Quantitative Investment Management
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Lecture 08
Binomial Interest Rate Tree

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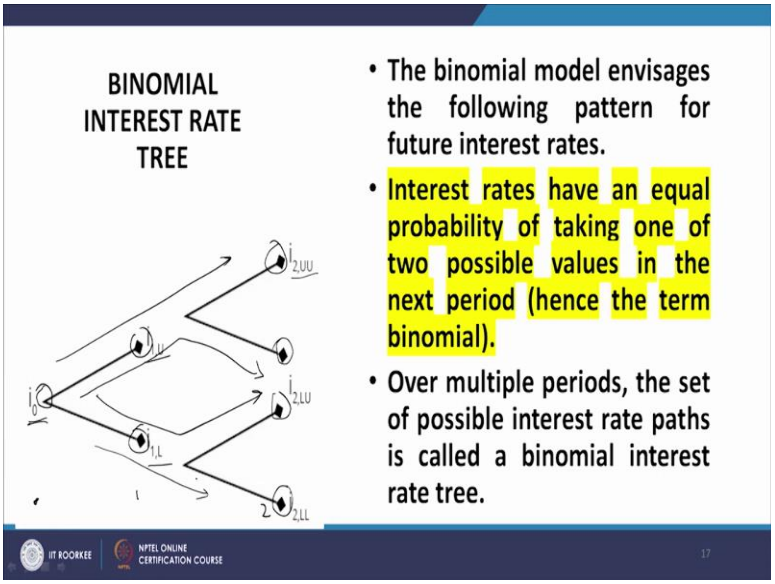

The template of the binomial interest rate tree is shown in the left hand panel of this particular slide. I will explain the tree first and then I will read out the adjoining nodes for your convenience. Let us, start with the extreme left hand point, the point from which the branches originate is called the node.

So, this is a node here, this is a node this is a node and similarly, these are nodes. Because they would again extend to if the tree extends further and these would be nodes from where further branches would originate. This is a 2 period binomial tree from t equal to 0 to t equal to 1 and then from t equal to 1 to t equal to 2, the interest rate that prevails as of now, that is the current spot rate is the rate that is represented at this point, and this would apply for the period from t equal to 0 to t equal to 1 year.

Now, from t equal to 1 year to t equal to 2 years, we assume that the rate that would prevail that is the forward rate. Please note this is the forward rate, because, we are sitting here at t equal to 0 and we are looking at the rate that would be prevailing at t equal to 1 year for the period from t equal to 1 to t equal to 2 years.

I repeat, we are sitting at t equal to 0 we are here we are at this point at this point, this is 1 and this is 2. So, we are sitting here at t equal to 0 and we are preparing this tree, this is tree is being prepared at t equal to 0. So, the rates that are relevant to us are the forward rates. So, what we assume here in this particular example of the binomial tree I repeat, I reiterate the word example. It is not a generalized tree, it is a typical example of the binomial tree.

So, what we assume here in this tree is that at t equal to 1 the forward rate that will prevail from t equal to 1 to t equal to 2 could be either i1,U or I1,L. So, there are two possible we are able to incorporate two possible values of the forward rate that we believe to represent borrowing or landing from t equal to 1 to t equal to 2 years.

So, in the case of the framework that we had earlier, this flexibility was not there, we had to choose a particular forward rate and we had to stick to that forward rate that there was only one possible valuation of the estimate of the forward rate that could go into our valuation. However, here we have the flexibility, this is a simplest possible tree where we have two forward rates.

To reiterate, this is a specific example, we can have 2, 3, 4, 5 a multitude of forward rates also that will introduce more complications into the tree. So, we will work with the simplest tree where at equal to 1 year, the forward rate that we believe to exist when we are sitting at t equal to 0 is could take either the value upper value that is i1,U 1 represents that it is the rate that will start at t equal to 1 or i1,D which is a lower value i1,L rather i1,L which represents the lower value that the forward rate could take.

So, we now have a flexibility to incorporate 2 rates in our analysis, one the upper estimate and the second the lower estimate. Again I repeat, this is not general we can have a multitude of forwards rates if you like. Here we are making another simplification, we assume that the probability of occurrence of i1,U and i1,L are both equal this both are 50 percent probable of occurrence at t equal to 1 year.

Then we move to t equal to 2 years, then t equal to 2 years we can have interest rates IU,U, i2,UU this is this path. If we move along this path we will get i2 because this rates starts at t equal to 2 years and this comes from the first part, which is 1U and then a second upper jump that is 2U. So, it is i2,UU and if it moves along this path that will give you i2UL first upper jump and second lower jump and if it moves along this path then it gives you i2LU and it moves along this path, it gives you i2,LL.

So, this is how this tree is formed, it is pretty parallel to what we had when we did the binomial valuation of (())(05:48) option in the course on financial and derivatives. Almost identical scenario. There of course, we do not assume any probabilities of the occurrence of stock prices. Here we assume that the up jump and the down jump of interest rates are equally likely.

But again that is an assumption that goes into the model, it needs not necessarily be so. So, let me after explaining this tree, let me read out the notes on this, the binomial model envisages the following pattern of future interest rates. Interest rates have an equal probability of taking one of two possible values in the next period, hence the term binomial.

Now, please note when I am talking about interest rates, I am talking about forward rates, the rate which is known at t equal to 0 or which is expected at t equal to 0, but which is likely to occur for deposits, or borrowing from t equal to 1 to t equal to 2 or t equal to 2 to t equal to 3 and so on. Over multiple periods, the set of possible interest rate paths is called a binomial interest rate tree as this diagram in the left panel represents.

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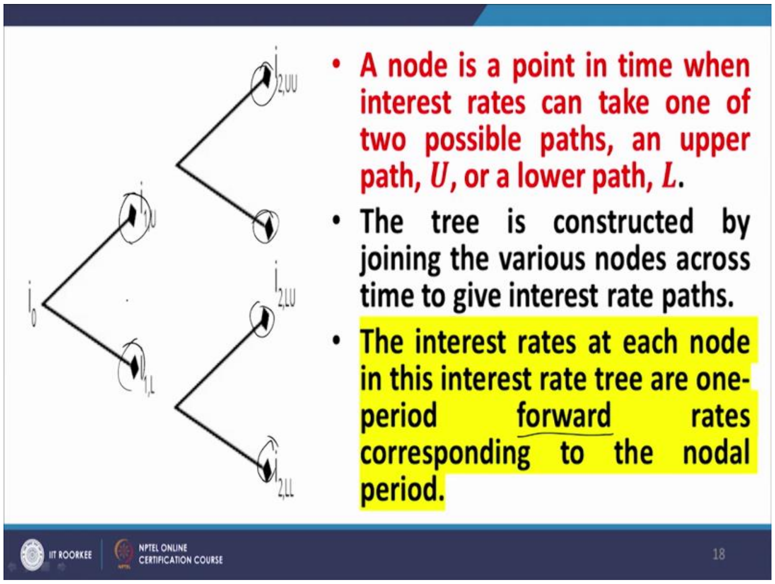

A node is a point in time, where interest rates can take one of two possible paths an upper path or a lower path. So, these are the possible nodes here, this is a node this is a node, this is a node, this is a node, all these are nodes here. The tree is constructed by joining the various nodes across time to give the interest rate paths.

Now, interest rates at each node in this interest rate tree are one period forward rates, this is important, this is Cardinal, these are not expected spot rates, they are forward rates corresponding to the nodal period. So, at this point say t equal to 1 and when you are at this node, the rate is i1,U or i1L. If the rate makes an up jump from i0, it will go to i1,U and if it makes a down jump from is i0, it will go to i1,L but both these rates are estimated to be 50 percent probable and both these rates are forward rates, that is the rates are determined at t equal to 0 at this point.

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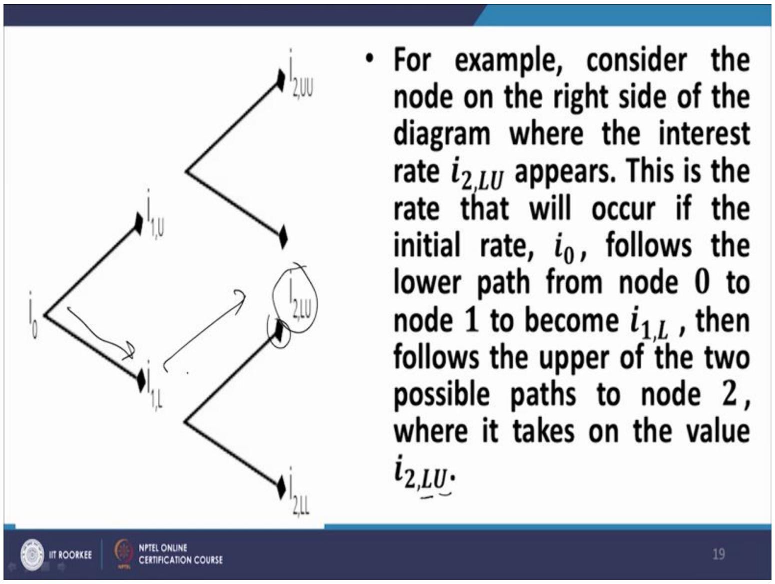

For example, consider the node on the right hand side of the diagram, where the interest rate i2,LU appears this particular rate. Obviously, this rate will occur if in the initial rate is i0, there is a down jump there is a down jump here and then there is an up jump here. So, this is the rate that is i2,LU is the rate that will occur if in through the initial state that is i0, there is a lower path from node 0 to low node 1 to become i1L.

Then and then it is followed by the upper of the two possible paths from node to node 2 from node 1 where it takes the value i2,LU. The first suffix L represents that it took the lower path from 0 to 1 and the second suffix U represent that from t equal to 1, to t equal to 2 took the upper path. So, it is this particular identification.

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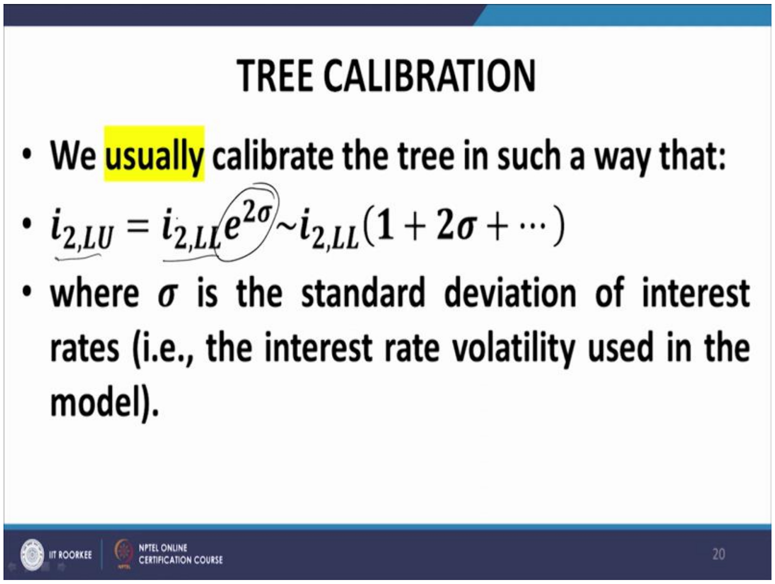

Now, tree calibration is very interesting. Again this is a specific example and we can have more generalized situations also the calibration can be done in more generalized manner. But for the purposes of our study for the level that we are going to be concerned with these are pretty much exemplary they form the (())(10:05) of most valuation frameworks.

So, we usually calibrate the tree in such a way that i2,LU is equal to i2,LL e to the power 2 sigma that is the any 2 corresponding to any point in time any 2 nodes, carry interest rates, which are 2 sigma apart or which are multiples of 2 sigma apart. So, i2,LU is equal to i2,LL e to the power 2 sigma.

So, this rate can be expressed in terms of this rate with e to the power 2 sigma and similarly, if you look at i2,UU it will be i2,LU e to the power 2 sigma and that will be equal to I,2LL e to the power 4 sigma. What is sigma? Sigma is the standard deviation of interest rates that is the interest rate volatility used for calibration of the tree.

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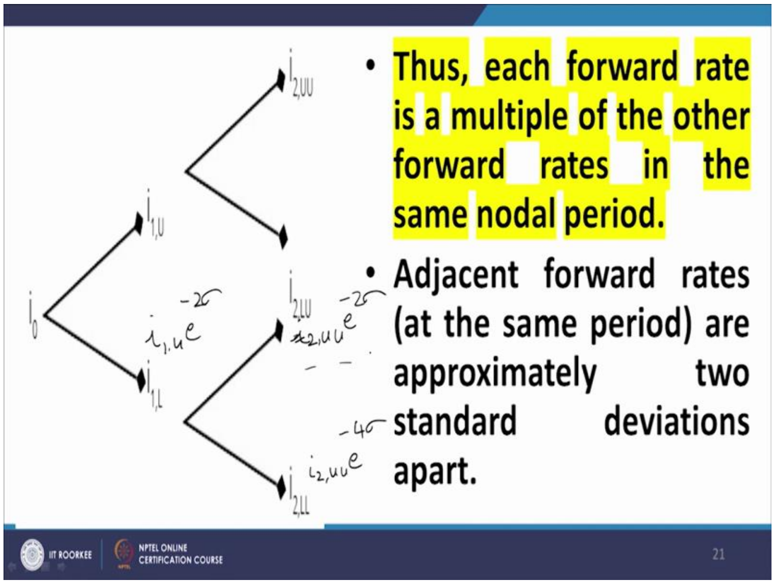

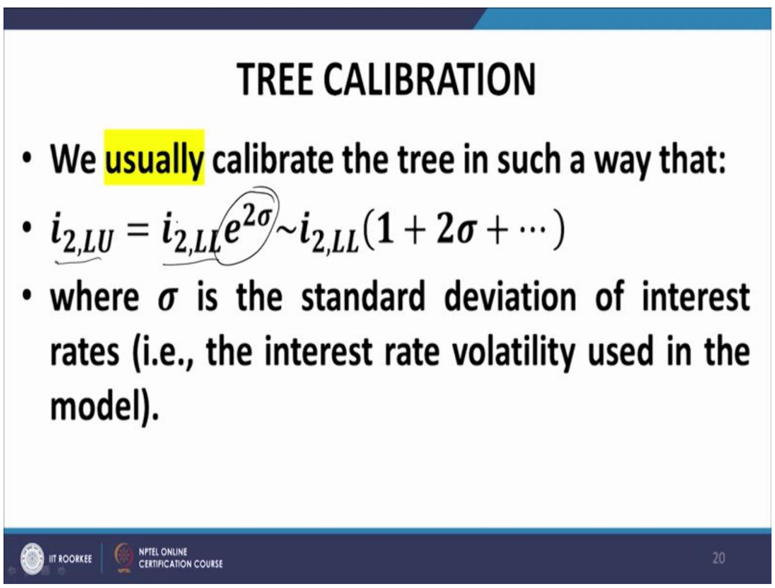

So, thus each forward rate is a multiple of the other forward rates in the same nodal period. So, as you can see here, i1,U will be equal to i1,L e to the power 2 sigma and similarly or conversely i1,L will be equal to i1,U e to the power minus 2 sigma and as far as the nodes t equal to 2 are concerned this would be equal to say in terms of this let us say if we work out in terms of this, this is i2,UU this will be equal to i2,UU e to the power minus 2 sigma and this will be equal to i2,UU e to the power minus 4 sigma and so, on.

Therefore, each forward rate is a multiple of the other forward rates in the same nodal period as you can see here, as you can see here as well, this will be equal to i1,U e to the power minus 2 sigma. Adjustment forward rates at the same period are approximately 2 standard deviations apart, as you can see approximately 2 standard deviations, because you can expand them this 1 plus 2 sigma. So, to the first lowest order, there would be 2 sigma apart, although more precisely, they would be multiples of e to the power 2 sigma, any 2 adjacent values.

As you can see here in this diagram also, any 2 adjacent values would be e to the power 2 sigma multiples of e to the power 2 sigma. Conversely, if you want to write I2,UU in terms of i2,LL, it will be i2,LL e to the power 4 sigma and this one the middle one will be i2,LL e to the power 2 sigma. So, they are multiples of 2 sigma adjacent values of the interest rates are multiples of 2 e to the power 2 sigma I am sorry.

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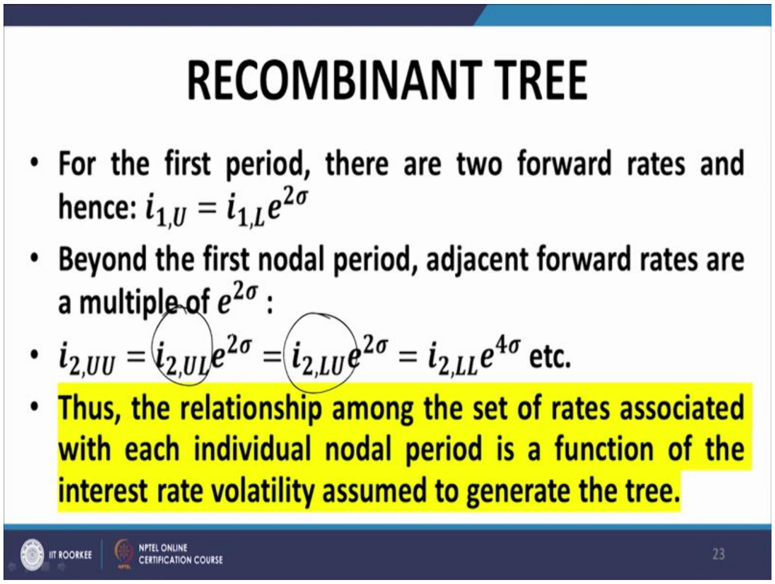

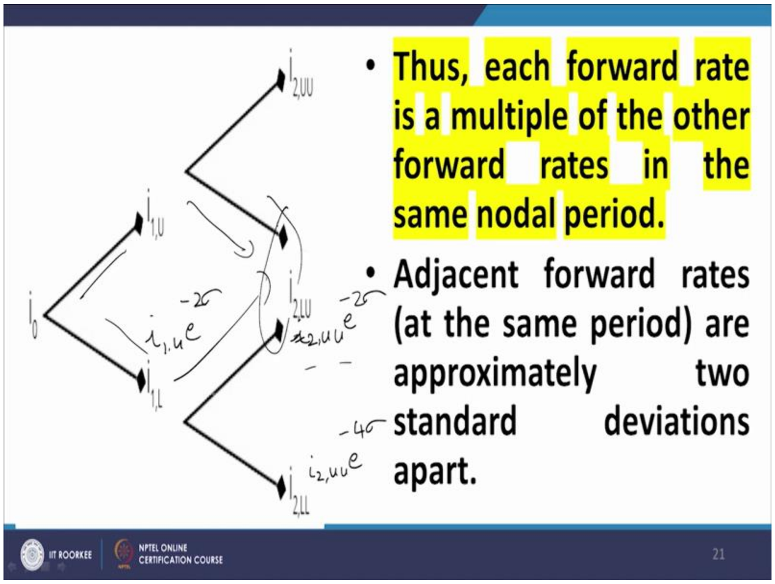

Recombinant tree, this is the most common model that is a adopted for the binomial interest rate tree. In this model an upward move the definition is very interesting very simple in this model an upward move followed by a downward move or a down upward move, this is the upward move and then we have a downward move that is we are arriving at this node.

And a downward move this downward move and then this is followed by an upward move that is this node. In the case of recombinant tree these 2 nodes coincide. In other words, what we have is iL,U, i2,LU for example, in this particular tree i2,LU is equal to i2,UL and this holds and this holds for further periods also, i3, i4, i5 and so on a future time periods as well. This is called a recombinant tree. Because there is a recombination of branches as you can see here. This branch here, this branch here and this branch here are combining again these 2 nodes are converging. So, that is what is called a recombinant tree.

A interestingly if you calibrate the binomial tree the interest rate tree in terms of what we did just now, what we discussed just now, that is this particular model, you end up with a recombinant tree, because as you can see here in this will be this if you move from here to here or if you move from here to here, you end up with the same value.

So, these two will converge. So recombinant tree in the for the first period there are two forwards rate and hence, I1,U is equal to i1L e to the power 2 sigma. Beyond the first nodal period adjusted forward rates are multiples of 2 sigma i2,UU is equal to i2,UL e to the power 2 sigma is i2, LU. So, this is equal to this e to the power 2 sigma is equal to i2,LL e to the power 4 sigma and so on. Obviously, you can reverse this relationship as well. Thus the relationship among the set of rates associated with each individual nodal period is a function of the interest rate volatility assumed to generate the tree.

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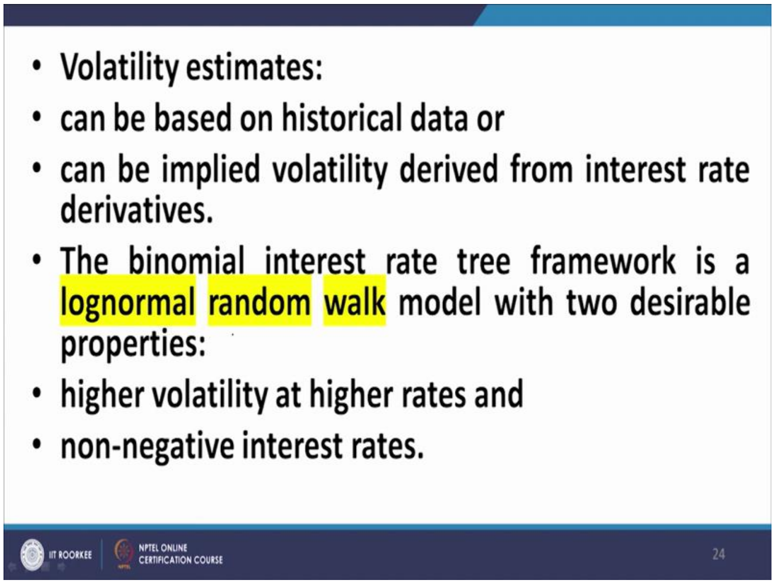

What about the volatility from where do we get this figure? Volatility estimates can be based on historical data or can be implied volatility derived from interest rate derivatives. So, volatility estimates can be based on historical data or can be implied volatility derived from interest rate derivatives.

The binomial interested tree framework is a log normal random walk, I discussed this, when I talked about the stock price model in the financial derivatives course, I shall talk a little bit about it again, when I talk about stock price modelling. For the moment, we can leave it as it is. And but, this particular model has two desirable properties, higher volatility at higher rates and non negative interest rates. In other words, we can factor out negative interest rates in this factor out negative interest rates in this model, we can confine ourselves to purely non-negative interest rates.

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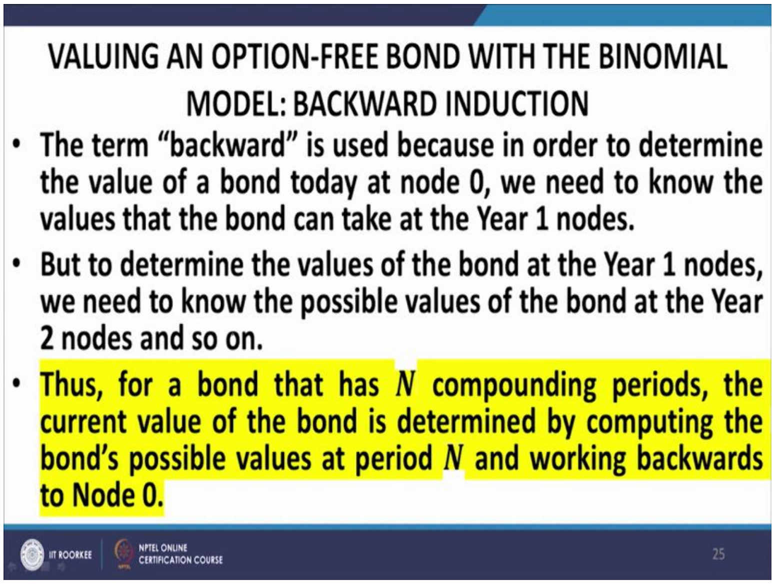

Now, valuing an option free bond with the binomial model, backward induction. You see, when we do the valuation, as which will see in example, very soon, in the next few minutes, the valuation process proceeds from the right to the left. First of all, we get the value of the bond at the right extreme nodes and then, using those values at the right extreme nodes, we move to the next layer from there to the earlier layer from there to the earlier layer, until we reach t equal to 0.

So, the evaluation precedes the process of backward induction, it starts from the right, it start from the highest time point and that is given in the binomial tree and from their own from the using the values at the highest time point, we move backwards towards the next earlier time point towards the next earlier time point, and so on until we reach t equal to 0.

Let me quickly read out the slide for you. The term backward is used because in order to determine the value of a bond today at node 0, we need to know the values that the bond can take at the year 1 nodes. But in order to determine the values of the bond at t equal to year 1 nodes, we need to know the possible values of the bond at year 2 nodes and so on.

Therefore, this is the cardinal point. Therefore, for a bond that has N compounding periods, the current value of the bond is determined by computing the bonds possible values at period N and then working backwards to node 0, from the values of the bond that it is going to take a t equal to

N we work out the values of the bond that it could possibly take at t equal to N minus 1 and then t equal to N minus 2 until we reach t equal to 0. Therefore it is called backward induction.

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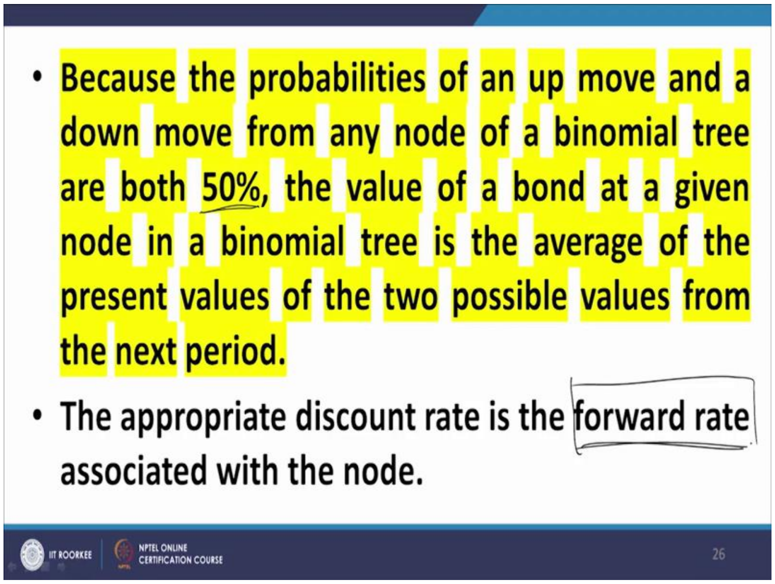

Because the probability is no this relates to the specific case that we are discussing in detail. That is where the up jump of the interest rate and the down jump of the interest rate. The up move and the down move are equally likely. The probability of both of them is 1 by 2. Because, the probabilities of an up move and a down move from any node of a binomial tree are both 50 percent.

Or this relates to the model that we are studying. It is not fully generalized. The value of a bond at a given node in a binomial tree is the average because both the tree have equal probabilities. So, naturally when we work out the expected value, it turns out to be the average of the present values of the 2 possible values from the next period. This will be clear from the following example. The appropriate discount rate with the forward rate this is important, associated with the node. Thank you.