Quantitative Investment Management
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Lecture 07
Forward Rates, Bond Pricing with Forward Rates

Forward Rates: In order to understand forward rates, what we mean by forward rates, it is necessary, it is opportune to go back and understand what is the forward contract. If you recall, a forward contract is a contract which is negotiated at t equal to 0, all the terms of the contract are negotiated at t equal to 0.

However, the actual execution of the contract the settlement under the contract is done at a future date that future date is also agreed to t equal to 0. In other words, all the terms that are related to the unambiguous settlement of the contract negotiated and agreed at t equal to 0 and then we have a settlement at a future date that is also pre specified, that is also agreed at t equal to 0, forward rates a similar.

Suppose you go to a bank and you want to make a deposit at the end of 1 year from now, but you do not want to be exposed to the interest rate risk during this intervening period t equal to 0 to t equal to 1 year, because the interest rates keep on fluctuating, the interest rates may change between t equal to 0 and t equal to 1 to your detriment. If you are making a deposit, let us say the interest rates go down.

And as a result of it, the possible income that you are going to derive from your investment which you will make a t equal to 1 will be lower than the rates that at which are prevailing at t equal to 0. So, what you do is you go to your banker and you say that, I will receive a certain amount of money from a transaction which has already been executed, for example, and I will make the money will be received at the end of 1 year from now. And when I received the money, I want to make a fixed deposit with your bank.

However, I want to be protected I want to be immunized against interest rate risk, during this intervening period, I want to be immunized against the change in the interest rates during this intervening period between t equal to 0 and t equal to 1 year when I will make the deposit then the banker may quote up certain rate, okay, I will give you this rate, let us say the rate is 6 percent per annum for the deposit that you make at the end of 1 year from now, for a period of 1 year.

This rate which you get for a deposit that is to be made in the future is called a forward rate. (())(02:59) tandem with the terminology that we use for a forward contract, the forward rate is the rate of interest that applies to forward loan, that applies to a loan, which is negotiated at t equal to 0 the terms of the loan including the rate of interest and the repayment and the amount of principal, all these conditions are agreed at t equal to 0.

However, the actual disbursement of the loan takes place at a future date. And the forward rate relates to the rate of interest that would apply from the date of disbursement to the rate of repayment, but this forward rate, I reiterate, this forward rate is also agreed at t equal to 0. So, this is what is the forward rate.

So, to repeat once more, a rate which relates to a loan either actual or conceptual at t equal to 0 agreed at t equal to 0 negotiated a t equal to 0 agreed to t equal to 0, but which operates for a loan that is going to take place in a future date say t equal to 1 year, 2 year whatever the case may be and for a certain maturity again, whatever the case may be.

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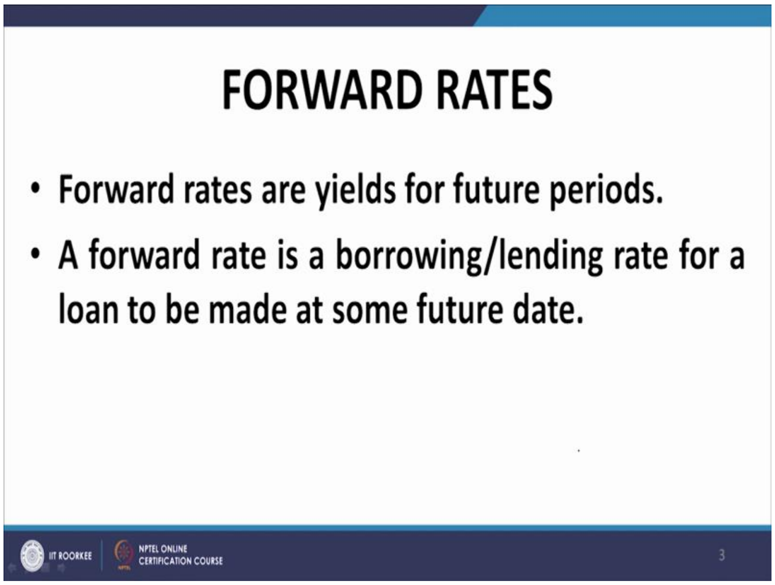

So, forward rates represent yield for future periods, forward rates represent useful future periods, but corresponding to loans that are agreed, negotiated and agreed at t equal to 0. A forward rate is a borrowing public lending rate for a loan to be made at some future rate. Just as I explained a few minutes back.

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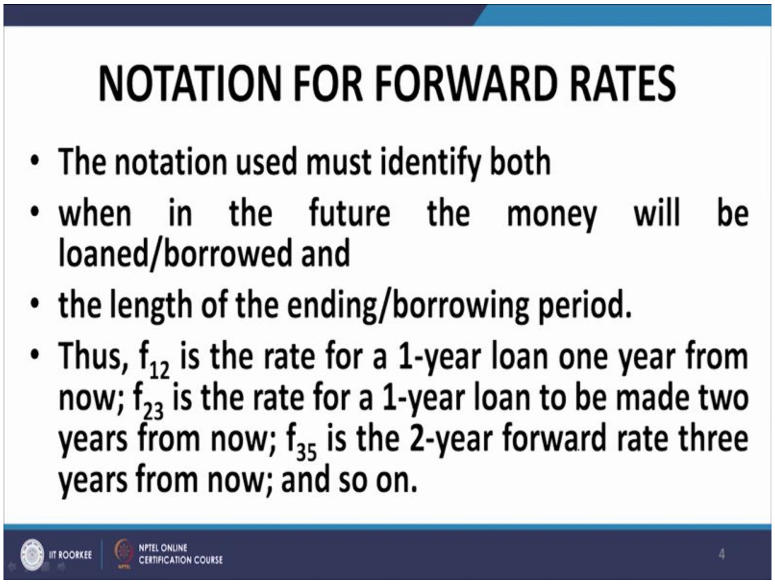

In the case of forward rate we have to be very careful about the notation that you have we are going to use we assume that the forward rate is agreed at t equal to 0. So, that part is kept away from the notation, imply the underlying assumption is that the agreement in relation to the forward rate is taken present at t equal to 0.

However, we should know at fourth point the loan is initialized and at fourth point the repayment is made. So, we need at least two indices, one representing the point in time at which the loan is dispersed or the loan is initialized or, and the second index representing the point in time at which the loan is to be repaid or recovered. So, the difference between the two indices will obviously represent the tenure of the loan.

One more thing I would like to emphasize in the event that we want to quote a rate or a rate is not agreed at t equal to 0, but at a later point in time, the point in time at which the rate is agreed, the rate is negotiated and agreed would require a third index for its identification. So, in essence, for example, if I use the symbol f12, it means the forward rate that relates to a loan that is initialized at t equal to 1 year, it is a 1 year maturity, and it will be repaid at t equal to 2 years.

Similarly, if I use the term f35, it represents the rate that would apply to a loan that would be initialized at t equal to 3 years from now and that would be of a maturity of 2 years, it would be repaid at t equal to 5 years. So, this is as far as the terminology goes as forward rates. As I mentioned, we normally assume that the rates, the forward rates are agreed at t equal to 0.

And therefore, we do not identify their point in time at which the rates are negotiated and agreed, we confine our identification to the point in time at which the loan is to be released, and the point in time at which the loan is to be repaid or return. Obviously, the difference will give you the tenure of the loan.

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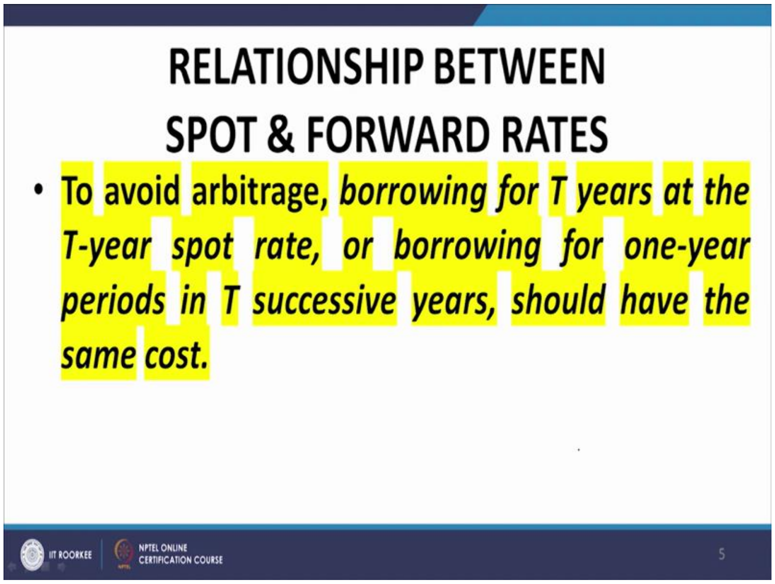

Now, we come to a very important relationship, the relationship between spot and forward rates. What are spot rates just to recall? Spot rates are the current market rates for deposits. If you go to the bank today and make a deposit today, the interest rate that you will get for the appropriate maturity that we are looking at is called the spot rate for that maturity.

But if you are negotiating the rate today, but the deposit is to be made at a future date, not today, then it turns out to be or it is called a forward rate. So, let us now look at arbitrage free relationship between spot rates and forward rates. Let me read out the relationship then we will talk about its appropriate derivation. To avoid arbitrage, borrowing for T years at the T-year spot rate, or borrowing for 1 year periods in T successive years, should have the same cost.

In other words, if you are making a loan, or if you are making a deposit for T years or let us say 5 years, at t equal to 0 at one stretch, if you get a rate, let us say that is S05. And the rate that you would get by rolling over that deposit for, let us say for t equal to 0 to t equal to 1 year, and then t equal to 1 year to t equal to 2 years and so on up to t equal to 5 years should give you the same cost that is the underlying no arbitrage or the arbitrage free condition that we need to show. Let us look at the justification or the rationale behind this no arbitrage or arbitrage free condition.

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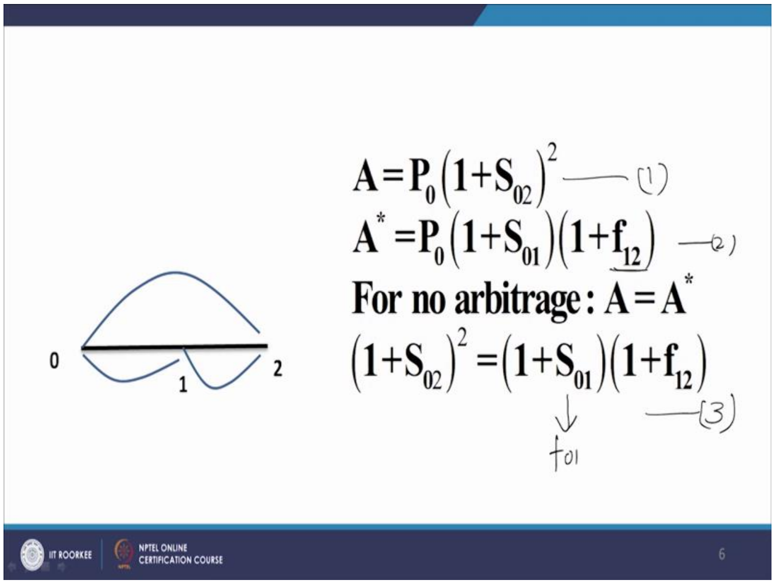

Let us assume that I make a deposit of P0 rupees in this diagram, it is is P0 rupees for a period of 2 years, I make it at the current spot rate for 2 years, I make a deposit at one go of 0 rupees for 2 years. And the rate let us assume that the bank gives me S0 to the spot rate, because I am making the deposit today. So, it will be the spot rate, the spot rate for a 2 year deposit, we call it S02.

And therefore, the amount that I would receive at maturity of this deposit would be equal to, would be given by this equation. Let me call it equation number 1. Now, suppose instead of making this one time deposit for 2 years, what I do is I make a deposit for 1 year, and then at the end of 1 year, I rollover the deposit for another 1 year. The question is, should I get the same amount or not?

The answer is very interesting. The answer is very intriguing. The answer is not necessarily at all. Why? There is a justification behind it. What did I say? I said that I make a deposit at t equal to 0 for 1 year that is from t equal to 0 to t equal to 1 year. And then at the end of the first year, I will take the rate that is prevailing in the market at that point in time, the rate that the bank is willing to give me at t equal to 1 year and then make a deposit for another 1 year from t equal to 1 to t equal to 2 years.

And the question that I asked was whether I should get the same amount as I have in equation number 1, the answer is not necessarily at all, why is that, that is because, in the second case, I am keeping a position open at t equal to 1 years. In other words, I am exposing myself to the possibility of the change in rate between what it is today for a deposit from t equal to 1 to t equal to 2 years to what it could be when I reach that point in time at t equal to 1 year and make a deposit of 1 year at that point in time.

Let me repeat, the rate that I am going to get for the second leg of my rollover deposit that is from t equal to 1 to t equal 2 years is not fixed, is not fixed at t equal to 0, it is only fixed at t equal to 1 year, therefore, I am exposed to additional risk in the second approach, when I make a rollover deposit. However in the first deposit where there is no rollover, the rate is crystallized at t equal to 0.

And that applies over the entire term of the deposit of t equal to 2 years. Therefore, because the risk content of the two legs is different in the second leg, or in the rollover leg, I am exposed to the possibility of the rate dropping or the rate increasing, in fact, because obviously, it could also increase, there is nothing to prevent rates from increasing as such. So, the rate could change that is the important thing.

And I would be exposed to that change, I may benefit out of it, I may undergo detriment out of it. But the important thing is I will be affected by that change. Therefore, in this case, the principle of no arbitrage or the arbitrage pricing will not hold. And there would not necessarily be any relationship between S02 and the rates that I get in the second leg or the rollover leg, or the rollover approach for making the same deposit.

Now, let me modify the condition a little bit, what I do is I settle the rate at t equal to 0 for the deposit that I am going to make a t equal to 1 or for the rollover deposit that I am going to make a t equal to 1 from t equal to 1 to t equal to 2 years. I repeat, I am not exposing myself in this new approach, I am not exposing myself to the fluctuation in rate or the actual rate that will prevail at t equal to 1, I fixed the rate at t equal to 0, just like I fix the forward price of a commodity, I fix the rate at t equal to 0, at t equal to 0, whatever I get at t equal to 1, I will make the deposit at this rate the bank agrees with me to give this rate for this deposit at t equal to 1 year.

In other words, what I am trying to say is I made the second leg of the deposit that is from t equal to 1 to t equal to 2 years at the forward rate f12. Now, what happens, now, obviously, I know the amount that I will get at maturity under both the legs. And assuming that obviously the conditions of default freeness prevail in so far as the lending and borrowing with banks are concerned, then the two legs will obviously be susceptible to arbitrage restrictions.

And if I do not get the same maturity at the end of 2 years, the arbitrage free pricing will operate or the arbitrage process will operate rather and people will try to extract the arbitrage profits out of the system. And as a result of which at maturity, what we will have is A is equal to A star where A star is this equation. Please note this very fundamental thing, very important thing.

We are considering a forward rate, a rate that is fixed at t equal to 0 so that the position at t equal to 1 of my deposit of my rollover leg is not open, it is not open, it is crystalline it is fixed at t equal to 0, it is negotiated and fixed at t equal to 0. And as a result of it, this leg that is leg represented by equation 2 has the same level of risk as equation 1, both of them are agreed upon at t equal to 0 and both of them the outcomes are under both of them are known with certainty.

So, that being the case, A must equal to A star. And that gives leads us to the relationship which is given in equation number 3. And this is a very, very important relationship. On the left hand side we have this portrait for 2 years, that is S02 and on the right hand side we have the spot and forward rates, obviously, S01 is equal to f01. So, I can also replace this S01 by f01 if I like, it does not really matter. But the bottom line is this relationship arises out of arbitrage free pricing. And this relationship is very, very important.

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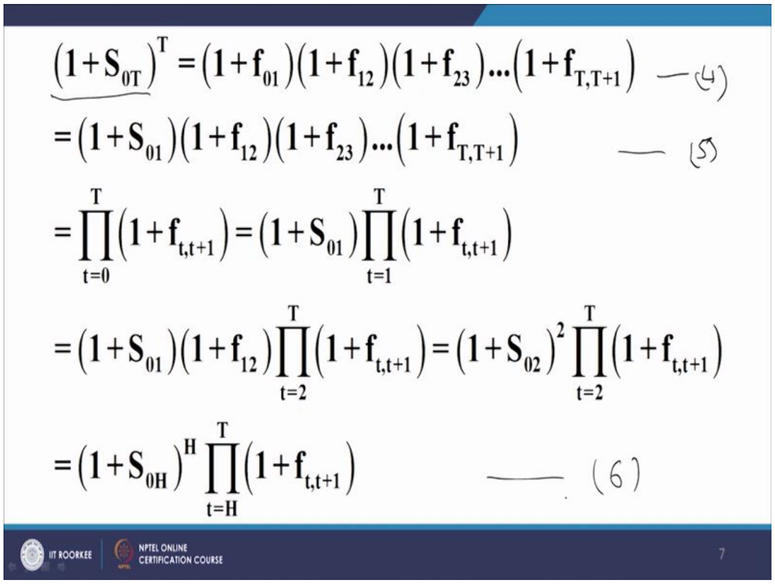

Now, we can generalize this relationship, I have talked about a deposit of 2 years, we can extend this deposit instead of 2 years, there is nothing sacrosanct about the deposit being of 2 years, it can be of 3 years, 5 years, 50 years or 100 years, whatever the case may be. So, let us generalize it to 2 years. And on that basis, I get equation number 4 here.

And equation number 4 can further be written in the form of equation number 5 by replacing f01 by S01, obviously, the forward rate between t equal to 0 and t equal to 1 is the spot rate. And therefore, we can replace f01 by f05. And this gives us equation number 5. And these equations can be further put in different forms, and we arrive at the various combinations on the right hand side for this expression that we have at the left hand side and you get this as equation number 6.

Please note these are very fundamental relationships between spot rates and forward rates, but the important thing is, they arise out of the conditions of arbitrage free pricing.

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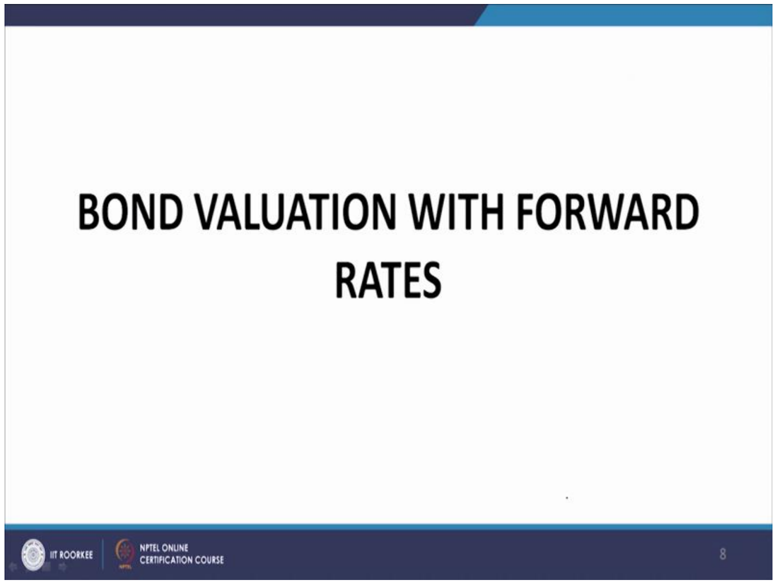

So, we shall use these relations to value the bond. We have already valued the bond using the spectrum of spot rates. We already know that the intrinsic value of a bond or the arbitrage free value of a bond is equal to the present value of all future cash flows emanating from the bond discounted at the appropriate risk adjusted and discount rates. But these, please note, these rates are spot rates, so, discounted at the appropriate risk adjusted spot rates that is important. So, now, we make use of this arbitrage relationship or no arbitrage relationship between spot rates and forward rates to express the same formula in terms of the forward rates, here is the expression.

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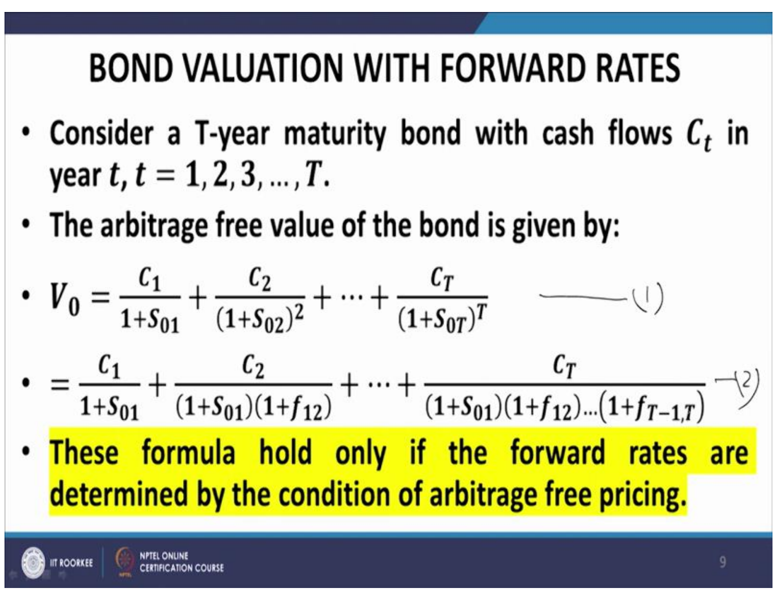

This expression relates to a T year maturity bond with cash flow Ct, capital T year maturity bond, the cash flows are C small t at the point in time small t, small t is any arbitrage point in time at which cash flows occur during the life of the bond, which is capital T years. Then from the arbitrage free pricing model, this is what we have this is equation number 1, this is brought forward from the earlier lecture on this particular topic V0 is equal to C1 upon 1 plus S01 plus C2 upon 1 plus S02 whole square and so on.

Using this condition that we have just not derived in the last few minutes about the relationship between the spot rates and the forward rates, we can write equation number 1 in the form of equation number 2, which these, please note, the equivalence between these two equations that is equation number 1 and equation number 2 emanates from the arbitrage free relationship between spot rates and forward rates of interest.

So, obviously, this would hold in an arbitrage free condition, arbitrage free market, where arbitrages are highly active or highly efficient markets, let us say, where arbitrages are extremely active, and any arbitrage opportunity is quickly siphoned away.

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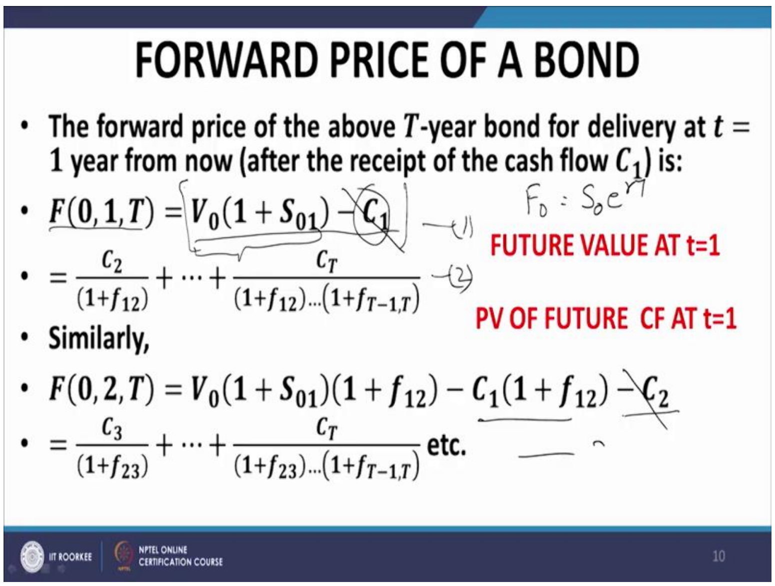

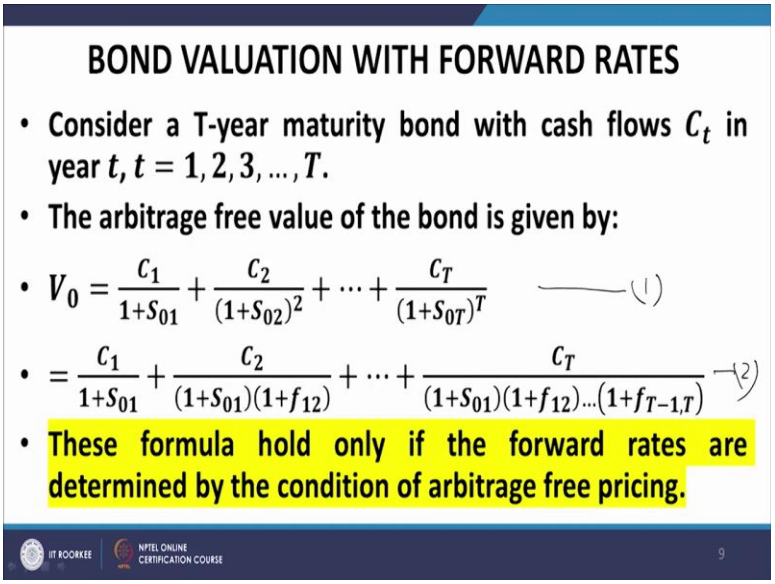

Now, we talk about the forward price of a bond, the forward price of underlying asset of a stock was discussed when I discuss certain examples about the process of arbitrage, how arbitrage free pricing takes place, and discuss the example of the forward price of a commodity or of an asset, which forming the underlying asset of a forward contract. The same rationale applies to the forward price of a bond that is the price of a bond at a future date.

But the price worked out as on today for the delivery of the asset at a future date, delivery of the bond at a future date. And what we have, it is simply an extension of that formula. If you recall, what was the formula for the forward price of an underlying asset it was given by F0 is equal to S0 e to the power rt. In other words, it is simply the future value of the spot price. The same rationale applies here.

And as you can see here, this is nothing but the future value of the spot price, V0 is the spot price, and V0 into 1 plus S01 is the future value of that price. Of course, if there is any cash payment during the life of the forward contract, then obviously the future value of that cash payment or that interest or dividend as the case may be, has to be deducted. And that represents this number.

So, the important thing please note here is that the price that I am working out here, the price that I am getting here is the forward price of the bond immediately after reiterated after emphasized after the coupon payment is made, if the price is to be worked out or the forward price is to be worked out immediately before the coupon payment is made, then this term would not appear, then the price would be given by only this expression.

And if the price is to be worked out after the coupon payment, then it would be this whole expression that is here, then the coupon obviously has to be deducted because the coupon has already been paid out. So, why is the rationale behind this is? Because if you are going to buy the bond after the coupon payment is made, then you are not going to get the coupon payment and the price has to be reduced by the amount of that coupon payment.

And this if we use the no arbitrage relationship between the spot and future prices or the forward rates, spot rates and forward rates and this equation, let me show you this particular equation, this is equation number 2 here, if we use this equation here, in equation number 1 on this slide, if I use equation number 2 of the previous slide, what I get is this expression, that is equation number 2 on this slide.

And please note, this shows what, this shows that the forward rate of a particular instrument of a particular bond is equal to the value at that point in time the present value at that point in time that is a T equal to 1 year of all future cash flows arising from the bond discounted at the appropriate forward rates.

I repeat, let me reiterate, the forward price of a bond or the forward value of a bond at a future date let us say t equal to 1 year is equal to the present value that is the value at t equal to 1 year, please note present value is not at t equal to 0 not as of today, the present value worked out at t equal to 1 year of all future cash flows that is C2, C3, C4, C5 and so on, entire tenure of the bond discounted at the appropriate forward rates in the manner that is shown by equation number 2 in this expression.

Similarly, the forward rate for a 2 year could be equal to the future value of the value of the bond at t equal to 0 minus the future value of the cash flows arising out of this bond up to the date of valuation. These are represented by these two figures. Please note again that this value of the bond is equal to the value of the bond after the second coupon, first and second coupons both are paid.

So, this is the post interest valuation of t equal to 2 years, if the second coupon is not paid, the evaluation has to be done before the second coupon is paid, then this term goes out of the reckoning and the value is given by the remaining expression. Again as you can see here, this equals the present value at t equal to 2 years of all future cash flows discounted at the appropriate forward rate in the manner that is given in equation number 3.

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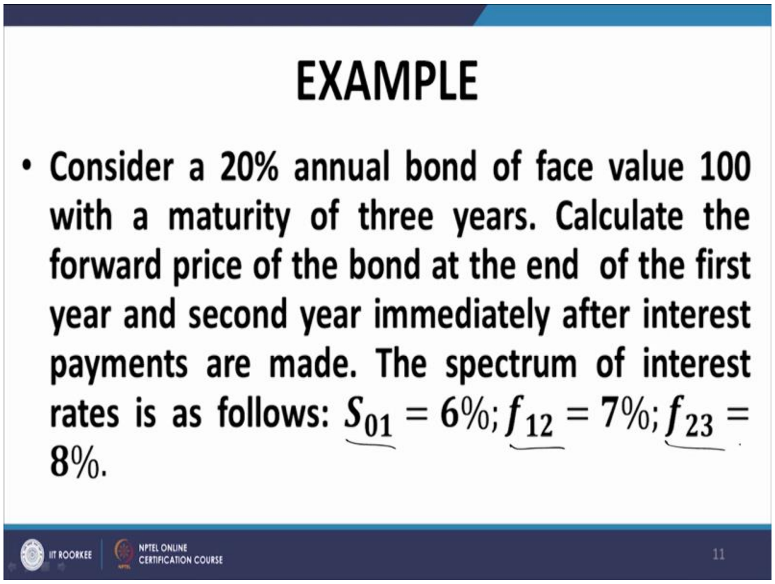

Let us do an example to explain whatever I have discussed so far. Consider a 20 percent annual bond, 20 percent is the coupon rate, the payments on the bond are annual and the face value of the bond is 100. Let me repeat, it is a 20 percent coupon bond, the face value is 100, the payment the frequency of payment of the coupon is annual, the remaining maturity of the bond is 3 years.

Calculate the forward price of the bond at the end of the first year, second year immediately after interest payments are made, the spectrum of interest rates are as follows S01 is equal to 6 percent, f12 is equal to 7 percent this is the forward rate, this is today's spot rate. And for the 1 year deposit and f23 this is the forward rate for a 1 year deposit to be made at the end of 2 years and that is equal to 8 percent.

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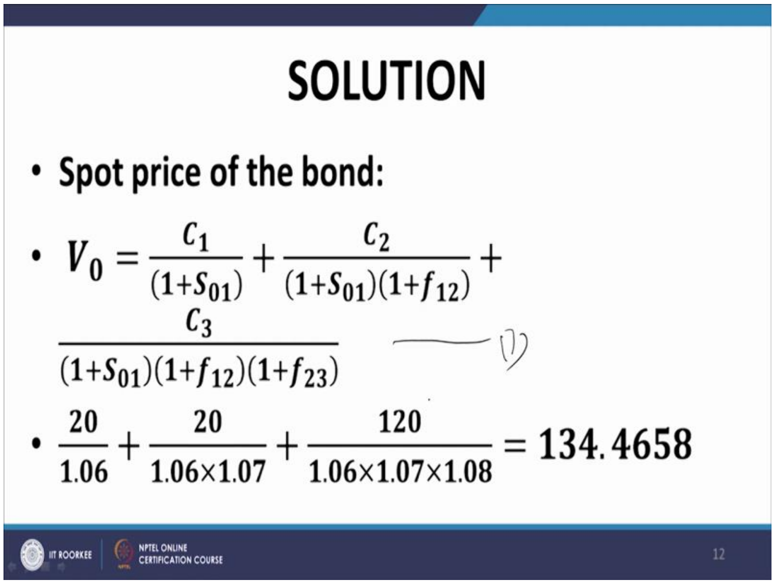

The current value of the bond or the current price of the bond V0 is equal to C1 upon 1 plus S01 plus this is given by this expression here, we have already done it earlier. So, this expression when you put the various values, we find that this is equal to 134.4658. This is the spot price of the bond or the value of the bond at t equal to 0.

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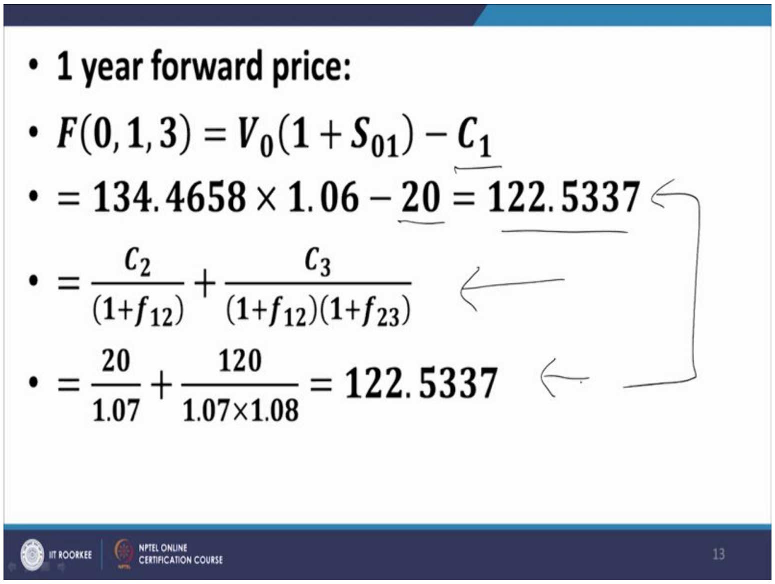

Now, for t equal to 1 what do we have? The forward price at t equal to 1 year, that is at the end of the first year. What is the forward price? Please understand this point, it is the price worked out at t equal to 0 for delivery of the bond at t equal to 1 year, actual delivery of the bond would be at t equal to 1 year after the first coupon is made.

Please note this expression means that it is after the first coupon is made. So, the future value of the spot price that was equal to 134.4658 into 1.06 minus 20, this 20 is the coupon at t equal to 1 year, 122.5337 is the answer. This is the 1 year forward price. And if you want to work out the 1 year forward price immediately before the coupon is paid, immediately before C1 is paid, it will obviously turn out to be 142.5337.

The amount of C1 will be added to this expression. And you can also work out the same forward price t equal to 1 year at t equal to 1 year by using this expression here, by using this expression. And again we arrive at the same result, this coincides with this. So, this is the example, which illustrates the forward price of a bond.

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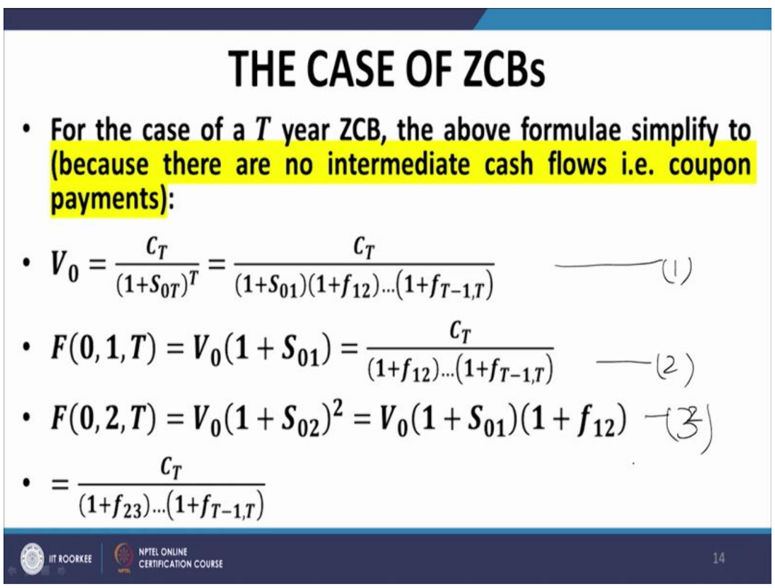

Now, in the case of zero coupon bonds, in the case of zero coupon bonds these formula simplify a lot. Why, because we do not have any intervening cash flows. What is a zero coupon bond? A zero coupon bond does not pay any coupons during his lifetime, there is an initial investment, and then the redemption of the investment value plus the return they are all whatsoever that interest figure maybe.

So, zero coupon bond is a bond, which has no coupon payments during the life of the bond, which is sold or which is issued at a certain value and redeemed at the value plus interest for the period of the life of the bond. And there is no intervening cash flow during the life of the bond, there is no coupon payment, I reiterate this fact. So, all the intermediate terms in the valuation formula disappear.

And we get this equation number 1, which is obviously much simpler than what we have in the earlier slide. But it is a simply a modification of that by putting the intervening cash payments equal to 0, this is equation number 2 represents the forward value at the end of 1 year, and equation number 3 represents the forward value at the end of 2 years, equation number 3 and so on. So, the expressions on the right hand side simplify, because there are no intervening coupon payments, they are 0.

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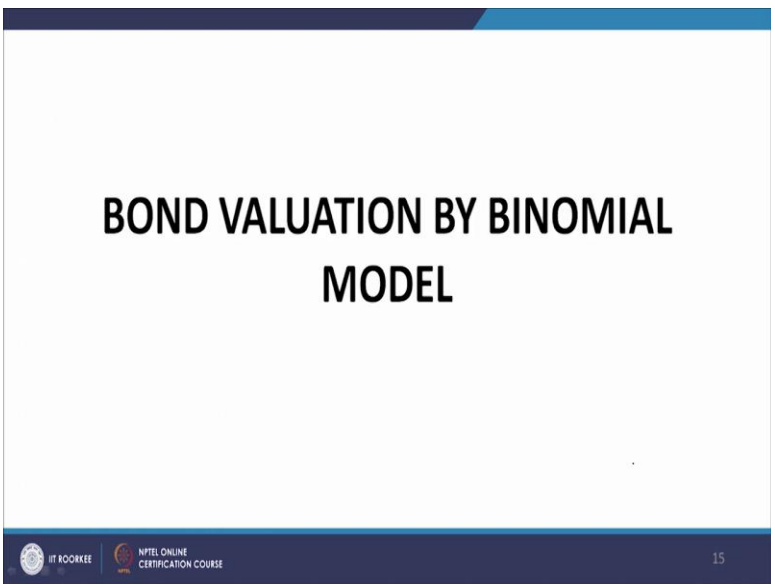

Now, bond valuation by the binomial model, this is a very interesting approach to the valuation of a bond. So, for the valuation of the bond that we have discussed is through using the arbitrage free pricing model. Now, we come to a slightly different model, which is called the binomial model. Why it is called the binomial model? We will come back to it. Why do we need this binomial model? There is a very interesting rationale behind it.

The rationale behind this is that in so far as simple bonds without embedded options are concerned plain or straight bonds are concerned, bonds which do not have any options attached to them. I will come back to this in detail as we proceed with this topic. But the basic thing is that in the case of bonds which are straight in nature, or which do not have any embedded options, the cash flows from the bond are pretty much fixed, because the neither the issuer of the bond nor the bond holder has any discretion to make early buying of the bond or early selling of the bond as the case maybe returning of the bond.

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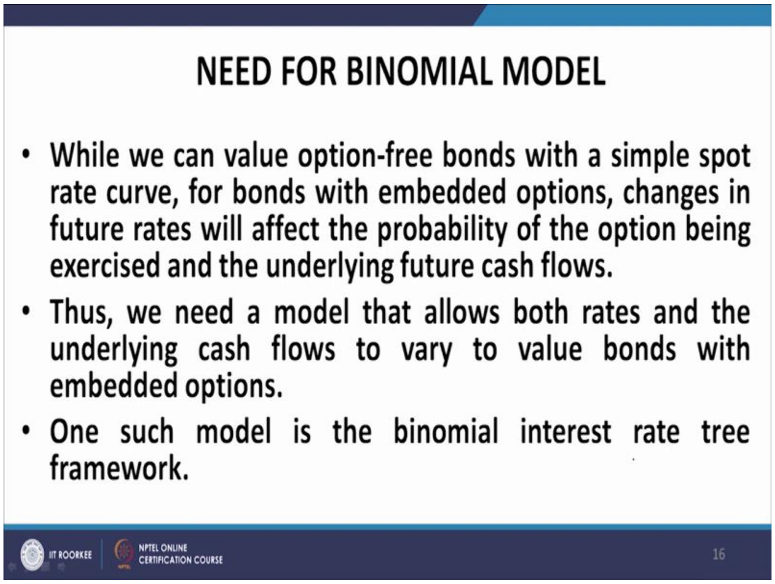

So, in that case, the cash flows are pretty much fixed. However, when we talk about bonds, which have options attached to them, for example, if a bond has a call option attached to it, that means what, that means the issuer of the bond has the right has the discretion and to buyback or to call back the bonds to recover the bonds from the bond holder by paying an appropriate sum of money at an appropriate point in time, which is specified in the issue document.

Similarly, in the case of bond, which has a put option attached to it, the seller of the, bond holder rather, the bond holder has the option to sell back the bond to the issuer of the instrument, if it is appropriate in terms of the issue contract issue document. Now, in this case, when we have such instruments, what happens is that the cash flows that arise during the life of the bond tend to vary, the cash flows during the life of the bond tend to vary.

And obviously, the maturity of the bond would also not be fixed, because in the case of a call option, the issuer of the instrument has the right to call back the bonds at an earlier date compared to its original, undescribed maturity. And in the case of a portable instrument, the bond holder has the right to sell back the bond to the issuer at an earlier date.

So, neither the cash flows from the instrument nor the maturity of the instrument remain fixed because of the excess of the. So, we need an approach which is relatively more flexible, which we can use for the valuation of this option bonds also, option embedded bonds also that is where the importance of this binomial model comes into play.

However, what we will do is, we will use the binomial model first to value state bonds that is bonds which do not have any embedded options. And then we will extend this model with through examples to consider cases where the model can be used for valuation of option embedded bonds. So, let me quickly read out this slide, while we can value option free bonds with a simple spot rate curve, like we have done earlier in the last few slides.

For bonds with embedded options, changes in future rates will affect the probability of the option being exercised and the underlying future cash flows. So, in the case of bonds, which have embedded options, any change in interest rates may induce the seller of the bond with the call option to excise the call option bond, or the buyer of the bond the bond holder to excise the bond if the interest rate change to his benefit.

Thus, we need a model that allows both rates and the underlying cash flows to vary to value bonds with embedded options. One such model is the binomial interest rate tree framework.

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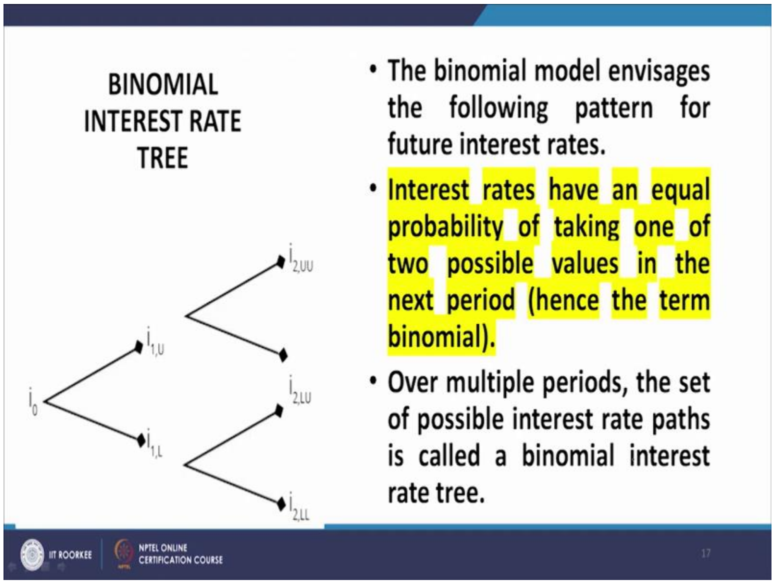

So, as we explain the binomial interest rate tree and then we use it in examples to value the option free bonds and then the bonds with embedded options. Thank you.