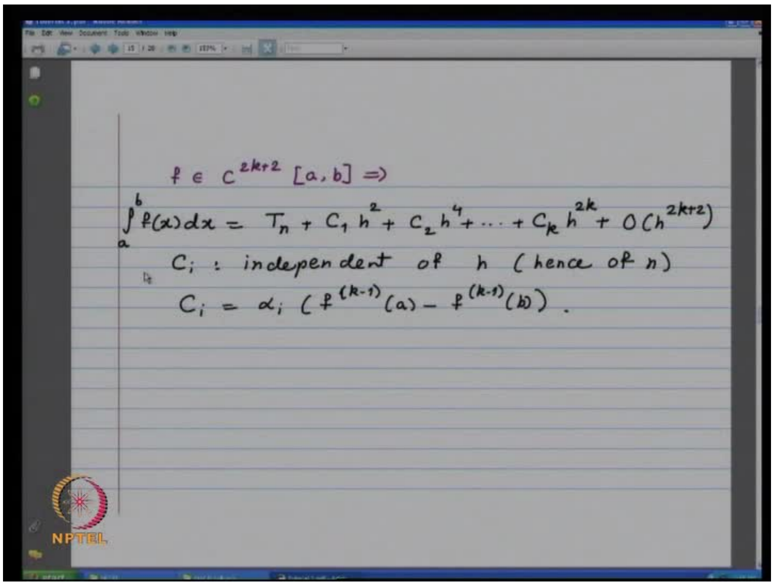

So, we have got a asymptotic series expansion integral a to b f x d x is equal to T n plus C1 h square plus C2 h raise to4plus C k h raise to 2 k plus term of the order of h raise to 2 k plus 2. So, we want the function to be sufficiently differentiable to k plus 2 times differentiable these C i's the coefficients they are going to be independent of h and hence independent of n and; that means, it is independent of the partition.

In fact, this C i's are some constants alpha i's multiplied by the k minus first derivative at a minus k minus first derivative at point b now this asymptotic expansion it is known as Euler Maclaurin series expansion and the coefficients alpha i's. So, for that what comes into picture are Bernoulli polynomial, but I do not want to get into details of those things it is just i want to tell you that if the function is sufficiently differentiable then we have got a asymptotic series expansion.

Now, using composite trapezoidal rule we got rid of the term h square and then obtained a result which has got error to be less than or equal to constant times h raise to4.

So, this asymptotic series expansion it tells us that it is possible now to get rid of the term h raiseto4and obtain a error to be less than or equal to h raise to6and. So, on

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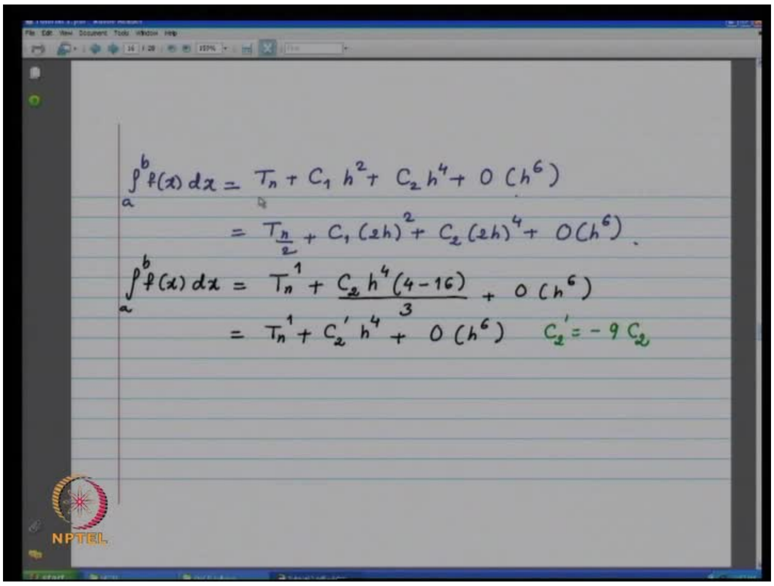

So, this is known as Romberg integration. So, look at this expansion T n plus C1 h square plus C2 h raise to4plus term of the order of h raise to6when instead of considering n intervals I look at n by 2 intervals I have T n by 2 plus C1 now over intervals are of length 2 h. So, it is 2 h square plus C22 h raise to4plus term of the order of h raise to6now what we are doing is we are considering4times this minus this and divided by 3. So, that was our T n 1. So, when you do that you are going to get integral a to b f x d x is equal to T n 1 plus this term gets cancelled because you are multiplying here by4and subtracting. So, nothing here.

This term is going to have C2 h raise to4common4minus this term so4minus 16 divided by 3 plus term of the order of h raise to 6.

And thus you have T n 1 plus C2 dash h raise to4 plus term of the order of h raise to6

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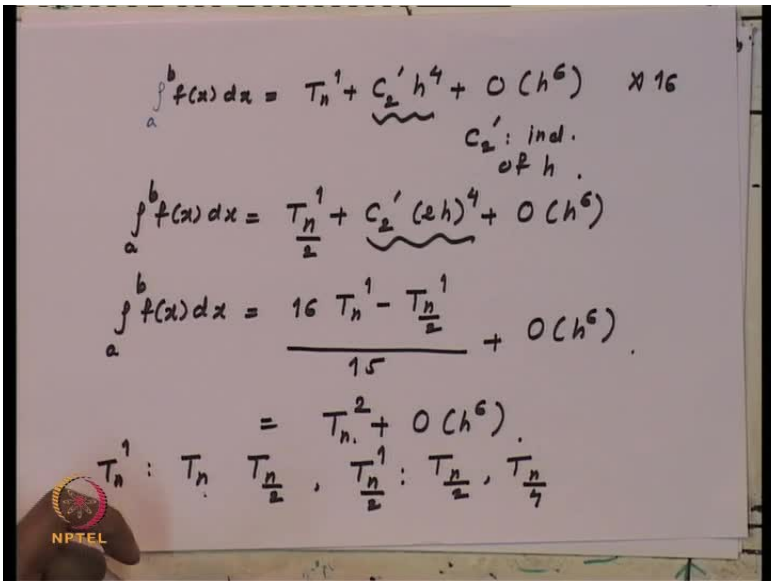

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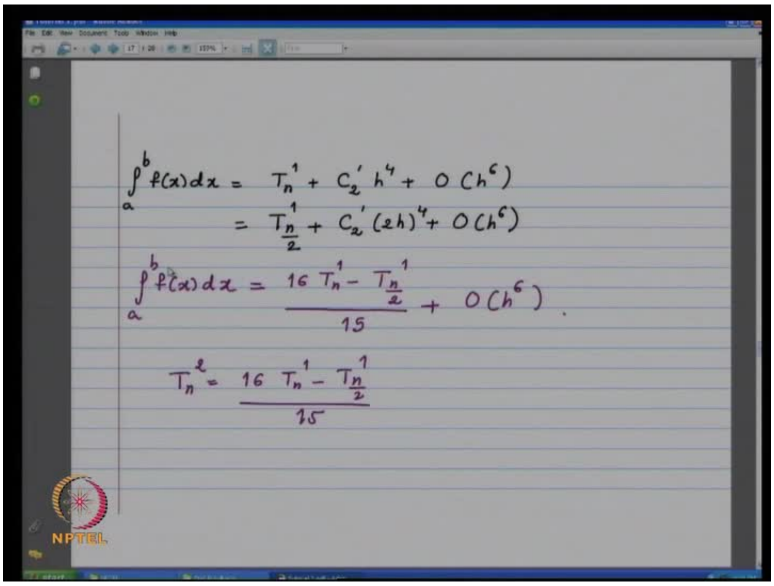

Where now C2 dash is minus 9C2. So, you have integral a to b f x d x is equal to T n 1 plus C2 dash h raise to4plus term of the order of h raise to6 C2 dash independent of h. So, integral a to b f x d x will be equal to T n by 21consider partition with n by 2 intervals C2 dash will not change what will change will be instead of h you will have 2 h raise to4plus term of the order of h raise to 6.

Now, we want to get rid of these 2 terms. So, we will multiply this equation by 16 subtract these 2 equations and divide by 15. So, you will have integral a to b f x d x is equal to 16Tn 1 minus T n by 21 divided by 15 this term will go away and you are left with term of the order of h raise to6so this is going to be your T n 2 plus term of the order of h raise to6 T n 1 what it involves is T nT n by 2 these are the composite trapezoidal rules t n by 21 will involve t n by 2 and t n by4.

So, Tn2 it is going to be based on trapezoidal rule based on n intervals trapezoidal rule based on n by 2 intervals and trapezoidal rule based on n by4intervals and their combinations that is going to give into give as a formula which has error to be less than or equal to h raise6so integral a to b f x d x is 16T n 1 minus t n by 21 divided by 15 plus term of the order of h raise to 6.

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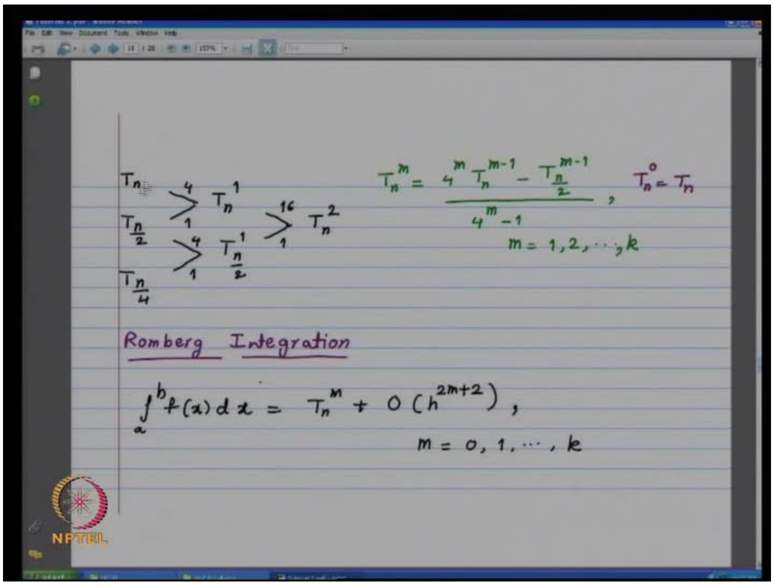

So diagrammatically here you have T n here you have T n by 2T n by4these are all composite trapezoidal rules then consider4times this minus 1 times this divided by 3 that will give you T n 1 next consider4times this minus 1 times T n by 4divided by 3 that will give you T n by 21 after obtaining this consider 16 times this minus this term divided by 15 that is going to give you T n 2.

More general case will be T n 0 is equal to T n is composite trapezoidal rule defineT n m by this formula that T n m is equal to4raise to m T n m minus 1 minus T n by 2 m minus 1 upon4raise to m minus 1.

M is equal to 12 up to k. So, the final thing T n k is going to give you the error to beless than or equal to constant times h raise to 2 k plus 2 or integral a to b f x d x is equal to T n m plus term of the order of h raise to 2 m 2.

If m is equal to 0; that means, the composite trapezoidal rule that it is of the order of h square then m is equal to 1 order of h raiseto4and. So, on so thus if our function f is sufficiently differentiable then based on composite trapezoidal rule associated with different partitions we can construct a approximation to integral a to b f x d x with the error of the order of h raise to 2 k plus 2 where k is equal to 012 and. So, on

So, next time we are going to consider numerical differentiation thank you.