Quantitative Investment Management
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Lecture 49
Stock Price Modelling

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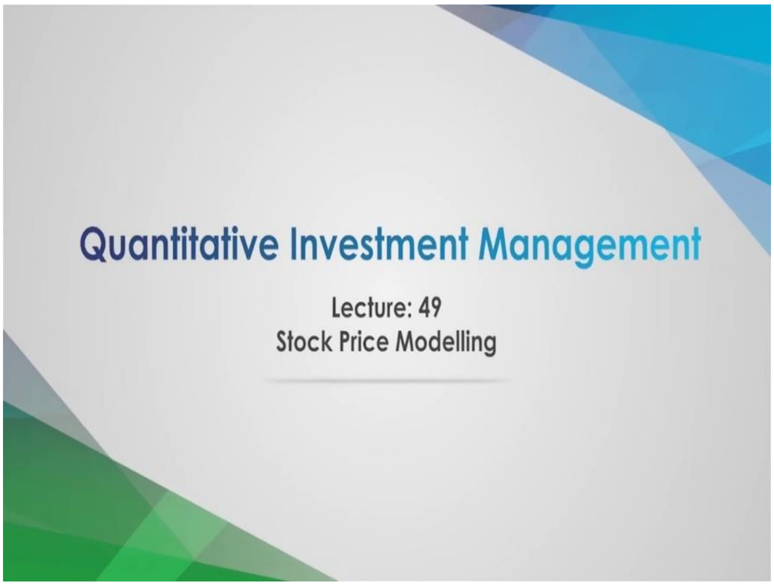

Welcome back. So, towards the end of the last lecture, I had introduced the concept of Brownian motion. Today we will take up from there and we will see how this concept of Brownian motion can be used for the Modeling of Stock Prices. Let us now move to the assumptions that underlie this model, which is based on the concept of Brownian motion that we discussed earlier.

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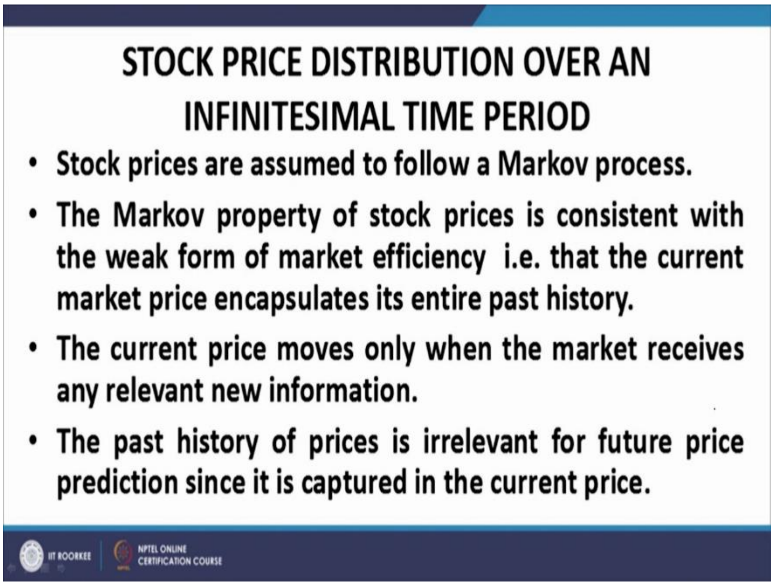

So, the postulates of this model are given on the slide. Stock prices are assumed to follow a Markov process. Markov process you would recall is a process that has no memory. In other words, the future evolution of the process depends on its present state and not and not on the path and that has been followed in arriving at the present state that is what is a Markov process. The Markov property of stock prices is consistent with the weak form of market efficiency. That is that the current market price encapsulates the entire past history.

This is what I mentioned just now, the current price moves only when the market receives any relevant new information this is a part of the efficient market hypothesis the weak form of the efficient market hypothesis and the past history of prices is irrelevant for future price prediction, since it is captured by the current price.

In other words, what the weak form of the efficient market hypothesis which is which is the principal or which is the hypothesis on which this model is based is that the entire past history, how the price has evolved over the past is and the entire information about the prices or the behavior of the prices is encapsulated in the current price.

And therefore, whatever has happened in the past is all encoded in the current price. And whenever some new information percolates down to the market, the price undergoes a change in response to that piece of information and the past history is irrelevant.

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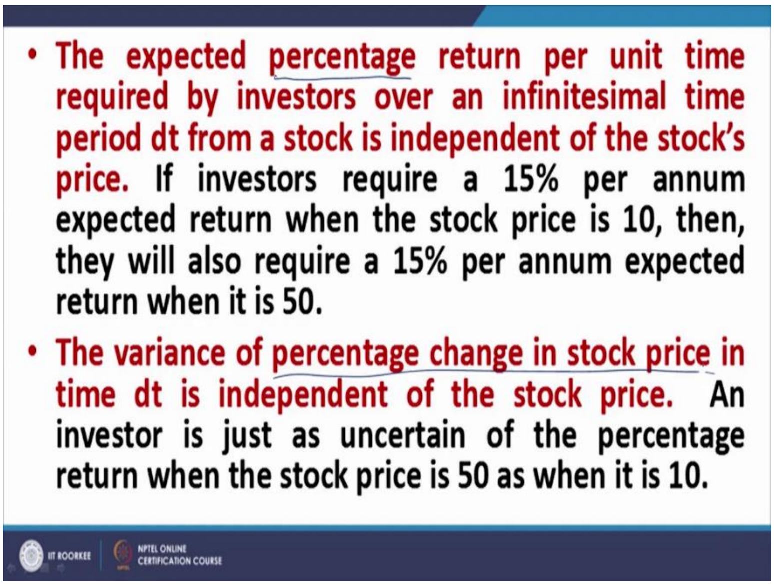

Now, we talk about the quantitative assumptions that go into the modeling of the stock price as per this model, the Brownian motion model. The expected percentage return per unit time, your return per unit time is sometimes abbreviated as the return required by investors over an infinitesimal time period dt from a stock is independent of the stock price. I repeat the expected percentage return per unit time required by investors over an infinitesimal time period dt from a stock is independent of the stock price.

Please note the word percentage return we are not talking about the value appreciation we are talking about the percentage appreciation. Percentage appreciation in the price per unit time and that is independent of the stock price. So, irrespective of whether the stock price of a particular stock is 100 or it is 150 the return there from in a short period of time dt infinitesimal period of time dt would be the same as per the expectation of the investors. The investors require a 15 percent per annum expected return when the stock price is 10 then they will also require a 15 percent per annum expected return when the stock price is 50. That is precisely what I mentioned just now.

The variance the second postulate, the variance of percentage change in the stock price in time dt is independent of the stock price. Please note the variance of what again, the variance of percentage change in stock price. This is important it is not the absolute stock price which is relevant here. It is the percentage change in stock price which is relevant. So, the variance of the percentage change in the stock price in a small time in time interval infinitesimal time dt is independent of the stock price.

An investor is just as uncertain of the percentage return when the stock price is 50 as is when the stock price is 10. Both of these assumptions seem to be rational, seem to be logical and therefore, they form the cornerstone of the backbone of the pricing of the modeling of stock prices.

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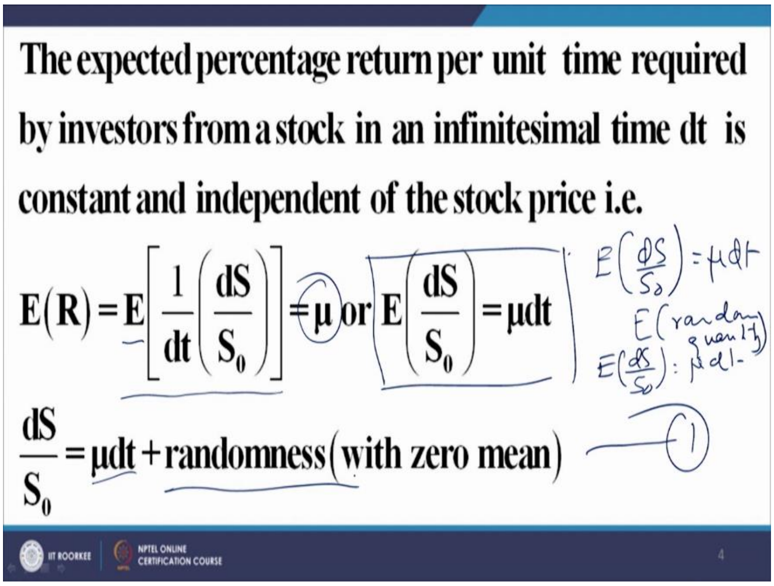

So, let us now put them into the math or let us introduce the math consistent with the about 2 quantitative postulates. The expected percentage return per unit time, this the first postulate the expected percentage return per unit time required by investors from a stock is an infinitesimal time dt is constant as an independent of the stock price.

What is the expected percentage change per unit time, percentage return per unit time is given by this expression which is there in the square bracket's dS by S0 is the percentage change in the stock price divided by dt gives you the percentage change in the stock price per unit time.

Let me repeat percentage return is the percentage change in the stock price per unit time and that is equal to dS upon S0 which is the percentage change in stock price divided by dt which gives you the percentage change in stock price per unit time and this E is the expectation operator. So, this whole expression that we have here that I have underlined represents the expected percentage change in the stock price or expected percentage return per unit time of the stock price.

So, this is independent of the stock price this can be represented by a number usually we call it mu, usually we represent this expected return as mu and therefore, we can write by transposing dt to the right-hand side we can write expected value of the percentage change in stock price as mu dt. Expected value of the percentage change in stock price is mu dt, it is the expression that we have in the box.

So, let me quickly repeat what we have, what I have done. What does the postulate say, the postulate says that the expected percentage return per unit time that is the expected change in the, expected percentage change in the price per unit time required by investors from a stock in an infinitesimal time dt is independent of the stock price.

So, expected percentage change in the stock price per unit time. This this the expression within the round bracket represents the percentage change in stock price. You divided by dt you get the percentage change in the stock price per unit time, you take the expectation there off, you get the expected percentage change in the stock price per unit time.

And this is this is a constant quantity the model says it is a constant quantity we represented by mu and this gives us that the expected percentage change in the stock price is equal to mu dt. In other words, we can write dS upon S naught where dS upon S naught is the percentage change in this stock price is equal to mu dt plus some random factor plus some statistically random factor, some statistical random number which is, which has a 0 mean.

You can see here, if I take the expectation value of let me call it equation number 1, if I take the expectation value of equation number 1, what I get is E of dS upon S naught is equal to mu dt because there is no randomness embedded in the first term and the expectation value of some that random quantity, random quantity. And this term by definition or by assumption is has a 0 mean so, we get E of dS upon S naught is equal to mu dt.

So, in other words, what I can say is, the solution of this equation which is there within the box is given by equation number 1. Let me repeat the solution of the equation that we have in the box given by equation number 1 where this underlined term that I have now is a random term we shall talk more about it, we shall be exploring the structure of this term in greater detail as we progress with this model.

And the basic thing that we need for the moment is that it should have a 0 mean, so that when we take the expectation value of the expression on the right-hand side of the equation number 1 we get the expression within the box.

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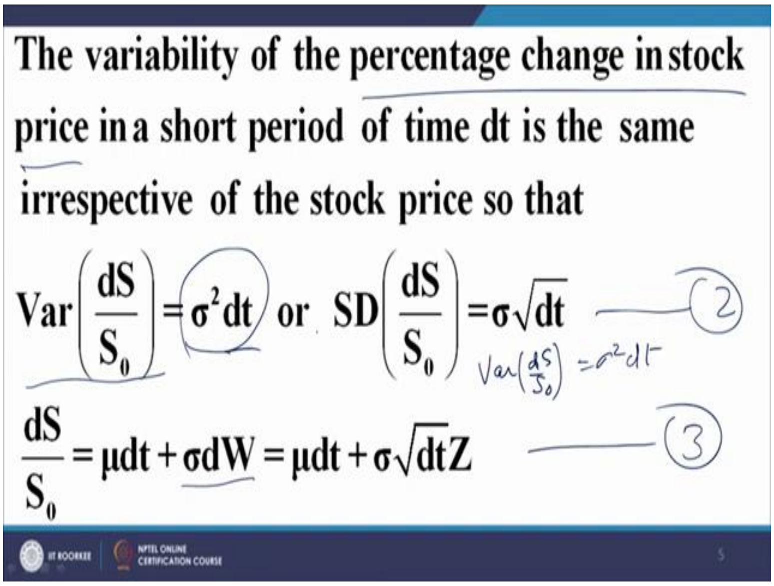

And then, we talk about the second assumption. The variability of the percentage change in stock price in a short period of time dt is the same irrespective of the stock price so that we can write this as the variability of the percentage change in the stock price. This is given by the expression that I underlined just now.

Percentage, please note again, the word uses percentage change in the stock price not absolute change in the stock price. And therefore, because we are using the percentage change in the stock price, we use variants of dS upon S naught the dS upon S naught gives us the percentage change in the stock price and you take the variance thereof and that is modeled as sigma square dt. That is independent of S naught that is the important point.

So, what it says, the model says is that it should be independent of the stock price and it is independent of the stock price. In other words, you can write the standard deviation of dS upon S naught that is the standard deviation of the percentage change in the stock price as sigma under root dt.

Now a model which is consistent with both the assumptions this equation number 1 that is on the previous slide and this let is call this equation number 2 is given by equation number 3 that we have here. This can easily be checked, if we take the expectation value of the left-hand side dS upon S naught mu dt contributes to the expectation value, but sigma dW does not contribute because expectation value of dW is equal to 0, and therefore, we get the expectation value of dS upon S naught is equal to mu dt or the expectation value of 1 upon dt dS upon S naught is equal to mu which was the first assumption.

Then, we look at the variance of this expression, variance of, variance of ds upon S naught if you look at this from the right-hand side mu dt is deterministic, so, it will not contribute to the variance and the entire variance will be encapsulated in the second term sigma dW and the variance of sigma dW is equal to sigma square into dt, which is precisely what the second, the second equation demands.

So, the bottom line of what I have been speaking so far is that we model the stock price by virtue of equation number 3. This is a very important equation and this will be the corner stone or the backbone of what we are going to study more in the context of stock prices modeling.

And but I must emphasize here that we are at the moment talking about infinitesimal time periods everywhere use it d, especially dt and dt represents an infinitesimal time period. So, the model that we are use, we have arrived at as equation number 3 is a model that is appropriate for small time periods, for infinitesimally small time periods. This model would hold good over infinitesimally small time periods for longer time periods like 6 months or 1 year or thereafter, thereafter say 2 years or 5 years. We need to modify this expression which we will do in the course of today’s lecture.

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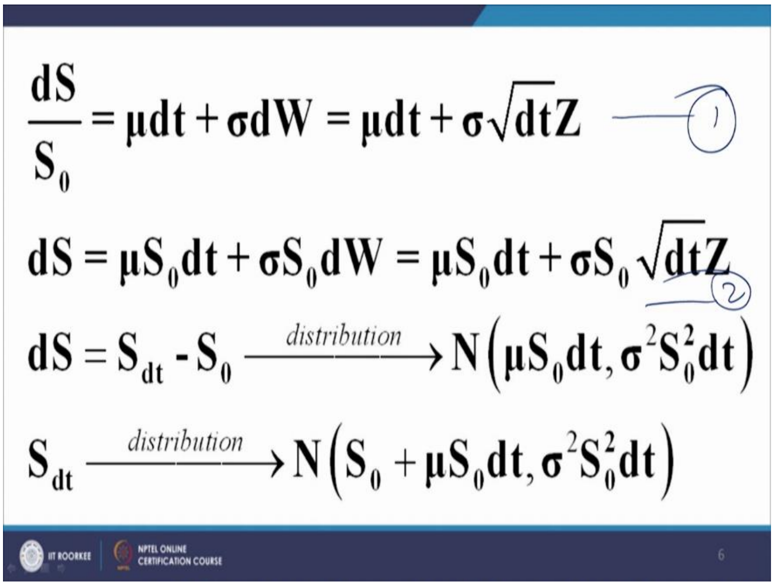

So, let us know explore the cardinals of this particular model. This is what we had from the previous equation I write it as equation number 1 on this slide. This represents our model of the stock price which is consistent with the assumptions that we have made about the stock price.

And from this expression, we can write dS in the form of equation number 2, which is a straight forward cross multiplication throughout by S naught. Now, if we look at this right-hand side of equation number 2, the extreme right-hand side what does it give us? It shows that dS, which is equal to the stock price at time dt minus the stock price at t equal to 0 that is this expression dS is equal to this expression S dt minus S naught, S naught is today's price.

And S dt is price at a time dt from now, and dt is small, of course, that I have mentioned just a couple of minutes back. So, S dS is normally distributed with a mean which is equal to mu S naught dt and with a variance which is equal to sigma square S naught square dt. So, as per this model that we are talking about at the moment, which is appropriate for small time periods for predicting price over small time periods, for example of days or weeks, and not much more than that.

So, dS is equal to S dt minus S naught and S, and dS is normally distributed with a mean of mu S naught dt and a variance of sigma square S naught square dt. In other words, we can say because S naught is known to us S naught is predetermined. We can say that the stock price at a small interval of time dt from now, at an infinitesimal time dt from now is normally distributed as with the mean of S naught plus mu S naught dt and a variance of sigma square S naught square dt. Let me repeat S dt is normally distributed with a mean of S naught plus mu S naught dt and a variance of sigma square S naught square dt.

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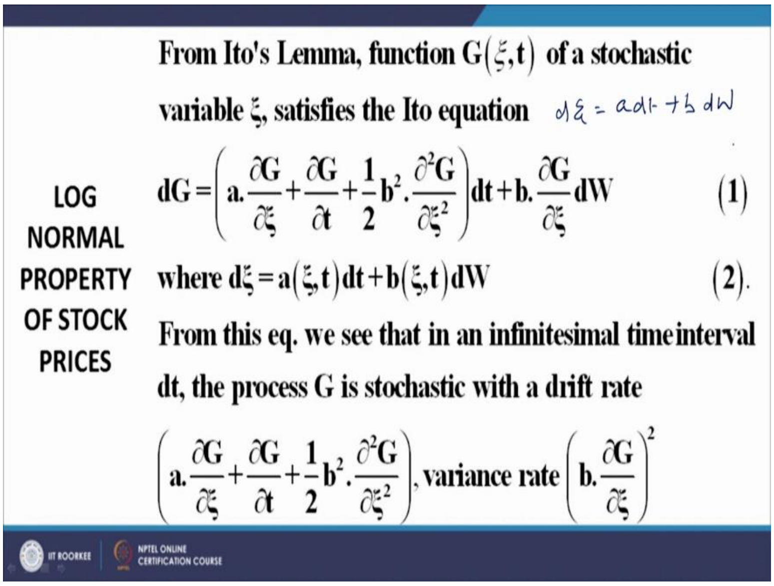

So, this is as far as the infinitesimal time period model of stock price s is concerned, if you are required, if you are given the exercise of predicting or calculating or projecting the stock price at an infinitesimal time period from now, you can use this model appropriately. Now, we try to extend this model with for a finitely long time period.

Let us say you have to predict or estimate the stock price 6 months from now or 1 year from now or more than that for that matter. Then, what kind of model do we use, that is the next question. And for that purpose, we invoke Ito's Lemma. Ito's Lemma is given in equation number 1 if G xi comma t is a stochastic, is a function of the stochastic variable xi, which satisfies the equation a d xi, d xi is equal to, d xi is equal to a dt plus b dW.

This is the stochastic differential equation for the xi and G xi comma t is a function of the xi which is which is captured by this SG and is also an explicit function of time. Then the infinitesimal increments a dG is distributed or follows the stochastic differential equation which is given by equation number 1, this was Ito's Lemma.

So, let me repeat Ito's Lemma the function G xi t of a stochastic variable xi satisfies the Ito equation given by equation number 1 where xi follows the stochastic differential equation which is given by equation number 2 as I have mentioned. Now, from this equation we see that an infinitesimal time interval dt the process G is stochastic.

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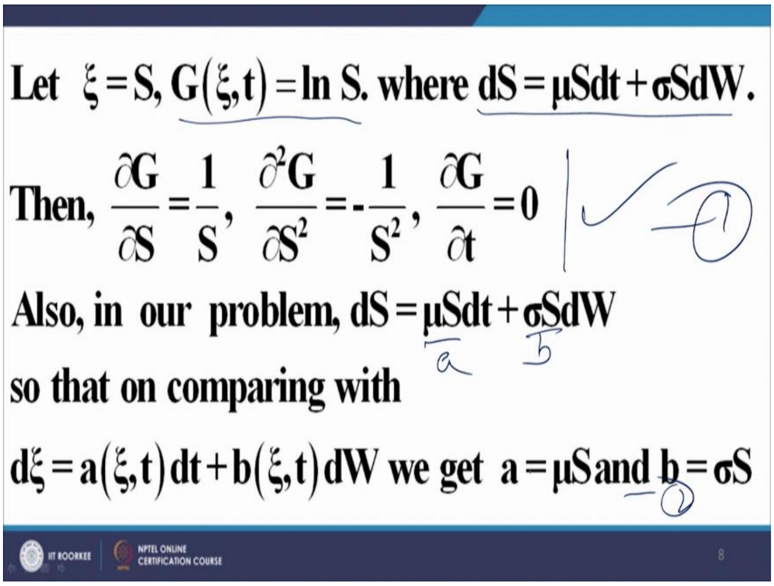

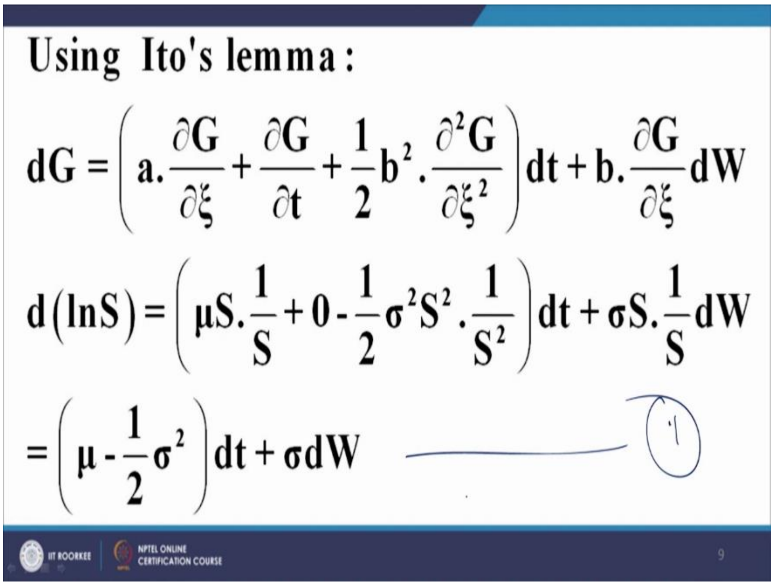

Now, our objective is to use this model, to use this equation which is the Ito's equation the celebrated Ito's equation Ito's Lemma to determine or to create, to formulate a model of the stock prices which is valid or which would hold good over a sustained period of time over a finite period of time rather than the infinitesimal model that we have discussed just now. For that purpose, we let xi equal to S and we define the function G xi t as log of S where S is the stock price, please note this S is the abbreviation for this stock price. G xi t is equal to log S where dS is equal to mu S dt plus sigma S dW.

This is the stochastic differential equation for our stock price as we have worked out few minutes earlier. Now, clearly what we find is that the first derivative partial derivative of G with respect to S is equal to 1 upon S and the second partial of G with respect to S is equal to minus 1 upon S square. And since, G is not an explicit function of time, the partial derivative of G with respect to time is equal to 0, so, this is elementary. Also, in our problem dS is given by mu S dt plus sigma S dW. So, a is equal to mu S and b is equal to sigma S.

So, using all these values, using all these expressions that we have got here, in equation number 1 let us say, and equation number 2 let us say, we substitute this in the Ito's equation and we have kept the expression for the stock price as equation number 1 on this slide.

Please note all I have done here in this equation, in this, I have substitute the various values G is equal to log of S, a is equal to mu S, b is equal to sigma S and del G by del xi is equal to 1 by S. The second partial as minus 1 upon S square and the time partial S0. That is all that has been done here. And when you simplify it a little bit, what you find is that d of log S is equal to mu minus 1 by 2, sigma square dt plus sigma dW.

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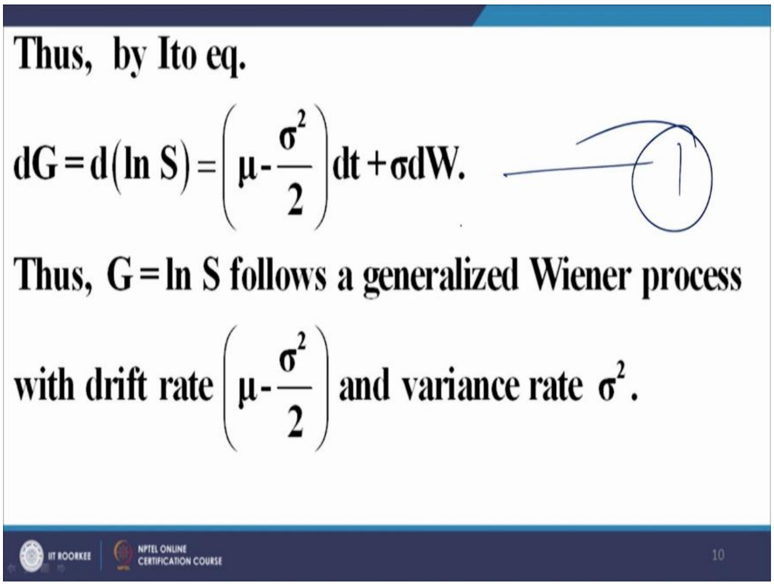

So, what do we infer, we find that d of log S, d of log S is normally distributed or rather it is follows a Wiener process, generalized Wiener process with a drift rate which is given by mu minus 1 by 2 sigma square and a variance rate which is given by sigma square.

Now, let me repeat from this equation the equation number 1 that we have on this slide, we infer that log of S is it follows the equation that is going to equation number 1 or it follows generalized Wiener process with a mean equal to mu minus 1 by 2 sigma square. Drift rate rather not the mean, drift rate mu minus 1 by 2 sigma square dt and a variance rate equal to sigma square. I shall come back to what is the mean and variance in the next slide for the moment let me repeat the drift rate is equal to mu minus 1 by 2 sigma square and the variance rate is equal to sigma square.

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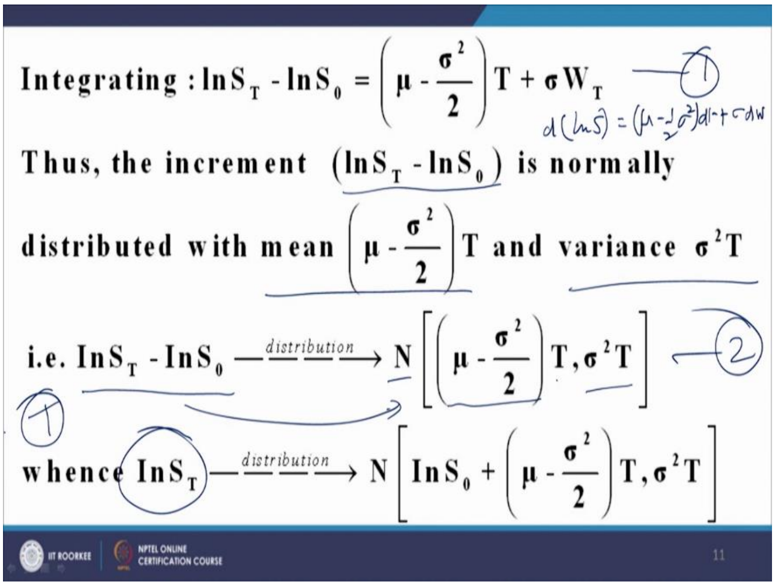

And now, from the previous equation what expression what did we have, we had d of d of log of S is equal to mu minus 1 by 2 sigma square dt plus sigma dW. So, this is what we had for the expression d of log of S. Let us integrate this both sides of this equation within the limits ST and S0 and what we get here is equation number 1 on this slide.

On integrating this expression that I have written here in pen in ink, what we get is equation number 1. Therefore, from this we find that the increment log of ST minus log of S 0 is normally distributed with a mean which is equal to mu minus sigma square upon T bracket into a mu minus 1 by 2 sigma square bracket T and a variance that is equal to sigma square capital T.

Let me repeat this is very important that increment log of S capital T minus log of S naught is normally distributed with a mean which is equal to mu minus sigma square upon 2 into T plus and with a variance equal to sigma square T. This is what we have here, mean is this expression and variance is this expression. Therefore, I can write log of ST minus log of S0 in this form that we call it equation number 2. Normally distributed with a mean mu minus sigma square upon 2 into T and a various equal to sigma square upon sigma square into T.

Now, I can transpose this log of S naught to the right-hand side to within the normal function, normal distribution function and therefore, I can conclude that log of ST is normally distributed with a mean of log S0 plus mu minus sigma square upon 2 into T and a variance equal to sigma square T.

I repeat the log of ST where capital T is not infinitesimal, capital T is not infinitesimal, capital T is a finite time interval between 0, therefore, this model holds for a finite time interval. We have log of ST is normally distributed with a mean of log S0 plus mu minus sigma square upon 2 into T and a variance equal to sigma square.

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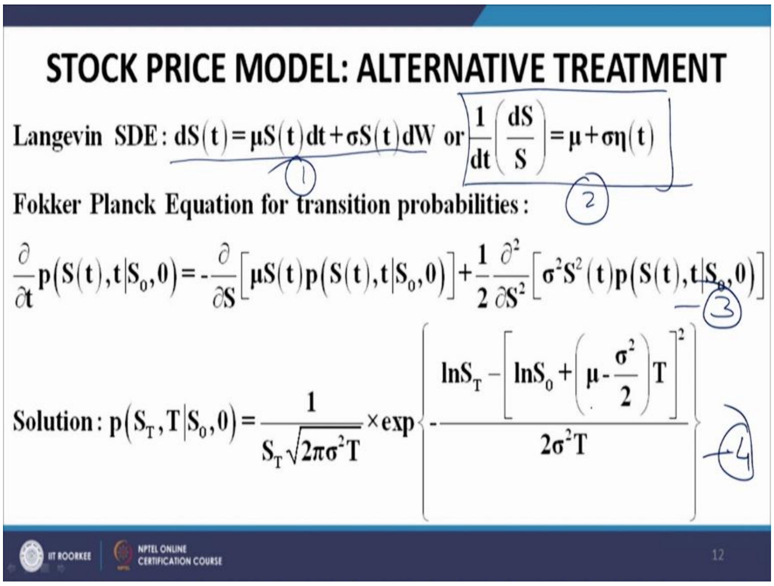

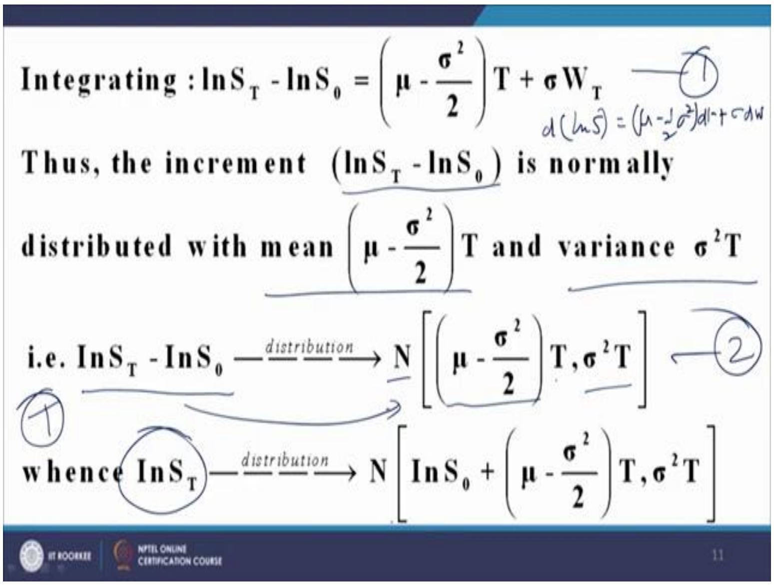

There is an alternative approach to this, which employs some interesting equations that we encountered in mathematical physics, namely the Langevin Equation, the Fokker Planck Equations. We can write the Ito equation for the stock price that I have underlined here in the form of a Langevin equation which is given in this inbox that I am creating just now. What is eta t, eta t is called the white noise.

It is in some sense not exactly so not mathematically correctly so, but roughly so it is a derivative of the Brownian motion, roughly and not precisely. So, corresponding, we can write the equation that let us call it equation number 1 in the form of equation number 2. And corresponding to equation number 2, which is a Langevin equation for the stock price. We can write the corresponding Fokker Planck equation which is the equation followed by the transition probabilities corresponding to the variables which are captured by the Langevin equation number 2 in the form of equation number 3.

And if you solve equation number 3, it is a tedious process I will not get into the nuances of solving this equation, although it can be solved exactly. For that matter, what we find is that we get the expression, which is equation number 4 for the probability density function. And we recognize this probability density function as the probability density function of a log normal distribution.

So, the stock price in the finite time model follows a log normal distribution, and this is precisely what is this equation is telling you, if you see this equation, what do you find we find that the log of ST is normally distributed with a mean which is given by log S0 plus mu minus sigma square upon 2 into T and a variance of sigma square t, that means what. That means ST follows a log normal distribution, log ST follows a normal distribution. Therefore, ST follow the log normal distribution if log x follows the normal distribution, the next follows the log normal distribution, so that is the definition of log normal distribution.

And here again, we point that the probability density function for transition probabilities turns out to be the one that we have for the log normal distribution showing again, that the stock price infinite, at finite times over finite intervals follows a log normal distribution.

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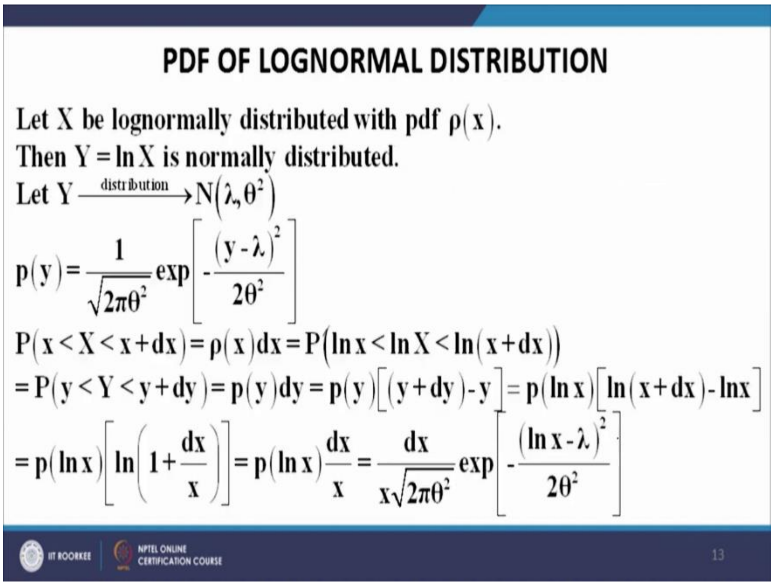

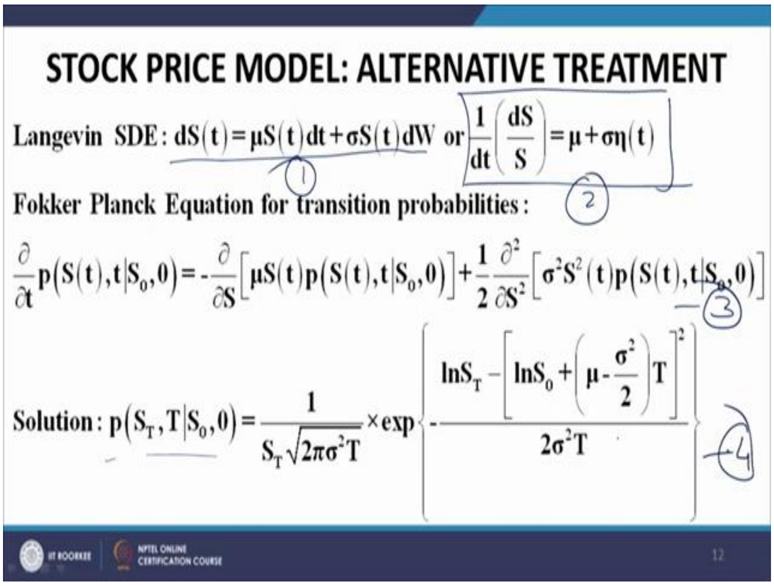

This gives you the PDF of the log normal distribution. Again, I will not devote time to this, this is a bit of a digression. So, I will retain it in the notes, I will retain it in the PPT, but it gives you the PDF of the log normal distribution, which is here, equation number 1 and you can compare this equation number 1 that you have on the slide with the expression that you have here. This is equation number 4, and you will find literally 1 to 1 correspondence showing that our stock price follows a log normal distribution over finite time.

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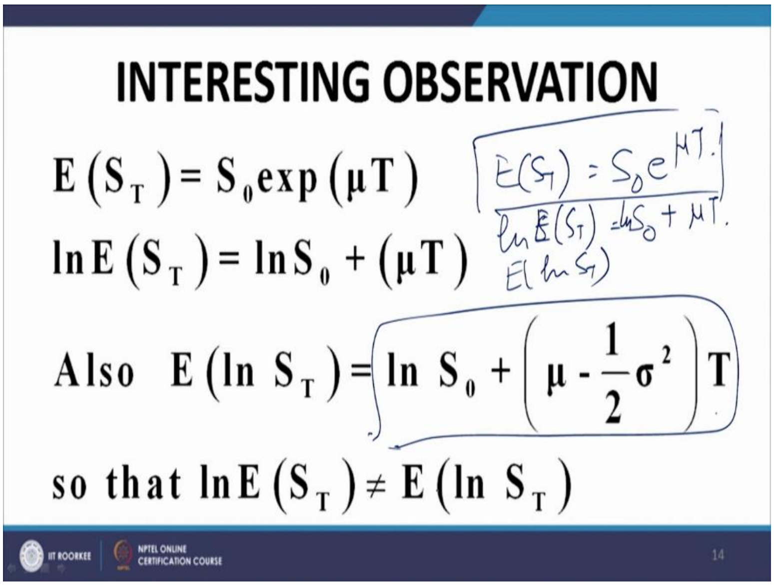

And there is an interesting observation here, if you look at this carefully, we have if mu is the expected return of the stock price, then we can write E of ST is equal to S naught e to the power mu T. Mu is the expected return then we can write this expression straight away provided mu is of course, continuously compounded and so on. If I take the log of both sides, what do I get? I get log of ST, logs sorry, log E of ST is equal to S naught plus mu T. I am sorry, log S naught plus mu T.

Now, let us look at what is E of log ST. This is log E of ST, log E of ST is equal to log S naught plus mu T. Now, let us look at what is E log of ST, we have just seen that E log of ST is normally distributed with a mean which is equal to log S naught plus mu minus 1 by 2 sigma square. And the mean of this expression E of log is St is equal to this expression that I have encapsulated in the box.

So, let me repeat log ST is normally distributed with a mean of this expression and variance of sigma square T and therefore, the expected value of log ST is nothing but the mean value and that is equal to the expression that I have put in the box. The outcome of what I have tried to establish here, is that the log of E of ST is not equal to E log of ST. They are not, in some sense they are not connotative.

So, let me repeat log E of ST is not equal to E log of ST that is an important observation. You can see here the proof it is quite straightforward, it is quite simple. So, this is an interesting observation.

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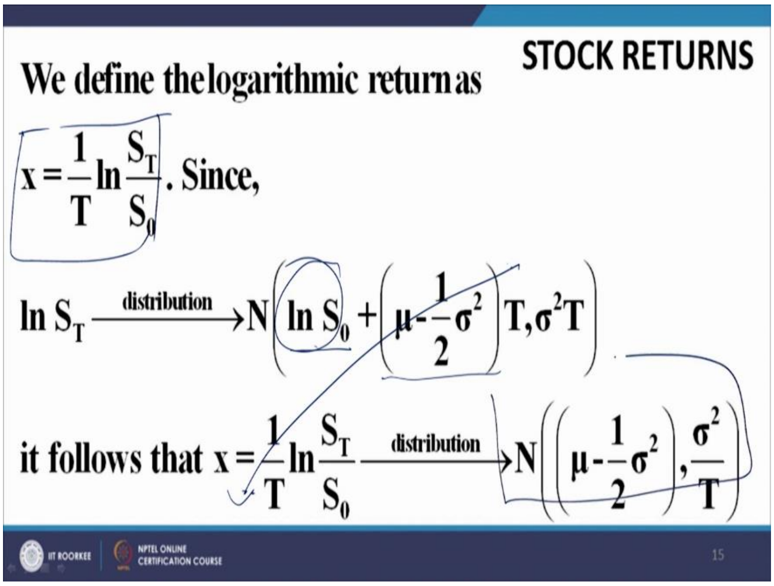

Now, we talk about logarithmic return. Now, we define logarithmic return in terms of the expression here, x is equal to 1 by T log ST upon S naught and when we use the fact that log ST is normally distributed with a mean of log S0 plus mu minus 1 by 2 sigma square T. And a variance of sigma square T we can straight away bring in the T to the left-hand side and take this log S naught also to the left-hand side divide throughout by T and what we get is that x equal to 1 by T. Log ST upon S naught is normally distributed with a mean equal to mu minus 1 by 2 sigma square this T will not be there this T has gone to the left-hand side. This T has been, has come here and a variance when you work out the variance you will divide throughout by T square.

And so, the variance of the return will be equal to sigma square upon T. So, let me repeat what I have done is simply a rearrangement, nothing more, nothing less. We know that log ST is normally distributed with a mean of log S naught plus mu minus 1 by 2 sigma square T and a variance of sigma square T. We take this log S naught to the left-hand side and we divide throughout by T. And when we take T within this normal function the mean is divided by T and the variance is divided by T square and that is precisely what we have in this box.

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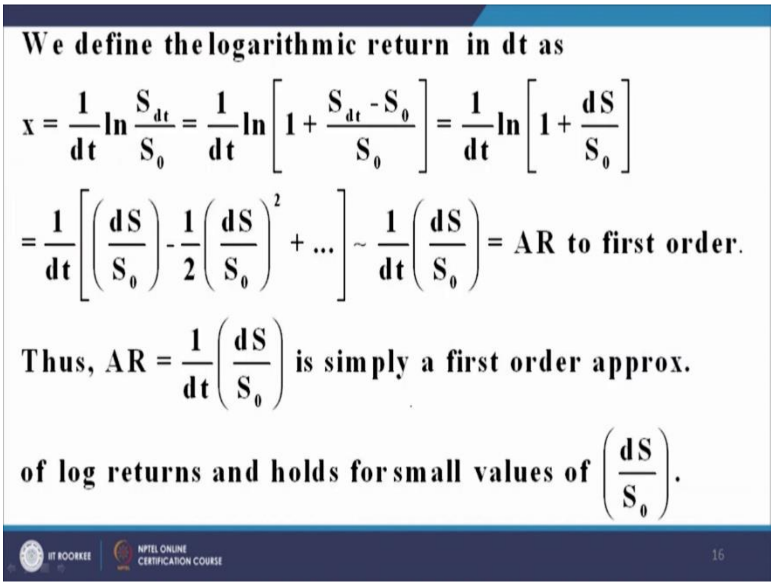

So, from here I will continue in the next lecturer. Thank you.