Quantitative Investment Management
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Lecture 47
Brownian Motion

Welcome back. So I was in the process of arriving at a description of the mathematical structure called Brownian motion which is essentially the analog, the continuous-time analog of what we have in a discrete random walk.

I started with a single step random walk which was represented by a single random variable X1 where X1 could take values plus 1 and minus 1 with equal probabilities and we found that the mean of the process is 0 and the variance of the processes equal to 1. We increase the number of steps to 2 and then we found that the variance increases to 2 and so on. As we increase the number of steps, the variance also increases.

Now our objective is to arrive at the continuous time analog by decreasing the step size to the infinitesimal level. In other words, we want to take the limit that n tends to infinity. Now as n tends to infinity and please note that n is also the variance of the process of which we have, which I have just outlined.

So as n tends to infinity in the other words, as we move towards the continuous time regime, the variance of the process blows up and that makes it un-compatible with the applications that we envisage to use the process for, whether they be in the world of physical sciences or financial sciences or economics for that matter.

We want to model processes that have a finite variance. So towards that objective, we rescaled our variable random variables that constitute the stochastic process, that constitute the random walk by as follows as is shown here.

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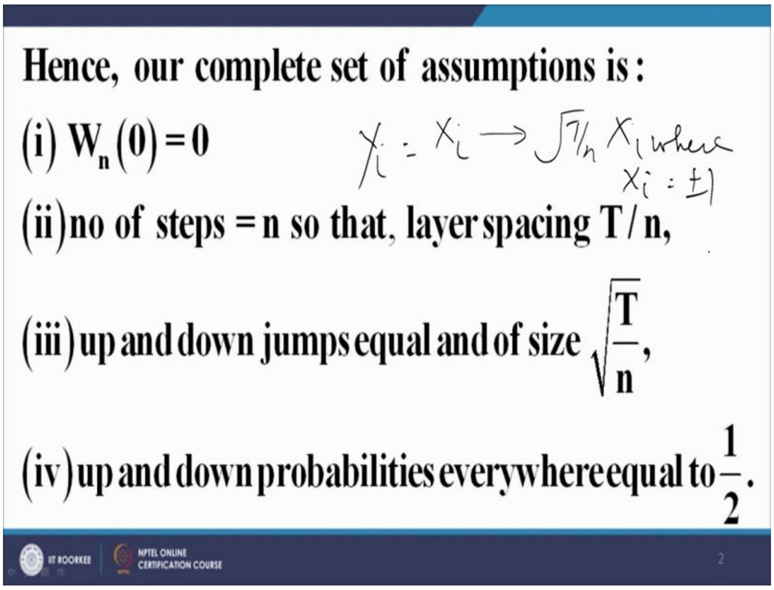

What we did was we moved Xi or rescaled Xi to under root T upon n Xi and then we found that the process does not blow up. The variance of the process is proportional to or equal to rather the time of evolution and here in this slide, I have the complete set of assumptions that we have for. This was the Y process.

Please note, we call it Yi so this is the Y random walk and the Y random walk consists of the following assumptions Yn(0) is equal to 0. That is the process starts at the origin. The number of steps is equal to n so that the layers spacing or the step size is equal to T upon n.

The up and down jumps are equal to under root T upon n because if you look at here, Xi can take the values plus minus 1 x where, where Xi can take the values plus minus 1 with equal probabilities. Therefore, Yi can take the values plus minus under root T upon n with equal probabilities.

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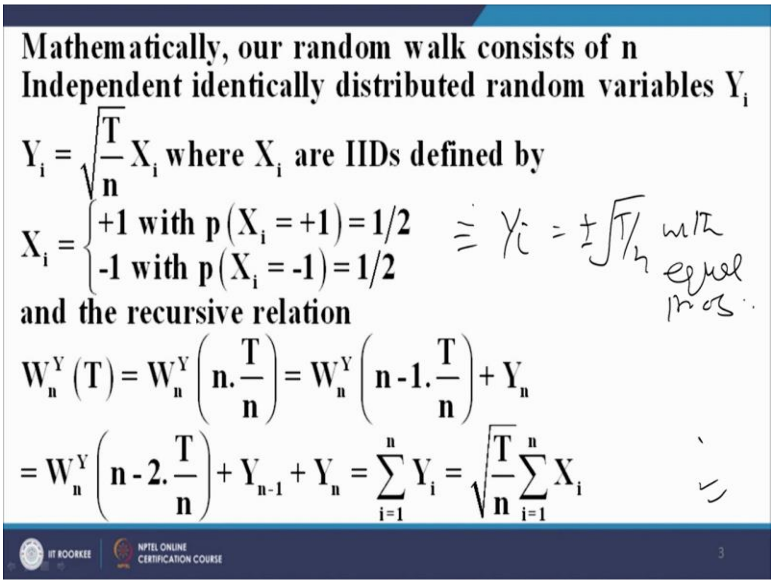

So mathematically, our random walk consists of that is the Y random walk. Please note now I am talking about the Y random walk. So the Y random walk consists of n independent identically distributed random variables. It is a sequence of n independent identically distributed random variables Yi where i ranges from 1 to n, where Yi is equal to under root T upon n Xi and Xi is distributed as follows, it can take only two discrete values. It can take the value plus 1 or the value minus 1 with equal probabilities.

Correspondingly Yi can take the values. This is equivalent to saying that Yi can take the values plus minus under root T upon n with equal probabilities. And the recursive relations for the Y random walk are given here and it is easily seen that it is a sum of all the constituent random variables Y1, Y2, Y3, Y4 up to Yn where n is the number of steps and T is the time length of, the total time length of evolution of the process.

Please note capital T is the total time length of the evolution of the process and n is the number of steps. So the layer spacing or the time difference between any two steps is equal to T upon n.

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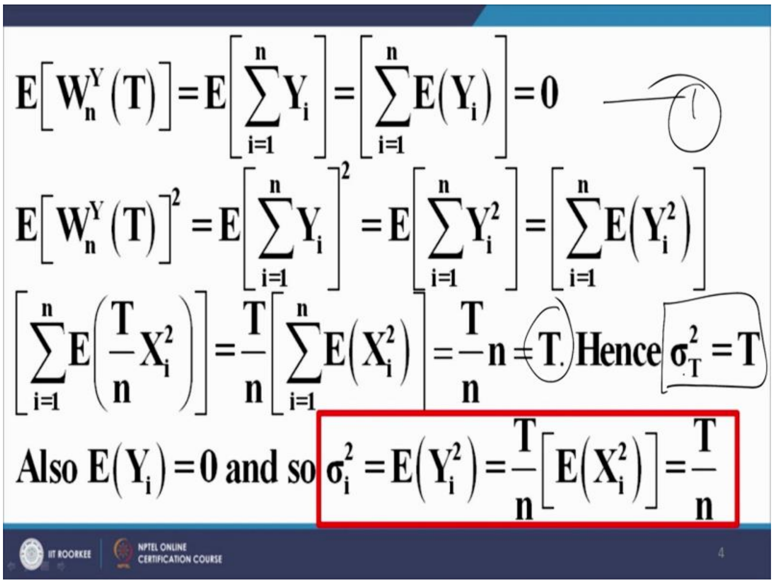

The cardinals of the Y random walk are given in this slide. The variance, the mean of the process is 0 as is seen from equation number 1. The variance of the process is equal to capital T which is the time of evolution.

So the variance of E and T is equal to capital T. You can see here in this box and the square of the expected value of the square of the process is equal to capital T as well and because the mean is 0, the variance turns out to be T.

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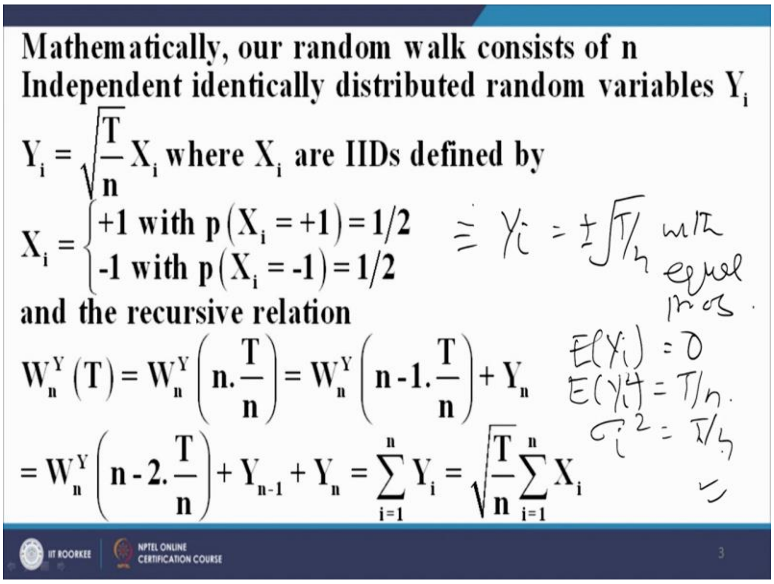

Now each Yi has a mean of 0 as well and that can be easily calculated from this particular expression that we have in this slide, Yi can take the values plus under root T upon n with probability half and minus under root T upon n with probability half. So expected value of Yi is equal to 0.

And as far as the expected value of Yi squared is concerned, Yi squared is concerned, what do we have? Yi square can take the values T upon n with probability half and T upon n with probability half. So the expected value of Yn squared is equal to T upon n and the variance sigma of Yi squared is equal to T upon n.

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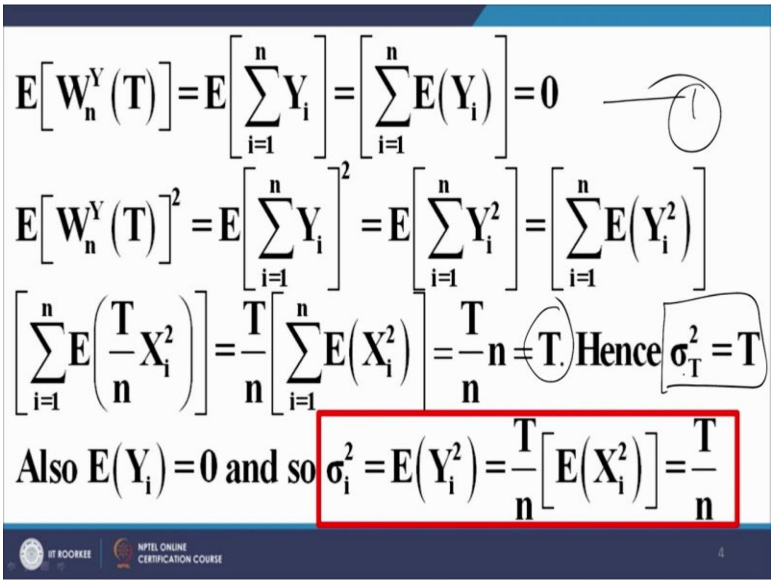

Now this is also shown in this red box. This is important. This will be carried forward when we talk about applying the central limit theorem to this process.

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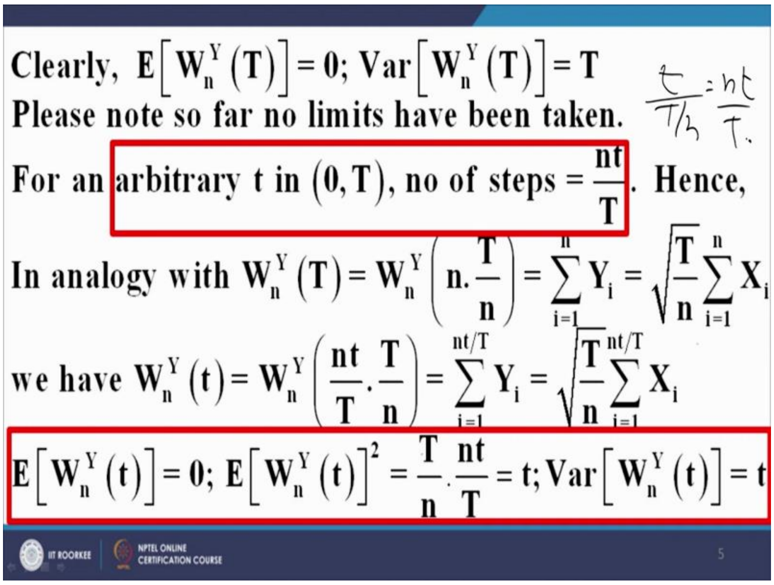

So let us know, let us know look at a situation where we are not talking about the total time evolution of the process but we are talking about observing the process at any intermediate time between 0 and capital T. So small t here is any arbitrary point in time at which we make the observation on the process.

Obviously, to arrive at small t with a step length of T upon n, we need how many steps? We need small t divided by T upon n number of steps and that is equal to small nt divided by capital T. So we are making the observation at the end of small nt upon capital T.

Now each step is modelled by a single random variable Yi. First step by Y1, second step by Y2 and so on. So nt upon capital T, number of steps will be modelled by nt upon capital T number of random variables. In other words, in our case, Yi will move from 1, 2, 3 up to nt upon capital T.

When we make these substitutions, what we find is that the variance of the process, the mean remains 0 by the way, the variance of the process is equal to the time of observation that is it is equal to the small t. At whatever point we are making the observation, the variance of the process is equal to that length of time. This is a very important and a very significant result.

Let me repeat. If small t is any arbitrary point in time between 0 and capital T where capital T is the total time of evolution of the process, then to arrive at small t we need to the step needs to make, the process needs to make n small t upon capital T number of steps.

And when we incorporate that into the analysis as shown in this slide, what we find is that the mean continues to be 0. In other words, the expected value of W(T) continues to be 0 but the variance of W(T) at time small t is equal to small t. That is an important result and that is why I have highlighted it in the red box.

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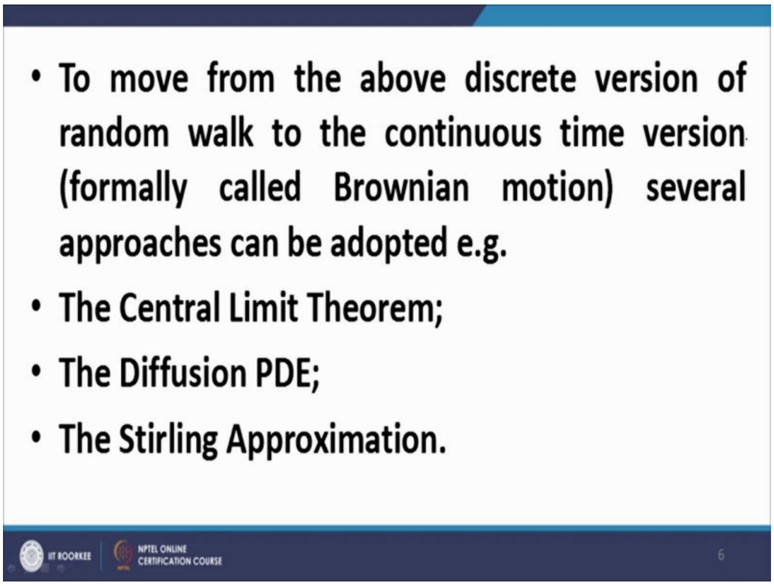

So to move from the above discrete version of the random walk to the continuous time version (normally called Brownian motion or formally called Brownian motion), several approaches can be adopted. We can use the central limit theorem. We can make use of the diffusion partial differential equation and we can use the Sterling approximation. I shall confine myself due to the paucity of time to the central limit theorem approach.

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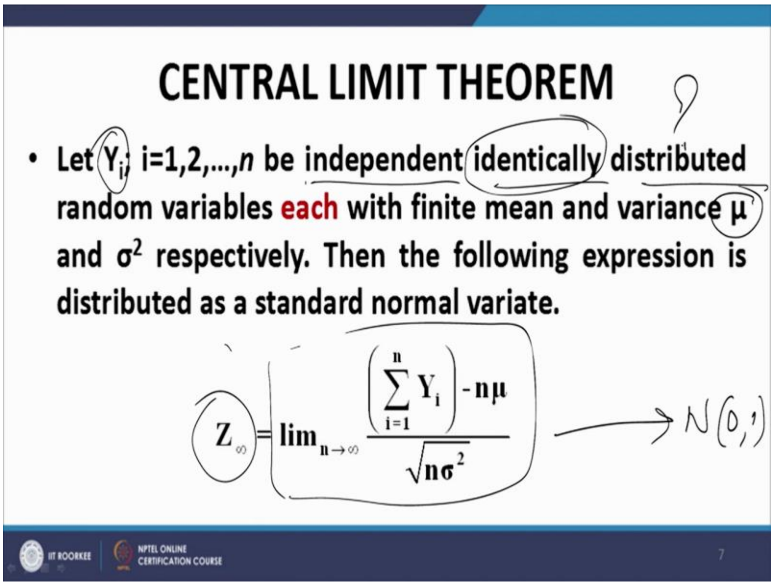

Now so let me first state the central limit theorem. There are many versions of the central limit theorem. I will state here the version that is relevant to us. So, let Yi (i equal to 1, 2, 3 up to n) be independent identically distributed random variables, IID random variables each with a finite mean and variance mu and sigma square. So each Yi has a mean of mu and a variance of sigma squared, then the following expression which is given below is distributed at a standard normal variate.

Now let me repeat this. Let Yi, i equal to 1, 2, 3, that is Y1, Y2, Y3 up to Yn all be independent identically distributed random variables with finite mean and variance mu and sigma square. Mean is mu and variance is sigma squared, then the following expression is distributed at a standard normal variate.

This Z infinity which is defined by this expression. This is distributed as N(0, 1). A standard normal variate is a normal variate with a mean of 0 and a variance of 1. Now the very interesting feature about the central limit theorem which I must mention is that we have nowhere mentioned that nature of distribution of Yi.

Yi need not necessarily be normally distributed that is very fundamental. We are not concerned with the distribution of Yi, it may be rectangular distribution, it may be a binomially distributed variable, it may be a Poisson distribution, what exponential whatever the case may be, we are not concerned with the distribution of Yi. All we want is they are independent identically distributed random variables. That is the only requirement.

Of course, the finiteness of the mean and variance are also required. But as far as the distribution is concerned, as far as the variables themselves are concerned, the only thing that is mandated for the application of the central limit theorem is that they should be independent identically distributed.

They should be identically distributed but what the distribution is, what the distribution is, is irrelevant. It may be normally distributed, it may not be normally distributed but this quantity that we have in the box here, this quantity is distributed as a standard normal variate. That is it is normally distributed with a mean of 0 and a variance of 1.

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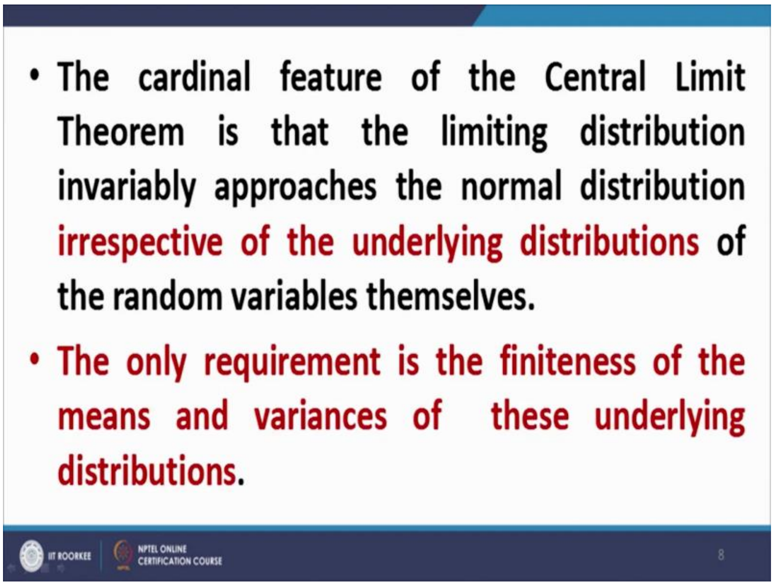

So the cardinal feature of the central limit theorem is that the limiting distributions invariably approach the normal distribution irrespective of the underlying distributions of the random variables themselves. How Y1, Y2, Y3, Yn are distributed is not the issue.

Whatever is their distribution, so long as all Y's are identically distributed, have the same distribution, same parameters of mean and variance, this limiting quantity that we that I defined by Z infinity will be normally distributed with a mean of 0 and a variance of 1. The only requirement is the finiteness of the mean and variances of these underlying distributions as I mentioned just now.

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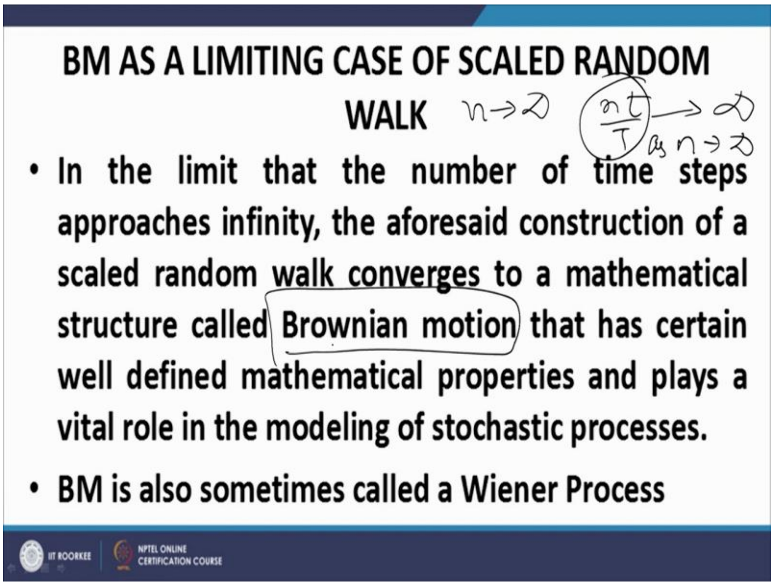

Brownian motion has a limiting case of scaled random walk. Now in the limit that the number of time steps approaches infinity, that is n approaches infinity. Please note n is the number of time steps. So n approaches infinity, if we are considering the total time of evolution of the process.

And there is one more thing that I must point out before we go further, that as n approaches infinity, this quantity n small t upon capital T also approaches infinity. So the limits as far as the limits are concerned, taking the limit with n tending to infinity would give you identical results as taking the limit nt upon capital T tending to infinity.

Because small t upon capital T is a finite quantity and therefore, when you multiply a finite quantity by a quantity which is approaching infinity the result remains unchanged. The total quantity, this quantity will also approach infinity as n tends to infinity.

Let us continue. In the limit that the number of time steps approaches infinity, the aforesaid construction of a scaled random walk converges to a mathematical structure called Brownian motion. This is a very important name.

Whenever we are talking about stochastic processes, almost all stochastic processes that are of relevance either in the physical sciences or in finance or economics, are modelled in terms of Brownian motion and variance of Brownian motion that has certain well-defined mathematical properties, and plays a vital role, plays a cardinal role in the modelling of stochastic processes. Brownian motion is also called Wiener process.

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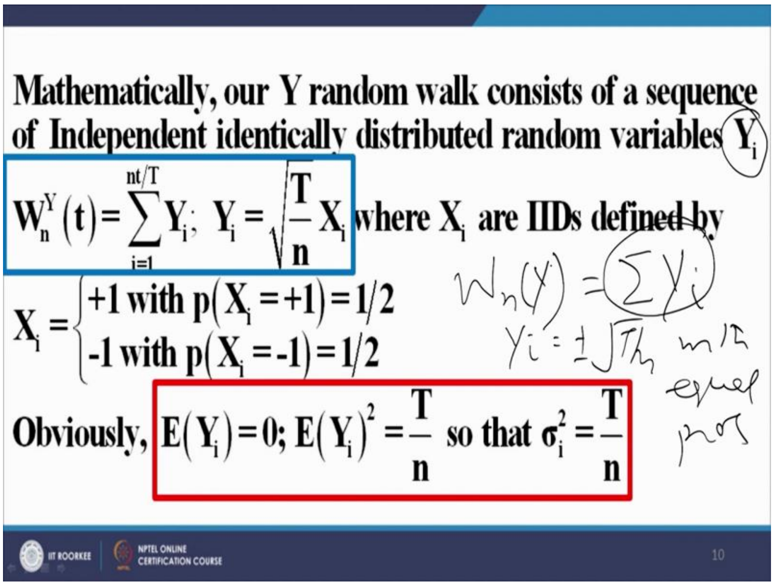

Mathematically, the Y random walk consists of a sequence of independent identically distributed random variables. I have already explained that in a lot of detail. The Y random walk consists of a sequence of independent identically distributed random variables. Which are those random variables?

Yi with the property that the random walk is given by the expression that is given the blue box and Xi's are of course, taking the values plus minus 1 with equal probabilities Yi's are taking the values under root plus minus under root t upon n with equal probabilities and each E(Yi) is normally, is not normally distributed.

Please note that. It is distributed with a mean of 0 and a variance of T upon n. It is a discrete distribution. It can take only two values, plus under root T upon n and minus under root T upon n with equal probability that means it has a mean of 0 and a variance equal to T upon n.

Now, we apply the central limit theorem to this collection of Y variables where each Yi is defined in terms of Yi is equal to plus minus under root T upon n with probabilities equal probabilities. So Wn(T) is a sum of Yi. We apply the central limit theorem to the sum of Yi in the form that I showed you a couple of slides back.

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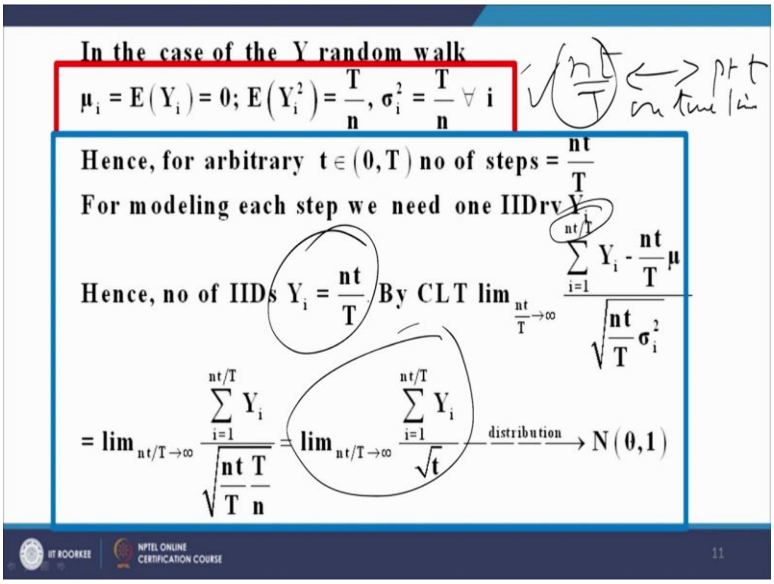

So when we do that, when we do that, what we get this and please note the number of steps that we will talk about, we are now talking about an arbitrary point in time t between 0 and capital T which is nt upon capital T time steps away from the origin. This corresponds to the point t on the timeline.

This is the number of steps, nt upon capital T and the time elapsed from t equal to 0 is equal to small t because one step length or one time step is of capital T upon n. So if you have to traverse a time of small t, the number of steps that the process will have to traverse is small t divided by capital T upon n that is small n into small t divided by capital T which is the quantity that I have encircled.

So, nt upon capital T, number of steps will give you a point which is small t away from t equal to 0 on the timeline. So we are applying the central limit theorem using this point as the reference point and this point is nt upon capital T number of steps away from the origin.

Time steps, please note this, and therefore we will need nt upon capital T number of Y variables to model the entire stochastic process from 0 to small t.

Each time step has to be modelled with one random variable because the process will undergo a transition at the end of each time step and then the transition needs to be modelled by a random variable.

So because there will be nt upon capital T number of transition points between 0 and small t, therefore, the number of random variables Yi's that we require for modelling the stochastic process is equal to nt small t upon capital T.

Therefore, we use the central limit theorem with this as the reference corresponding to the time t on the timeline, small t on the timeline. And therefore, the number of variables is equal to this.

And then when we use the central limit theorem, the summation will extend from 1 to n small t upon capital T because this is the number of variables that are there in the stochastic process up to times small t which is this small t is an arbitrary point between 0 and capital T. So using this what we get on simplification is that this expression that we have here is normally distributed with a mean of 0 and a variance of 1.

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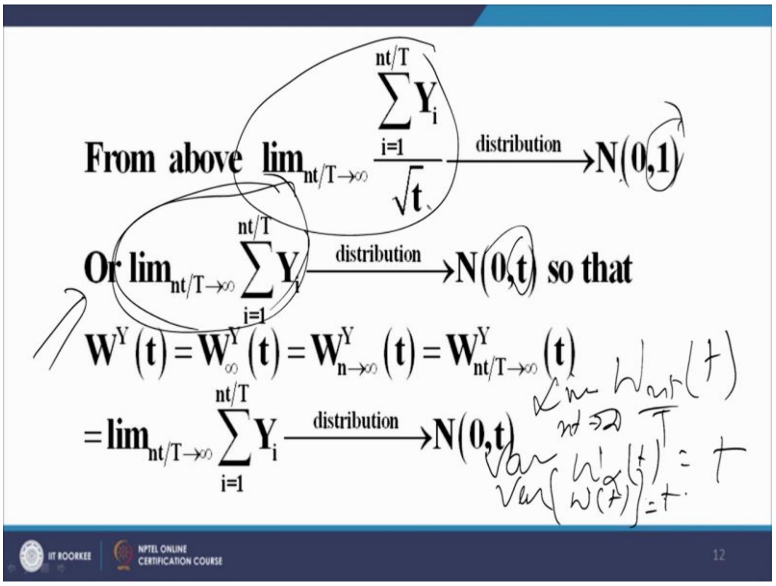

Let us continue from this. This is what we have from the previous slide limit n small t upon capital T tending to infinity summation of Yi upon under root T is normally distributed with a mean of 0 and a variance of 1 that gives us that the limit of limit n small t upon capital T tending to infinity summation from 1 to nt upon capital T summation Yi is normally distributed with a mean of 0 and a variance equal to small t.

Please note this point. Here the variance is 1 but as soon as you consider only the numerator, you see the stochasticity is contained in the numerator. The denominator does not contribute to the randomness, does not contribute to the stochasticity and the flow of time is deterministic, totally deterministic.

So because the flow of time is totally deterministic, when we take this to the right-hand side, take this into the distribution, the variance will be multiplied by small small t and therefore, summation of small yi in the limit that y tends to that nt upon capital T tends to infinity is equal to small t and this entire expression is normally distributed with a mean of 0 and a variance of small t.

But this expression is nothing but Wnt upon capital T into of small t and E, please note in the limit of course, I have not mentioned limit and t tending to infinity. So I must write here- limit nt tending to infinity. I can simplify this and write it as W infinity t is equal, the variance of this expression is equal to a small t.

I can drop out the infinity because it is understood and I can then write it as W(t) as a variance of variance, let me write down variance here, variance here, variance of W(t) is equal to small t.

Now this is the fundamental feature of the variance and please note, the mean of the process continues to be 0 because it is a sequence of variables each of them having a 0 mean, so that the mean of the entire process continues to be 0.

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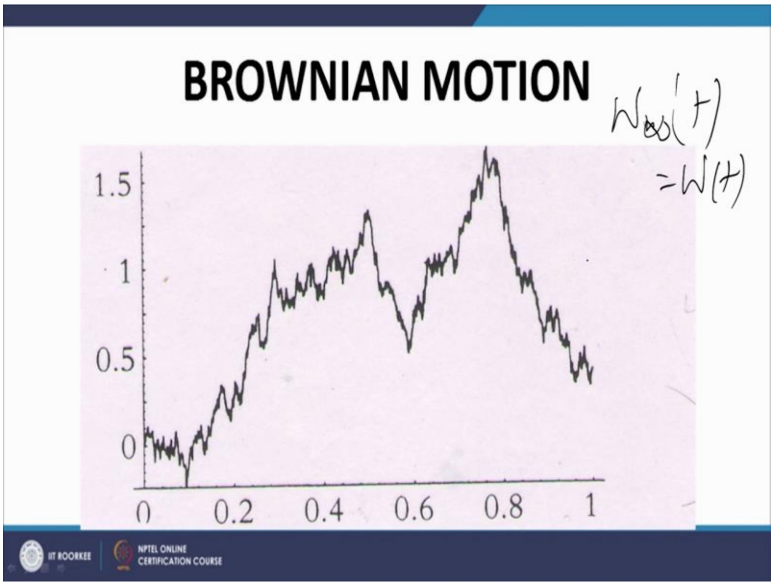

This is what the process looks like. The process that I have just outlined Wt, Wt which is in fact, an abbreviation of W infinity t, this is equal to W(t) looks like what you see on the slide. This is called Brownian motion.

The Brownian motion is the continuous-time analog of a random walk which is unbiased, where we start with a finite number of steps and then take the limit the number of steps tends to infinity, so that the steps that squeezes a completely and becomes infinitesimal and we have a continuous time regime.

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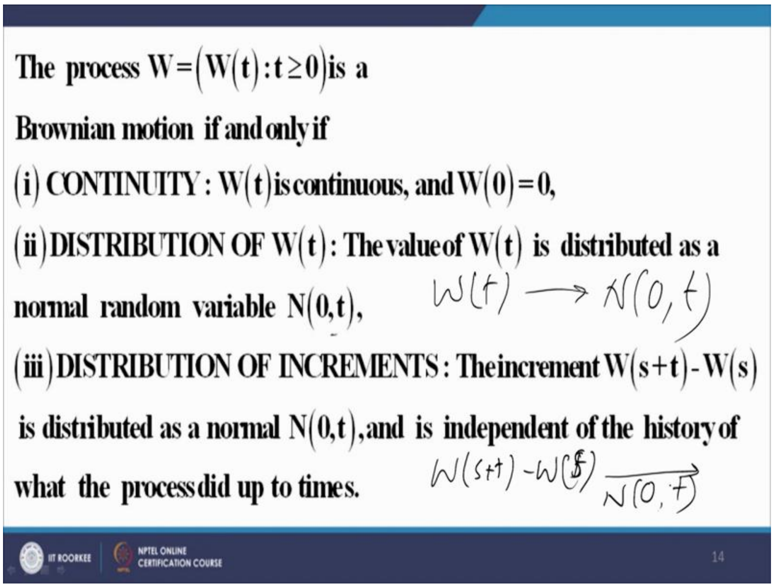

So let us summarize the properties of this process W(t). W(t) is called Brownian motion and it has the following properties. First of all, W(t) is continuous. You can see from this diagram in fact, you can make out that this process is continuous but it is nowhere differentiable.

I repeat the process is continuous, you can see from the from the diagram that the lines are continuous but the lines are so jagged that at any point you take on this particular process at any process point you will find that it is not differentiable and therefore this process is a special process where it is continuous but nowhere differentiable.

Now the distribution of W(t). The distribution of W(t) as I have mentioned, W(t) is normally distributed, normally distributed with a mean of 0 and a variance of small t. Distribution of increment. This is another interesting feature of this Brownian in motion.

If you talk about an increment of Brownian motion, that is if you talk about say W(s plus t) minus W(s) is also normally distributed. And what is the distribution? The distribution is has a mean of 0 and a variance equal to the difference of the two variances. It is t plus s minus s and which is t.

So the increment W(s plus t) minus W(s) is distributed normally with a mean of 0 and a variance of small t and is also independent of the history of the process up to the time s. So whatever was the status of the however the process came whatever path it followed up to t equal to s is irrelevant for the future evolution of the process.

The future evolution of the process will only depend on the point at which the process is at t equal to s. How it arrived at that particular point which at which it is, at t equal to s is of no relevance in determining the future trajectory of the process. So that is the important part. This is a Markov process. That is what we call a Markov process. The Brownian motion is a Markov process.

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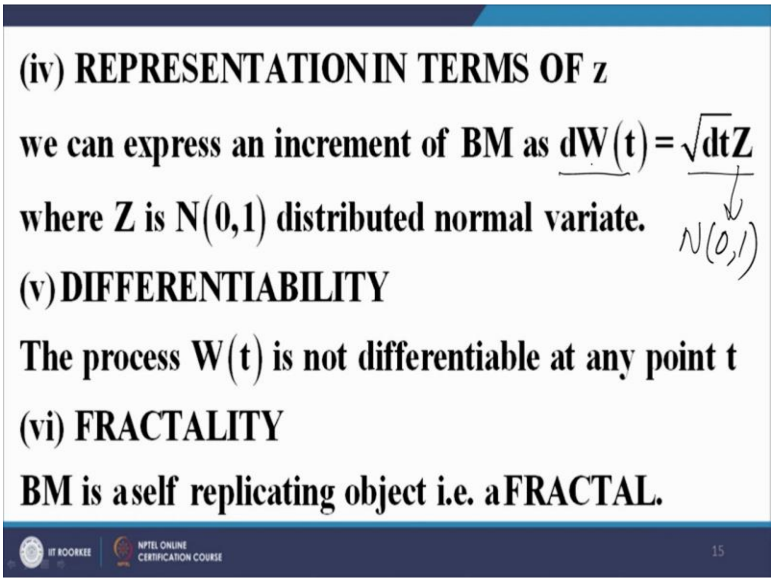

Representation in terms of the standard normal variates that now we can write the infinitesimal increment, dW(t) as in the form of Z into under root dt. I repeat we can write the infinitesimal increment of Brownian motion, dW(t) in the form of Z under root dt.

You check from this that from by taking the mean and variance of both sides that the mean and variance of the left-hand side coincide with the mean and variance of the right-hand side. Mean of the left-hand side by definition by that very definition of Brownian motion that we saw just now is equal to 0 and the variance is equal to dt.

If you look at the mean and variance of the right-hand side, it is also 0 because Z has a, this Z is normally distributed with a mean of 0 and a variance of 1. Therefore, the mean and variance of under root dt into Z is equal to 0 and dt. Because please note that dt is a non-stochastic quantity. It is the deterministic quantity, it is a parameter, deterministic parameter.

And there is another point here that I must mention that whenever we talk about a normal distribution, a normal distribution is completely specified by knowledge of its mean and variance. So because the mean and variance of this, these two quantities on the left-hand side and the right-hand side of this equation are identical, it follows that the distribution is also identical.

In other words, we can perfectly represent, we can exactly represent the infinitesimal increment of Brownian motion by the expression given on the right-hand side in terms of the standard normal variate.

Differentiability I mentioned already, the process W(t) is not differentiable at any point or differentiable with 0 probability or we can say the process is nowhere differentiable with probability 1.

Fractality, the Brownian motion is a fractal. What is a fractal? Let me quickly outline because of paucity of time if you blow up the structure of Brownian motion, for example, let me go back to this structure.

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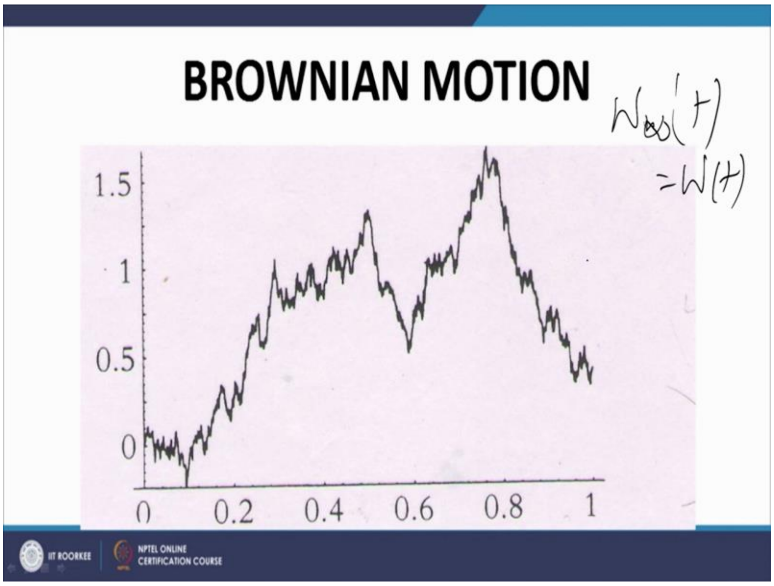

If you take any segment of this Brownian motion, let us say you take this segment and you blow it up under a microscope, what you will find that the structure remains intact. This is what is a fractal is. It is a self-similar evolution which we call a fractal. So Brownian motion is a Markov process. Brownian motion is also fractal. Brownian motion is also continuous but Brownian motion is not differentiable.

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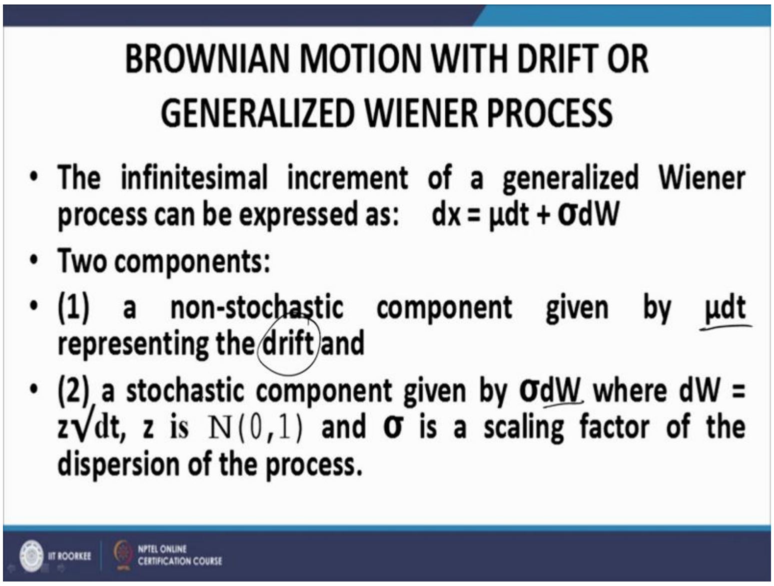

Now we talk about Brownian motion with drift. We generalize the concept of Brownian motion. You see the Brownian motion in itself is can be is a very well-defined mathematical object (no doubt about that) but the point is that the structure as it is can be used to model only a very limited class of processes.

We need something more general than Brownian motion to model processes which have which have variances and which have means, which are non-zero or variances which are not exactly equal to time but are proportional to time for that matter.

For these kinds of processes, what we move on to is generalized Brownian motion. The generalized Brownian motion comprises of two parts. Basically, it is a superposition of two types of motion. One is the standard Brownian motion which also the amplitude of which can be scaled by adding a scaling factor and the second is a drift or a trend that you can incorporate, that you can add on to this Brownian motion to further generalize it.

So let me repeat, the generalized Brownian motion consists of two parts, a non-stochastic component given by mu dt which is which does not contain any randomness which is simply showing a trend capturing a trend or capturing a drift in the process.

For example, when we model stock prices, due to the inflationary trends which have been observed throughout history, we need a factor to incorporate the inflation. That can be done by this particular component.

And then we have a stochastic component given by sigma dw. This dw is standard Brownian motion increments and by multiplying it by single sigma, what we are doing it? We are able to manipulate the amplitudes of vibrations of the Brownian motion such that we get the variance of the process gets scaled in accordance with the variance that is empirically observed for the process that we are trying to model.

So that is by incorporating a scaling factor in front of the standard Brownian motion and we also can add on a factor of drift that is in order to incorporate certain trend behavior of the Brownian motion.

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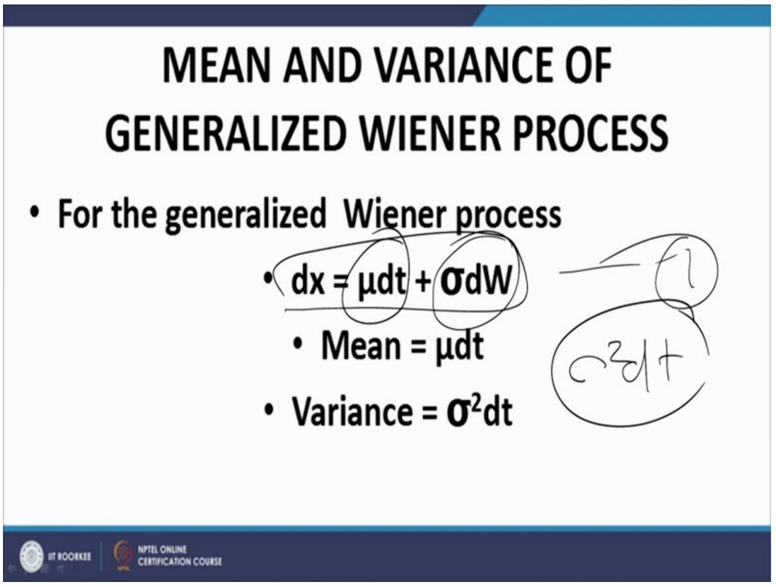

So this particular Brownian motion is represented by this equation number that I have encircled. Let us call it equation number 1. dx is equal to mu dt plus sigma dW. mu dt is the non-stochastic part; sigma dW is the stochastic part; sigma is the scaling factor. The factor which influences or which modulates the amplitudes of vibrations of the Brownian motion and mu is the setting of the trend. It is the trend per unit time or the drift per unit time and that is multiplied by the time factor to capture the full trend.

So dx is equal to mu dt plus sigma dW. What is mean? Mean of this expression must be mu dt because the mean of dW is equal to 0, expected value of dW is 0. We have done that already. What about variance?

Now, mu dt will not contribute to the variance of this process. Why? Because it does not have any stochastic component. mu is a constant and dt as I mentioned just now is a deterministic parameter.

So mu dt will not contribute to variance. The entire variance is captured by sigma dW and we assuming that sigma is a constant, the variance becomes sigma square into the variance of dW. The variance of dW is dt. So the total variance is equal to sigma squared dt. So let me repeat. The mean of generalized Brownian motion which is given by equation number 1 is mu dt and the variance of this motion is given by sigma square dt.

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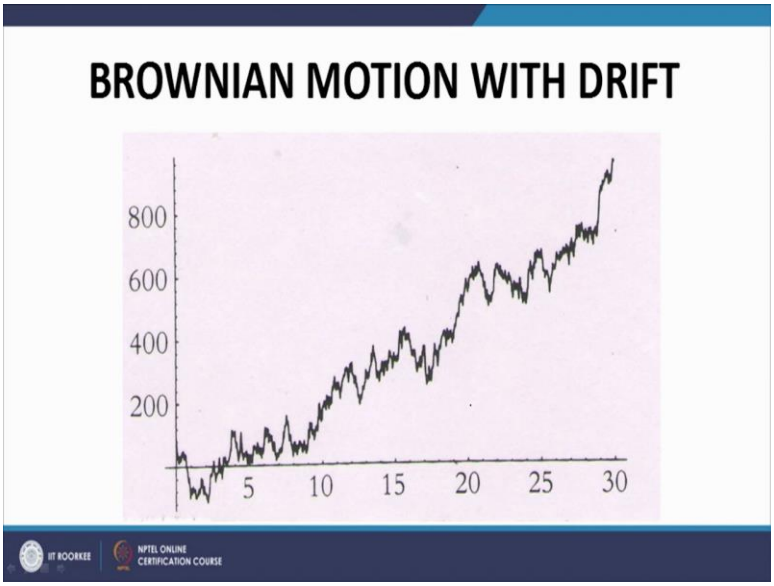

This is an example, this is an illustration of Brownian motion with the drift component in it. You can see the upward trend clearly manifest in this diagram and you can also see the vibrations is stochastic or the random vibrations that are occurring around that upward trend trendline.

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RANDOM WALKS	TIME LENGTH /LAYER SPACING	JUMP SIZE	EXP VALUE	VARIANCE
SINGLE STEP RW	T	± 1	0	1
TWO STEP RW	T/2	± 1	0	2
n-STEP RW	T/n	± 1	0	n
n-STEP SCALED RW	T/n	$\pm\sqrt{(T/n)}$	0	T
n-STEP SCALED RW	T/n	$\pm\sigma\sqrt{(T/n)}$	0	$\sigma^2 T$
n-STEP SCALED RW WITH DRIFT	T/n	$(\mu T/n)\pm \sigma\sqrt{(T/n)}$	μT	$\sigma^2 T$
Brownian Motion	->0	->0	0	T
Scaled BM	->0	->0	0	$\sigma^2 T$

So this is a chart that shows the whatever we have discussed so far which captures in a nutshell what the time steps are, what the scaling factors are, and what is the mean and variance of the process. So this is for reference of the learners. I will not spend time on it.

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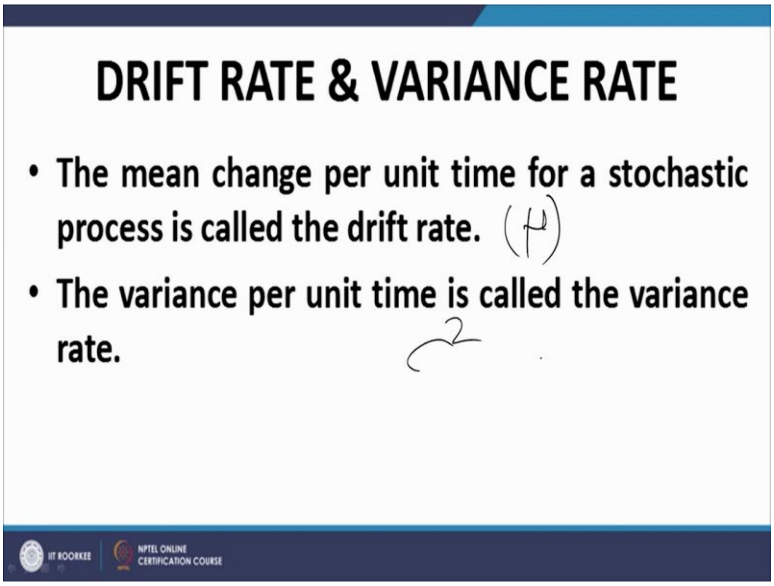

Then I briefly touch upon drift rate and variance rate. The mean change per unit time for a stochastic process is called the drift rate that is mu in our example. The variance per unit time is called variance rate that is sigma squared. So from here I will continue in the next lecturer. Thank you.