Quantitative Investment Management
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Lecture 45
Option Pricing – American Options

Welcome back. So, let us continue from where we left off. At the end of last lecture, I was talking about a problem, I was discussing a problem on the two step binomial model. So let us revisit that problem, and then we will proceed therefrom.

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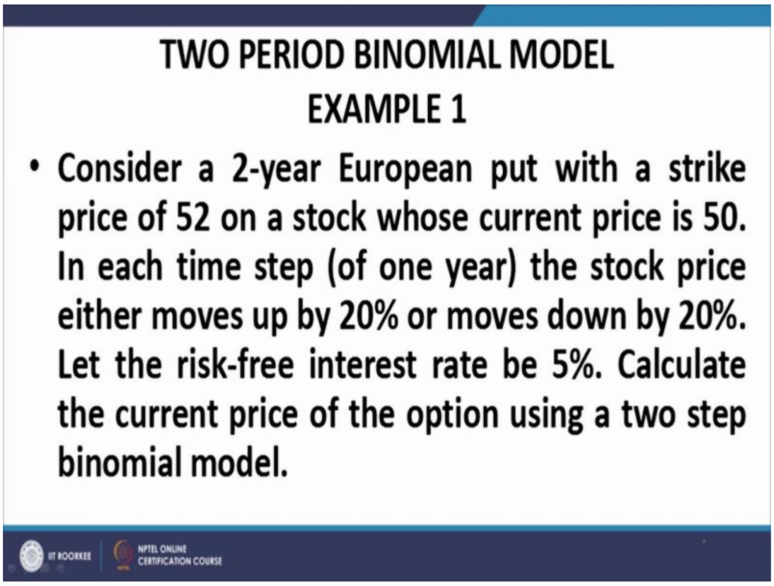

Consider a 2-year European put with a strike price of 52 on a stock whose current price is 50. In each time step of one year, the stock price either moves up by 20 percent or moves down by 20 percent. Let the risk-free interest rate be 5 percent per annum continuously compounded. Calculate the current price of the option using a two step binomial model.

So, the data that is given to us is the current stock price and the up and down swings are given in terms of percentages, the life of the option is two years and we have to model it as a two step binomial tree. And the risk-free rate is also given as 5 percent per annum continuously compounded.

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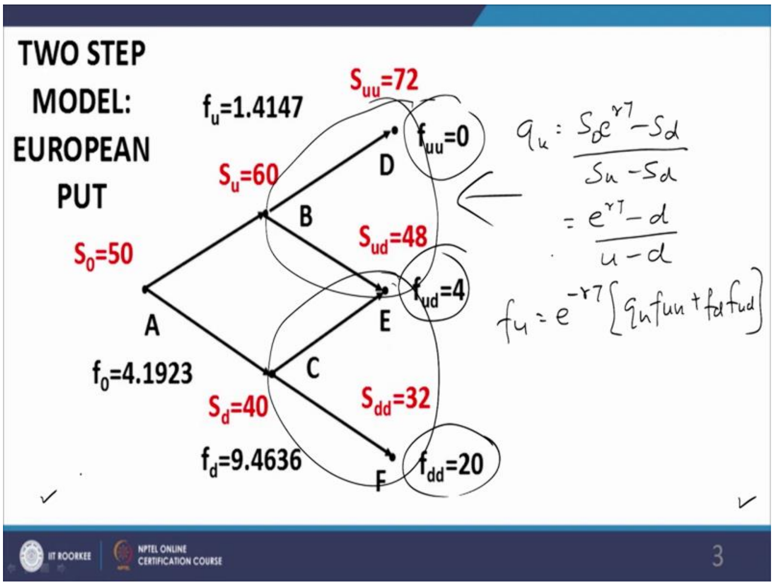

So, our first step is to work out the synthetic probabilities. Please note, because the size of the up jump and the down jump is the same across the entire tree, the synthetic probabilities or the q-probabilities corresponding to the different branches of the tree will be the same. And we can work them out as q of u is equal to S 0, e to the powerr T minus S T upon S u minus S d. Or, in terms of percentages or price changes, size of jump, in terms of percentages, if I divide throughout by a 0, what I get is e to the power r T minus d divided by u minus d.

So, this is the formula that we use for q u. q d will be equal to 1 minus q u as you all know, and the rest of the, then we also need of course, to calculate the value of the option at the various terminal nodes. Now, if the stock price at maturity turns out to be 72, then the put option will not be exercised, and therefore, the value of the option will be 0. If the stock price turns out to be 48, the put option will be accepted and the payoff is equal to K minus S T that is 52 minus 48, that is 4.

And if the stock price turns out to be 32, then the payoff from the option will be equal to 52 minus 32, that is 20. These are the figures that are given in the, adjoining the nodal points D, E, and F. Now, the first step is to work backwards. For example, we can work backwards for this branch of the tree or this segment of the tree and what we arrive at is that the value of the option at the node B, it turns out to be 1.4147. Similarly, using the data that we have for this branch of this option, or the segment of the tree I am sorry, what we get is f d is equal to 9.4636.

For example, we can work out f u as e to the power minus r T. q u f u u plus q d f u d. Now putting in the various values, q u and q d we have already worked out, f u u for this segment of the tree is 0, and f u d is equal to 4, r S equal to 5 percent continuously compounded, t is equal to 1 year and that enables us to work out f u. Similarly, we can work out f t. And using f u and f t and again the same values of q u and q d, we can work out the value t equal to 0, which turns out to be 4.1923.

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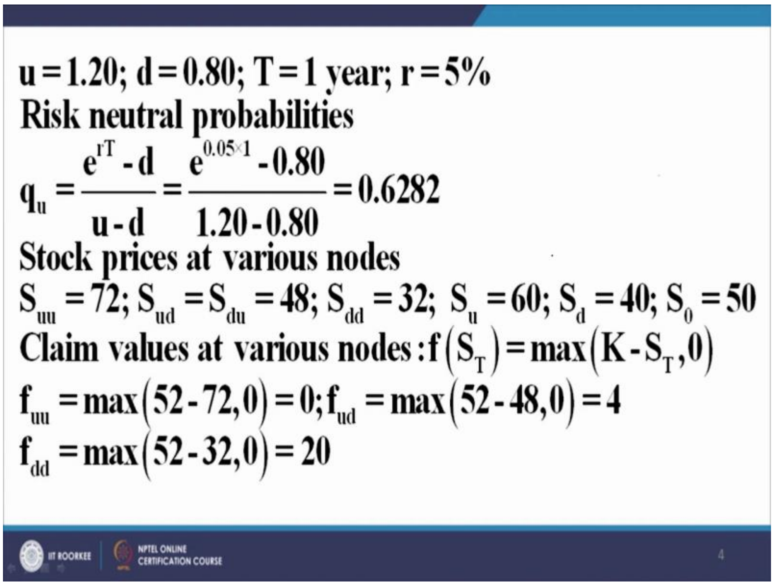

So, this is the working of this example. If you work out the risk neutral probabilities, q u turns out to be 0.6282 and q d is 0.3718. The stock prices at the various nodes are easily worked out. You are given that the current stock price is 50. That t equal to 1 jump is either 20 percent upwards or 20 percent downwards. So it would either be 60 or it would be 40. And then there is another jump at t equal to 2, which would again be of 20 percent upwards or 20 percent downwards.

So, the possible or the spectrum of prices at t equal to 2 would be would be 72, would be 48 and then 32. That is what is given here, S u u is 72, S u d is equal to 48, S d d is equal to 32. Please note here one thing that S u d is equal to S d u, that is equal to 48, this is called a concomitant tree. A tree where the branches recombine is called a recombinant tree, I am sorry. So, please note this particular feature of this type of binomial tree. S d d is equal to 32, S u is equal to 60, S d is equal to 40 and S 0 is equal to 50.

Claim values, I have already explained, at the various nodes. It is 0 for the upper node when the stock price is 72, 4 the middle node, when the stock price is 48, and the claim value is 20 for the lower node when the stock price is 32.

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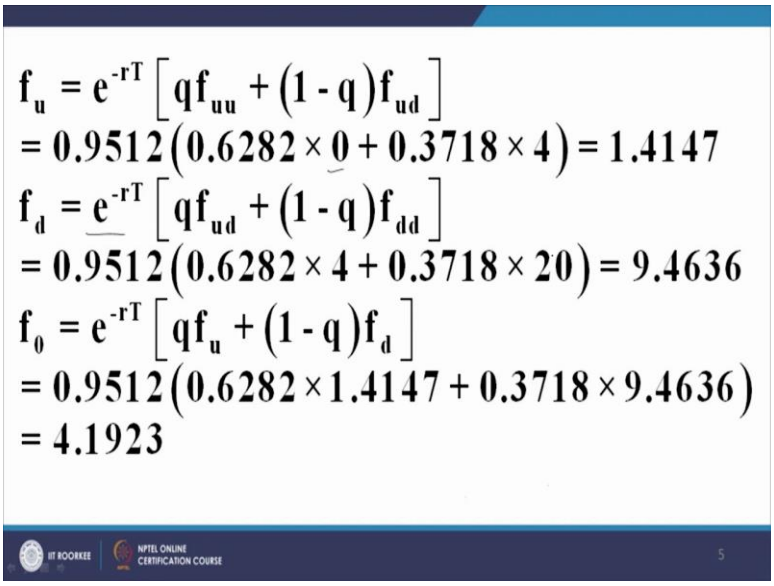

Then this is the working for f u as I explained. f u is given by e to the power minus r T q into f u u plus 1 q d, that is 1 minus q u into f u d. Substituting the various values, we get the value of f u u as 1.4147. Similarly, f d is 9.4636. And using these two values and working backwards, we get the value of f 0 as 4.1923. So in essence, we are rolling back the tree. From the terminal, from the data that is there at the terminal node, we rolled back the tree step by step and then we arrive at the value at t equal to 0, which is f 0.

Now we modify this methodology to work out the value of American option. Recall, that what is the difference between the American and the European option. The European option is exercisable only at maturity. For example, in the option that we just considered, the maturity of the option was two years. So the option could only be exercised at t equal to 2 years and not earlier. However, in the case of American option, the option holder has the discretion, has the choice to exercise the option at an earlier date as well.

So the implication of this property on the methodology that I have just discussed, is that we need to examine at every node, whether the option exercised would be optimal or not. In other words, whether the payoff generated from the exercise of the option is higher than the computed value of the option. And if the payoff generated from the exercise of the option is higher, we will substitute that higher value at that particular node in the tree. Why? Because the logic is simple.

The logic is that if the exercise of the option, which is now allowed, because it is American, results in a higher payoff, the investor would exercise the option and claim the higher payoff rather than the implicit or the intrinsic value of the option, which is the calculated value as per the binomial tree. So, let me repeat. When we are evaluating a American option using the same binomial tree method, methodology, the only difference will be that at every node, we need to check whether the exercise of the option gives you a higher payoff than the computed value.

And if it so happens that at any particular node that the exercise of the option gives you a higher payoff than the computed value, then we will substitute that higher payoff for the computed value and proceed as earlier, working backwards, until we arrive at the value t equal to 0. Even at t equal to 0, we will check whether the earlier exercise of the option at t equal to 0 or the immediate exercise of the option gives you a higher payoff than the payoff that is represented in terms of the backward calculation of the binomial tree. So let us, as an illustration, let us do the same problem that we did just now on the premise that the underlying put option is a American put.

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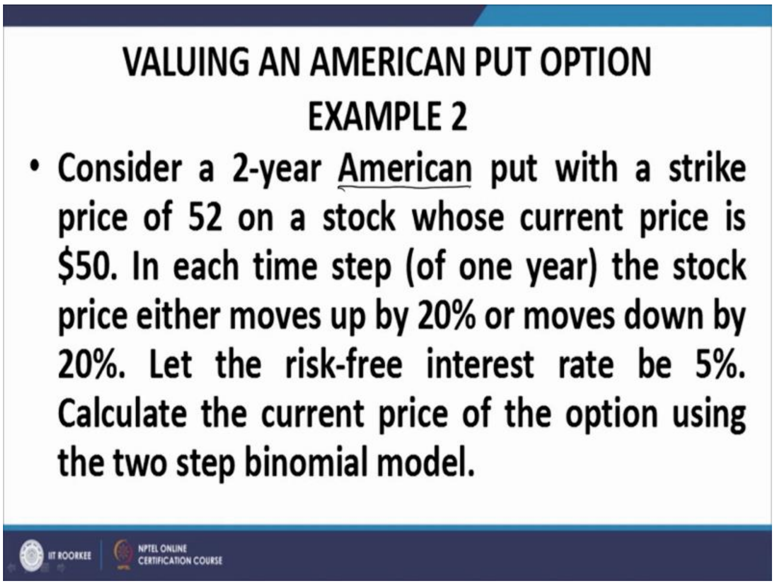

So we consider a two year American put. Now please note this particular feature. American put with a strike price of 52 on a stock whose current price is 50. In each time step of one year, this is a two year put. So we are having a two step binomial tree model, just as in the previous case. In each time step of one year, the stock price either moves up by 20 percent or moves down by 20 percent. The risk-free interest rate is 5 percent per annum continuously compounded. Calculate the current price of the option.

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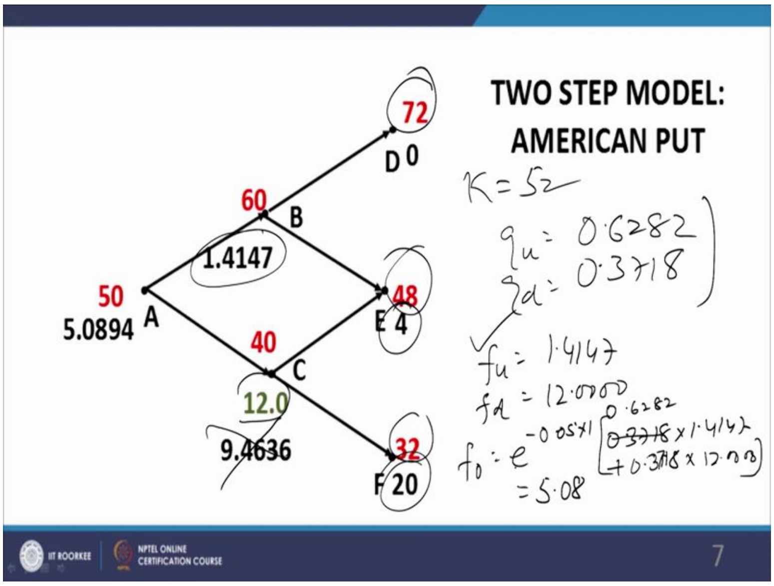

Now, this is, this is what we calculated in the previous example. These were the stock values in red at the terminal nodes, and corresponding to the stock values, because it is a put option, the payoffs are given at the node d by 0, because recall that the exercise price is 52, K is equal to 52. So, if the stock price is higher than the exercise price, the output will not be exercised and the payoff will be 0. At this point, because K is 52, the stock price is 48, the option will be exercised yielding a payoff of 4.

And similarly, the payoff at the point F will be 20. Now, using this data and using q u and q d if you recall, this is equal to 0.622 and 0.3718, these are the values of the probabilities and these are the same across the various segments of the tree because of the reason that u and d, that is the percentage up jump and the percentage down jump are same in each case. And using this q u and q d and f u u and f u d, we arrive at this value, 1.4147.

Now at the node B, we need to check whether the put option exercised will give you a higher payoff, than the computed value, which is 1.4147. The stock price is 60, the exercise price, please note, is 52. So obviously if you exercise the option, you will get a 0 payoff, because you recall, it is a put option and the computed value is 1.4147, which is higher and therefore we will carry backward the computed value. We will ignore the value or the payoff that is arising out of the exercise of the option because it results in a lower payoff. In this case it is a 0 payoff.

Now, let us look at the other scenario, scenario represented at the node C. The computed value at the note C using the same formula as we have here is 9.4636. The price of the stock at this point is 40. The exercise price is 52. Therefore, if I exercise the option at the node C, I will get a payoff of 12. The computed value is 9.4636. So obviously at this point, at the node C, it is optimal to exercise the option.

Should the system evolve in such a way that node C materializes, I would be better off exercising the option at this node, rather than carrying it forward for the next year, for the second year. So I should exercise the option at the end of the first year if the stock price goes down to 40 because I will get a payoff of 12 in distinction to the value of the option at this node which is given by 9.4636. So, what we do is we substitute this 12 figure instead of this 9.4636 when we work out the value of the option at t equal to 0 or at the initial node.

I repeat. Instead of 9.4636 we shall use 12 when we calculate the value of the option at the node A. f u, therefore, what will be the values? q u and q d will be the same of course, and f u will be equal to 1.4147 and f d will be equal to 12.0000. And therefore we will arrive at f 0 is equal to e to the power minus r T, that is minus 0.05 into 1 into q u, that is 0.3718 into f u and that is 1.4147 plus 0.3718. I am sorry, this is 0.6282 into 1.4147 plus 0.3718 into 12.00. This will be the value and this will be equal to 5.08. Now please, you will notice change in value. In the case of the European option, the value was 4.19 Here we are arriving at a value of 5.08.

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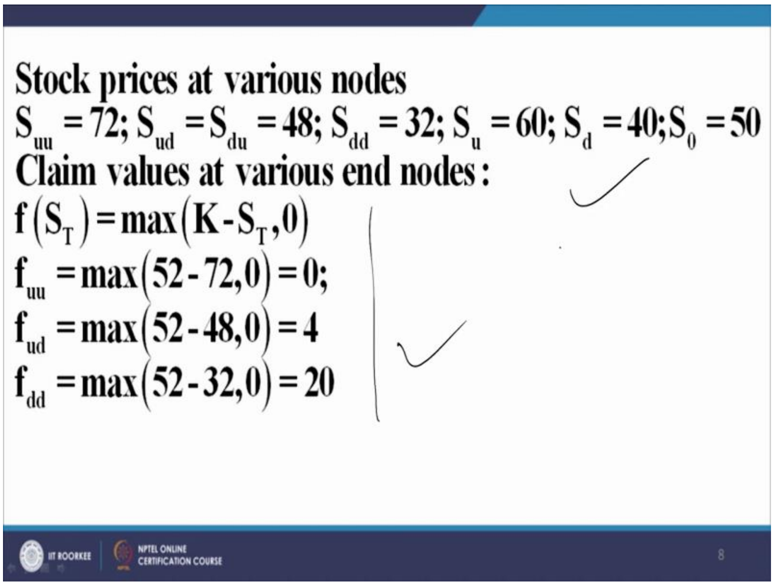

So, this is the working. These are the stock values at the various nodes and these are the claim values at the terminal nodes.

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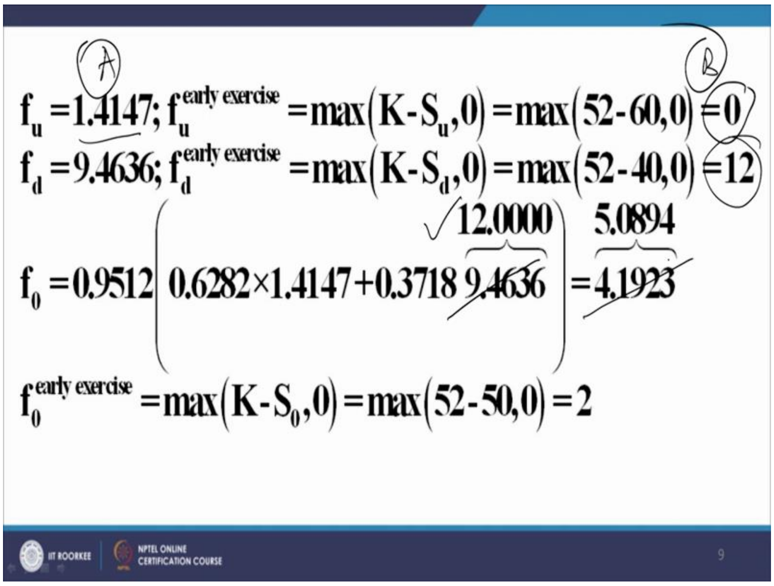

This is the working backwards as you can see here. f u is equal to this expression, which is f u is equal to 1.4147. But f u early exercise is equal to 0, because the stock price is equal to, at the point B, the stock price is equal to 60 and the exercise price is 52. So, the option will not be exercised. So, the early exercised value is 0. Therefore, the maximum of this, maximum of A and B, that is the computed value and the payoff is equal to 1.4147.

Then we talk about f d. The computed value f d is 9.4636. The exercise value is equal to 12. And in this case, you can see here that the computed value is lower, the payoff is higher. And therefore, for the next step, we will carry the value of f d as 12 and not as 9.4636. So, this is shown here. This is replaced 9.4636 is replaced by 12, and 4.1923 is replaced with 5.0894. So, this is how we work. And please note, finally, we also check for early exercise at t equal to 0 that is at the node A

Now at the node A, the stock price is equal to 50, the exercise price is 52. Therefore, the payoff from the option is only 2. However, the price of the option turns out to be 5.0894. Therefore, we will go along with the computed value of the option at node A, that is 5.0894, which will represent the worth of the option, which will represent the price of the option.

So at every node we need to check for early exercise. If on early exercise, is it beneficial? Is it more beneficial to the investor, to the party holding the long position? And if it so happens, that the payoff is higher compared to the computed value, we replace the computed value by the payoff, which is the higher value, and we work as earlier.

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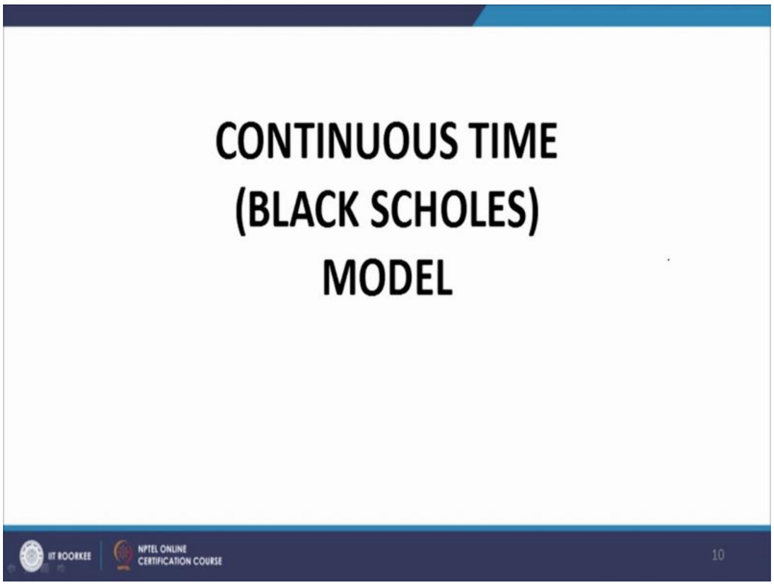

Now, we move from the binomial model to the continuous time model, the continuous variable model, which is the celebrated Black Scholes model, which was propounded by Fischer Black and Myron Scholes, for which they were awarded the Nobel Prize as well. So, we shall, but before we get into this model, we work out the Black Scholes partial differential equation, the solution of which gives us the value of the derivative.

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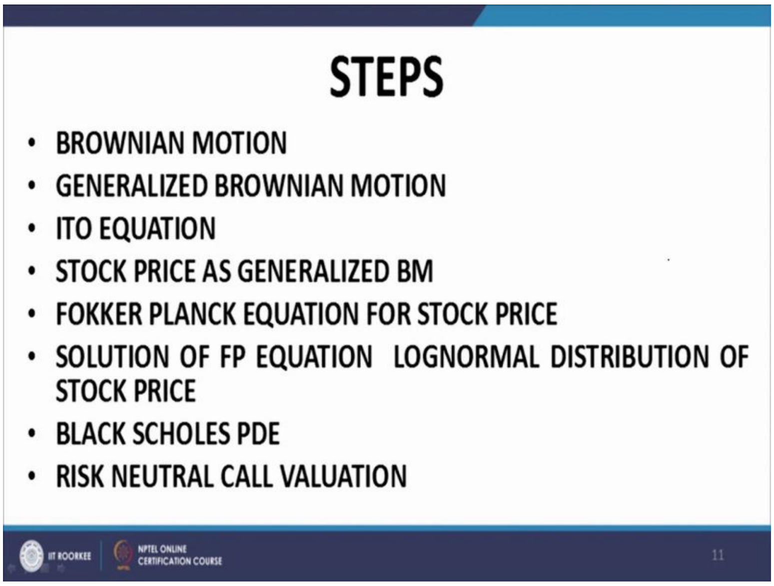

We need to follow, we need to introduce certain concepts and we need to follow a certain, stepwise methodology which is explained or which is specified in this particular slide. We start with explaining the concept of Brownian motion, we generalize the concept of Brownian motion and then we derive or we obtain the Ito's equation and our next step is to model the stock price because you see, we cannot arrive at the value of the option unless we have some model for the behavior of the stock price.

So, the next step is to model the stock price. And then, we will show that the assumptions that go into the modeling of the stock price imply that the stock price will follow a lognormal distribution. So our next step would be to derive the Black Scholes partial differential equation. We will also talk about, one thing is, let me clarify that the Black Scholes partial differential equation's derivation follows the same methodology or same philosophy as the no arbitrage principle in the one step or two step binomial model.

However, we also saw in the context of the binomial model that the no arbitrage or the arbitrage-free binomial model is similar to or is equivalent to the risk-neutral evaluation of the underlying option. And we will also do the risk-neutral valuation of the option using the same methodology as we did for the, in the case of the binomial model. So without much ado, let us proceed to the first step. Let us now move to the discussion on Brownian motion.

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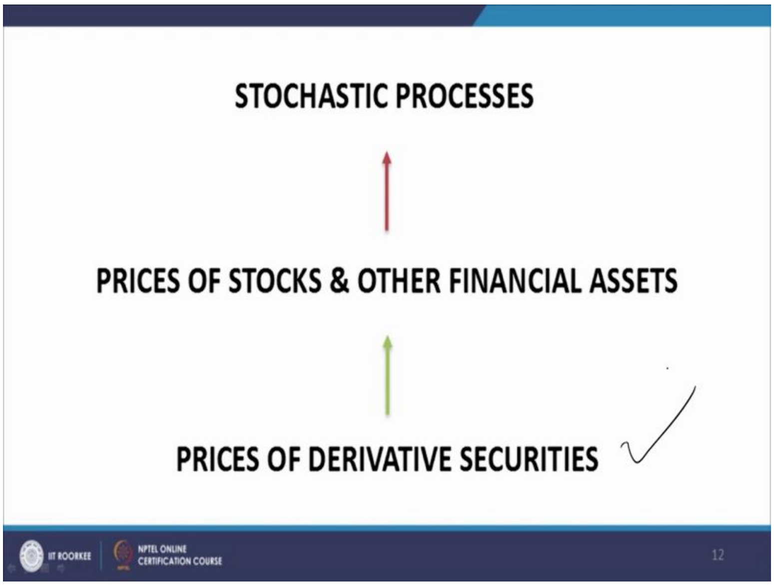

Before that, you see why do we need all that? You see, that is explained in this slide very succinctly. This is our objective. This is our objective. What is our objective? Our objective is to ascribe a price to derivative securities. Now derivative securities, what are derivative securities? They are functions or they are prices or functions or prices of the underlying asset. Therefore, we need to have a model for the prices of underlying assets.

And the underlying assets follow, have a content of randomness in their prices. As I mentioned once during this course of this lecture series, that we need necessarily to have some element of randomness in the behavior of the underlying assets, the price of the underlying asset, if we are to construct derivative securities or derivative instruments on those assets, and if the derivative instruments are to be worthwhile, are to be useful.

Otherwise, if the behavior of the underlying asset is deterministic, if it is absolutely known, how the system or how the asset is going to evolve over the future, there is no point in writing derivative contracts on such instruments. Therefore, the underlying assets that we have in practice, always derivative contracts are written are invariably assets whose evolution is embedded with a certain amount of randomness.

And such processes, which are not perfectly predictable or whose evolution, whose future evolution is not perfectly predictable, they have an element of randomness in their evolution are called stochastic processes. So, prices of derivatives are functions of prices of underlying assets, which are stochastic processes.

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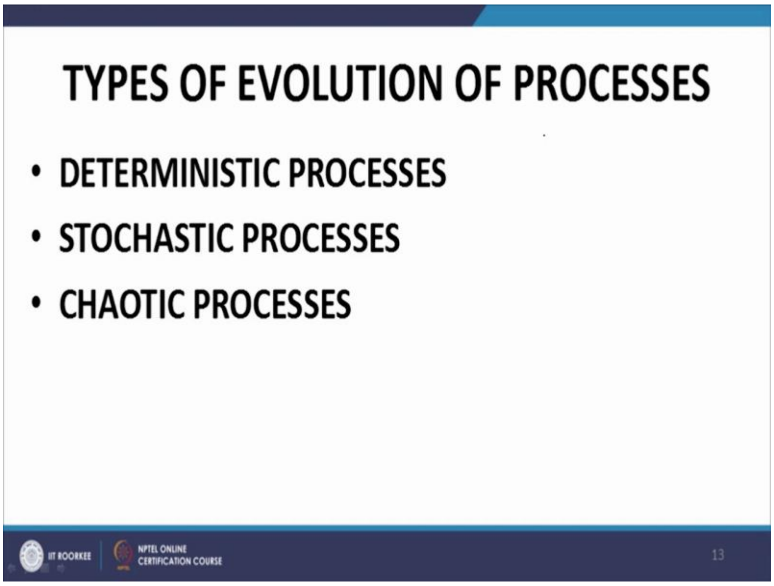

So, a quick touch on the types of evolution of processes. First of all the simplest or the deterministic processes where given a knowledge about the initial conditions of the process, we are able to precisely predict the future evolution of the process with perfect precision, that is the important part. So, given the knowledge of the initial conditions, t equal to 0 status of the process, we can precisely predict the future evolution of the process. These are called deterministic processes.

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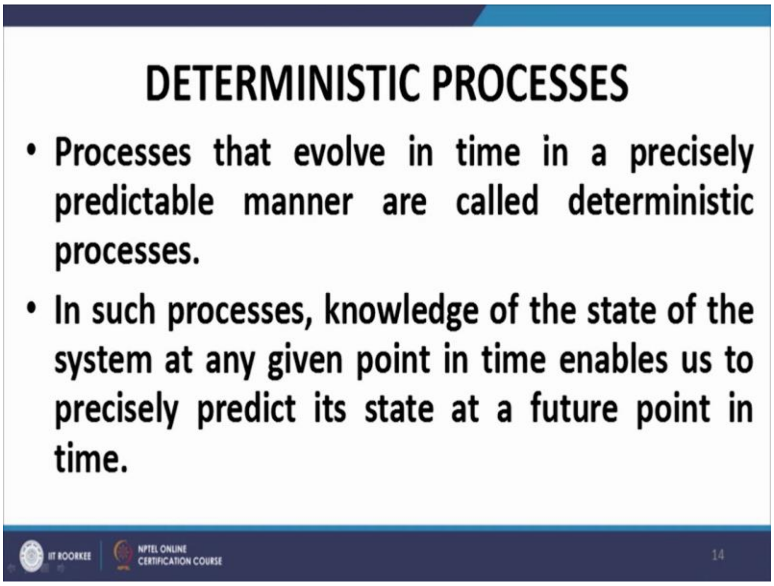

So, let me read it out for you. Processes that evolve in time in a precisely predictable manner are called deterministic processes. In such processes, knowledge of the state of the system at any given point in time enables us to precisely predict a state at a future point in time. There is no randomness in their behavior. If you know their status, you know their cardinals at a given point in time, let us say arbitrarily that, you know the cardinals have the process at t equal to 0, then you can precisely predict how the process is going to evolve over the future time period for which the experiment or observation is to be made.

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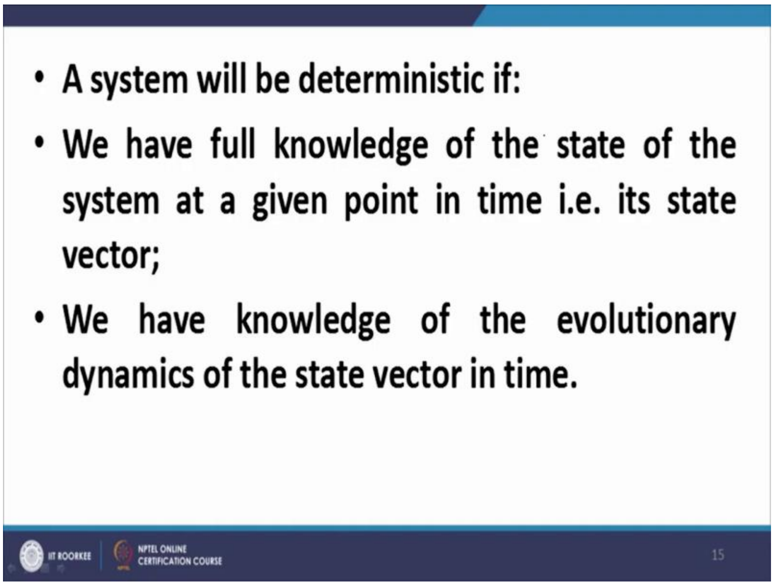

A system will be deterministic if we have full knowledge of the state of the system at a given point in time, arbitrarily t equal to 0, not necessarily so, but usually, we have it as t equal to 0. It can be any arbitrary point in time. And knowing that the status of the process at any arbitrary point in time, we are able to precisely predict the behavior of the system thereafter, that is the important thing.

And therefore, we have full knowledge of the state of the system at a given point in time, the state vector and we have knowledge of the evolutionary dynamics of the state vector in time. So how the state vector or how the system is going to evolve in time? What are the physical laws or financial laws or other laws that the system is going to obey as it evolves in time, we have knowledge about that, and we have the initial conditions known to us. If we have information on both of these, then we can precisely predict and the system would be deterministic.

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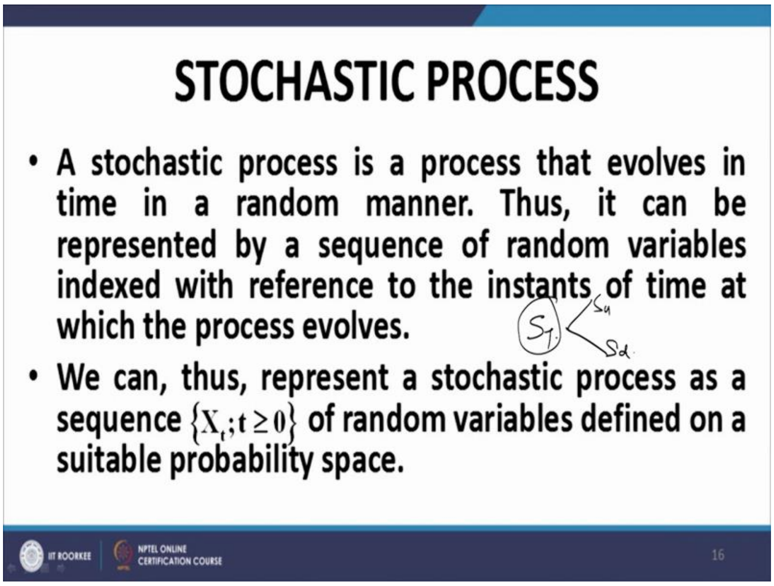

Then we have a stochastic process. A stochastic process is a process that evolves in time in a random manner. There is an element of randomness in its evolution. Therefore, what is the important characteristic of a stochastic process? The important characteristic of a stochastic process is that its evolution or its trajectory cannot be precisely ascertained, cannot be precisely determined, knowing its initial conditions at t equal to 0.

How it is going to evolve in future, we do not know with absolute certainty. We encountered an example just in the last lecture when we talked about the binomial model. At t equal to capital T, that was a t equal to 1 step, one time step, the underlying asset could either jump up to S u or it could jump down to S d, but whether it is going to jump up or down is not known to me sitting here at t equal to 0. That is, the evolution is random. We do not know what value it is going to take at t equal to capital T when it is going to undergo a transition.

So a stochastic process is a process that evolves in time in a random manner. Thus, it can be represented by a sequence of random variables indexed with reference to the instants of time at which the process evolves. Just as we did in the binomial model, at t equal to capital T, the stock price, let us say the stock price could either go up to S u or could either go down to S T. This is, thus, we can model, the behavior of the stock price at t equal to capital T by a random variable S T, where S T can take the value S u or it can take the value S d with certain well defined probabilities.

So, this is called a random, S T is the random variable and S u and S 2 are the possible values that the random variable could take with of course, the respective probabilities, whatever they may be. We can thus, now, this was a single step model. This was a model where the system or the stock underwent a transition only at one point in time. And if we have a two step model, then we can model it using two random variables, one random variable to capture the jump at t equal to 1 and the second random variable to capture the jump at t equal to 2.

So, if it is an n step process, then we need n random variables and the entire process can be represented or can be collectively known as the sequence of random variables S 1, S 2, S 3 and so on. So, we can thus represent a stochastic process as a sequence X t t greater than equal to 0 of random variables defined on a suitable probability space. That means, you can ascribe appropriate, well defined probabilities to each of those random variables that constitute the stochastic process.

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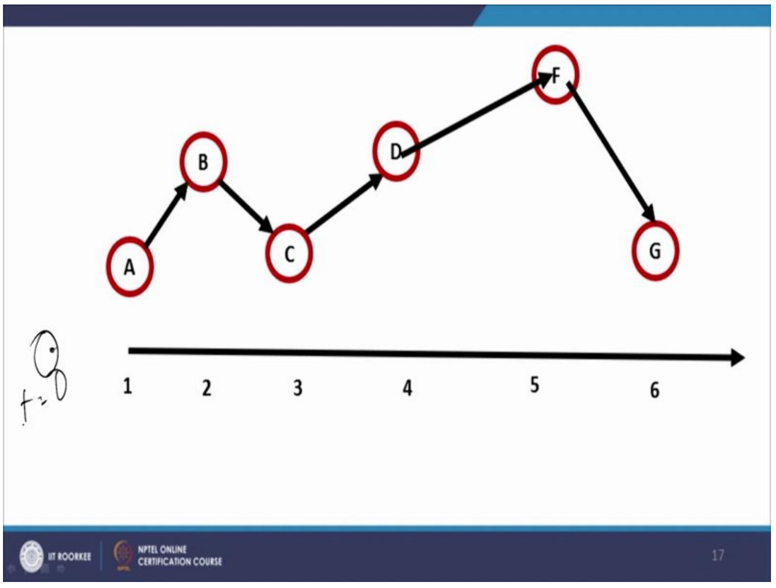

This is an illustration. At t equal to 1, we have the random variable A. A could take certain values. At t equal to 2 again, when the system undergoes a transition, we can represent it by the random variable B, we can again have a spectrum of values with respect to probabilities. You see, please note, we are sitting here at t equal to 0, this is, let us say, this is t equal to 0. So, at t equal to 0, we want to forecast or we want to predict or we want to study the behavior of the system as it evolves over time from t equal to 0 onwards.

The system can make transitions at t equal to 1, t equal to 2, t equal to 3 and so on. And at each point at which the system can make a transition, it can take one of the values of a spectrum of values and therefore, we can represent it by a single random variable at that particular point in time. Similarly, at equal to 2 again the system will undergo a transition. So, we need another random variable to describe the behavior of the system or the jumps of the system at t equal to 2 and so on. So, this collection of random variables A, B, C, D, E and F, these will constitute the stochastic process that we are trying to study.

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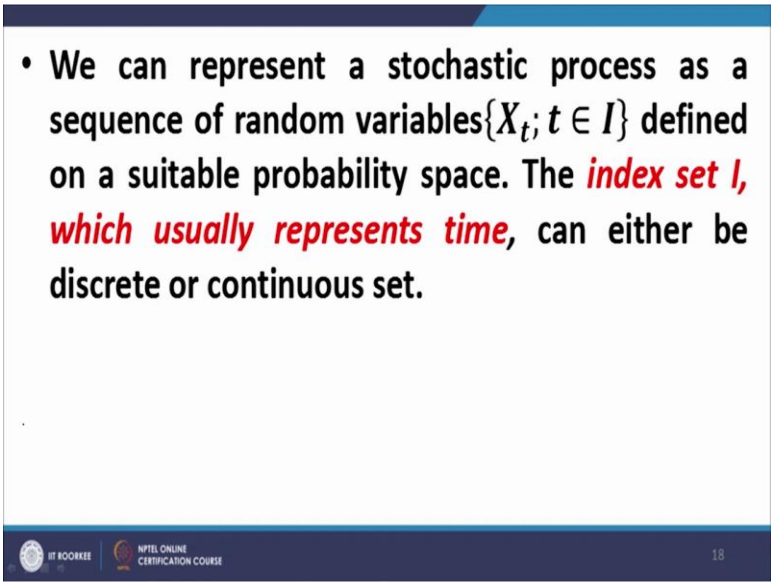

We can represent a stochastic process as a sequence of random variables X t, t contained in I defined on a suitable probability space. The index at I which usually represents time, not necessarily so, but usually, it represents time, and it can either be discrete or continuous.

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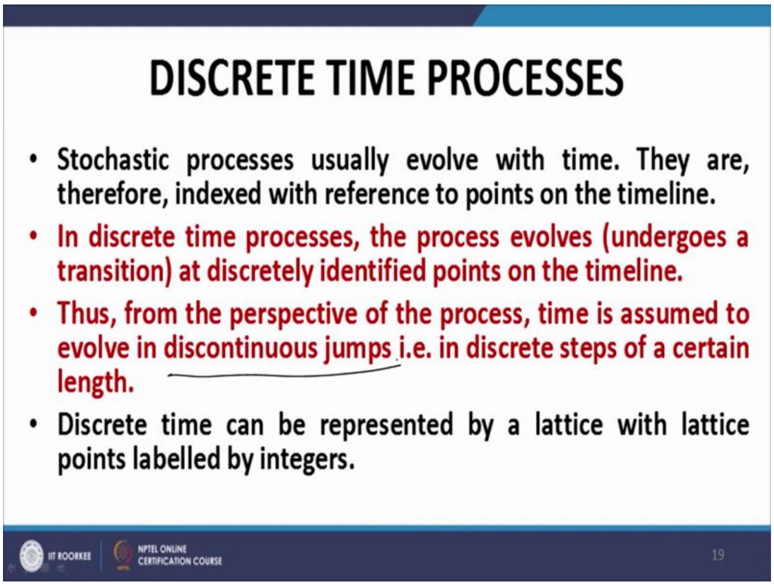

Therefore, we can have discrete time processes and continuous time processes. Talking briefly about discrete time processes, stochastic processes usually evolve with time. They are therefore, indexed with reference to points on the timeline. In discrete time processes, the process evolves, undergoes a transition at discreetly identified points on the timeline. Please note the timeline itself may be continuous, there is no restriction on the timeline not being continuous.

The point is that the system will evolve, the system will undergo a transition at discrete points on the timeline, discrete, well defined marks on the timeline at which the system will evolve, the system will undergo a transition. So, in discrete time processes, and these are called time steps, by the way, at which the system undergoes a transition. In discrete time processes, the entire process evolves, undergoes a transition in discretely identified points on the timeline.

Thus, from the perspective of the process, time is assumed to evolve in discontinuous jumps. This is the important part. From the perspective of the system, because it is only going to make a transition at discrete points. So, if you look at it from the perspective of the system, time is not flowing continuously. Time is flowing in jumps, time is flowing in discrete chunks. And at every jump, the system wakes up and makes a transition.

When the next jump occurs, system again wakes up and makes a transition. So these are discrete time processes. So thus, from the perspective of the process time is assumed to evolve in discontinuous jumps, that is in discrete steps of a certain length. Discrete time can be represented by a lattice, with lattice points labeled by integers.

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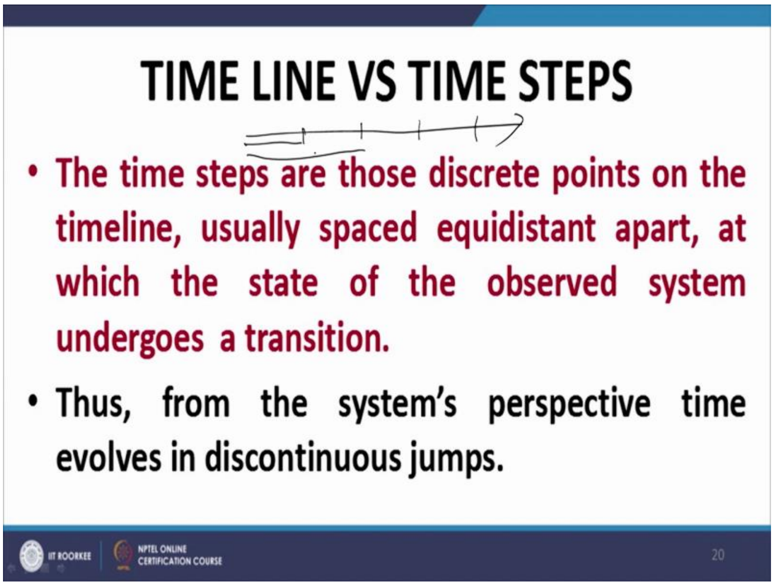

Now time line vs time steps, I have emphasized this point already, timeline is the underlying continuous flow of time. However, on the timeline, you can have discrete points at which the system undergoes a transition and these are called time steps. This will be one time step, this will be two time steps, and so on. So the timestamps are those discrete points on the timeline, usually spaced equidistant apart, at which the state of the observed system undergoes a transition. Thus, from the system's perspective, time evolves in discontinuous jumps.

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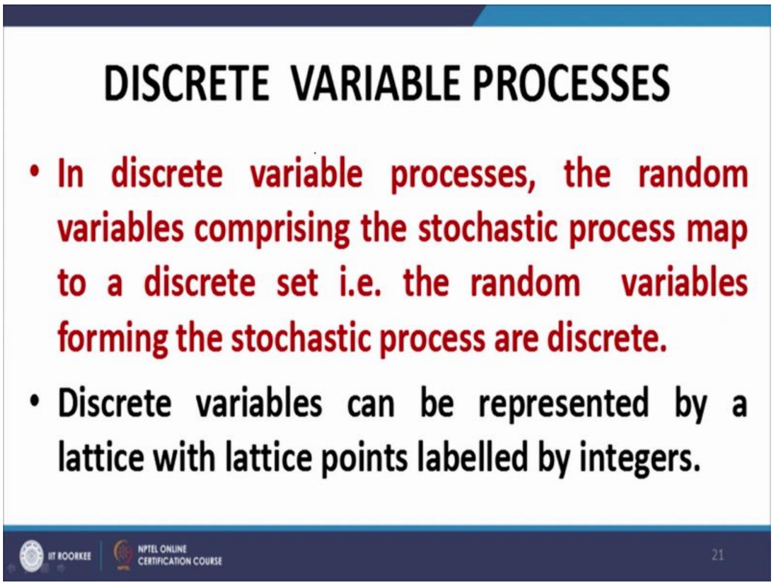

Then we have discrete variable processes. In discrete variable processes, the random variables comprising the stochastic processes mapped to a discrete set. In other words, simply stated, the random variables the sequence of which forms the stochastic processes have a discrete spectrum of possible values.

They cannot take any value within a continuous interval or a continuous set, they can take only discrete values within a set of possible values. That is a discrete variable process, that is the random variables forming the stochastic processes are discrete. Discrete variables can be represented by a lattice with lattice points labelled by integers.

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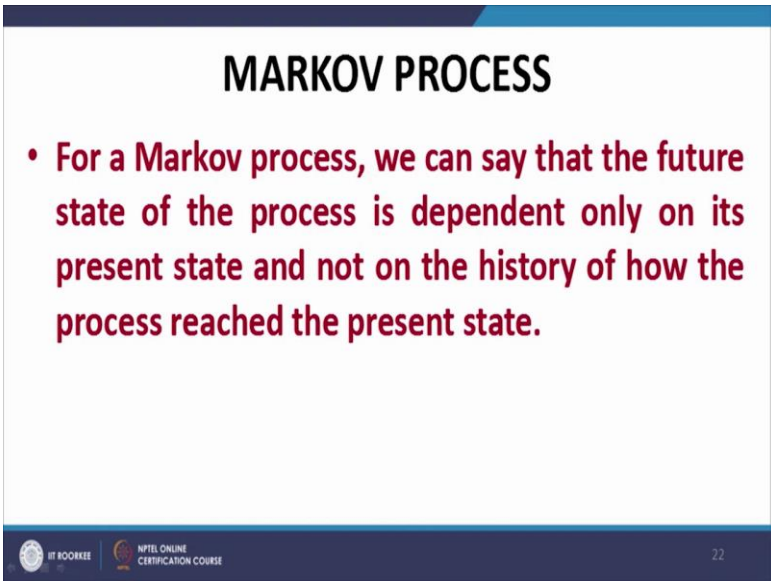

Then, we talk about Markov processes quickly. Markov processes are those processes, whose memory is confined to the preceding state alone. For example, if you look at the future evolution of a particular process, immediately, the present state of a particular system, the evolution of the present state of the particular system, will depend only on where the system was at the previous state and not earlier.

How the system reached that previous state is irrelevant. What path or what is the sequence of values that the random variables took, what path the system followed in arriving at this particular point is irrelevant. It is only if you are studying the value of the process at t equal to 1, that is, the value of the process at t equal to 0 is the only criterion that will determine which of the possible values the system could take at t equal to 1. So that is the Markov process. A process whose memory is confined to the immediately preceding value.

So let me quickly read it out. For a Markov process, we can say that the future state of the process is dependent only on its present state and not on the history of how the process reached the present state. This is important. The path is irrelevant. The path is irrelevant. The path followed by the system thus far is irrelevant. It is only the current values which is relevant for knowing which of the following possible values the system could take in the next step.

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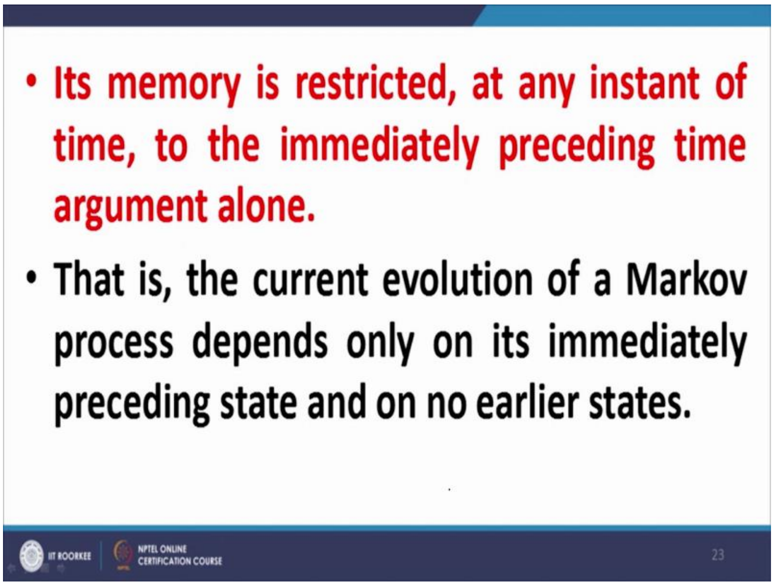

Its memory is restricted at any instant of time to the immediately preceding time argument alone. And what it is going to do at t equal to 1 will depend on where it is at t equal to 0 and not earlier, and not earlier. That is, the current evolution of the Markov process depends only on its immediately preceding state and no earlier states.

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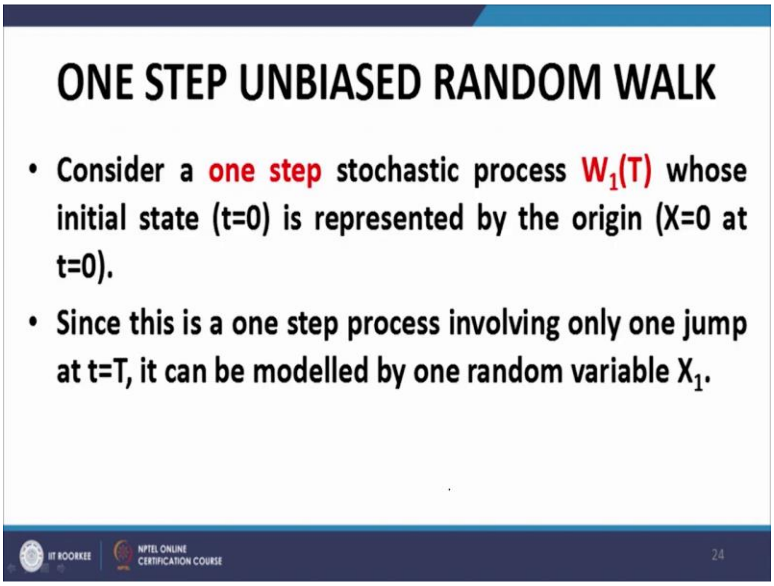

One step unbiased random walk, we will take it in next lecturer. Thank you.