Quantitative Investment Management
Professor J.P. Singh
Department of Management Studies
Indian Institute of Technology, Roorkee
Lecture 50
Black Scholes Model

So, let us now look at an example to illustrate the two models that we have discussed prior to the break the Infinitesimal time model and also Finite time model.

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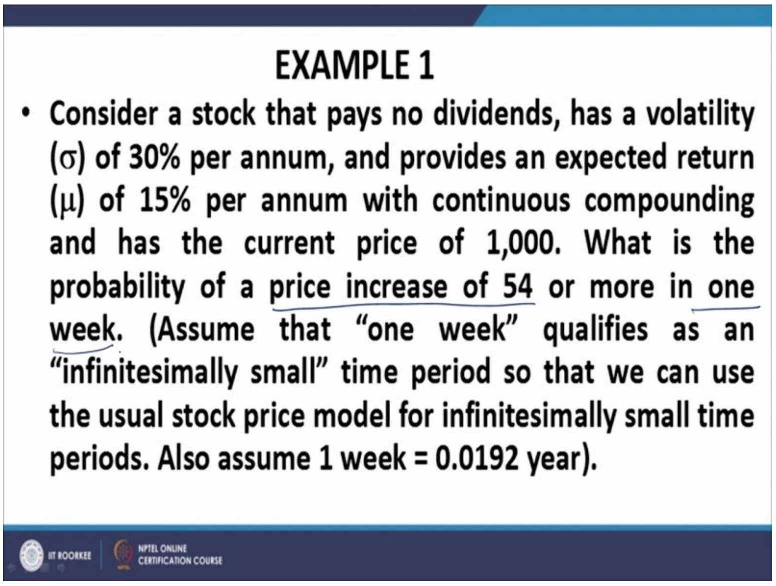

Let us start with the first one, consider a stock that pays no dividends as a volatility sigma of 30 percent per annum and provides an expected return mu of 15 percent per annum with continuous compounding, it has the current price of 1000. What is the probability of a price increase of 54, price increase of 54 or more in one week, this is very important to take, you need to take note of this fact that we are talking about a time of one week.

Assume that one week qualifies as an infinitesimally small time period. So, that we can use the usual stock price model for infinitesimal time period also assumes that one week is equal to 0.0192 of a year. So, here it is explicitly given that we need to use the infinitesimal model we did not use the finite time model.

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VOLATILITY	0.3	
EXPECTED RETURN	0.15	
CURRENT PRICE	1000	
TIME	0.0192	YEARS
IN THE INFINITESIMAL MODEL	$\mu S dt$	$\sigma^2 S^2 dt$
CHANGE IN STOCK PRICE (dS) IS NORMALLY DISTRIBUTED		
WITH MEAN μSdt	2.88	
SD σS√(dt)	41.56921938	
DESIRED PRICE INCREASE	54	
CORRESPONDING Z VALUE	1.229756073	
REQUIRED PROB	0.1093	

So, volatility is given at 30 percent, that is 0.30, expected return is given as 15 percent on continuous compounding basis, that is 0.15. The current price is given 1000, the time is given as one week that is equal to 0.0192 of an year. In the infinitesimal model, the change in the stock price dS is normally distributed with a mean is equal, with a mean equal to mu into S into dt, this is the mean and variance is equal to sigma squared, S square dt standard deviation is equal to sigma S under root dt where S is the current stock price, please note that fact.

So, the mean is equal to mu into S into dt which works out to 2.88 using the values that we have here, mu equal to 0.15, S is equal to 1000 dt is equal to 0.0192. And the standard deviation is equal to sigma S under root dt. Again, we use this value, respective values and we find that the standard deviation is equal to 41.5692. The desired price increase is 54 and therefore, the Z value that we get because please note these are normally distributed dS is normally distributed. So, we do calculations by calculating the Z value and then using the standard normal distribution tables.

So, we calculate the Z value. How do we calculate Z value? Sigma is given to us, this is sigma or standard deviation, this is the standard deviation, this is the mean and this is the increase in value. So, we can find out the Z. What is Z, Z is equal to x minus x bar upon sigma.

So, desired price increases 54, the mean of dS is equal to 2.88 and the standard deviation is 41.56, so 54 minus 2.88 divided by 41.56 is equal to 1.23 approximately, now, we can use the normal distribution tables and we find that the required probability is equal to 0.1093. So, that is how this problem is to be worked out.

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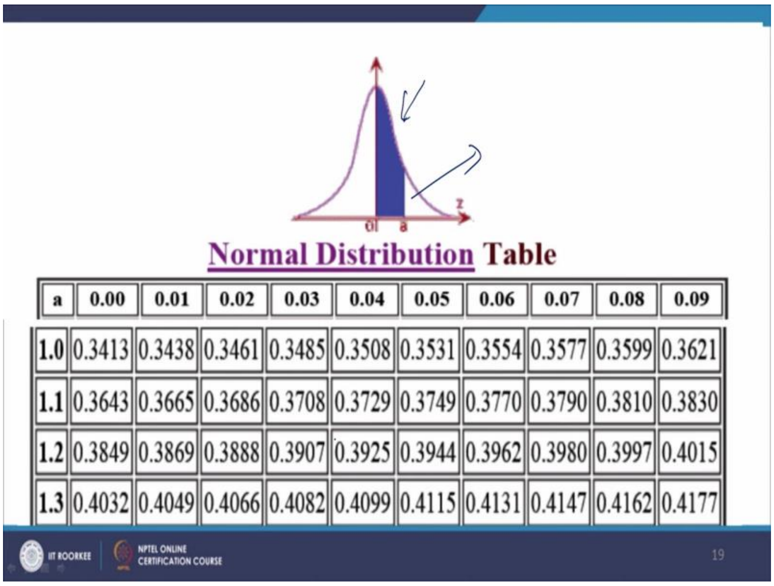

a	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177

This is the normal distribution table which you can use and you can see that 1.23, 1.23 corresponds to a value of 0.3907, but this is the value of the shaded area, what we are concerned with is this area and therefore, we subtract this expression that we have here 1.23 0.3907 is subtracted from 0.5 and we arrive at the required value 0.1093.

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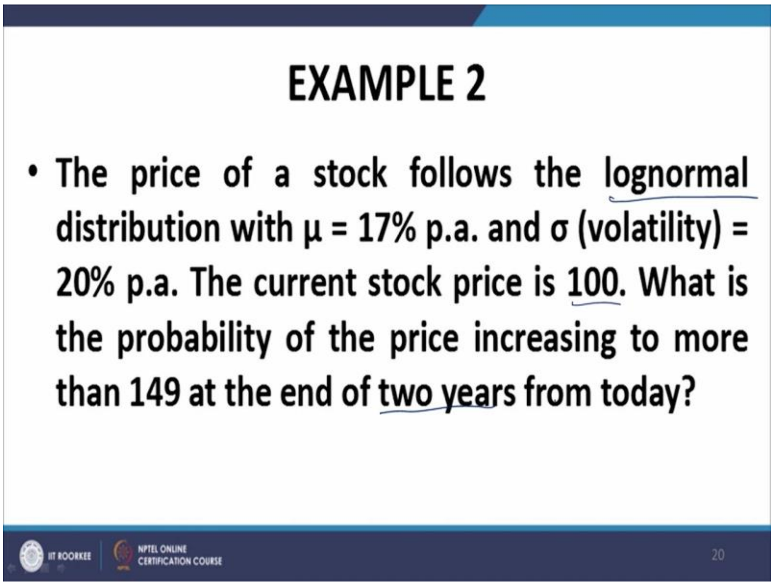

Now, let us do another example. The price of a stock follows the lognormal distribution that mu, log normal distribution this is again important you have to be very careful about understanding the problem, the language that is used in the problem. The price of a stock follows the lognormal distribution with a mu of 17 percent per annum and a volatility sigma of 20 percent per annum. The current stock price is 100. What is the probability of this price increasing to more than 149 at the end of two years for now, now this is two years.

So, obviously, there is no question of using the infinitesimal model, we have to use the log normal model. Please note even if the question did not specifically mention that they stock price follows the log normal model, we should have used the log normal model because the time that is given to us is two years, which is a substantially long period of time, and certainly by no means can be classified as infinitesimal.

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EXPECTED RETURN μ		0.1700
VOLATILITY σ		0.2000
CURRENT STOCK PRICE S(0)		100.0000
PROJECTED STOCK PRICE S(T)		149.0000
TIME HORIZON T		2.0000
EXPECTED VALUE OF LN S(T)	LN S(0) +[μ-0.5σ^2]T	4.9052
SD OF LN S(T)	σ√T	0.2828
LN S(T)	LN 149	5.0039
Z VALUE		0.3492
P(Z>0.3492)		0.3632

So, here the expected return is given as 17 percent, volatility is given as 20 percent, the current stock price S naught equal to 100 and the projected stock price is 149, we want to find the probability of the stock price going beyond 149. The time horizon is given as two years. So, the expected value of log ST is equal to log S 0 plus mu minus 1 by 2 sigma square T.

And when you substitute all these values, we get the expected value of log ST as 4.9057. And the standard deviation of log ST is equal to sigma root T, which works out to 0.2828. So, we have the mean value, we have the standard deviation, but we need to work out log of ST where ST is the target value, the target value in our case is 149.

So, log of 149 is given as 5.0039. So, using the three values, 1, 2, and 3, we can calculate the Z value, how do we calculate the Z value? It is equal to 5.0039 minus 4.9052 divided by 0.2828 and that turns out to be 0.3492, then we will use the normal distribution tables to find out the value of the standard normal variate exceeding the value of 0.3492 and we find out this probability to be 0.3632.

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ANALYSIS OF STOCK PRICE MODELS	
BASIC DATA	
INSTANTANEOUS RETURN	0.15
STD DEVIATION PER ANNUM (Vol)	0.3
CURRENT STOCK PRICE	1000
EXPECTED STOCK PRICE	1054
TIME FRAME	0.0192308

Now, we look at the difference between the two models. This is something which is interesting, how much inaccuracy percolates into our analysis, when we use the infinitesimal model in lieu of the finite time model. So, I have worked it out in two situations where firstly where the time is equal to one week, and secondly, where the time is equal to one year.

So, I have used this data which is here on the slide, instantaneous return is 0.15, standard deviation is equal to 0.3 that is 30 percent per annum, and that is volatility, current stock price is 1000 and expected stock price is 1054, the timeframe is one week for the first illustration.

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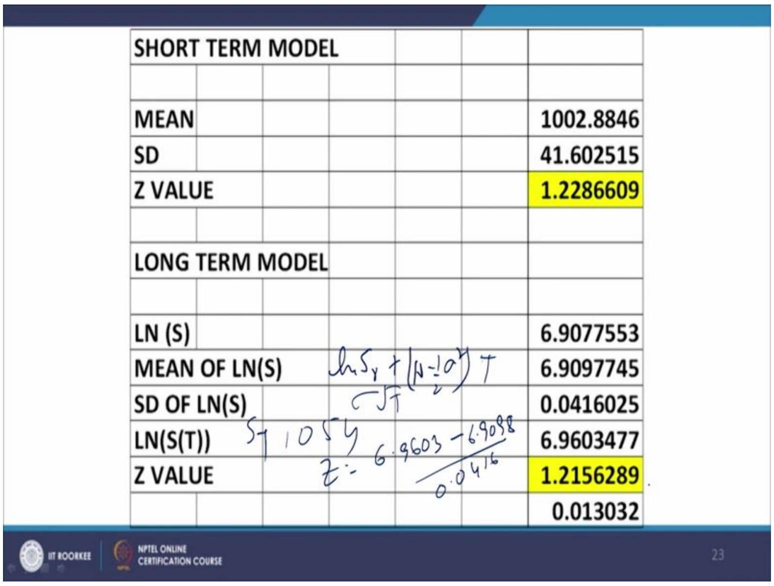

Using this data we have the mean as 1002.8846. Please note the mean is given by mu into S into dt and the standard deviation is equal to 41.6, we have worked it out earlier, sigma into S into under root dt and that gives us the Z value corresponding to our chosen price of 1054 as 1.2286.

Now, let us use the finite time model for doing the same analysis, the log of S is given as 6.9077 where s is equal to the stock price is 1000. So, log of 1000 is 06.9077. Mean of log S is equal to the user expression log of S naught plus mu minus 1 by 2 sigma square into T where T is equal to one week, mu and sigma are already given to you, log of S naught is equal to 6.90.

So, mean of log of S is equal to 6.9097. Standard deviation of log S is equal to sigma under root dt in our case, you can write it as T as well, that is equal to 0.04160, log of ST, what is ST, ST is equal to 1054 ST. So, log of 1054 is equal to 6.9603 and that gives us the Z value of 6.99603, 6.9603, Z is equal to 6.9603 minus 6.9098 divided by 0.0416, that gives that is equal to 1.2156. So, if I use the finite time model for the same analysis, as I used for the infinitesimal model, I get a Z value of 1.2156 compared to the Z value of 1.2286 if I use the infinitesimal value.

Let us see. So, these are pretty close to each other. So, if you, if our time horizon is 1 weak, we can as well do a reasonable job using the infinitesimal model without bothering about the finite time model. But the situation changes radically. When we look at a time frame of 1 year.

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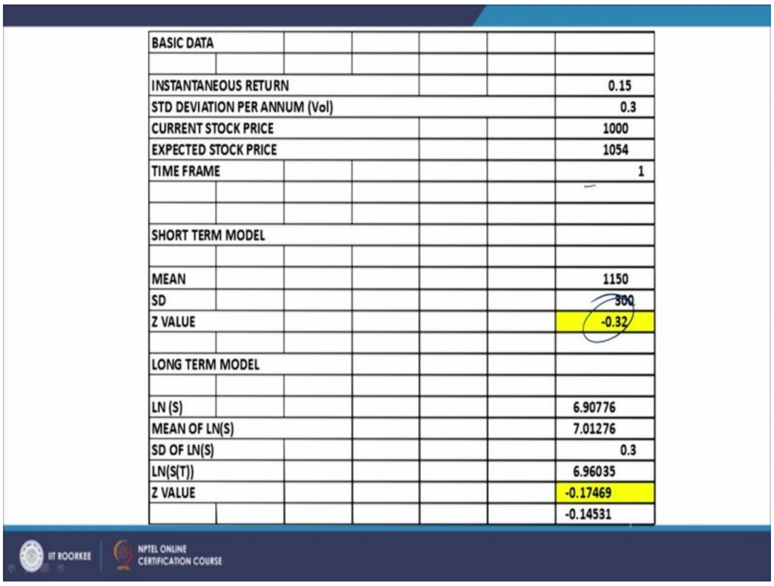

BASIC DATA	
INSTANTANEOUS RETURN	0.15
STD DEVIATION PER ANNUM (Vol)	0.3
CURRENT STOCK PRICE	1000
EXPECTED STOCK PRICE	1054
TIME FRAME	1
SHORT TERM MODEL	
MEAN	1150
SD	300
Z VALUE	-0.32
LONG TERM MODEL	
LN (S)	6.90776
MEAN OF LN(S)	7.01276
SD OF LN(S)	0.3
LN(S(T))	6.96035
Z VALUE	-0.17469
	-0.14531

That is illustrated in this example here, when you find that in the short term model, the Z value turns out to be minus 0.32, whereas in the long term in the finite time model, it turns out to be minus 0.174. So, there is a huge difference between the Z values that we calculate if the timeframe is increased from one week to 1 year. That is the reason that when we talk about a long duration, for predicting the price, we should be using the finite time model, for very short period that infinitesimal model is good enough.

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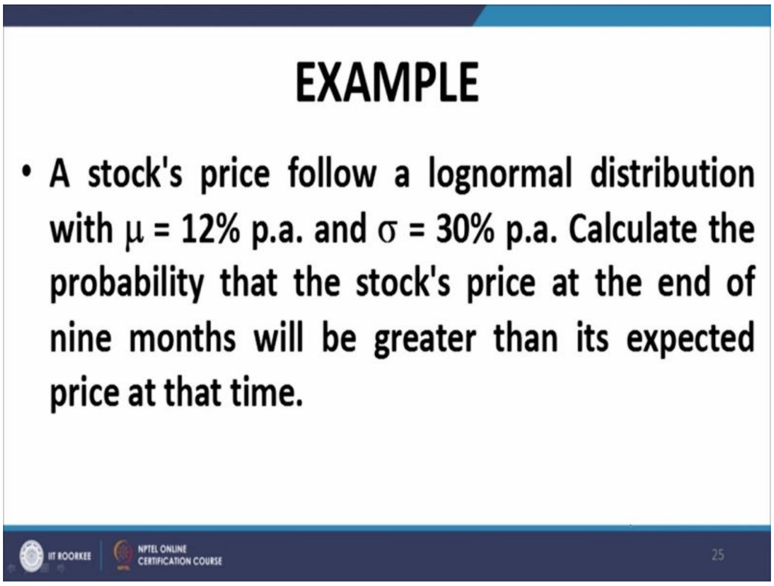

Another example, a stock price follows a log normal distribution of the mu of 12 percent and a sigma of 30 percent. Calculate the probability that the stock price at the end of nine months will be greater than its expected price at that time.

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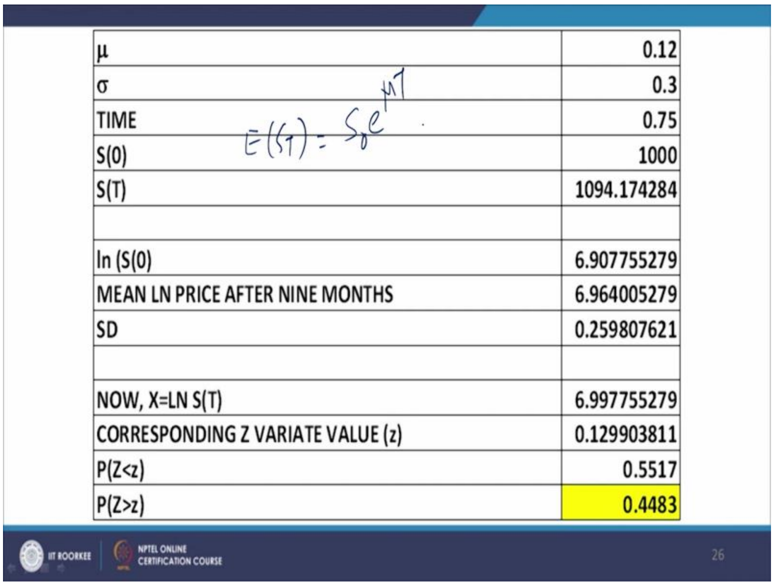

The first step is to work out the expected price, when the expected price to find is equal to 1094.17. How can you find out the expected price? We are given the current price S naught which is equal to 1000, we are given the continuously compounded return mu is equal to 12 percent, the time is equal to 0.75. So, the expected price is equal to E of ST is equal to S naught e to the power mu t, where everything is given to us, T is equal to nine months, mu is equal to 12 percent, S naught is equal to 1000. So, that gives us E of ST as 1094.

Now, knowing the expected price, it is quite simple. We have worked it out in the previous example, how to find out the probability corresponding to a given target price. And using that methodology, what we find is that the probability is equal to 0.4483. So, so far, so good.

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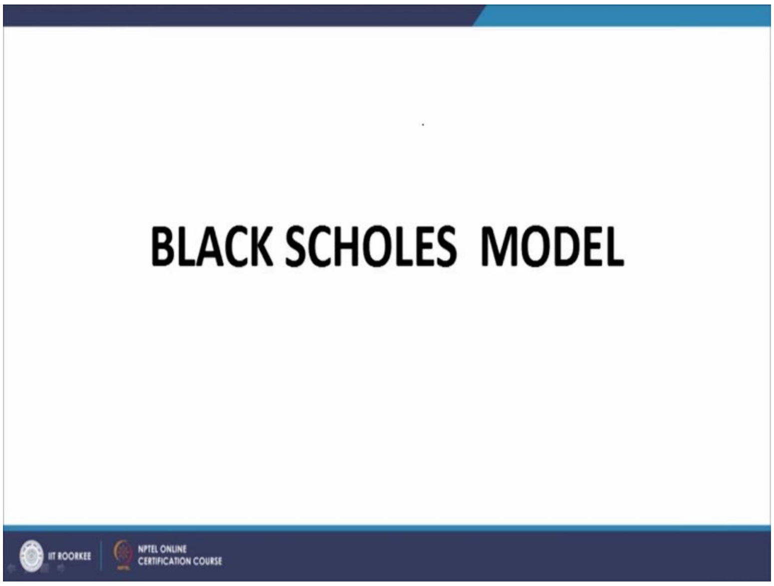

We have talked about the stock price modeling part of the course, we talked about the infinitesimal model, where the stock price or the change in the stock price was normally distributed where a mean equal to mu, S, dt and variance equal to sigma square, S squared dt, then we using the Ito equation or for that matter, for Fokker Planck equation, we found that over a finite time period the stock price follows the log normal distribution.

In other words, log ST follows the normal distribution, ST follow the log normal distribution, and the mean is given by a log of S naught plus mu minus 1 by 2 sigma squared into T and a variance equal to sigma square S square into, a sigma square into T, I am sorry, variance equal to sigma square into T.

Now, our next step is to use this model, use these models that we have discussed so far, in order to ascribe a value using arbitrates considerations as well, we try to ascribe a value to the derivatives, to a particular derivative. We start with the simplest form of derivatives, that is the European derivatives and, of course, they can be extended to the American derivatives as well. Although we do not have time, sufficient time in our course structure to cover the pricing of American derivatives, we shall confine ourselves to European derivatives, then in that we shall confine ourselves to the Black Scholes model.

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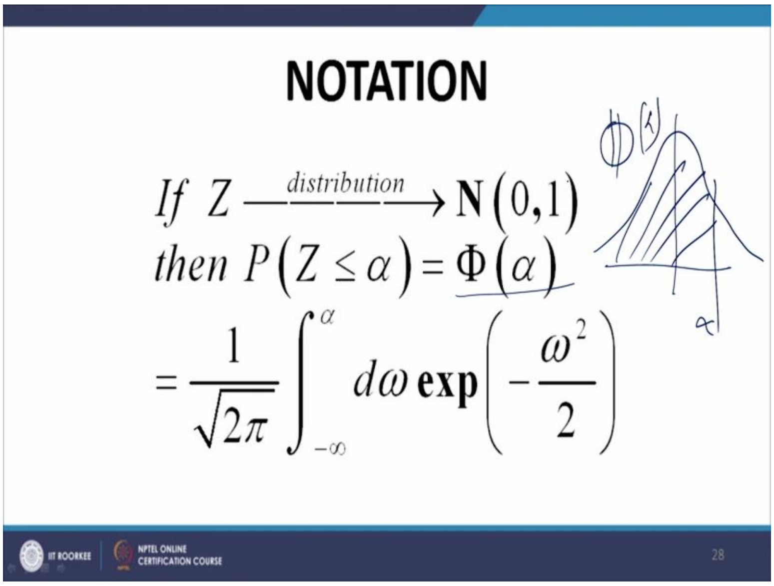

So, let us start with the assumptions of the Black Scholes model, but before I go into the assumptions of the Black Scholes model, there is a notation which I shall be following throughout, if Z is normally distributed with a mean of 0 and a variance of 1. That is, if Z is the standard normal variate then the probability that capital Z is less than equal to any predetermined value alpha is written as capital phi of alpha.

Let me repeat, this is very important and this will feature again and again in what is going to follow, if capital Z is normally distributed or a standard normal variate then the probability that capitals Z is less than equal to alpha is represented as phi alpha.

So, basically what we are trying to say is, suppose this is normal distribution, if this is any point alpha, then the, this whole probability will be called capital phi of alpha, cumulate this is called the cumulative normal probability distribution function, cumulative normal distribution function.

Let me repeat again once more, if Z is normally distributed with a mean of 0 variance of 1 it is the standard normal variate and if it is, if Z is the standard normal variate, then given any predetermined value alpha, the probability of Z lying less than alpha is written as capital phi of alpha and using the probability density function, we can write it in this form.

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Pr *ove* $1-\Phi(\alpha)=\Phi(-\alpha)$

$$I=1-\Phi(\alpha)=1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha}d\omega\,\mathbf{exp}\left(-\frac{\omega^2}{2}\right)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{\alpha}^{\infty}d\omega\,\mathbf{exp}\left(-\frac{\omega^2}{2}\right).\ \textit{Set}\ \omega=-\theta,\ d\omega=-d\theta$$

$$\textit{so that}\ I=-\frac{1}{\sqrt{2\pi}}\int_{-\alpha}^{-\infty}d\theta\,\mathbf{exp}\left(-\frac{\theta^2}{2}\right)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\alpha}d\theta\,\mathbf{exp}\left(-\frac{\theta^2}{2}\right)=\Phi(-\alpha)$$

So, this is another simple proof that we need to keep track of, I will not get into the nitty gritty of the proof, but at the same time, this is what is proved here 1 minus capital phi of alpha where capital phi is a cumulative normal distribution function, 1 minus capital phi of alpha is equal to capital Phi of minus alpha. Let me repeat, 1 minus capital phi of alpha is equal to capital Phi of minus alpha. The proof is straight forward exercise in integration and I will leave it as an exercise for the learners.

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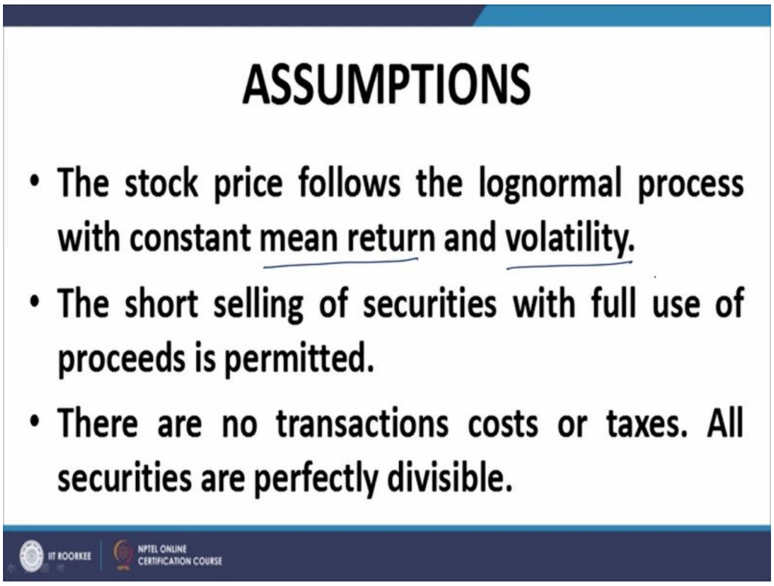

Let us now move over to the assumptions underlying the Black Scholes model. Number 1, the stock price follows the normal, log normal process with constant mean return and volatility. In other words, both the mean return and volatility are not functions of time. This is the first assumption, so they are, they follow generalized Brownian motion.

The stock price follows the log normal process with constant mean return and volatility. Short selling of securities with full use of proceeds is permitted, short selling of securities with full use of proceeds is permitted. There are no transaction costs or taxes. All securities are perfectly divisible.

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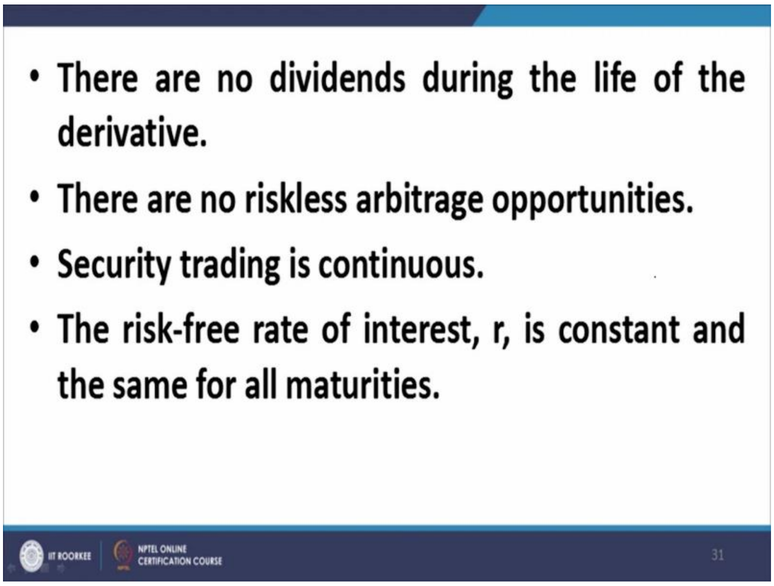

There are no dividends during the life of the derivative, there are no riskless arbitrage opportunities, security trading is continuous, the risk free rate of return r is constant and the same for all maturities that is the term structure of interest rates is flat.

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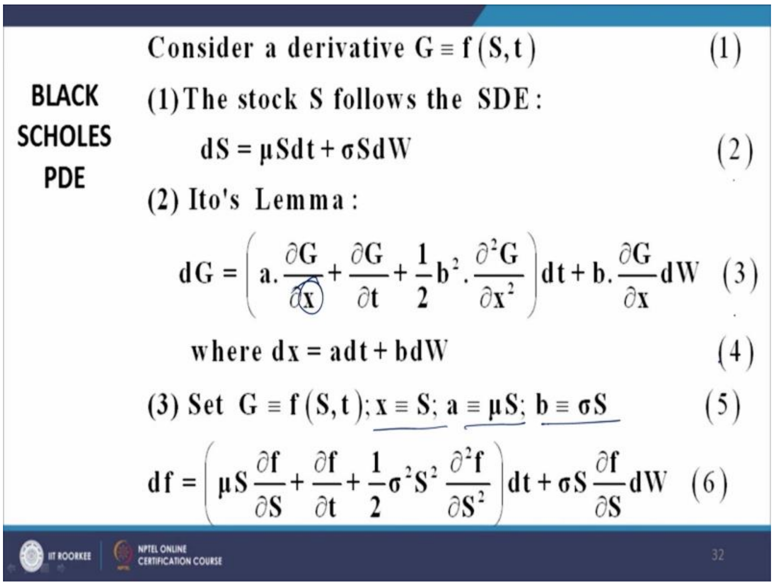

So, with these assumptions with (())(17:43) assumptions we now proceed to the derivation of the Black Scholes model. The stock price follows the stochastic differential equation given by this expression, we all know that, we have discussed it right at the beginning of the previous lecture.

dS is equal to mu S dt plus sigma S dW or ds is normally distributed with a mean of mu S dt and a variance of sigma square S square dt. The Ito's lemma also we are familiar with, given G as a function of a stochastic variable x, your x follows a stochastic differential equation given by expression 4 and is also an explicit function of time we get an expression for the differential of G as equation number 3.

So, this is given to us, the stock price model is given to us which is equation number 2 and the Ito's equation is given as equation number 3, dx is equal to adt plus bdW this is equation number 4, not much to explain here. Now, we set G equal to f of S comma t where f is a derivative, because it is a function of S and what is S, S is a stochastic variable and S is a stochastic process, so stock price and stock price follows a stochastic process. So, S represents a stock price and which is a stochastic process.

T is explicit time dependent, so if required, we shall be coming back to it in later point, x is equal to S, a is equal to mu into S, b is equal to sigma into S all these are obtained by comparison of equation number 2 with equation number 4. x is equal to S, a is equal to mu S, b is equal to sigma S, G is equal to f of S, t common, then when we substitute all these values into the equation or the Ito equation, that is equation number 3, what we get is equation number 6. So, so far it is only a manipulation of quantities, algebraic manipulation of quantities, nothing else, but we now come to the more important part.

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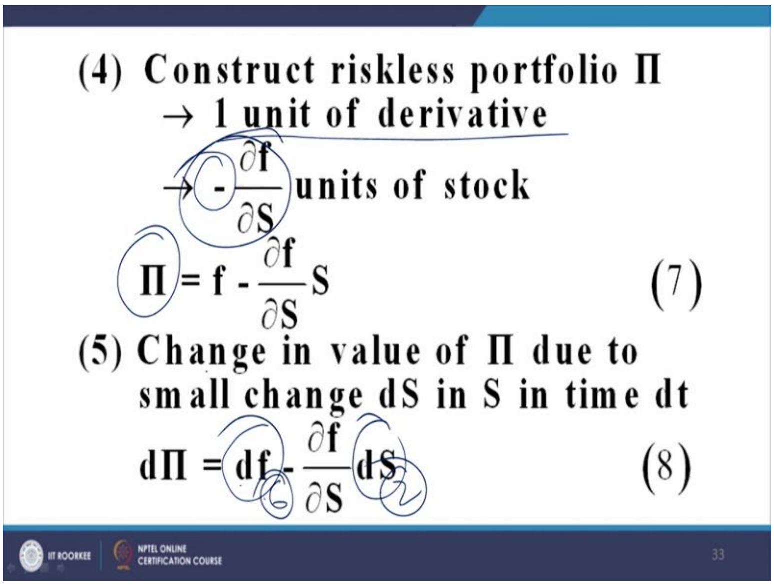

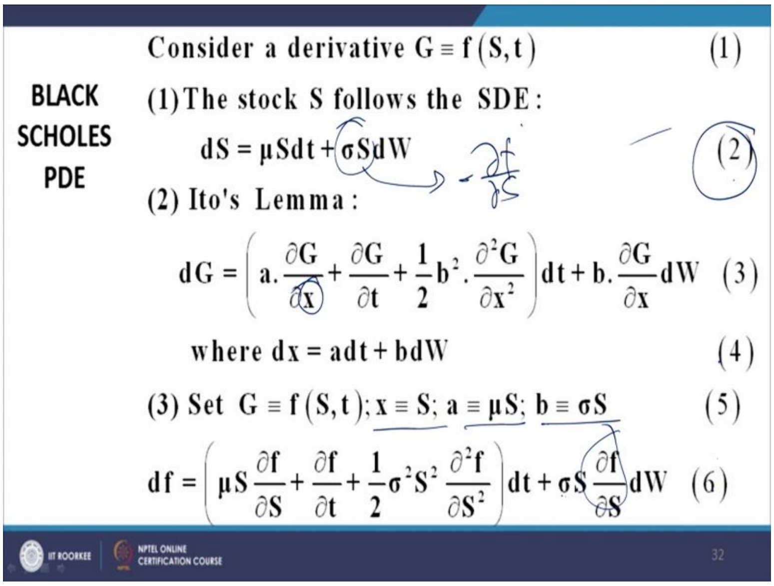

If you recall, when we talked about the pricing of derivatives using the binomial model, what did we do, we created a portfolio which comprised of the derivative and the underlying asset, opposite position in the underlying asset, delta units of the underlying asset. And then we showed that we can construct such a portfolio and it has a no risk embedded in it.

In other words, the stochasticity or the randomness in the derivative is neutralized by the randomness in the underlying asset. And as a result of which, what we get is a product or combination, a portfolio that is devoid of any randomness that is devoid of any stochasticity and therefore, that should give us by arbitrage considerations that should give us the risk free rate of return, precisely the same methodology we are going to follow here.

We construct a portfolio which consists of 1 unit of the derivative and minus del f upon del S units of the stock minus del f upon del S units of the stock, let us call this portfolio pi, capital pi. So, capital pi is a portfolio that consists of 1 unit of the derivative, let us say a long position in 1 unit of the derivative and a short position, that is what the minus sign represents. So, short position in del f by del S units of the stock. From where did we get this del f upon del S, I will come back to it in a minute.

But the value of this portfolio is given by equation number 7, f, f is the cost of the derivative minus, del f upon del S, this is the number of units of the underlying asset into S, S is the price per unit of the underlying asset. Change in the value of pi, due to a small change in dS, small dS in S in time dt is given by expression number 8, this is straightforward differentiation. So, we do not have much to explain here.

Now, this d in this expression, df, we have already worked out, where have we worked out, we have worked out in equation number 2, this shows equation number, now we have worked out in equation number 6. And as far as dS is concerned, dS, we have got equation number 2. So both of these expressions, df and dS are available to us. This is here in equation number 6. And this is here in question number 2. Now, the important point you see, where did we get this del f upon del S quantity in this number for constituting the riskless portfolio?

If you look at this equation number 6, the coefficient of dW in equation number 6 is sigma S del f upon del S, the coefficient of dW, see why I am talking about dW, because dW gives you the randomness, dW is the only componant in these equations which captures the randomness, it is a infinitesimal Brownian motion increment.

So, it is where the randomness in these processes is embedded, so, the coefficient of dW in equation number 6, is sigma S del f upon del S, the coefficient of dW in equation number 2 is sigma into S. Now, if these two randomness terms are to cancel each other or to annul each other, clearly, we need to multiply this expression by del f upon del S and then also a negative sign, because then when we add the 2 terms, these expressions will cancel out, that is precisely you will see that, that is precisely what is happening.

So, this is the reason this is the source of this expression minus del f upon del S that we find here in this equation, this is not from, this is not arbitrary. This is given by the relations that we have between the process, the stochastic process followed by the underlying asset and the process followed by the derivative asset. So, the change in the value of pi due to small change in dS is given by this expression, this is a simple.

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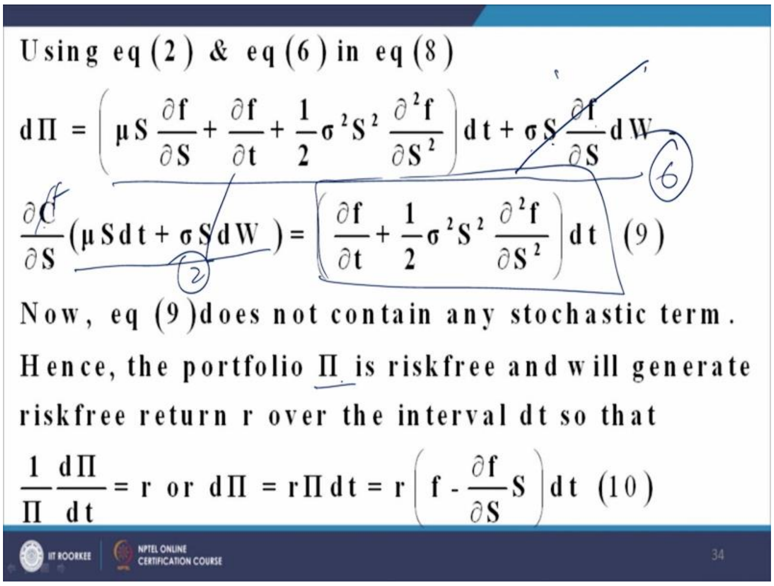

Now, when we put the values from equation number 6, this is equation number 6, please note, this part is equation number 6. And this part is, this part is equation number 2. We find, you can clearly see what I have been explaining just now, this part and this part cancels out, you can see here, it is what sigma S del f upon del S into dW and you can see here, del C upon del S or del C it should be f here I am sorry, del f upon del S into sigma S dW same term opposite sing.

So, they should cancel each other and they indeed cancel each other. And what we get is this expression, which is equation number 9. And please note, this expression contains no randomness, it does not contain any term containing dW, it does not contain any term containing Brownian motion, Brownian motion increment and therefore, there is no randomness in this term. This is, this is the equation which is followed by the risk free portfolio which is represented by a merger, a combination of the derivative and del f upon del S units of the underlying asset.

Now, equation 9 does not contain any stochastic term. Hence, the portfolio capital pi, the portfolio capital pi is risk free, and will generate a risk free rate of return over the interval dt. So, that this gives us if we can see it is quite elementary, when you substitute the respective values, you get the equation number 10.

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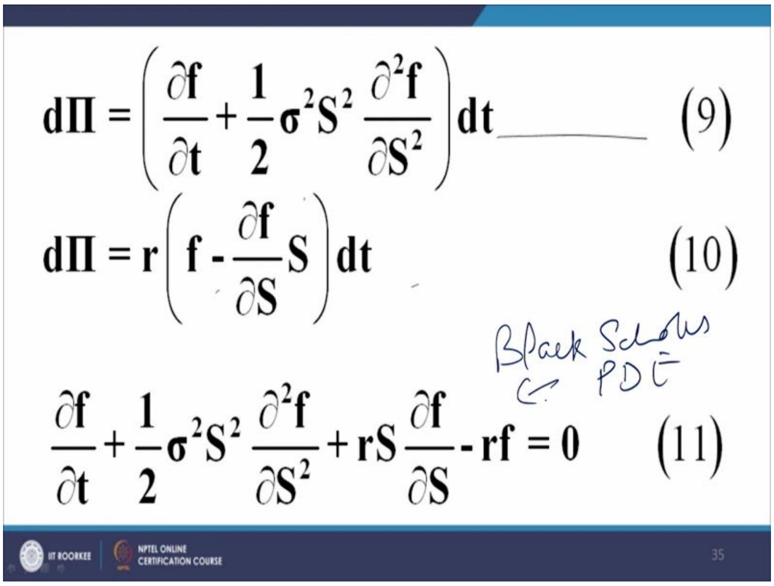

So, what do we have from here, what do we have, we carry forward from the previous slide? From the previous slide, we carry forward two expressions for the same d pi, one is equation number 9. The other is equation number 10. The job is now easy, we simply need to equate the two and what we get is equal to number 11 which is the celebrated Black Scholes partial differential equation. When you solve this equation for a derivative, the result that you get is the valuation is the price or the value of the derivative product. So, this is the Black Scholes PDE.

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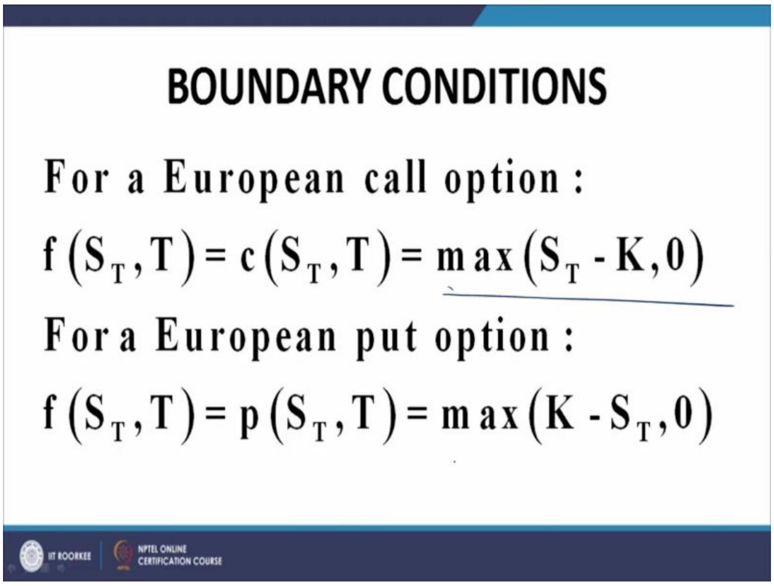

Now, for a European boundary conditions, these are basically terminal conditions for European call options, we have f of St comma capital T where capital T is maturity date of the option is equal to C, S capital T, T is equal to f, you know the payoff function. So, you see, the point is at the date of maturity of the option, the price of the option must equal its payoff that is precisely what is being used here similarly, for the put option.

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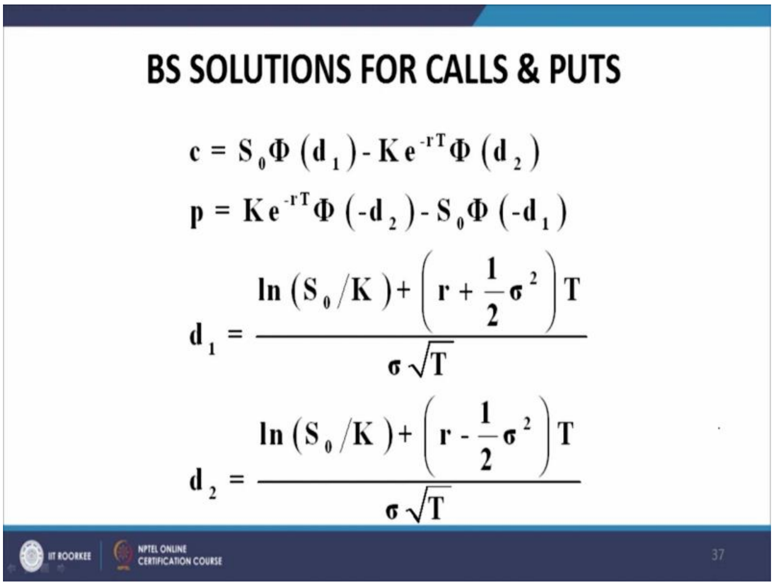

And when you use this boundary conditions and solve the partial differential equation that we have here, these are the results that you get, this seem slightly involved, but with a little bit of exercise, with a little bit of practice, you would be able to manage to recall these results, they are fundamental, they are very, very important and they would definitely constitute, they would definitely be problems based on these equations in your exercises as I have mentioned, and the final exam.

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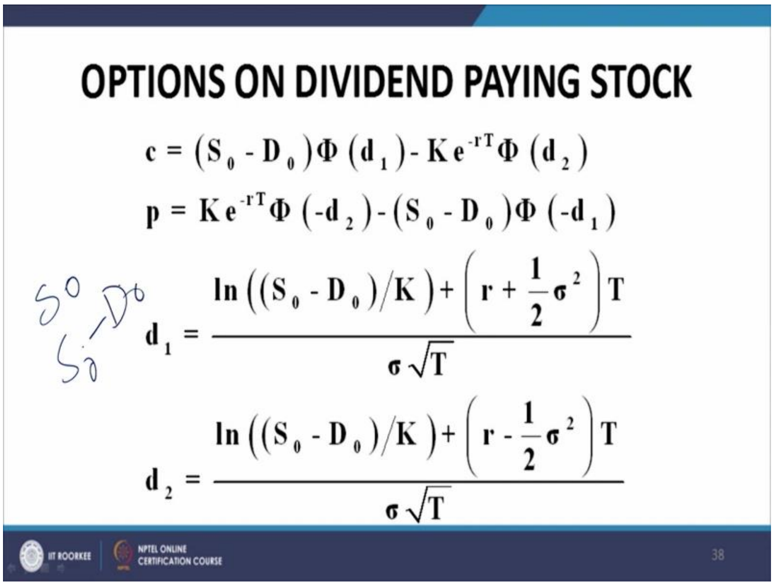

And this, this is the, these are the, these are the same results, but in the context of dividend paying stock. You can see here, the stock price S naught, is replaced by the effective stock price S naught minus D naught, where D naught is the present value of dividend. That is the only change that occurs here.

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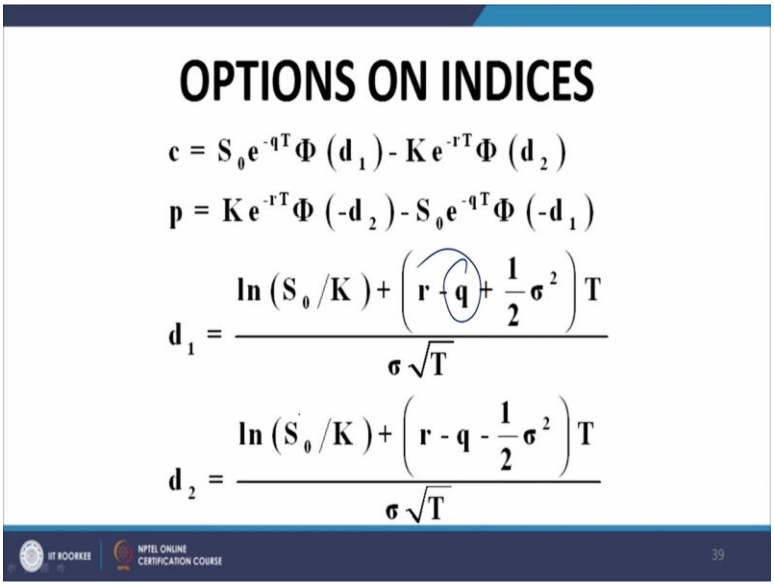

This is options on indices formula for options on indices, where you have a continuously compounded return given on the underlying asset, the q is the continuous compounded return on the underlying asset. So, in this case, this additional factor of q comes into play.

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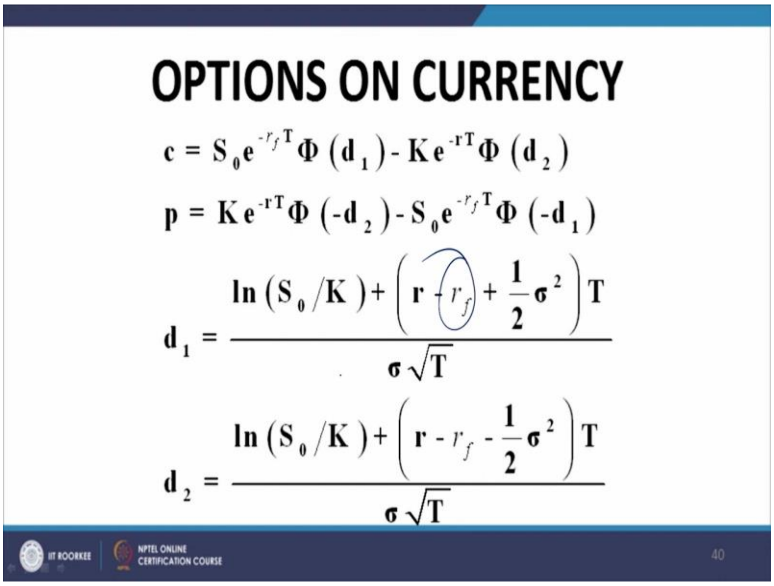

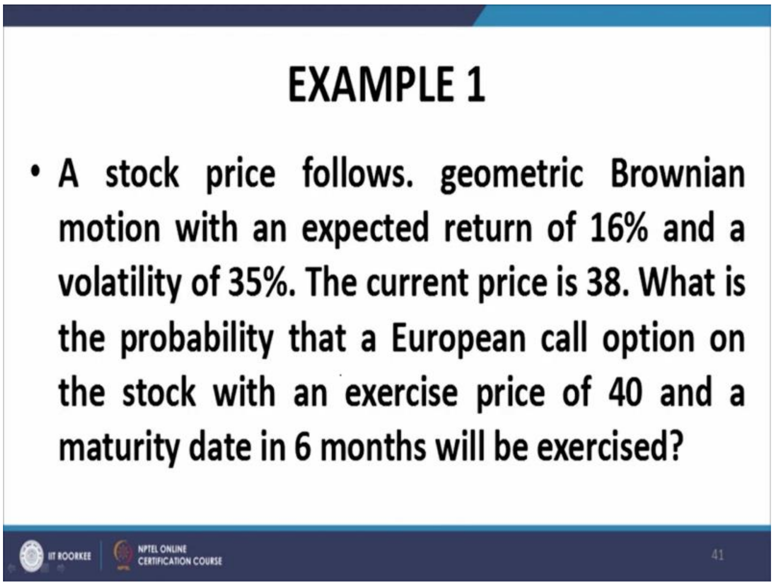

And similarly, in the case of currencies, this factor of rf which is the risk-free rate for the foreign country comes into play when you are talking about options or products on foreign currencies. I will continue from here in the next lecture. Thank you.