Quantitative Investment Management

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Lecture 09

Bond Pricing with Binomial Trees

In the last lecture, we discussed the binomial tree approach to ascribing a value of a bond, I described the methodology in the last lecture. Today, let us look at an example to clarify most of the things in relation to this binomial model.

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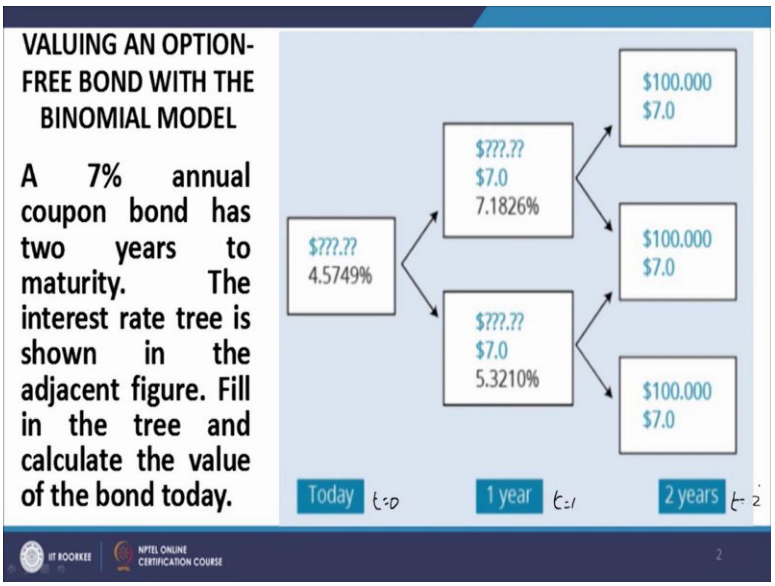

So, here is the example on your slide. A 7 percent annual coupon bond has 2 years to maturity, the coupon rate is 7 percent and the term to maturity is 2 years. The interest rate tree is shown in the adjacent figure you can see it on the right panel of your slide, you have to work out the value of the bond and fill in the appropriate blanks in the tree.

Now, before we proceed to solve this problem, let us look at and understand the binomial tree that is given to us. This is t equal to 0, this is t equal to 1 and this is t equal to 2. So, on a per user of this tree, what we find is that for a deposit or for a the spot rate that is or the forward rate starting at t equal to 0 that is obviously the spot rate.

For the period from t equal to 0 to t equal to 1 is 4.5749 percent then for the deposit from t equal to 1 to t equal to 2 from t equal to 1 to t equal to 2, the forward rate as assessed as estimated at t equal to 0 can take either of 2 values, what are those two values? 7.1829 percent or 5.3210 percent. So, this is the flexibility that is embedded in the binomial model, which is not there in the earlier model that we discussed.

We can put in or we can feed in our estimates of various values of this of the forward rates as well as the probabilities of their occurrence. Incidentally in this problem, and in the model that we have been discussing, the 2 rates that we have here are equally likely. So, the probability of the forward rate being 7.1826 percent or 5.3210 percent is 1 by 2 in each case.

Now, let us proceed with the valuation exercise for that purpose as discussed in when I explained the concept of backward induction, we must start from the right-hand side, our first objective is to calculate the cash flows at the right extreme time period or the maximum time period that we have here, which is 2 years because the life of the bond is 2 years.

So, we work out the cash flows at the end of 2 years. At the end of 2 years, the bond would mature for repayment, therefore, we will get back the principal of 100 and we will also get a coupon of 7 percent at the end of 2 years. So, the total cash flow at the end of 2 years will be 107 and this is shown here in this figure. Now, we look at the discounting part that is displayed in the next slide.

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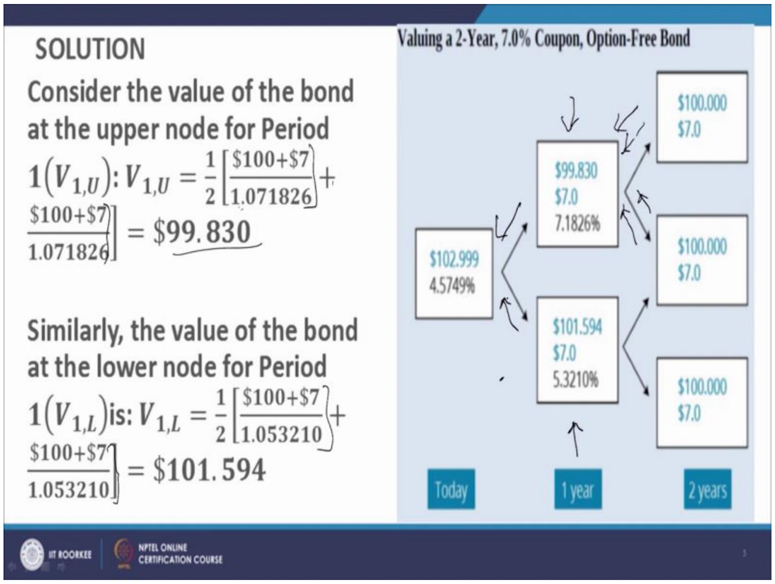

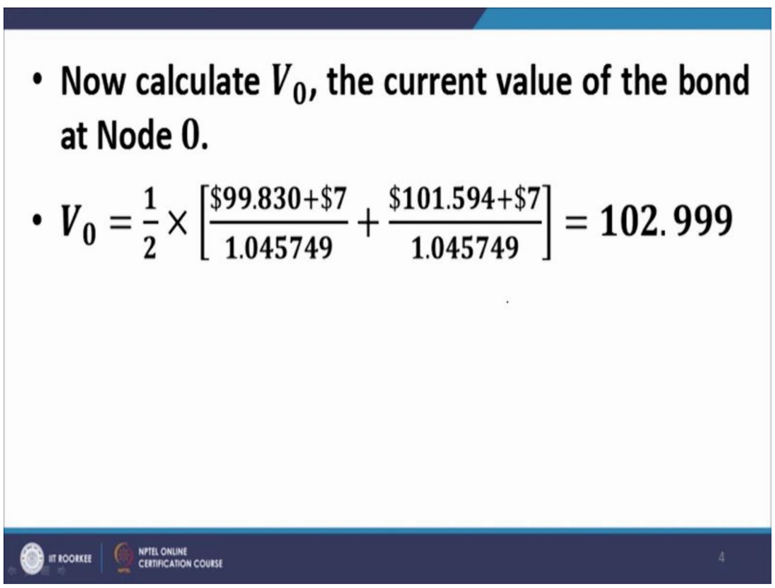

Here is the discounting part. So, just to recap the cash flow occurring at the at t equal to 2 in all the scenarios that is irrespective of what the discount rate is, what the interest rate is, will be 107 comprising of 100 of the principal repayment in 7 of the coupon payment. Now, what are the various possibilities? The 107 that is going to occur at t equal to 2 has to be discounted back to t equal to 1. For that, we use the forward rate at t equal to 1 which covers deposit, which covers the period t equal to 1 to t equal to 2 and we have 2 values for that rate.

So, corresponding to those 2 values, we have the tree that is shown on the right hand panel of your slide. Now, if we move along this path, that means what that means 107 is to be discounted at the rate of 7.1826 percent and that will give us a certain value that is this value 100 plus 7 divided by this value.

And then we can consider this path as well, for going back from t equal to 2 to t equal to 1 to this particular node, this node, this node can be reached from t equal to 2 to t equal to 1, the upper node can be reached in two ways. One way is along this path, and the second is in this particular path.

So, there are two paths by which I can reach from t equal to 2 to t equal to 1 upper node. And in the first case, we are discounting 107 at 1 at 7.1 at 7.1826 percent. And in the second case also, if you look at it carefully, that is an interesting feature of the tree I will come back to it, but in the second case also the discount rate is the same 7.1826 percent.

So, this is the first valuation and this is the second valuation, and because these are two paths leading to the upper node at t equal to 1, we get the value V1,U as the average of these two values, and that turns out to be 99.830. Similarly, the there are 2 paths for moving from t equal to 2 to t equal to 1 lower node, that is this particular node. In the first case, and the interest rate applicable is 5.3210 percent and in the second case also, the interest rate applicable is 5.3210 percent.

I mentioned this is a peculiarity of this particular problem. And as a result of which, we get these 2 values, which incidentally happen to be the same, and we take the average and the average works out to 101.594. Because these are the only 2 paths, we will be taking the average of only these 2 values.

So, what we get is the value at the upper node at t equal to 1 is 99.83, the value at the lower node of t equal to 1 is 101.594. Now, we need to move back from t equal to 1 to t equal to 0, the relevant rate is 4.5749 percent. So, discounting will be done at 4.5749 percent. And now, please note the two values or the two quantities that need to be discounted are different in contrast to what we had from when we move from t equal to 2 to t equal to 1.

In the case of t equal to 2 to t equal to 1, you find that all the three values that are there are 107. However, in this case, we have two different values at the upper node the value to be discounted is 99.83 plus, of course, the coupon payment of 7 and the, in the case of the lower node the value to be discounted as 101.594 plus again, the coupon rate of 7. The rate that is to be used in both cases turns out to be the same that is 4.5749 percent.

So, here is the working again, we take the average because we have two paths going from t equal to 1 to t equal to 0 we have this path, we have this path and we have this path. So, we have two paths going from t equal to 1 to t equal to 0 and therefore, we need to take the average of the 2 valuations corresponding to the 2 respective paths.

That is given in this slide and the discount rate incidentally, as I mentioned is the same for both the path and we ended up with the value 102.999 which is the value of the bond a t equal to 0 corresponding to the coupon rate of 7 percent, maturity of 2 years and the interest rates following the binomial tree that we saw in the right hand panel of the of this particular slide. So, that is how this problem is worked out.

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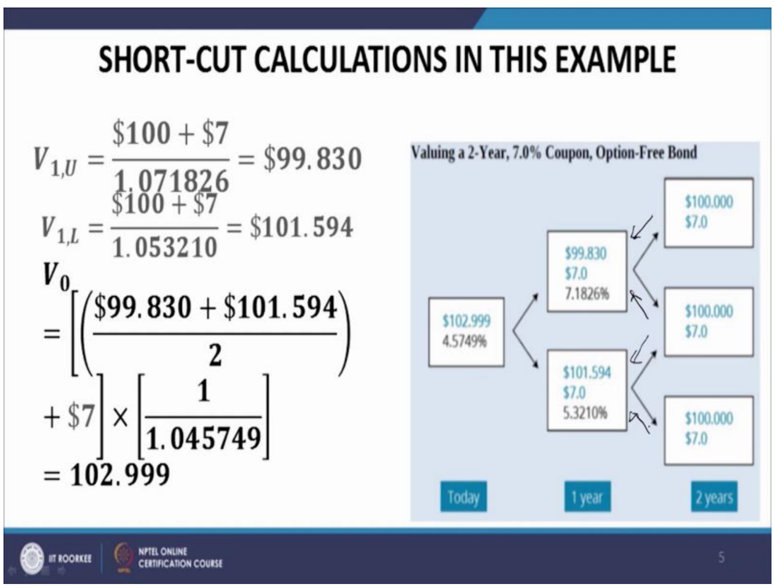

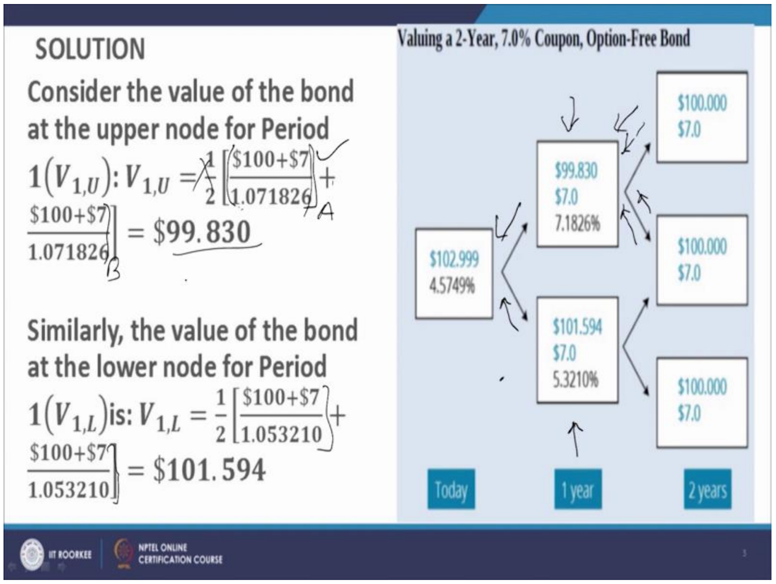

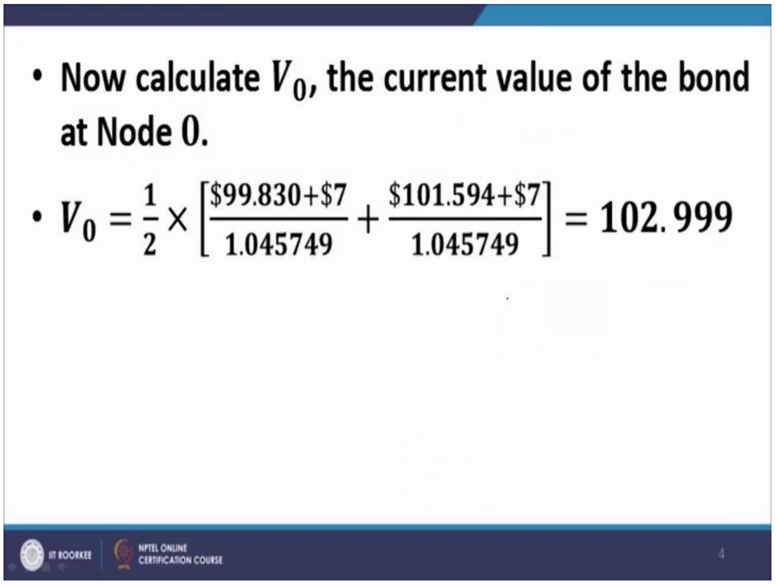

Now, interestingly, we can shortcut this problem in this particular case. If we can shortcut this problem a little bit. Why? Because we find that the 3 values that are occurring at t equal to 2 that are to be discounted are identical in all the thre cases. And the interest rates if you look at the intercepts carefully corresponding to this path and this path, it turns out to be the same.

And similarly, the interest rate corresponding to this path and this path also turns out to be the same and therefore, instead of taking averages, we can take one value as the appropriate valuation of the bond at the upper node and the lower node respectively, as shown in this diagram.

We can dispense with the 1 by 2 factor that is here, we can dispense with this 1 by 2 factor and use this factor alone this factor alone this particular factor, and we can dispense with this factor. Because, you can see here that, this is simply the same factor is repeated, this A and this B are identical.

And therefore, if you add A and B and then take the average you end up with the same quantity. That is a special feature of this tree, please note that is not universally applicable. In fact, you find that it does not apply when we move from t equal to 1 to t equal to 0. Why? Because in this case, the values that are to be discounted are different.

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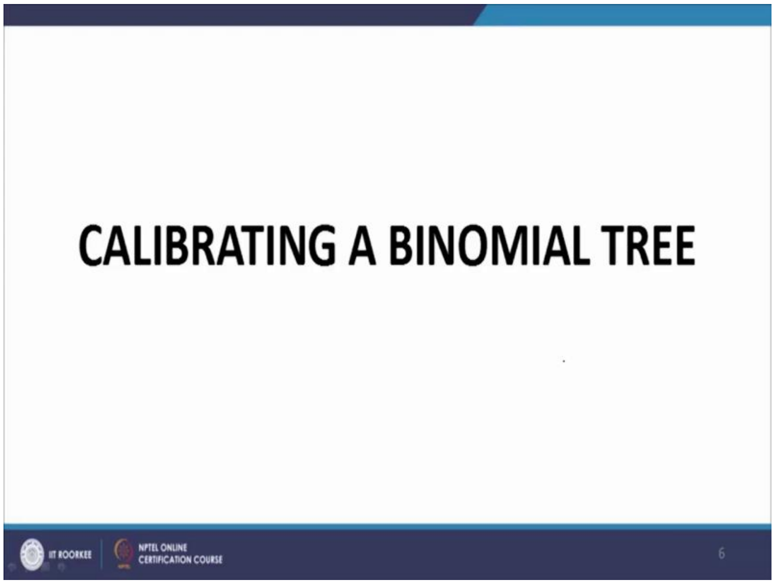

Calibration of a binomial tree:

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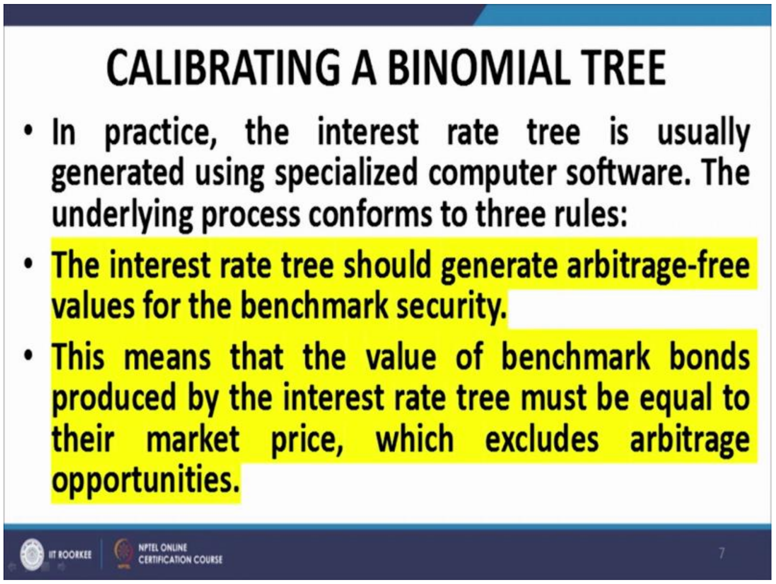

In practice, the interest rate tree is usually generated using specialized computer software, the underlying process conforms to 3 rules. This is very interesting, very important as well. The interest rate tree should generate arbitrage-free values for the benchmark security, whatever the benchmark securities are there, usually government bonds or government securities.

And the important thing is that if we try to work out the value of the benchmark security that is the government bonds, using the binomial tree, we should arrive at the same values as we arrived at when we use the arbitrage-free pricing model. So that is one test of our binomial tree, whether the binomial tree that we have constructed is satisfactory is in conformity with reality that the value that is generated by using the binomial tree for benchmark securities.

I emphasize points for benchmark securities corresponds to the value that we arrive at using the arbitrage-free pricing model for the same securities. This means that the value of benchmark bonds produced by the interest rate tree must be equal to their market price which excludes arbitrage opportunity. So, that is what I explained just now, let me repeat, this means that the value of the benchmark bonds produced by the interest rate tree must be equal to their market price which excludes arbitrage opportunities.

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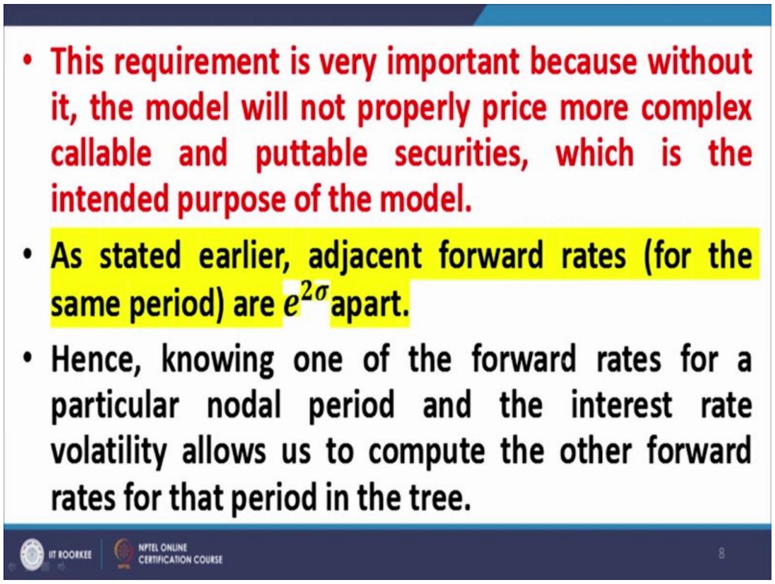

This requirement is very important. As I mentioned, just now this requirement is very important, because without it, the model will not properly price the more complex callable and puttable securities, which is the intended purpose of this model. In fact, this model has been devised primarily to ascribe appropriate valuations to securities which have embedded options in them like callable feature or puttable feature that we should talk about very soon.

As stated earlier, adjacent forward rates for the same period are e to the power 2 sigma apart. We have discussed this in the last lecture. So, adjacent forward rates are usually when we calibrate a tree, we usually assume usually assume again, it is not mandatory, it is a feature of this particular model that we are talking about that adjacent forward trees for the same period are e to the power 2 sigma apart.

Hence, knowing one of the forward rates for a particular nodal period and the interest rate volatility that is sigma, we are able to compute the other forward rates for that period in the tree, if we know any one single rate and we know sigma, because the adjacent points on the nodes at corresponding to the same time period are e to the power 2 sigma multiples of each other. We can devise the entire spectrum of forward rates and given one rate and given the value of sigma.

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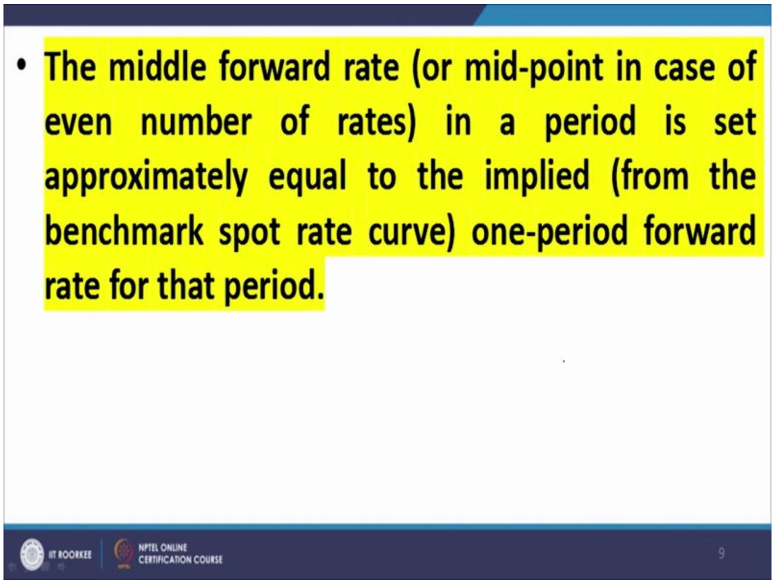

The middle forward rate or mid-point in the case of even number of rates in a period is set approximately equal to the that rate, which is implied from the benchmark spot rate curve, one period forward rate for that period. So, as far as the middle rate is concerned, we normally assumed that the middle rate corresponds to the rate that is obtained from the spot as spot rate curve or the spot rate curve of benchmark securities. And on that basis, whatever rate is implied by the spot rate curve, we use that as the middle rate or approximately the middle rate when we calibrate a binomial tree.

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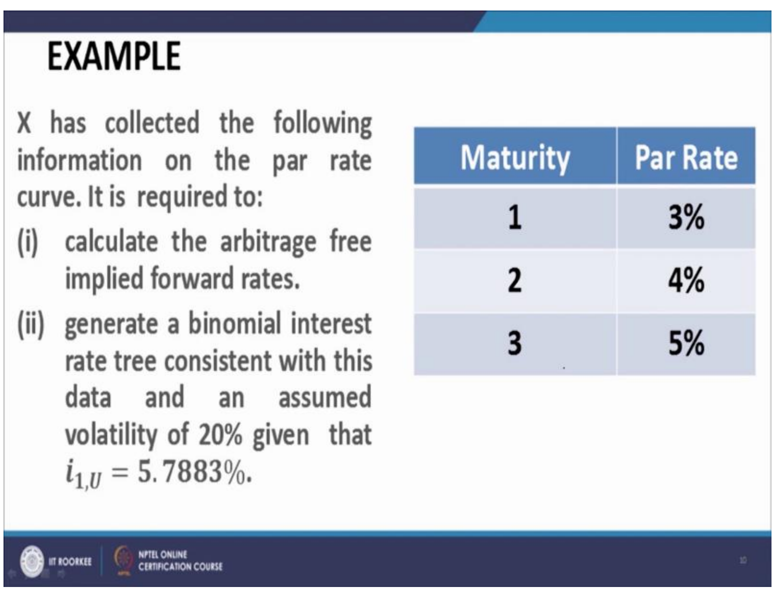

Let us, just do an example on the arbitrage calibration of the binomial tree. You are given the following. These are the set of par rates. Please I will come back to what we mean by par rates in a minute, but X is collected the following information on par rate curve, you are required to calculate the arbitrage free implied forward rates generate a binomial interest rate tree consistent with this data and an assumed volatility of 20 percent. Given that i1,U that is the upper node at t equal to 1, carries a rate of 5.7883 percent. What are the rates that are given to us par rates given to us? Maturity, 1 year it is 3 percent, maturity 2 years, it is 4 percent, maturity 3 years it is 5 percent

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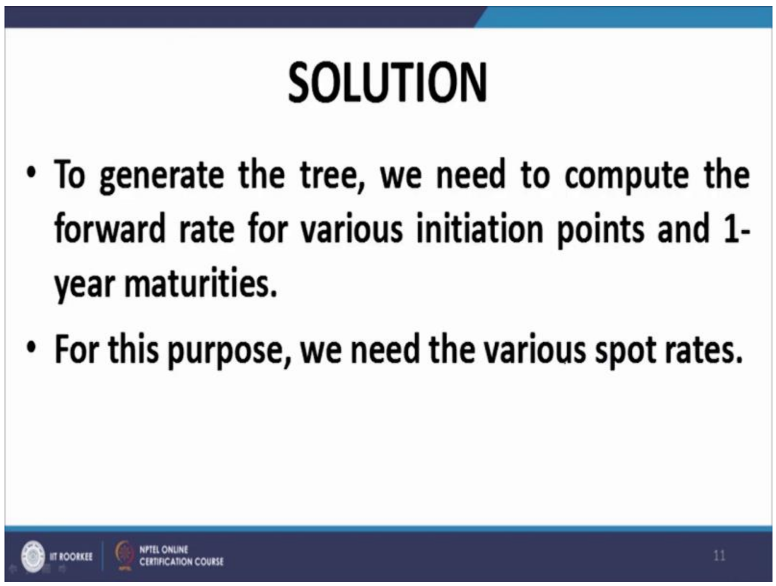

What we mean by par rates.

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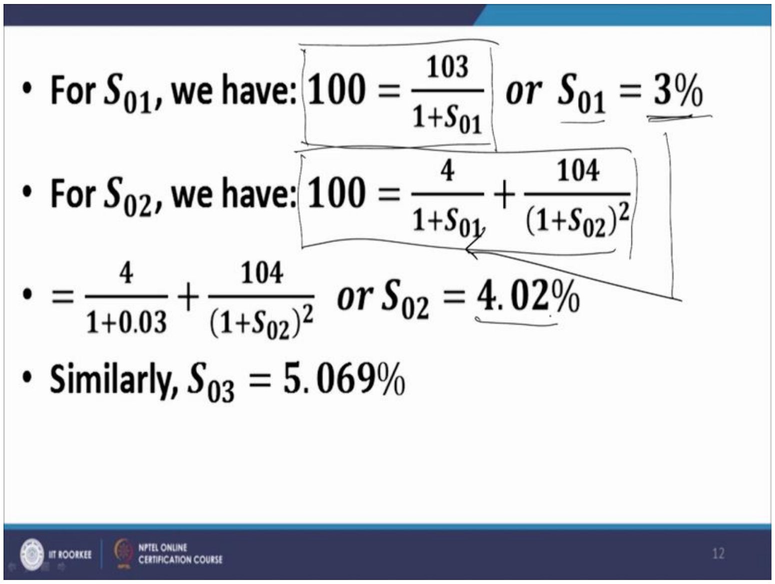

Let us look at this particular slide. When we talk about the par rate, what we are saying is that, if let us say the par rate is X, then if X is the coupon rate of a given bond to which this par rate is ascribed, then the bond should be quoting at par. Let me repeat what I said just know, if X is the par rate corresponding to a given bond, then if the coupon rate of that particular bond is X, X percent or X, then the bond should be quoting at par.

In other words, it is the coupon rate at which the given bond would quote at par. Let me repeat it is the coupon rate at which the given bond would quote at par. So, our in our example, what do we have? We have the par rate corresponding to 1 year is 3 percent. Therefore, if we have a bond with a 3 percent coupon, that should be trading at par that is that gives us this particular equation,

Because the cash flow that is going to occur at t equal to 1 because it is a 1 year bond. So, that cash flow that is going to occur at the end of 1 year is going to comprise of the principle of 100 we assume the principal to be 100 and the coupon payment of 3. So, the total coupon payment at t equal to 1 year is 103, we discounted at the appropriate spot rate and we arrive at the spot rate for 1 year, S01 is the spot rate for 1 year. It is not the par rate it is the spot rate.

Incidentally, it happens to be equal to the par rate in this case. But as you will see later, this need not necessarily be so. Now, let us work with the 2 year par rate. Again the 2 year par rate is given to us as 4 percent. Therefore, if we have a bond, a 2 year bond with a coupon rate of 4 percent, then that bond should be quoting at par with the given spectrum was spot rate. This gives us this particular equation.

You are going to get a coupon of 4 at because we assume that the coupon rate is equal to the par rate. So, we get a coupon of 4 at the end of 1 year t equal to 1 year, at the end of 2 years, we will get the redemption of the principle of 100 and a coupon payment of 4. So, that is 104. The first payment that is t equal to 1 year will be discounted at S0. Where S01 is the spot rate corresponding to the maturity of 1 year and the second payment of 104 comprising of principal and coupon would be discounted at S02.

Where S02 is the spot rate corresponding to a maturity of 2 years. This allows us when we solve this equation, what we find is that, using S01 is equal to 3 percent which is given here this is given here S01 is equal to 3 percent. We substitute this value here and we get S02. We get an equation for S02, one unknown and what we find is S02 is equal to 4.02 percent.

Similarly, we can find S03 and what we find is that S03 is equal to 5.069 percent. The equation would be 100 is equal to 5 divided by 1 plus S01 plus 5 divided by 1 plus S02 whole square plus 105 divided by 1 plus S03 whole cube. And S01 when we substitute equal to 3 percent, S02 we substitute equal to 4.02 percent, we get an expression for S03 which we can easily solve and we get S03 is equal to 5.069 percent. So, the par rate spectrum has been now translated to the spot rate spectrum. The spot rate for 1 year is 3 percent, spot rate for 2 years is 4.02 percent and the spot rate for 3 years is 5.069 percent.

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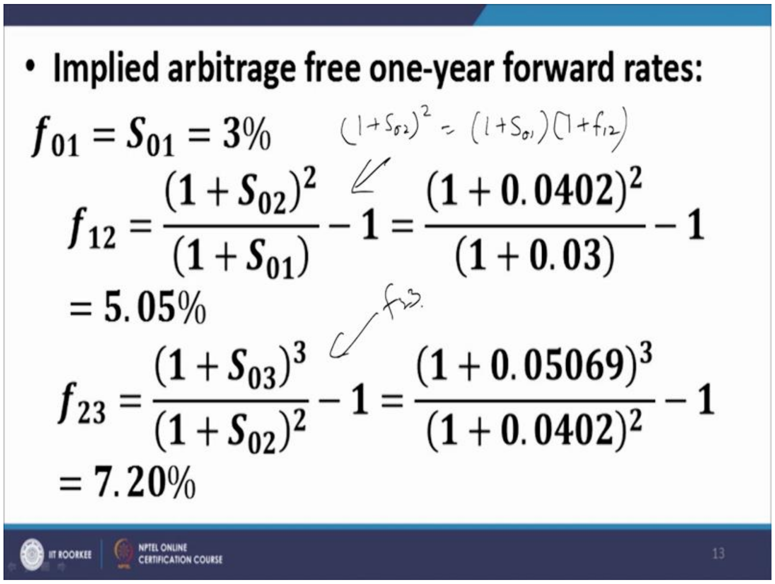

This enables us to arrive at the arbitrate-free forward rate spectrum. Obviously, the forward rate for S for the first period that is f01 is equal to S01 is equal to 3 percent. As far as f12 is concerned, that is the forward rate from t equal to 1 to t equal to 2, we use the principle of no arbitrage, what does the principle of no arbitrage tell us? It tells us that 1 plus S02 whole square is equal to 1 plus S01 into 1 plus f12 when we simplify this equation for f12, we get this expression.

And when we substitute the values S02 equal to 4.02 percent, S01 equal to 3 percent what we get is f12 is equal to 5.05 percent. And similarly, for f23, we can again invoke this arbitrage-free condition, and on the basis of this arbitrage-free condition, we arrive at this expression for f03 f23 I am sorry, and we substitute these values and we get f23 is equal to 7.20 percent. So, now, we have the spectrum of forward rates and we are equipped to start working on the binomial tree, calibrating the binomial tree.

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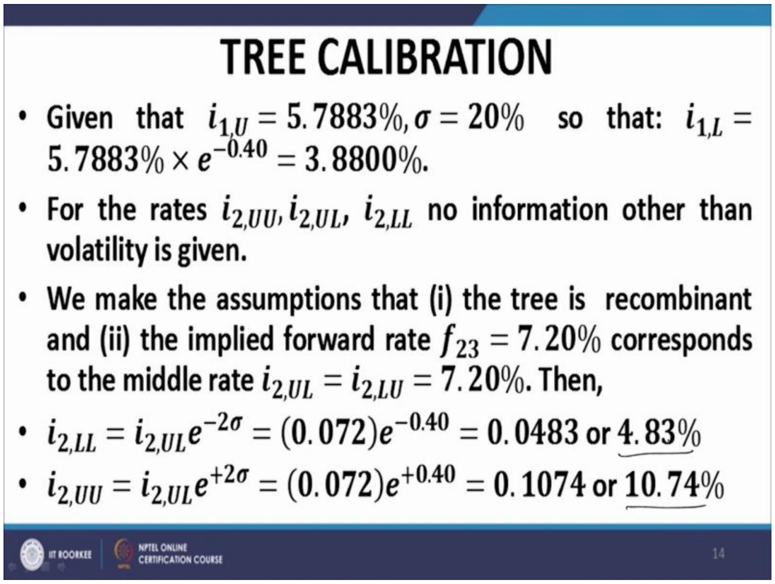

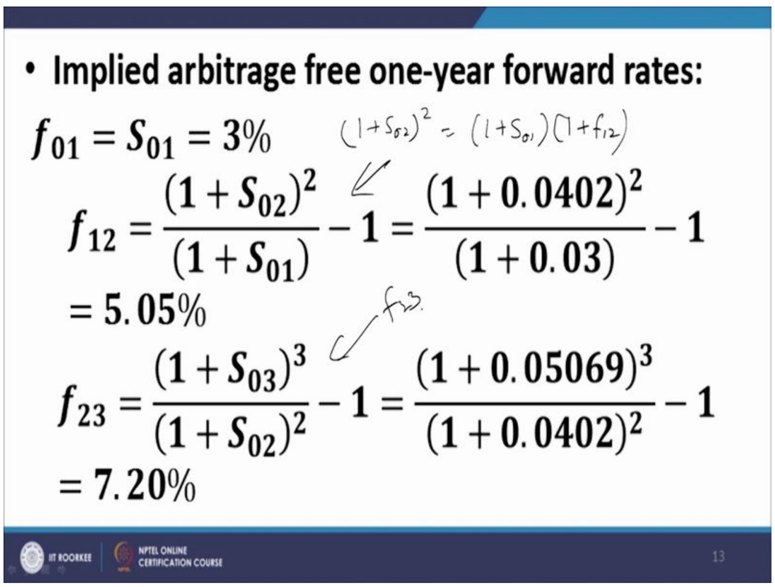

Here is the calibration we are given iU, i1,U is equal to 5.7833 percent. We are also given sigma is equal to 20 percent. So, i1,L is nothing but i1,U e to the power minus 2 sigma, sigma is equal to 20 person i1,U is equal to 5.7883 percent which is substitute the values you get i1,L equal to 3.88 percent.

This is quite simple trivial in fact, we are given the value i1,U, we are given the value of sigma and we know from the principle that applies to this particular model that i1,U and i1,L are multiples with a with a coefficient of e to the power 2 sigma, sigma is known to us therefore, we can find out i1,L that is nothing but i1,U e to the power minus 2 sigma substituting the value of sigma substituting the value of i1,U, what we get is i1,L is equal to e to the 3.88 percent .

Now, let us look at the calibration of the tree at t equal to 2. We have 3 rates that we have to look at, we have to look at i2,UU we have to look at i2,UL and we have to look at i2,LL. We have to find out three rates. Now, we are given one condition that usually the mid-rate or the middle rate is taken at approximately equal to the rate that we arrive at through the arbitrage free pricing or arbitrage free valuation.

That gives us i2,UL equal to i2,LU because this tree is a recombinant tree and therefore, i2,UL will be equal to i2,LU and both will be equal to 7.20 percent in line with our assumption that the middle rate corresponds to the rate that we arrived at through arbitrage free valuation, which we worked out earlier as 7.20 percent .

Here, let me show you where we done that, here is 7.20 percent. So, that is the arbitrage free rate, and we use it as the middle rate. For the calibration of the tree. Now, knowing the middle rate, it is very simple to go one step up, it will be a multiple of this middle rate with e to the power 2 sigma, sigma has given us 20 percent. So, we simply put the value of sigma and we arrived at 10.74 percent for the upper node at t uppermost node at t equal to 2.

And for the lowermost node at t equal to 2, what we have is, it is equal to I,LU e to the power minus 2 sigma I,LU we know 7.2 percent and e to the power minus 2 sigma, sigma is equal to 20 percent. So, that gives us the rate of 4.83 percent. So, this is how the entire tree can be calibrated.

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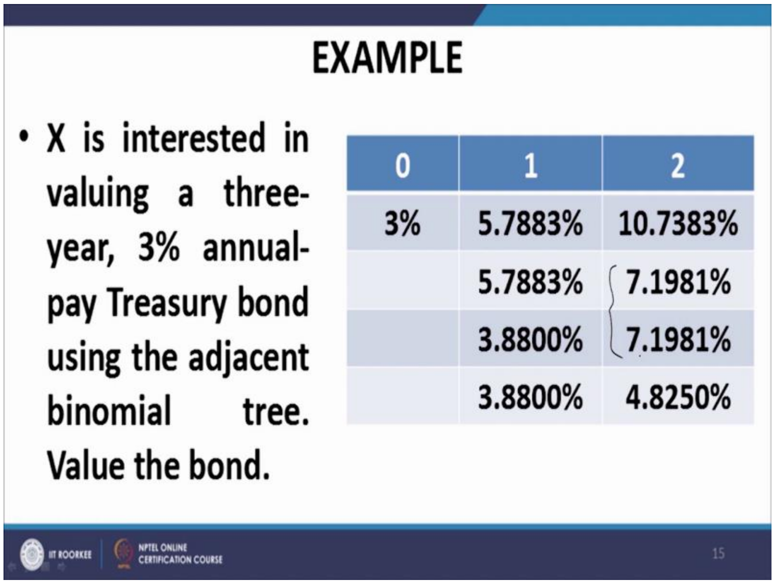

Now, what we do is, we do an example of a comprehensive example, on the valuation of a bond, which has a 3 year maturity, which has a 3 year annual pay treasury coupon, and it is a treasury bond. And we use the binomial tree which is given in this table on the right hand panel of your slide.

The S01 rate, if you look at it carefully, the S01 rate is equal to 3 percent. The rate from t equal to 1 to t equal to 2 can take either of 2 values that is the forward rate f12 can take either of 2 values 5.7883 percent or 3.8800 percent. And then at t equal to 2, again, the forward rate can take 2 values corresponding, please note it it is a recombinant tree.

So, the forward rate at t equal to 2 that is the rate from t equal to 2 to t equal to 3 that is f23. Now, f23 can take the values at 10.7383 from the upper node that is from the node 1U it can also go down to the value 7.1981 percent from the node 1U and from the node 1L, it can either go up to 7.1981 this is the recombinant part, and then it can also go down to 4.825 percent. So, that is the description of the tree.

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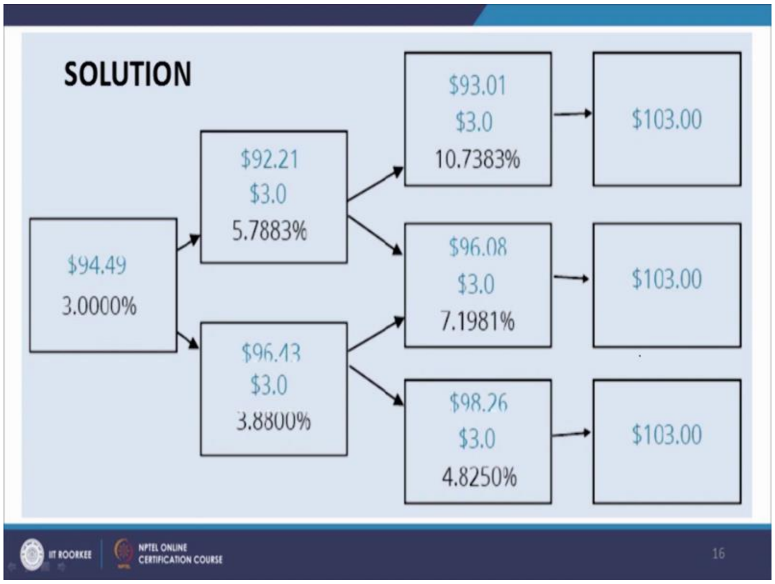

This is the this shows the tree on this particular slide shows the tree for t equal to 0 to t equal to 1, that is the spot rate at t equal to 0, we have 3 percent. For at t equal to 1 to t equal to 2 that is f12, we have two possible values. We have 5.7883 percent that is the forward rate for t equal to 1 to t equal to 2. And the other forward rate that may possibly occur with a probability of 50 percent is equal to 3.8800 percent.

So, at t equal to 1 the rates could either jump up from 3 percent to 5.788 percent or jump to 3.8800 percent. Each with probability 1 by 2, these are forward rates, which hold from t equal to 1 to t equal to 2, two possibilities, either both of them equally a likely. Now, at t equal to 2 again we have two possible values that the rates can take.

If it is at the upper node and two possible values that the forward f23 can take even if it is at the lower, if it is at the upper node, then it can take the values 10.738 percent or it can take the value 7.1981 percent and if it is at the lower node at t equal to 1, it can take the value 7.1981 percent or 4.825 percent.

So, this is the graphical description of the tree. Let me quickly recap t equal to 03 percent, this rate applies from 0 to 1 from 1 to 2, the rate set apply that is f12 can take 5.7833 percent or it can take 3.88 percent both with equal probability. And forward rate which will apply for t equal to 2 to t equal to 3, from t equal to 1 to t equal to 2 the interest rates could jump from 5.788 percent to 10.738.

This rate 10.738 percent will apply from t equal to 2 to t equal to 3 that is f23. So, f23 can take any of these three values 10.738 percent or 7.1981percent, from the upper node and from the lower node, it can take the values 7.1981 percent or 4.825 percent. Now, we proceed to the valuation. Thank you.