Quantitative Investment Management
Professor J.P. Singh
Department of Management Studies
Indian Institute of Technology, Roorkee
Lecture: 5
Arbitrage Free Pricing

Welcome back, so in the last lecture I was talking about risk and Arbitrage, so a quick recap about the contents.

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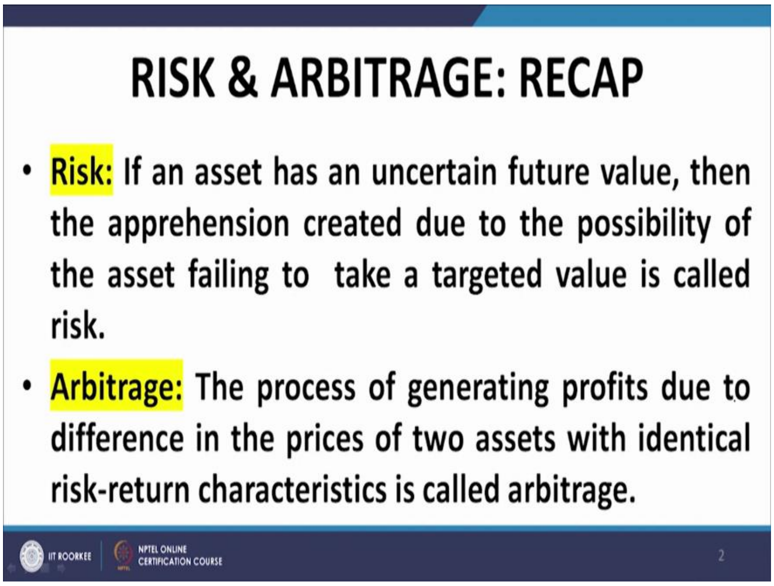

Risk, we define in general, if an asset has an uncertain future value, of course if an asset has a certain future value, then it is a riskless asset, so if an asset has an uncertain future value, then the apprehension created due to the possibility of the asset failing to take a targeted value is called risk.

So basically, risk arises from the possibility of the asset taking more than one value at maturity or at a future date. As a consequence of which some of the values may be undesirable from the perspective of the investors, some of the values may be desirable.

Now, the possibility or the probability of the asset taking those undesirable values creates an apprehension in the mind of the investor and this is called risk. Then we talked about arbitrage, also in some detail, how do we define arbitrage? The process of generating profits due to the difference in the prices of two assets with identical risk-return characteristics is called arbitrage. In other words, if you have two assets which give you identical cash flows with identical characteristics of risk and return, and they are priced differently at a point in time, then this creates an opportunity to extract profits from the system, and this process is called arbitrage.

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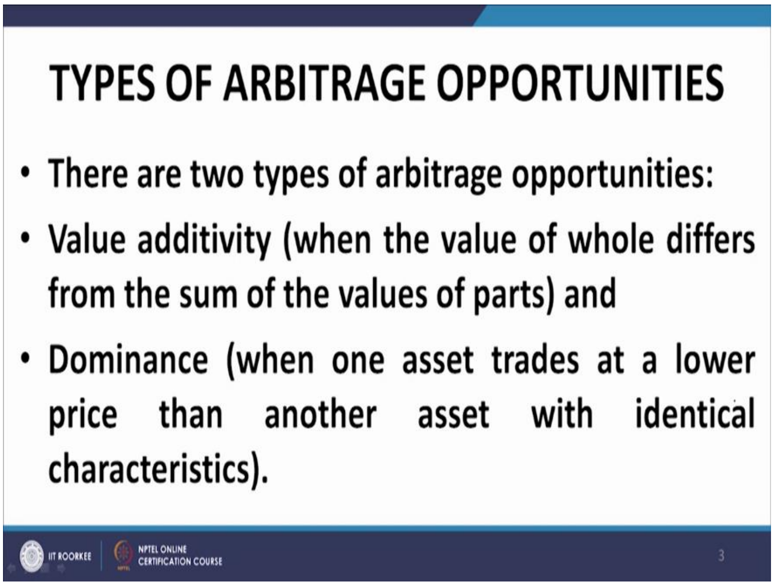

There are two types of arbitrage opportunities, one is called the Value Additivity and the other is called the Dominance. Value Additivity arises when the value of the whole differs from the sum of the value of its parts. And Dominance arises when one asset trades at a lower price than another asset with identical characteristics.

So, if two assets have identical payoffs and one asset is trading at a higher price than the other then, that is called dominance. And when an asset can be split up or decomposed into smaller constituents with respect to cash flows, then the total asset or the asset comprising of all those constituents, trades at a different price from the aggregate price of its constituents than it is called value additivity. Let us look at examples of both of these types of arbitrage opportunities.

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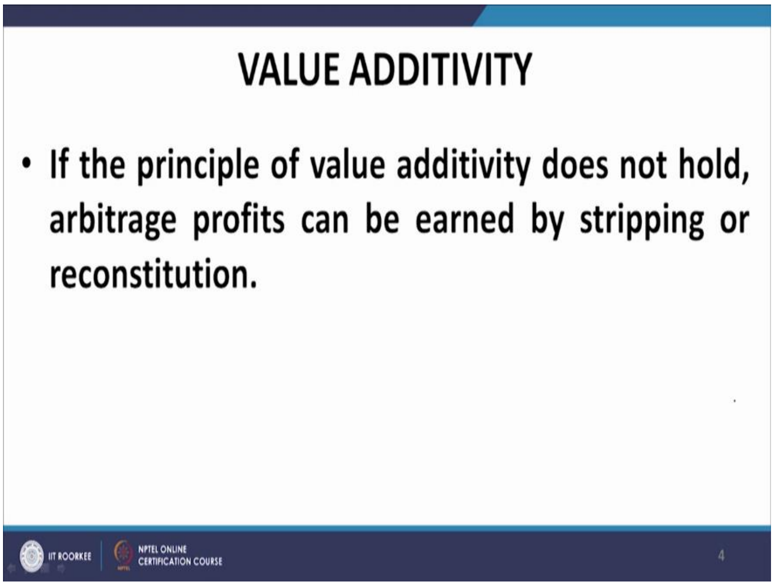

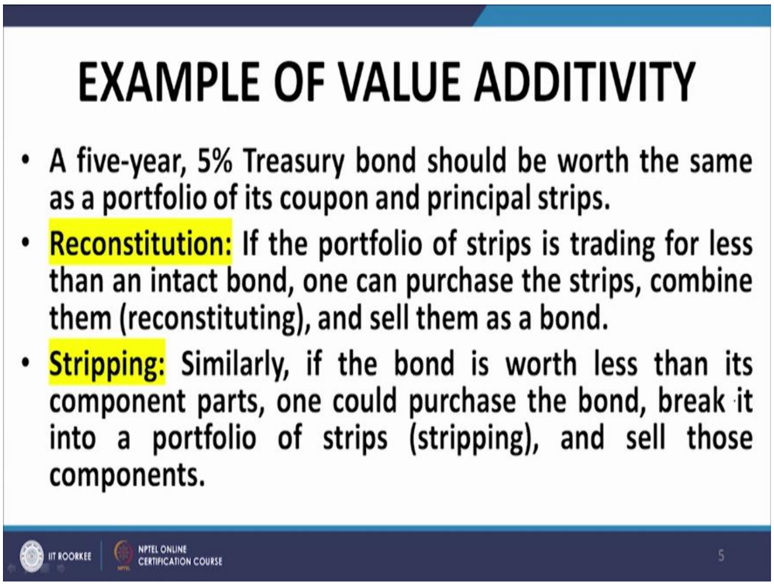

If the price of value, if the principle of value additivity does not hold, arbitrage profits can be obtained either by stripping or by reconstitution. How does it happen, let us go to the example. The five-year, 5 percent treasury bond should be worth at the same as a portfolio of its coupon and principal strips. So, if you have a treasury bond which gives you a certain coupon payment, let us say 5 percent and it has a maturity of five-years, then the price of that particular bond should equal in order that there be no arbitrage. The price of that particular bond should equal the aggregate price of all its coupons and the principal repayment. In other words, if you strip the bond into its coupon and the principle, the aggregate price of the combination of the strips and the coupon should be equal to the price of the five-year, 5 percent bond.

Now, how does the, if this particular principle does not hold. In other words, if there is inequality between the prices of the bond itself and the stripped constituents of the bonds, then you can extract arbitrary profits. For example, if the portfolio of strips is trading for lower price. You will buy all this, you will take a long position or you will buy all these constituent strips, combine them and sell them as a package, which would represent the bond and thereby you will be able to make a profit, because the strip themselves, the coupon strips, and the principal strips, together who are trading at a lower price, and as a result of which you buy all of them, combine them, make them into a package, and sell them at the price of the bond, which is higher.

Conversely, if the converse situation is there, that is the bond trading at a lower price and the strips are trading at higher price, you simply buy the bond, strip it into its constituent's coupon strips, and the principal strips and sell them in order to derive an arbitrage profit. So, this is how arbitrage profits can be generated, either by reconstitution, constitution, when the parts of the bond are being traded at a lower price than the whole bond, and or by stripping when the parts of the bond are trading at a higher price than the bond itself.

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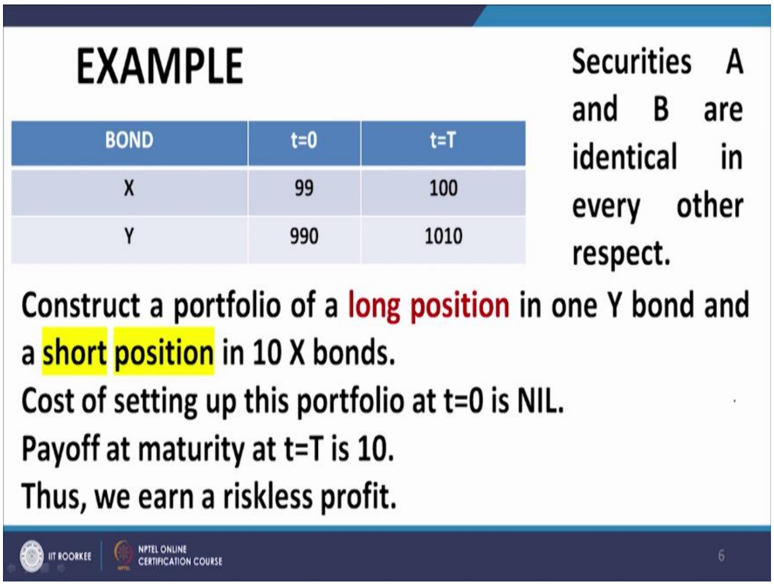

Here is an example to illustrate. We have two bonds X and Y, X has a price of 99, Y has a price of 990, and the maturity value of X is 100, and the maturity value of Y is 1010. Now, clearly you can see that if you have 10 X bonds, what situation are we able to get the price that we are going to pay for buying the bond is only 991 bonds is costing 99, so 10 bonds will cost 990, X variety that is and the payoff would be 1000.

On the other hand, if I invest the same 990 in bond Y, what do I get? I get a payoff of 1010. So obviously, what I will do is, now it will take a short position in the X, 10 X bonds and will take a long position in the Y bonds. If I take a short position in the 10 X bonds, that means I am selling 10 X 1, that means I will have a cash inflow of 990, and against that 990, a cash inflow I can buy 1 Y bond.

Now, taking a long position in the Y Bond, what do I get? I get a payoff of 1010, whereas against the short position in 10 X bonds, I have to make a payment of only 1000. In other words, I extract 10 rupees or 10 units of money of arbitrage profit at the date of maturity of the bond, this is called the principle of value additivity. When the value additivity of the constituents is different from the value of the whole of the instrument.

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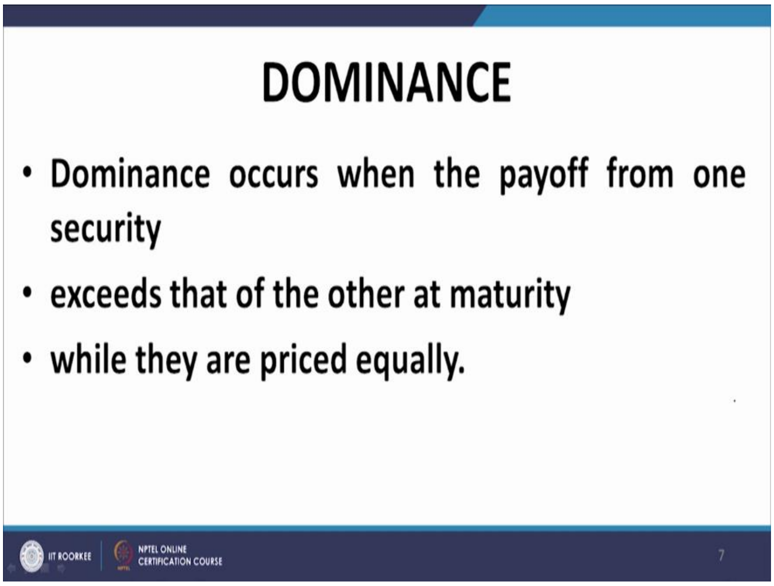

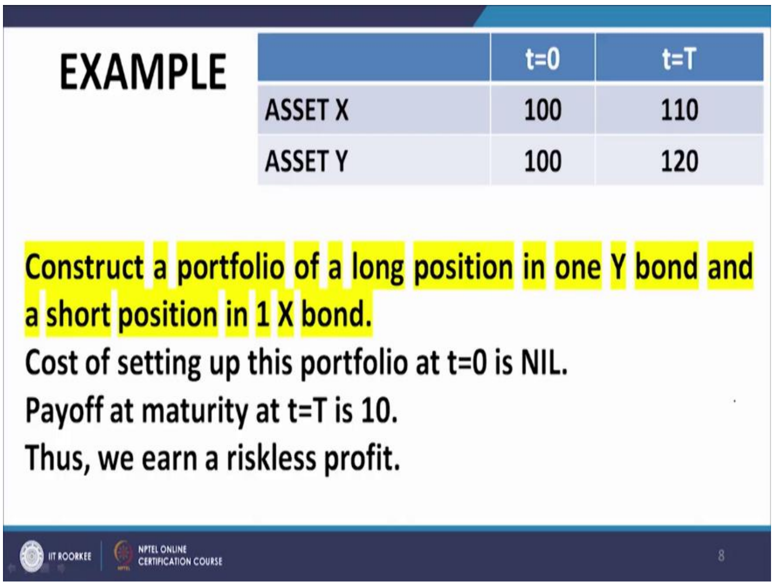

And dominance, dominance occurs when they pay off from one security exceeds that of the other at maturity while they are priced at t equal to 0, at the current point in time they are priced equally. So, example asset X and in asset Y are both being created at 100 as that X gives you a payoff of 110 at maturity, asset Y gives you a payoff of 120 at maturity. Clearly, as that X is giving you a lower payoff, so what will you do? you will take a short position in asset as X and you will take a long position in asset Y.

In other words, you will sell asset X and you will use the proceeds for buying asset Y and as a result of which when at maturity you will get 120 units of money, by the long position, because you are holding asset Y long, whereas against the short position in asset X, you will have to make a payment of 110 and you will make an arbitrage profit of 10 units.

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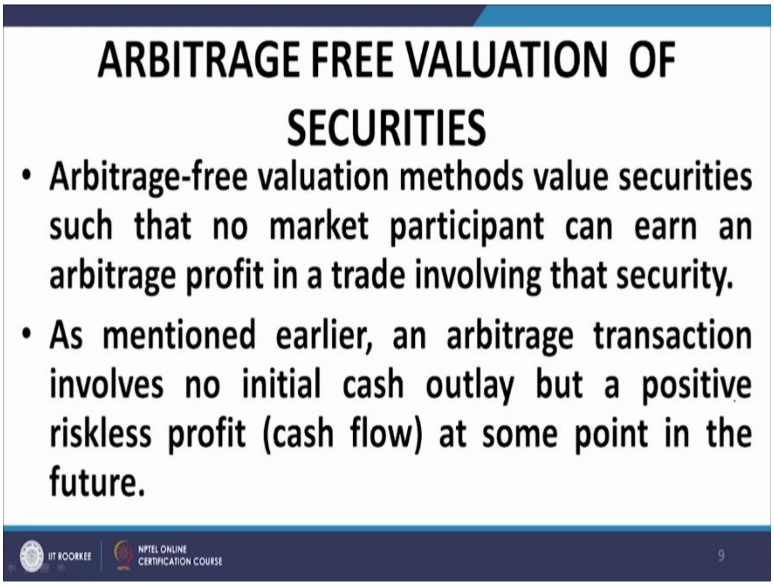

Arbitrage Free Valuation of Securities: Arbitrage-free valuation methods value securities such that no market participant can earn an arbitrage profit in a trade involving that security. Let me repeat, arbitrage-free valuation methods value security such that no market participants can earn an arbitrage profit in a trade involving that security. As mentioned earlier, an arbitrary transaction involves no initial cash outlay but a positive riskless profit at some point in the future.

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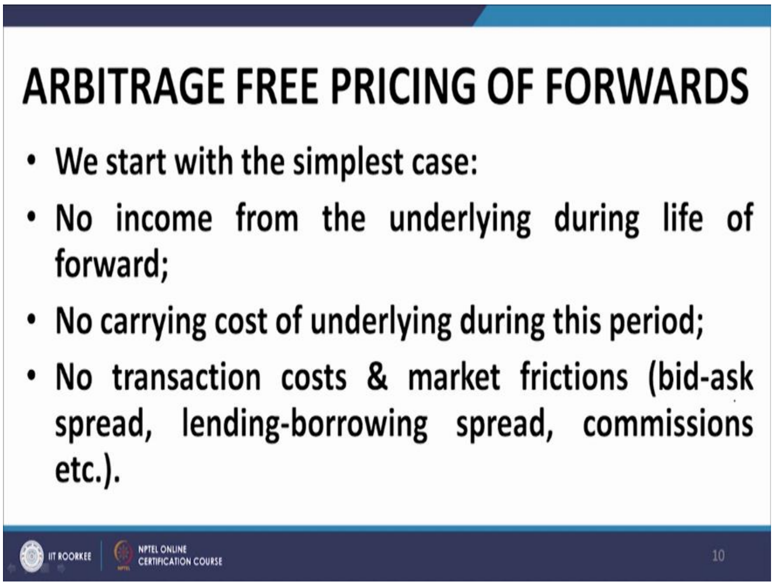

So, let us now take practical examples of how the concept of arbitrage-free pricing helps us to attribute a value, to attribute a price, to certain complex financial instruments. I will start the illustration with the with the attempt to value a forward, or to work out the forward price, under a forward contract.

Forward contract has just to recall as a contract which is negotiated at t equal to 0, the terms of the contract, all of them are negotiated at t equal to 0, to facilitate an unambiguous settlement, but the actual settlement of the contract, the actual receipt delivery of the underlying asset, and the payment of the price takes place at a future date. So, we talk about how to calculate the forward price under a forward contract.

We make the following fundamental assumptions, simple assumptions, no income from the underlying during the life of the forward. During a later section of this course, we will talk about relaxing these assumptions, so for the moment no income from the underlying asset during the forward. This is the basic illustration of the no arbitrage or arbitrage-free pricing, so I have tried to keep the exposition as simple as possible.

No carrying costs of underlying during this period, during the life of the forward contract. No transaction costs and market fractions bid-ask spread, lending-borrowing spread, and commissions, etc. So, we keep the exposition at the very basic level, at the very elementary level, no income, no carrying cost, and no fractions. Let us look at this table.

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	t=0	t=T
BORROW	$+S_0$	$-S_0\exp(rT)$
BUY STOCK	$-S_0$	0
SHORT FORWARD	0	F_0
TOTAL	0	$F_0-S_0\exp(rT)$

$$F_0=S_0\exp(rT)$$

We enter the forward contract is negotiated at t equal to 0 and the settlement is at t equal to capital T, which is called the maturity of the forward contract. What I do, S0 is the spot market price, not the forward price, please note forward market prices what we have to determine I have represented it by the by the expression F0.

This is the forward price, okay, and S0 this spot price, please note the difference S0 is this spot price at t equal to 0, F0 is the forward price at t equal to 0, our exercise is to determine F0 in terms of S0. So, what we do is, we borrow a certain amount of money S0, and we use that amount of money at two by one unit of the commodity, underlying commodity in the spot market. I repeat, I borrow a sum of money S0 and use that money to buy one unit of the underlying commodity in this spot market. So, the net cash flow is cancelled out, whatever I have borrowed, I have used it for buying one unit of the underlying commodity in the spot market.

I also had simultaneously entered into a short forward position, short forward position. Now, what happens at the date? this is the situation or these are the set of transactions that I do at t equal to 0, that is today, that is now. And at the maturity of the forward contract what happens? against the short position that I have in the forward contract, I will have to deliver the underlying asset, but I already have position of the underlying asset, so what I do is I deliver the underlying asset which I have bought at t equal to 0 by borrowing a money S0, so and against the delivery of the underlying asset, I will receive the forward price which is F0. So, this is the amount that I will receive at t equal to capital T, under the short position in the forward contract.

I will deliver this asset and underlying asset which I have bought at t equal to 0 by borrowing the money, and I will receive the cash, if forward price at maturity that is F0. However, because I have borrowed S0 at t equal to 0 I also have to repay this amount to the bank or from the party from whom I have borrowed and that as assuming that R is the continuously compounded a risk-free rate, why is risk-free? because we are assuming the forward contract will be default-free we are not factoring in the issue of default, in the forward pricing that we are talking about right now.

So, assuming that the forward contract is default free, this rate that we have used for borrowing can be the risk-free rate, because the if the forward is default free, I will receive the money and I can pay back to the bank right away, and therefore as far as the repayment of the principal to the bank is concerned, that is also default free, on the presumption that the forward contract is default free. Anyway, coming back to this, I have borrowed an amount F0 at the continuous compounded rate R, for a period 0 to capital T, and therefore the amount that I have to repay is equal to S0 exponential rT.

Now, you can see here, let us work at the total cash flows. The total cash flows at t equal to 0 is 0 as you can see here in this table, the aggregate cash flow at t equal to 0 is T0, you have borrowed at 0 and you use that same money to buy one unit of the underlying assets, so the net cash flow is 0, forward as I mentioned does not entail any cash flow at t equal to 0.

Now, as far as the situation at t equal to capital T is concerned, the total cash flow is F0 minus S0 exponential rT, now because this situation is risk-free, because I as I emphasize once again, we are not considering the possibility of default in the forward contract, and the initial cost of setting up this portfolio is 0, and therefore the maturity proceed should also be 0, because this is the riskless transaction and that gives us this equation, F0 is equal to S0 exponential rT. So, this is an example of forward pricing using the principle of arbitrage-free pricing, AFP.

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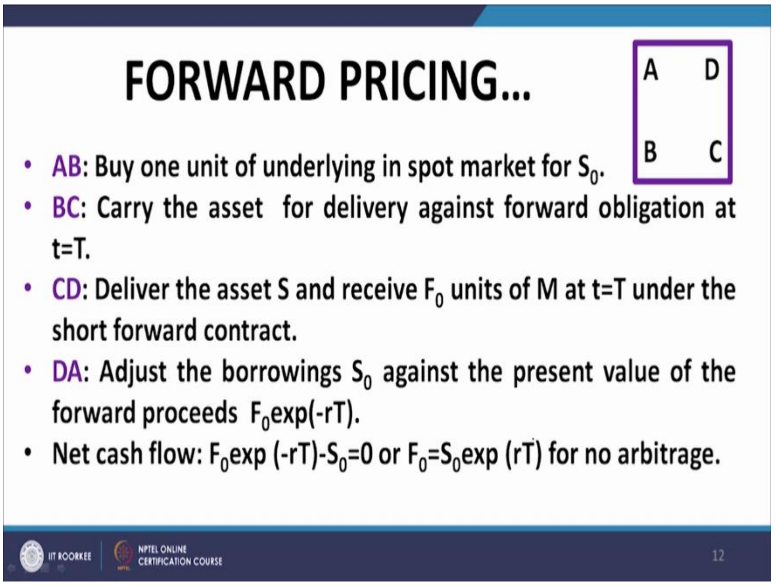

These are the steps that, this illustrates the steps that I explained in the previous table. You can represent it as a square, the process AB involves buying the stock buying one unit of underlying in the spot market for S0. BC involves carrying the asset unutilized carrying the asset with you against a short position in the forward contract between t equal to 0 and t equal to capital T.

CD invoice delivery of the asset at t equal to capital T, and received of the price F0 against the short position in the forward contract. And DA represents the adjustment of the borrowings at S0 against the present value of the forward process F0 exponential minus rT. You can also look at it from the perspective of AD, AD represents the future value of the borrowing that you have taken for buying the asset in this spot market. So, this is another way of representing the same set of transactions that are shown in the previous table.

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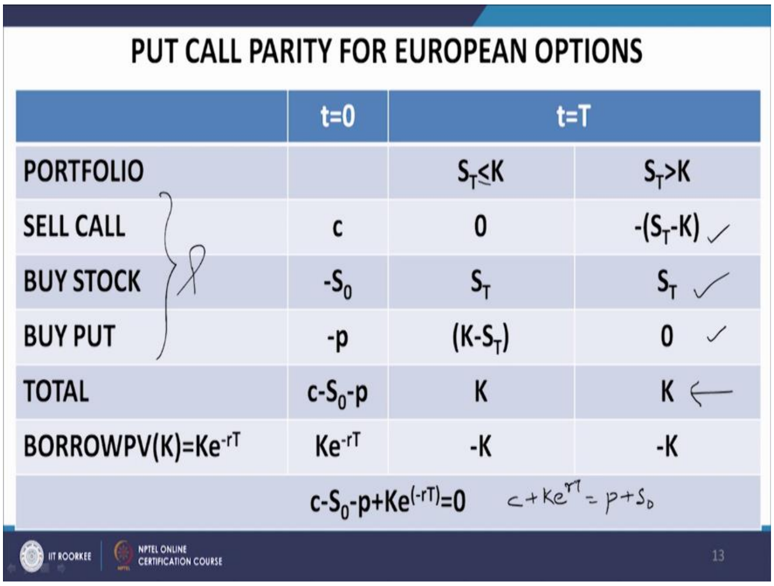

Now, we will talk about another example, which is slightly more involved. This is the Put Call Parity relationship that we have for European call and put options, so that as I mentioned, options are of two types, put options and call options, call options until a right to buy the underlying asset at a predetermined price, and put option until a right to sell the underlying asset at a predetermined price. European options are accessible on a particular date, we have another variety of options which is American options, which is accessible up to a particular date, the date is called the maturity of the option.

Now, let us look at this diagram, we are having a portfolio comprising of a short position in a call option, a long position in the stock, and a long position in the put option. If since you are short in the call, that is you have written the call, you will receive the call premium, therefore small c, because it is an European option, we represented by small c, small c is the cash inflow, we have written the call option, I repeat, and therefore you will receive the premium.

You are buying the stocks that will enter a cash outflow equal to the spot price of this stock which is S0, so it is minus S0, you are buying the put option again, you have to pay the premium, its a cash outflow and let us say the price of the put option, the price at which it is being traded is p, so the cash outflow is minus p.

So, the total cash flow at t equal to 0 is c minus S0 minus P. Let us now look at the total situation that arises at t equal to capital T which is the maturity of the two options call and put options, I repeat they have identical maturities and identical exercise prizes as well. So, both the call and put and European have identical maturities and have identical exercise as well. Now, as on the date of majority of the option, we can have two situations, St can be less than

St as less than K, I am sorry, and S capital t can be greater than K, you can have equal to either with the first one or the second one, it does not really matter.

In other words, you can have two situation St less than equal to K, or St is greater than K, these are only two possible states of nature that can arise. The stock price at maturity can either be less than equal to a given predetermined number that is K, or it can be greater than a predetermined number that is K, so we can have only two states of nature.

St less than equal to K or St greater than a greater than K, where S capital t is this or is the price of the stock at maturity. Please note this is a random variable, we do not know what value exactly, St is going to take at t equal to 0, this analyse analysis is being done at t equal to 0, please note this point. S capital t is the future price of the stock, so we do not know what it can take, but we have a predetermined number K, such that, and therefore, St can either be less than equal to K, lie between 0 and K, or it can be greater than K, so these are the only two possible states of nature.

Now, if St is less than equal to K, the call will not be exercised, the person who has bought the call, you are short in the call, so you have sold the call, the party who is long and they call the party was bought the call, will not exercising rather by the asset from the market, then by the asset under the call option contract at the excess phase K, and you will not access the option.

And as a result of it, the payoff from the option A call option will be 0 in this situation. If St is greater than K, obviously he would exercise the, the person is long in the option will exercise the option, you would rather buy the asset at K under the option contact, and sell it in the market at St, and thereby make a profit at St minus K. I repeat, he will exercise the option by the underlying asset at K and sell it in the market at St, because St is greater than K will make a profit, and this net profit will be equal to St minus K.

Now, because you are short in the option, and this is a 0 sum game, this would be the your loss as well, therefore the loss for you in the event that St is greater than K turns out to be minus St, minus K. Now, as far as the stock is concerned, the stock will take the value St, whatever that a value St is, whether St is less than K, equal to K, or greater than K, it will remain at St.

And as far as the put option is concerned, you long in the put option, so if St is less than K, please note what is put option? put option is the right to sell, so if St is less than K, rather exercise the option, and sell under the put option at K, then selling in the market at St. Therefore, in the event that St is less than K, I would exercise the put option, and I would make a profit of K minus St.

Buy the asset from the market at St which is lower, and sell the asset under my long position in the put option at K, thereby making a profit of K minus St, and if St is greater than K, obviously I cannot do, that because K is smaller and I can, I would rather sell the asset if I have one in the market at St, then selling it under the put option contract at K. So, this is the situation as far as the long put option is concerned.

Let us examine the aggregate payoff of the two possibilities, that is St less than equal to K, and St greater than K in the two cases. If you sum up all of these, you find that the payoff in both the situation, St less than K, equal to K, or St greater than K turns out to be K. In other words, what is the inference?

The inference is that irrespective of what value St is going to take, whether St is going to take the value less than K, equal to K, let me put equal to K here as well. So, if St is going to take the value less than K, equal to K, or greater than K, the payoff of this portfolio, the payoff comprising of this portfolio, this set of three assets or three positions, rather short call, long stock, and long put aggregate to K.

The payoff is independent of the state of nature, I repeat, this is very important, very fundamental, this payoff is independent of the state of nature and it is equal to K, and therefore the present value of this particular portfolio, this particular portfolio let us call it p, this portfolio p, capital P, must be equal to the present value of K at the risk-free rate. I repeat, because the payoff is K, which is the predefined number, and it is independent of the state of nature, that means it is a certain payoff, and therefore the cost of this portfolio must be equal to the present value of this quantity that is Ke to the power rt.

And therefore, what happens? Therefore, we get this relation c minus S0 minus p plus Ke to the power rt is equal to 0, or we get it in the familiar form c plus c plus Ke to the power rt is equal to p plus S0, which is the well-known protocol parity relationship, which I believe you would have encountered if you have studied Financial derivatives, if you have studied options in any little detail.

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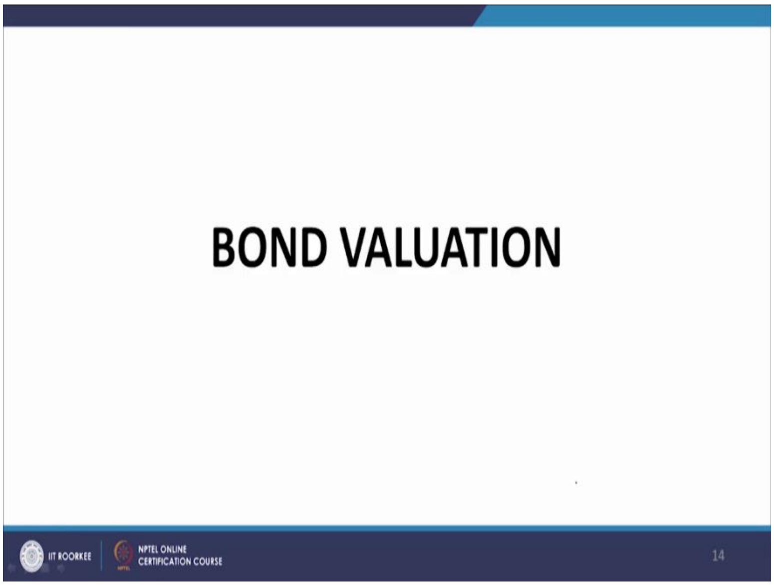

Bond Valuation: Here again, we will encounter arbitrage-free pricing, this is pretty much ubiquitous in financial derivatives and even in financial instruments when we talk about pricing of financial assets, very often we take recourse to the principle of arbitrage.

We have a complex financial product which we need to price and we try to split it up and reconstitute that the payoff of that complex financial product in terms of more rudimentary financial instruments and then try to equate the values of the two portfolios, one comprising of the complex financial product, and the other comprising of the parts of the, or the assembly of simpler financial products, that give you pay of equal to identical with that of the complex financial product. So, let us talk about Bond valuation now.

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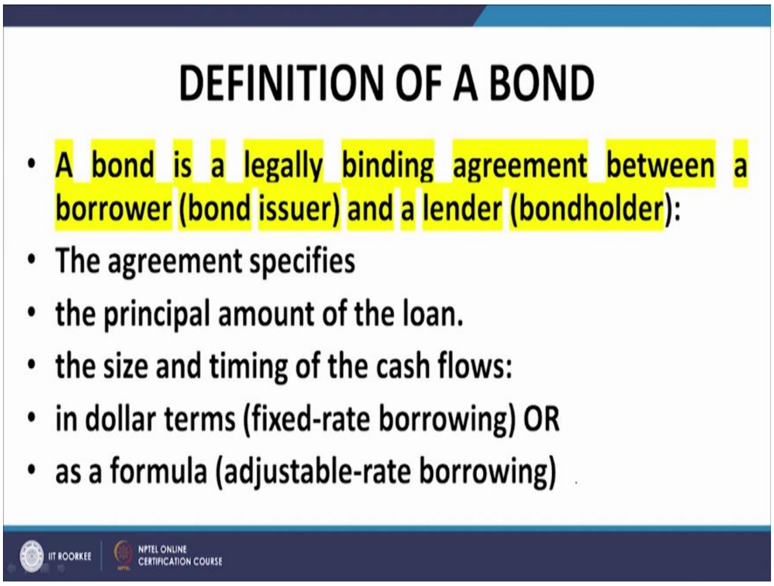

Definition of a bond, its quite trivial. I believe most of the learners would be familiar with it, and nonetheless will quickly run through it. A bond is a legal binding agreement between a borrower bond, issuer boundary, issuer is the borrower. In other words, a bond is an agreement between two parties, one party borrows the money and issues the bond, the other party lends the money and is the bond buyer or the bondholder or the bond purchaser.

So, a lender, and a lender who is the bondholder, the agreement specifies the principal amount of the loan, the size and timing of the cash flows, in dollar terms, in the case of a fixed statement dollar means any money, any currency you may, dollar epitomizes any currency that is it may be rupees, it may be pesos, it may be Euros, whatever the case may be or it may so happen that instead of specifying a dollar value or a percentage, the interest amount or the interest percentage may be left open, open in the sense that the formula for the calculation of the interest rate may be provided in the issue document, but the actual value of the interest or the percentage in terms of a percentage for the interest may not actually be made, may not be crystallized at the time of issue of the bond, this is called in fact a floating rate bond.

Where the interest rate is tagged to some other variable, some other market variable, and its revised at periodical intervals showing the life of the bond on the basis of that the value, that the market variable takes, depending on what value the market variable takes, you revise the interest rated periodical intervals, and this is called a floating rate bond.

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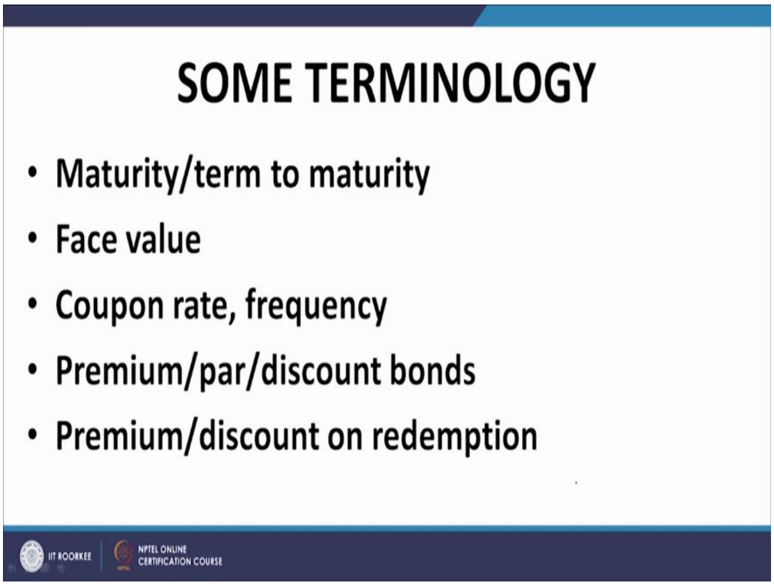

Some terminology, maturity term to maturity, well it is the time at which, it is the life of the bond in a sense, it is the period over which the bone remains active, and at the end of maturity, the principle borrowed against the bond against the issue of the bond is redeemed, it is repaired back to the bondholder, by the bond issuer. The face value, well the face value is the value on the basis of which all computations of Interest as well as the principal redemption are made. I have to emphasize here, that the redemption may not necessarily be at face value.

Although usually it is, but it need not necessarily be the redemption value of a bond may be more than, maybe less than, or maybe equal to phase value, but it is usually specified as a percentage of phase value. it may be redeemed at par, it may be redeemed at a certain percentage, above par or it may be redeemed at a certain percentage below par. So, and of course, coupon payments or they can the payments of interest are with the reference phase value.

So, face value is one benchmark, one standard that is contained in the issue document and that is the basis on which the actual cash flows on account of interest, the actual payments on account of interest are calculated by applying a certain percentage, which is called the coupon rate on the face value, and similarly the redemption value. The redemption value is also determined with reference to the face value, although it is specified in here, and how it is determined? it is specified in the offer document.

Coupon rate, coupon rate is the rate at which interest is spread over the life of the instrument or the annual rate at with the interest is spread to the lender by the borrower. And frequency of coupons, it is not necessary that all bonds that are you should have an annual interest payment, it is very often the case, in fact most of the government instruments, long term instrument, government bonds issue a half year, have half yearly coupon payments. In other words, coupons that they are paid at the end of every 6 months, interest is paid at the end of every 6 months, so the frequency is twice a year.

In most cases, not necessarily you usually it is a half yearly, it can be annual, it can be quarterly, there is no legal restrictions on how the coupons are to be paid over the frequency of payment of the coupons. Premium, par, discount, as I mentioned the bonds can be redeemed at par, the bonds can be redeemed at premium, the bond can be redeemed at discount. If a bond is redeemable at a value, redeemable at a value which is higher than the par value, we call it redemption at a premium, and if the redemption is at value which is lower than par value, or the face value then it is called redemption at a discount.

So, if a bond is being quoted in the market, below par it is said to be quoted at a discount, and if a bond is being quoted in the market at a price, at a value which is above par value, it is quoted at a premium, it is deemed to be quoted at a premium, and a bond which is quoted at par value is quoted at par.

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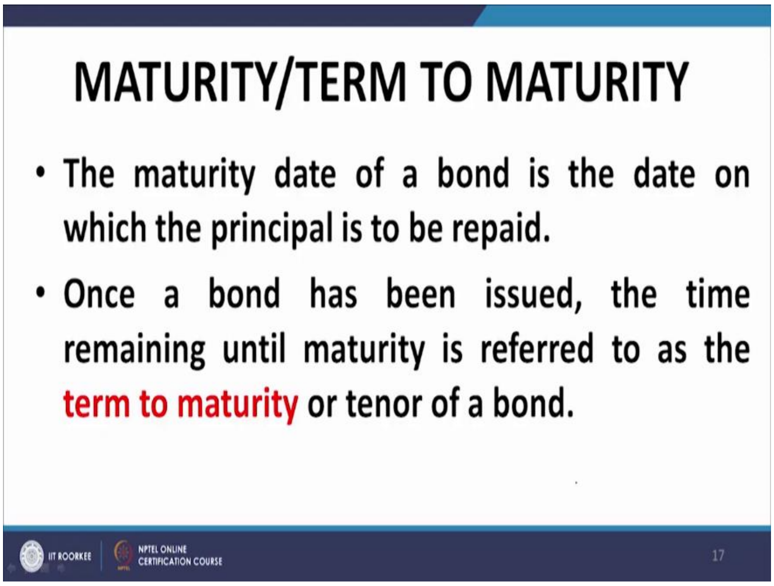

So, this is some terminology, thank you.