Quantitative Investment Management
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Lecture 48
Stochastic Calculus

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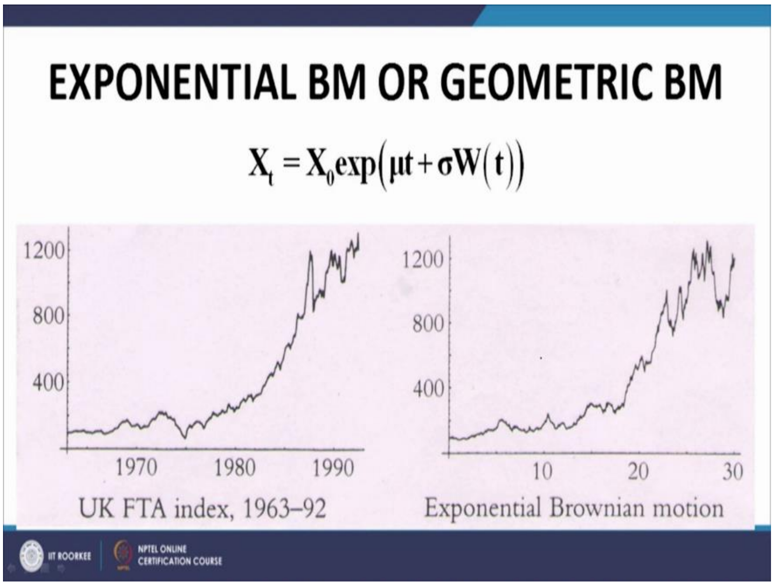

Welcome back. So, let us continue from where we left off before the break. This table that this figure that you have on the slide gives you the Exponential Brownian Motion which say normally used to capture an exponential trend. You can see here how beautiful this figure reproduces, or is able to model the behavior of prices in the UK. So, this is another approach to the modeling of prices where we find empirically that there is an exponential increasing trend or in addition to of course, the local fluctuations.

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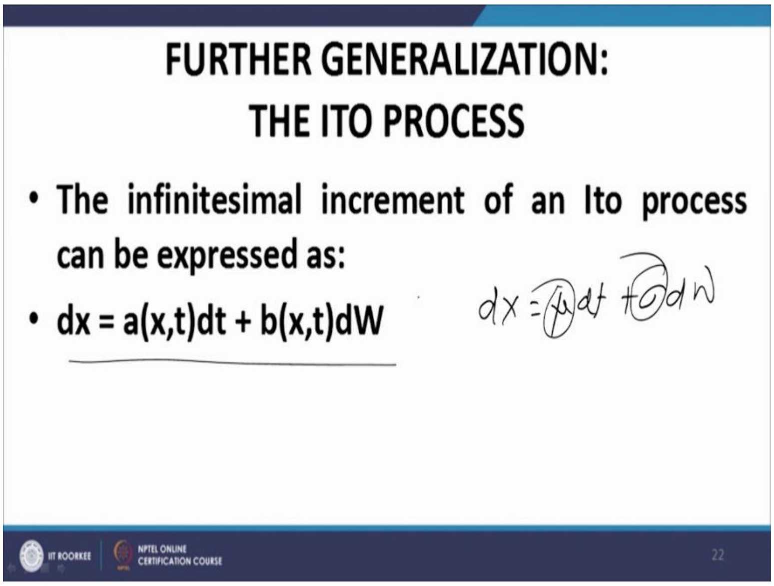

So, further generalizations of the Brownian motion how can we do it? You see so for when I talked about the generalized Brownian motion that was what, that was dx is equal to mu dt plus sigma into the infinitesimal Brownian motion incrementally dW. Now, in this process which was called the generalized veneer process or the generalized Brownian motion. Mu and sigma were constant but there can be processes where these both mu and sigma are functions not only of t but are also functions of x itself which is shown here in this particular equation. This process is called an Ito process. We shall not be dealing much with Ito processes, our discussion, in so far as the modeling of stock prices is concerned will be confined to the use of generalized Brownian motion.

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So, now, we come to a very very interesting and exciting area where we try to explore the ramifications of the incorporation of stochasticity or randomness into the functions of variables and how we can proceed or what changes we require, what modifications we require, when we move from the conventional Newtonian calculus to stochastic calculus. When we use calculus or while using trajectories which are not deterministic, but which are which incorporate an element of randomness and which are there for functions of Brownian motion.

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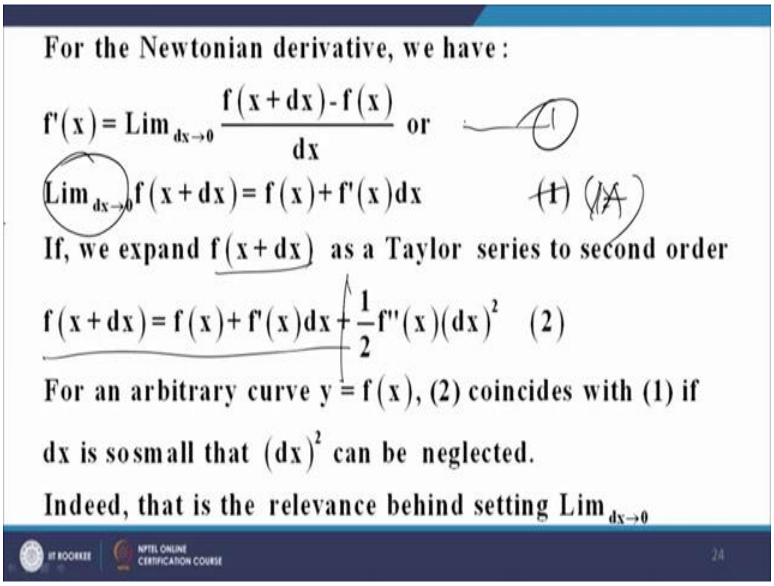

Let us start with a very simple exploration, equation number 1 here this particular equation either written in this form or form equation number 1 or this particular equation. Let me renumber it as equation number 1A whether it in any question form of equation 1 or equation 1A is the contemporary approach to Newtonian differentiation. This is what we have studied in our lower classes in our 11th standard or 12 standard whenever we get an introduction to differentiability or calculus.

Now, let us move a little bit ahead, let us move to the next step which is the Taylor expansion of functions. If we do a Taylor expansion of fx plus dx around the point x, what we get is equation number 2. fx plus dx is equal to fx plus f dash x into dx plus 1 by 2 f double dash x into dx square plus higher powers of dx squared dx for that matter.

Now, consider an arbitrary curve y is equal to fx. Equation number 2 will coincide with equation number 1 in what circumstances, it will coincide with equation number 1 in only one situation when the dx that we are talking about small enough so that we can neglect higher powers of dx that is dx squared dx cube and so on. These values become so small that they become insignificant and as a result of which we can truncate the Taylor series at the point after the first order expansion.

This first order expansion if you look at it carefully, is absolutely similar to equation number 1 and in fact, the use of this limiting criterion limit dx tends to 0, is only to justify what I have mentioned just now, the dx must be very small. In other words, the point represented by x plus dx must be very close to the point x. So that so that when we try to expand the curve around fx, we are not able to do a good enough job by truncating the Taylor expansion at the first point at the first degree.

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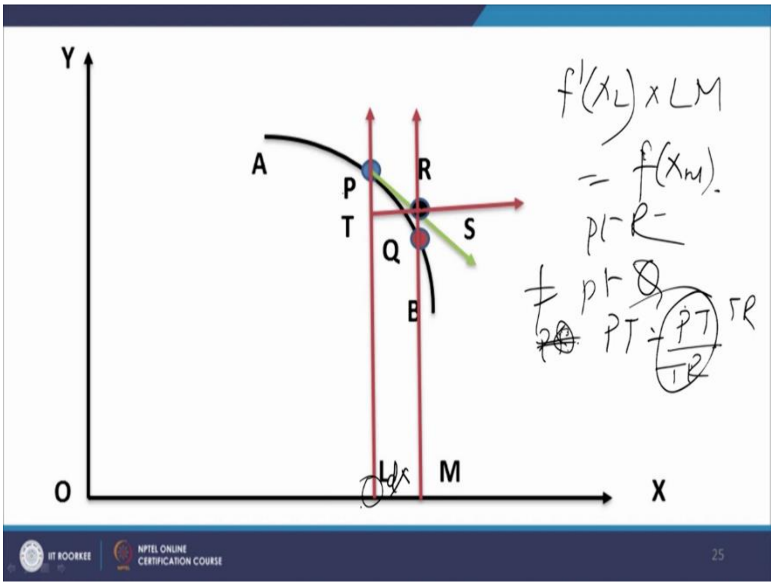

Let us look at it graphically or geometrically what it represents. Let us say we have a curve AB. Let us say PQ is a segment of that curve, we are at the point P, the point P corresponding to, correspond to the point to the X coordinate L, X equal to L and we want to determine or we want to estimate the value of the Y value of this curve the ordinate of this curve. When the X coordinate moves from L to M.

Let me repeat A B is a curve which represents the trajectory of a deterministic function P is an arbitrary point on that curve which is which is modeled or which is represented by X equal to L. Then, our objective is to calculate the value or calculate the trajectory value at the point Q which correspondence to report X equal to M.

Now, if we use the Newtonian calculus, how would we do it, first of all, we will work out the derivative of the curve at the point P let us call it f dash XL and then what we will do is we will multiply it by the distance LM. So, f dash x into LM will give me the value of the curve f of XM. This is what we would do as per the Newtonian calculus.

But if you look at it very carefully what we are doing by this method is we are assuming that the region or the portion of the curve between P and Q is a straight line and we are using and on the assumption on the premise that the distance the curve structure between P and Q is a straight line. We are working out the derivative or which represent the slope of the tangent and in actual fact, this value that we get is the point R and it is not the point Q.

Why is it so, because we have approximated PQ by a straight line and we are using the slope of the straight line and then we are working out the value corresponding to the shift from XL to XM or L to M and using that what we find is we arrive at the point R because what do we have, we have PR, we have PQ, I am sorry, we have P T is equal to P T divided by P divided by TR into TR and this is what is the slope PT upon TR is the slope at this particular point that is the point P.

So, we are arriving at the point R, we are not arriving at the point Q and this difference is the difference that is represented by the higher order terms in the Taylor series. If you use a Taylor series to make the same expansion, you will get a better and better results as you make more use of more and more terms in the Taylor series. If you use this the second order, if you include the second order correction represented by the second-degree term in the Taylor series, you will be able to account significantly for the curvature of this the concavity of this and then the point R that you get will be much closer to Q and your estimate would be much more accurate.

So, that is the fundamental thing in Newtonian differentiation when we talk about Newtonian differentiation the basic idea is to split up to truncate the curve into infinitesimal small straight lines. In other words, putting it the other way around, we can say that when we talk about Newtonian differentiation, we are assuming that we can truncate the or we can the straight line the curve that represents the trajectory of the deterministic function can be represented or can be equated to an assortment of infinitesimal small straight lines.

The deterministic curve that we differentiate can be assumed to comprise of an assortment of infinitesimal small straight lines. PR is some is a similar situation where we if the points a point if this, L and M are very close to each other, then the difference between R and Q is going to be smaller and smaller and smaller.

And if L and M are infinitesimally close to each other, then our first order approximation is good enough but as M moves away from L, what happens? The difference between R and Q also increases and that is the reason that the second order corrections become more and more significant as we talk about dx being larger and larger. This is basically dx LM is what we call dx. So, given this backdrop, let us know look at how the situation changes when we talk about stochastic calculus.

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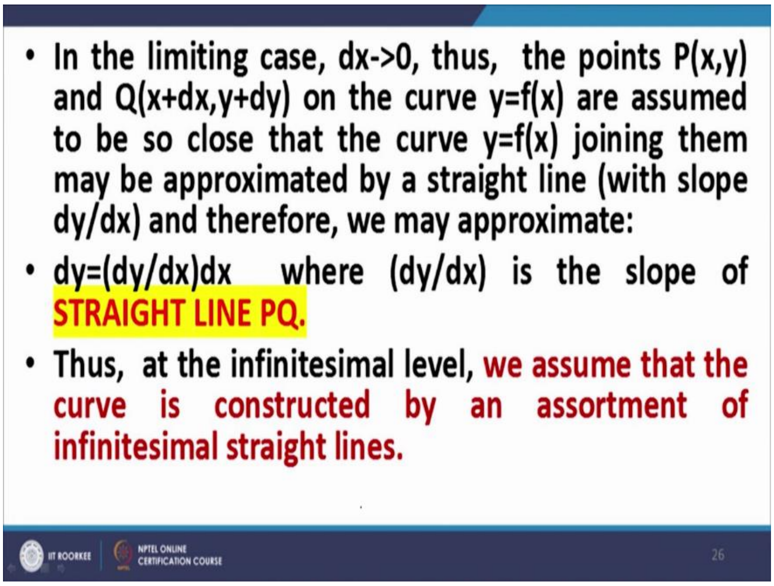

So, this is what I have explained just now, let me read it out. In the limiting case dx tends to 0, thus, the points P and Q on the curve Y equal to fx are assumed to be so close that the curve Y equal to fx, joining them may be approximated by a straight line the slope given by dy by dx and therefore, we can approximate the value of the point at the point Q by using that particular slope by using that particular slope dy by dx.

Although, which is a first order approximation where dy by dx is the slope of the straight-line PQ. Thus, the infinitesimal level we assume that the curve is constructed by an assortment of infinitesimal straight lines.

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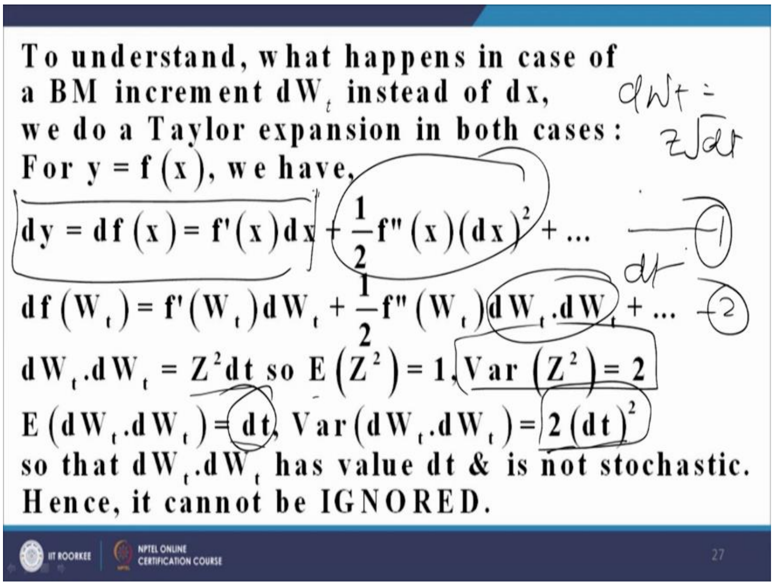

Now, let us see what happens if we replace the traditional deterministic functions of x by a function of a Brownian motion. Let us see what happens in an analogy with equation number 1 which is the Taylor equation for which is the Taylor equation for a deterministic function. We can write the Taylor equation for a function of a Brownian motion which is I will write as F of Wt and we can write this in the form of equation number 2 here, d of f Wt is equal to f dash Wt, d f Wt plus 1 by 2 f double dash Wt dWt d dot dWt.

Now, we saw that if we ignore this part if dx is small enough that we can ignore this part which is usually the case which is the case when we impose the condition limit dx tends to the 0. How does the dx tends to 0 limit manifests itself, it manifests itself by enabling us to ignore all higher values of all higher powers of dx, dx square, dx cube and so on and confine ourselves to the first order term dx.

Now, this is the situation as far as Newtonian calculus is concerned. Wherever we do Newtonian calculus, we make the assumption dx small we do not need to consider dx square, dx cube and so on. And we can confine ourselves to this particular part which I have put in a square.

Now, can we do the same thing for f W, or can we do the same thing for dW rather. Can we say that dW squared or dW dot dW can be ignored and thrown away? That is the question that we need to answer. Let us explore that. Now, we know that dWt, dWt can be represented in terms of the standard normal variate as Z under root dt. We have seen that it is one of the fundamental properties of Brownian motion.

So, we use that and what we get is dWt dot Wt is equal to Z square dt. What is the expected value of this, the expected value of this is equal to, expected value of this is equal to dt why it is dt, because the expected value of Z square is equal to 1 and therefore, expected value of dWt, dWt is equal to dt.

What about the variance? Now, first thing we notice here is that the expected value is not 0. Please note because we are talking about stochastic function, we are talking about functions of a stochastic variable, we have to consider the mean and variance as first representations of those variables. And we can see here immediately that the expected value of dW square is not 0 it is 1. It is a dt I am sorry, it is not 0, dt small but it is first order in t that is another important feature, dt may be small agreed, but it is first order in t and we cannot throw away first order terms without further exploration.

At the very outset, we find a difference between what we did in equation number 1 here, which was Newtonian calculus, and what was the scenario that we encounter when we talk about stochastic variables. Let us now see at how the variance behaves, the variance of Z square is equal to 2. You take it from me for the moment it has improved in the later section of this presentation, we will call it we will talk about it also for the moment we let us assume that variance of Z square is equal to 2.

Therefore, what is the variance of dW dot dW that will be equal to 2 into dt square. Now, when we see another observation. What we find is that the variance is not in first order in dt, we saw in the case of Brownian motion what was the various Brownian motion, the variance of Brownian motion was dt, it was first ordered in dt. Here, we find that the variances second order in dt it is dt square.

Now, in line with the convention the standard process that we have if dt is infinitesimal dt square becomes irrelevant, that is what we have been talking about when we talk about Newtonian calculus. We using the same logic what we find is that, because the variance of dW dot dW is second order in dt, it is much too small to be of any relevance in our analysis.

What, if the variance is very small, if the variance is so small that it can be ignored, what can we say about the process, what can we say about the stochastic process, we can say or the random variables, we can say that the random variable is deterministic, the random variable will not take random values because if there is any element of randomness, it must be captured by the variance. And if the variance is 0 that means, there is no randomness in the process.

Now, you can see here that the variances of second order in dt, dt squared and therefore, it cannot, it cannot be your significance, it needs to be ignored, it needs to be thrown away in line with our standard philosophy, standard procedure. And therefore, what we find is that the mean of dW dot dt is dt. The variance is 0 and therefore, what can we say about this particular quantity dW dot dW. It is deterministic and it has a value of dt because it variance is 0 means this value is deterministic. And if this value is deterministic, then that is also its expected value.

So, in this case what happens, this expression, dW dot dW has a deterministic value, which is first order in time and because it is first ordered in time you cannot throw it away and therefore, the process, the philosophy of throwing away the second and higher order terms will not operate when we talk about functions or stochastic calculus. That is the fundamental difference between functions of normal variables or conventional calculus or Newtonian calculus and (())(18:03) functions or what we call stochastic calculus.

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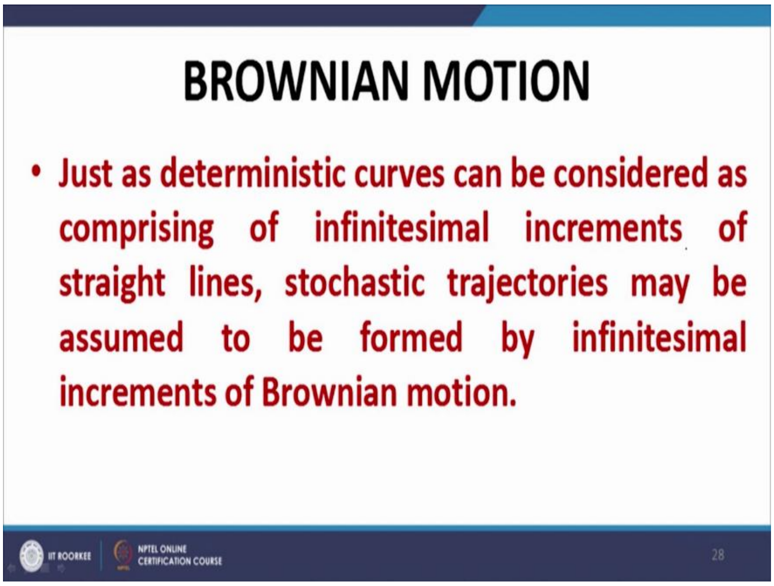

So, just as deterministic curves can be considered as comprising of infinitesimal increments of straight lines, stochastic trajectories may be assumed to be formed by infinitesimal increments of Brownian motion.

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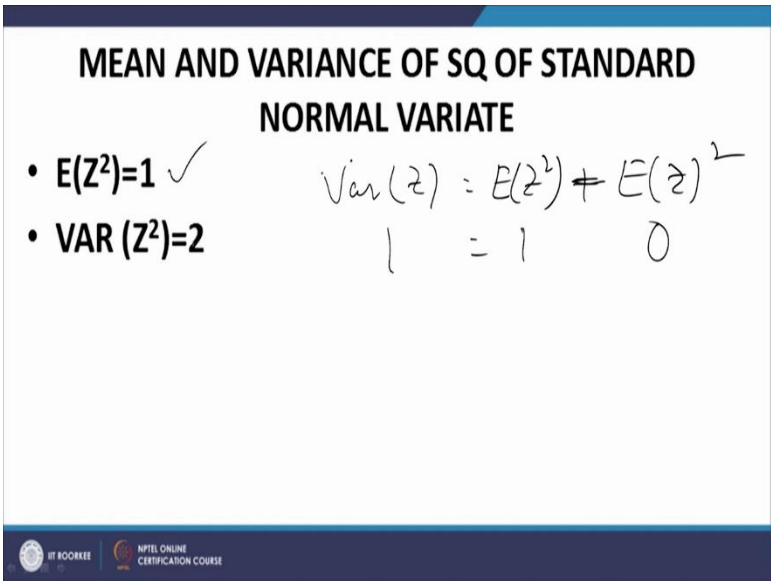

Mean of variance of square of standard normal variate I will not devote time to it, mean of Z square is equal to 1 this is quite simple. We have variance of Z is equal to E of Z square plus minus I am sorry E of Z whole square this is 0, this is this is 1 so, this has to be 1. So, that is this is elementary.

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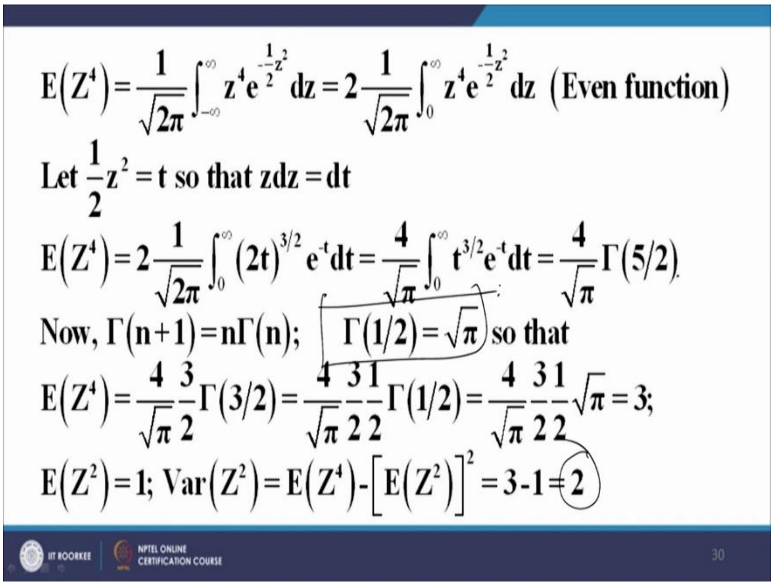

The calculation of the variance is slightly more involved, I have put it in the presentation, it is given here, we find that the variance is equal to 2 we need to use gamma functions for this purpose and gamma 1 by 2 is equal to root pi that also we need to use this result. Using this result, what we find is that the variance of Z square is equal to 2. I leave it as an exercise for the mathematically inclined learner and let us move further.

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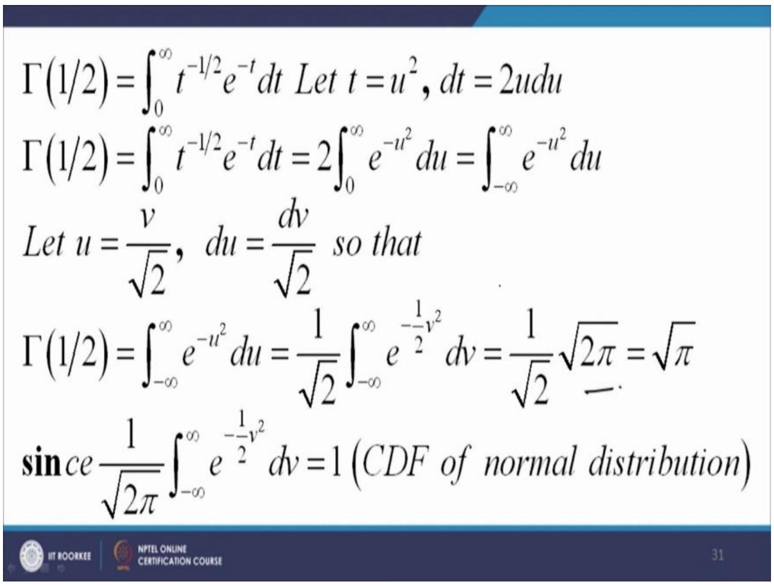

This is the proof that the gamma function of 1 by 2 is equal to under root pi. This is also for information, it is a digression on the continuation of this course, but nevertheless, I have put it in the presentation so that the interested learner can always refer to this.

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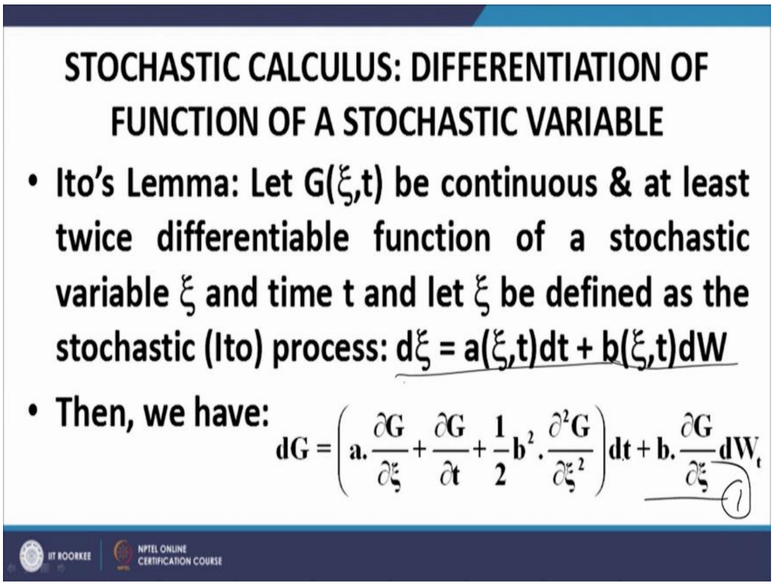

Now, we talk about differentiation of a function of the stochastic variable. So far, what we have found is that the Newtonian machinery that we have with us is not good enough for us to differentiate or to play around or doing calculus of functions or stochastic variables. Where does this discrepancy manifest itself, where do we go from here. We have agreed that it is not good enough, but what is good enough how do we arrive at the appropriate machinery, appropriate calculus for doing differentiation of stochastic processes is our next exercise.

Let that is this is a famous celebrated Ito's Lemma, what does it say? It says let G Xi comma t G is a function of Xi and also an explicit function of time. Be continuous and at least twice differentiable function of a stochastic variables xi and time t. And let xi be defined by the stochastic differential equation, d xi is equal to a dt plus b dW dt is the infinitesimal time increment and dW as you know is the infinitesimal Brownian motion increment.

So, d xi or xi is a stochastic process that is represented by the stochastic differential equation that you have in front of you, in front of you and G xi t is a function of this stochastic variable xi and time is an explicit function of time as well. Then, we have dG is equal to this expression, which let us call equation number 1. Our objective law is to ascribe a skeletal proof to this particular expression.

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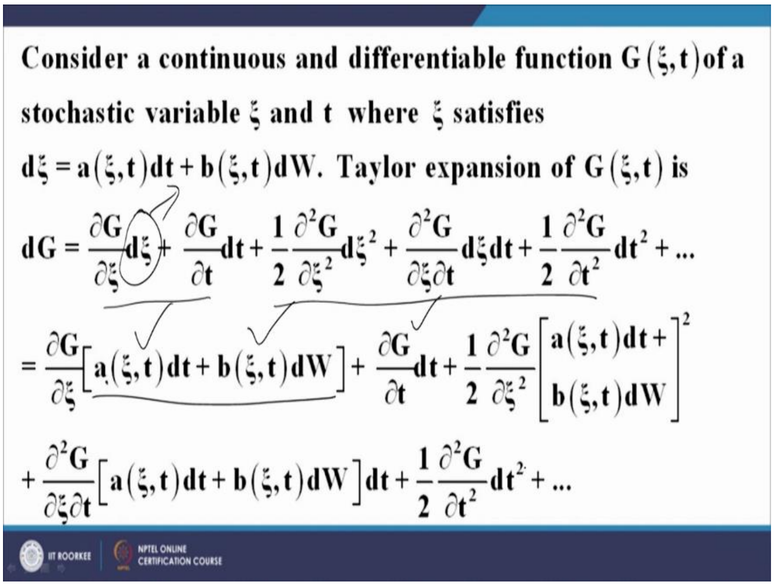

Consider a continuous and differentiable function G xi t of a stochastic variable xi and t, where xi satisfy the stochastic differential equation given by d xi is equal to a dt plus b dW where the symbols are the usual meaning. We do the Taylor expansion of G xi t and what do we get dG is equal to del G upon del xi into d xi, the first order terms these are the two first order terms and then we get the second order terms.

There are two first order terms, there are three second order terms. Now, as far as the first order terms, we have are concerned we substitute d xi by this expression that we have here a dt plus b dW. So, that is what we have done in this equation del G but del xi into this a plus b, a dt plus b dW please note the, this is first order in dt and this is in first order in dW. So, we have no issues with either of them they will be carried forward as it is.

So, this is okay and this is also okay. This is also okay. We have done nothing about this. Now we come to the important part. Let us look at d xi square. If you simplify this expression d xi square put using xi equal to a dt plus b dW, what do we get?

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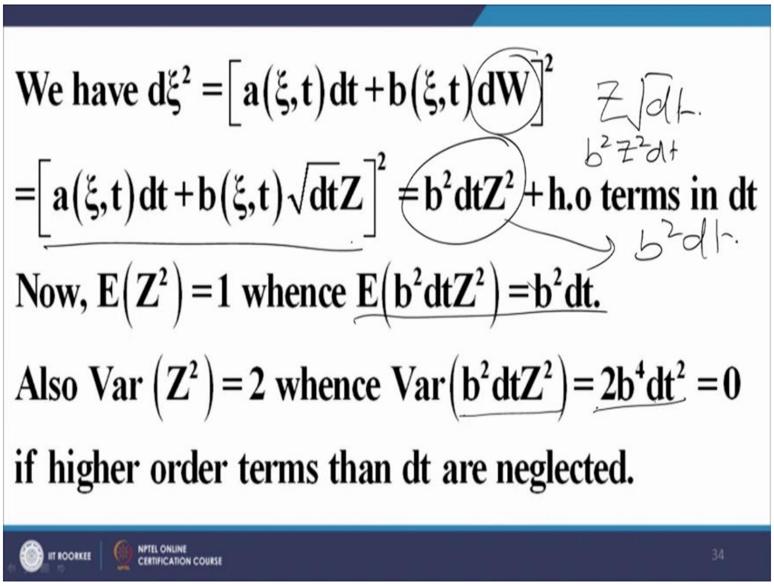

Let us look at it in this slide, what we get is this dW, I can write in the form of the standard normal variate Z under root dt. Now, suppose I open the square what do I get? The first term will be a square dt square which is of second order in dt. So, we throw it away we do not consider it. When we talk about this second or second term, the second term is b square, dW square and dW square is equal to b square into dW square is Z square dt, b square into Z square dt y because dW is equal to Z under root dt.

Now, let us explore this expression carefully. What is the mean of this? The mean of this is equal to b square dt because the mean of Z square which is the stochastic part is equal to 1 and therefore, E of b squared dt, Z square is equal to b square dt. What about the variance? If you look at the variance, we have already shown in the previous slides, that variance of Z square is equal to 2. Therefore, variance of b square dt Z square is equal to b to the power 4 dt square into the variance of Z square, which is 2. So, that is equal to 2 b to the power 4 dt square.

Now again, I invoke the discussion that I had a few minutes ago, dt square is small enough to be ignored. And that shows what that shows that the variance of this particular quantity that we have here b square dt, Z square the variance is almost 0, close to 0, close enough for us to ignore it.

And therefore, we take the variance as 0 the mean is equal to b square dt. So, therefore, this expression will manifest itself will continue to be relevant notwithstanding the fact that it is in second order in dW notwithstanding that fact, it will continue to be relevant, and it will be carried forward as b square dt.

What about the cross term? Cross term is dt into under root dt which is of the order 3 by 2 in dt which also we can throw away. So, this entire square term has only 1 term of significance, that is b square dt into Z square, so, this is I have discussed this.

Now, what about the rest of the terms? If you look at the rest of the terms let us go back, a dt into b dW into dt, dt into dt will give you dt square which you have which is second order in dt and can be thrown away. This is Z under root dt into dt that is a order Z dt to the power 3 by 2 that also you can throw away and then this term is also of dt square which again you can throw away.

So, what are we giving, what are we carrying away from this whole complex cumbersome expression, we are carrying this, we are carrying this, we are carrying this 3 terms first 3 terms and as far as this whole term is concerned 4 term is concerned, you will only carry 1 by 2 b square dt del square G upon del xi square.

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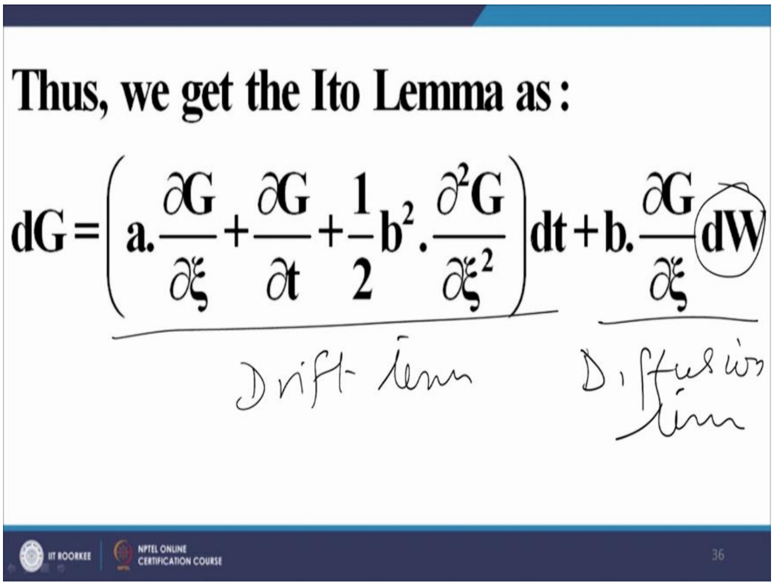

Making all these substitutions what we have is the Ito’s lemma as dG is equal to the expression that you have on the right-hand side. This is the drift term, this is the diffusion term, drift, this is the diffusion term, stochastic term this is also called the stochastic term, the random term, the diffusion term, the randomness is manifest in this dW. This is the manifestation of randomness. This is the representation of randomness.

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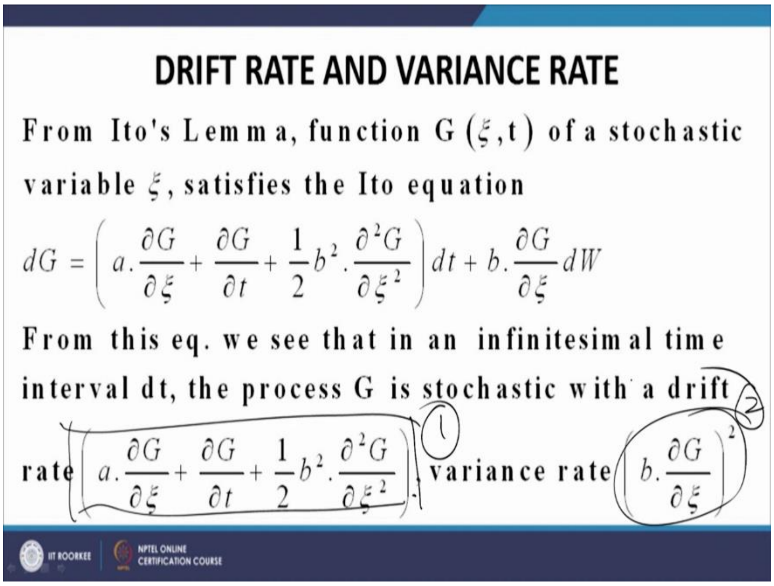

Even the drift rate I have also already mentioned that what is the drift term and what is the diffusion term. The drift rate is given by this expression here expression number 1 and the diffusion rate variance rate rather is given by the expression number 2.

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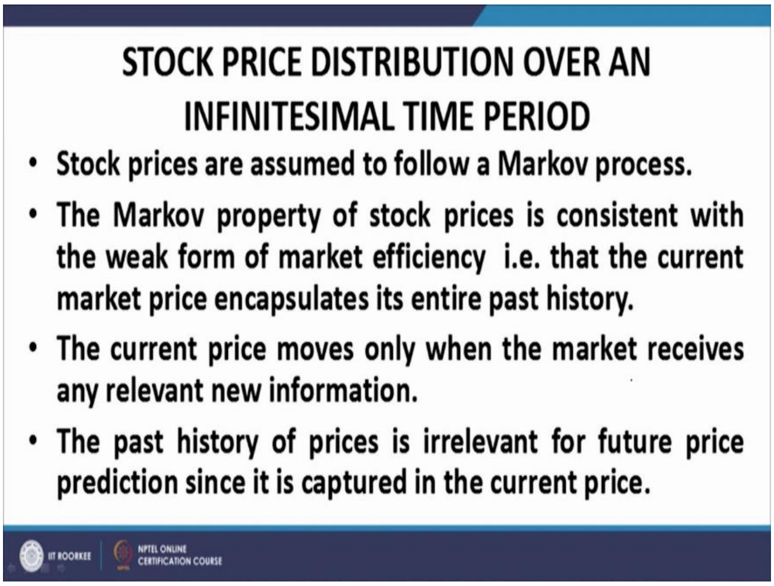

Now, we talk about stock price modeling over an infinitesimal time period. So, when we talk about stock price modeling, we base the modeling on certain assumptions about the uncertain observations rather about the empirical behavior of the stock. So, let me read out the assumptions that go into this model. Number 1 stock prices are assumed to follow a Markov process.

Number 2 the Markov property of stock prices is consistent with a weak form of market efficiency that is the current market price encapsulates its entire past history. The current price moves only when the market receives any relevant new information. These are standard assumptions that go into the efficient market hypothesis. Then, finally, the past history of prices is irrelevant for future price prediction, since it is captured by the current price.

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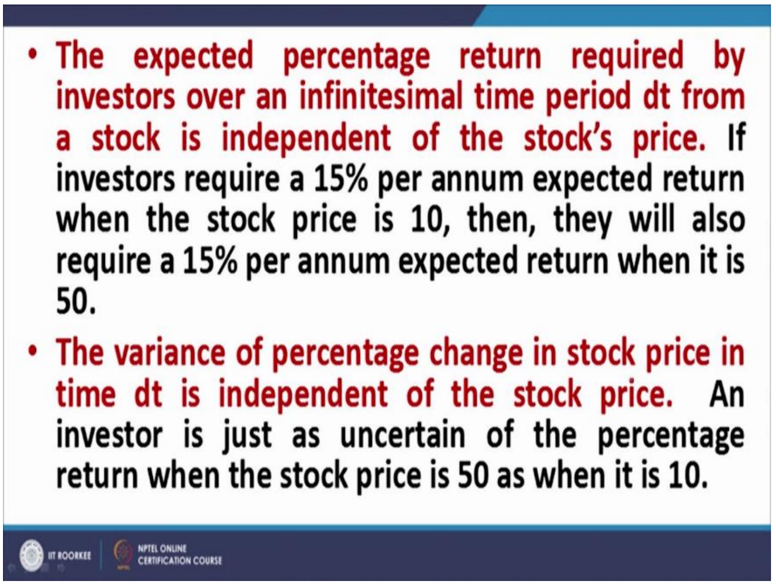

The 2 quantitative assumptions that we make and the ramifications of them, the model that emerges from there I will take up in the next lecture. Thank you.