Quantitative Investment Management
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Lecture: 44
Option Pricing – Binomial Model - 2

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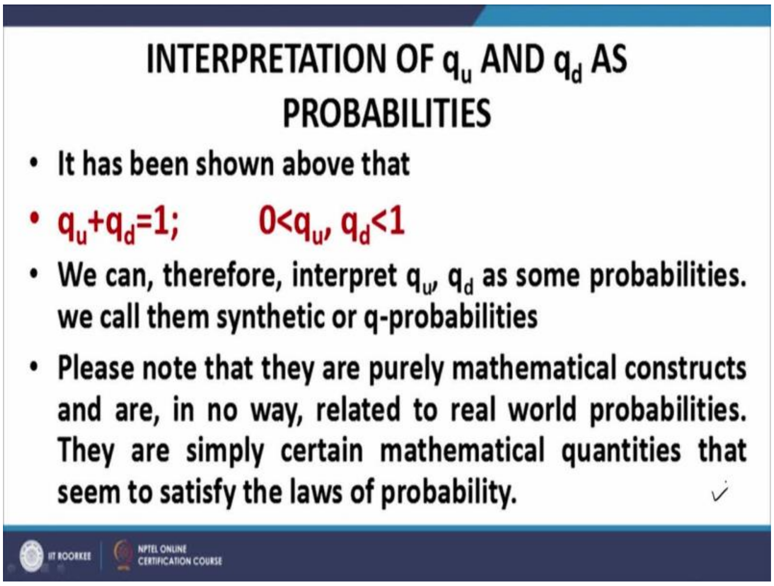

Welcome back. So, before the break, we saw that q u plus q d is equal to 1, this was the first expression that we arrived at, in respect of q u and q d, and the second expression was that values of q u and q d must lie between 0 and 1. That being the case, that being the case, it is reasonable or it is intuitive rather, that we consider this q u and q d as some sort of probabilities.

So, we can therefore interpret q u and q d as some probabilities, let us call them synthetic probabilities or q probabilities. Please note that they are purely mathematical construct. I repeat, they have not been obtained by actual experimentation, they do not represent real life upswing, probabilities of upswing and downswings of the stock price or of the derivative price for that matter. They do not represent that.

But they are simply mathematical constructs, they are numbers, which we arrived at, during our process of valuation of the derivatives, such that these numbers sum up to 1 and these numbers, each of them lies between 0 and 1 and therefore, we can call them as probabilities, let us call them as synthetic probabilities or q-probabilities. Please note that they are purely mathematical constructs and are in no way related to real world probabilities. They are simply certain mathematical quantities that seem to satisfy the laws of probabilities.

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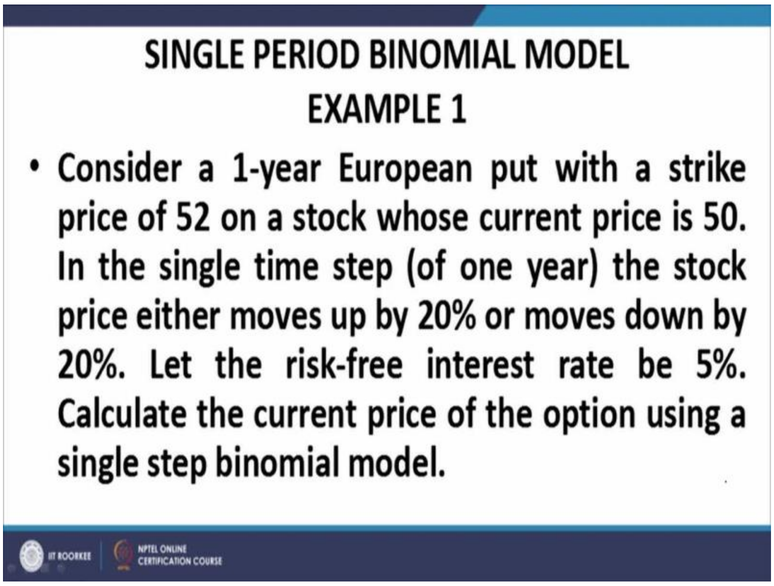

Let me do this example. Now, consider a 1 year European put with a strike price of 52 on a stock whose current price is 50. S naught is 50, the strike price that is K is equal to 52. In the single step, single time step of 1 year, we assumed that the time step is a 1 year. I repeat, I repeat again, that we, there is no restriction on what we assume as a time step. The length of the time step in terms of real time. The only thing is, it should be a single time step, this system should undergo a transition only at the end of the step and nowhere in between.

In the single time step of 1 year, the stock price can either move up by 20 percent or move down by 20 percent. Let the risk free interest rate be 5 percent. Calculate the current price of the option using a single step binomial model. So, all the data is literally given to us in the problem. Straight away, S naught is equal to 50, K is equal to 52 the capital T is equal to 1 year, the stock price can either move up by 20 percent or down by 20 percent, that means it can either go up to 60 or it can go down to 40, and the risk free rate is given as 5 percent. That is all that is really required for this problem. Let us solve this problem.

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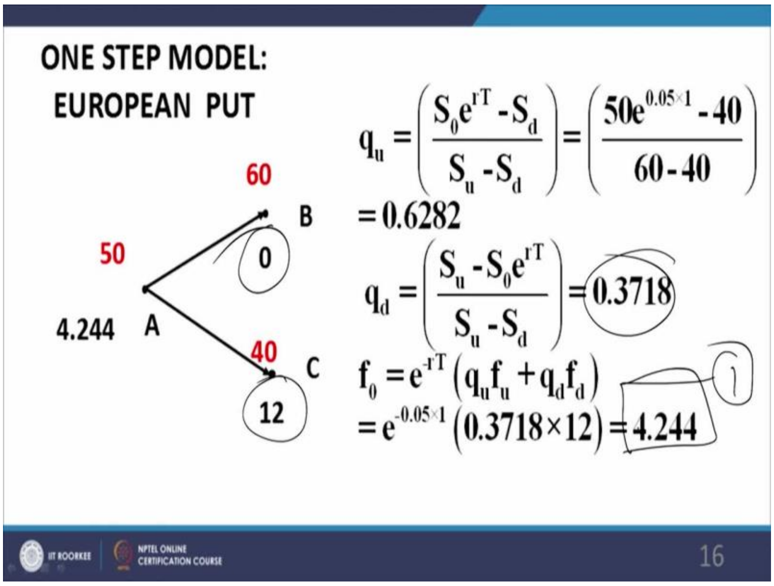

Let us first work out q u and q d. We are given, q u is given by a S naught e to the power r T minus S d upon S u minus S d. Now S naught is equal to 50, e to the power plus r T, I am sorry, e to the power r T is equal to e to the power 0.05, that is, r is equal to 5 percent, T is equal to 1 year, and S d is equal to 40 and S u is equal to 60. So, all the data is fed into this formula, and we get q u is equal to 0.6282. q d is equal to1 minus q u, and that is equal to three 0.3718. The value of the derivative of T equal to 0 is given by this formula, where we have to find out f u and f d

What is f u? f u is a derivative value if the stock price makes the up jump, and if the stock price makes the up jump, it goes to 60. And remember, what is the strike price of our option? The strike price of our option is 52. And this, and it is the put option there, and the actual market price is equal to 60. Therefore, the option will not be accessed, and therefore the payoff from the option 0 as you can see here.

And in the case, when the stock price is 40, the access price is 52, the option will be exercised, and there be a payoff of 12 in the case, when the stock price goes down from 52, 42, at the end of the time step of 1 year. So, that is what it is. q u is equal to 0.6282, q d is equal to 0.3718, f u is equal to 0, f d is equal to 12. And putting these values, what we get is f 0 is equal to 4.244. So, a straightforward problem.

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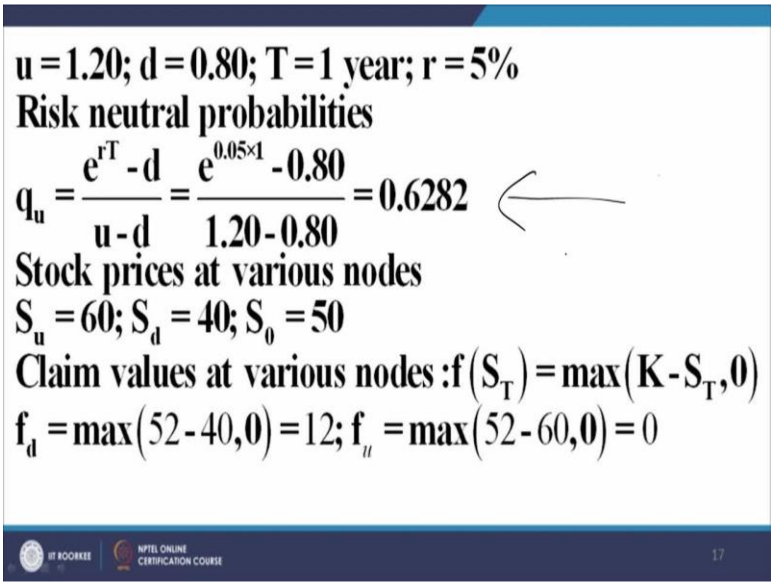

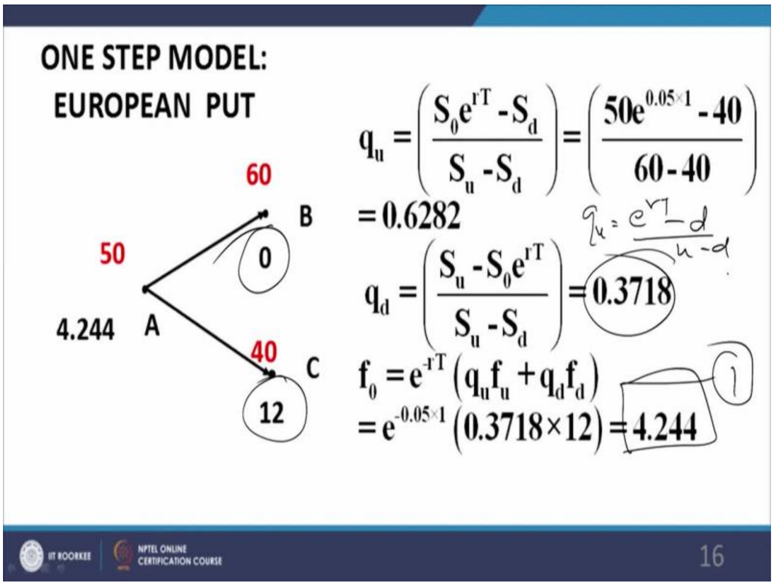

These are the calculations summarized in this slide, using u and d. I used S u and S d in the earlier slide, here I have used u and d. The result is obviously the same, we are dividing throughout by S 0. For example, if you look at, if you look at this particular expression, if you divide throughout S u, by S 0 what we get is q u is equal to e to the power r T minus d upon u minus d. This is precise with the formula that has been used in this expression. The rest is as we discussed in the previous slide. So, we arrive at the answer of answer 4.244.

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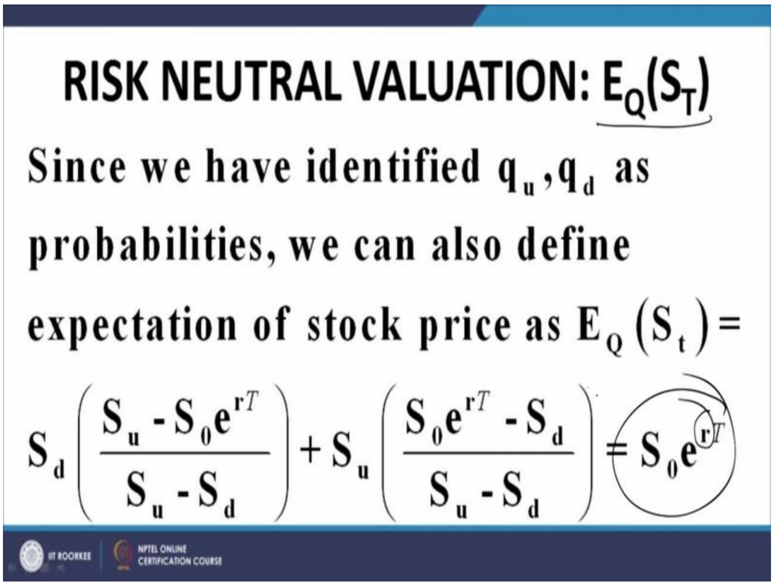

Now, let us investigate in greater detail, let us examine in greater detail, some more properties, some more features of this q-probabilities. In order to do that, what we need to do next is to find out this expression E Q S T. What is E Q S T? It is the expected value of the stock price under these q-probabilities. I repeat, it is the expected value of the stock price under the q-probabilities. So, what will it be given as? It will be given us q d into S d plus q u into S u. And when we put these values here, of S u and S d, this is what we get S 0 e to the power r T.

So, E Q of S T is equal to S 0 e to the power r T. What does this indicate? This indicates that the stock, the underlying asset of the derivative is giving us a risk free rate of return r if, or with reference to if the return is calculated with reference to q-probabilities. Let me repeat, the risky asset, the underlying asset, the stock is giving us expected return equal to the risk free rate, if we work out the expected return on the basis of the q-probabilities. This is another very striking feature of this q-probabilities.

I repeat once again, the risky asset, the asset which embodies the riskiness of our model, please note we are having to put two assets, the risk free asset and the risky asset, the stock or the underlying asset is the, is the representation of the riskiness or in this world or in this environment that we are doing the modeling. And this risky asset, why is this risky, because it can take two values, a T equal to capital T S u or S d and we do not know what value it is going to take setting here at T equal to 0.

So, that is a random variable. And that randomness manifests itself as riskiness in this particular asset, in this particular stock. We find that this asset, which is risky, is giving us an expected return equal to the risk free rate of return if that expected return is calculated with respect to q-probabilities. What does it mean?

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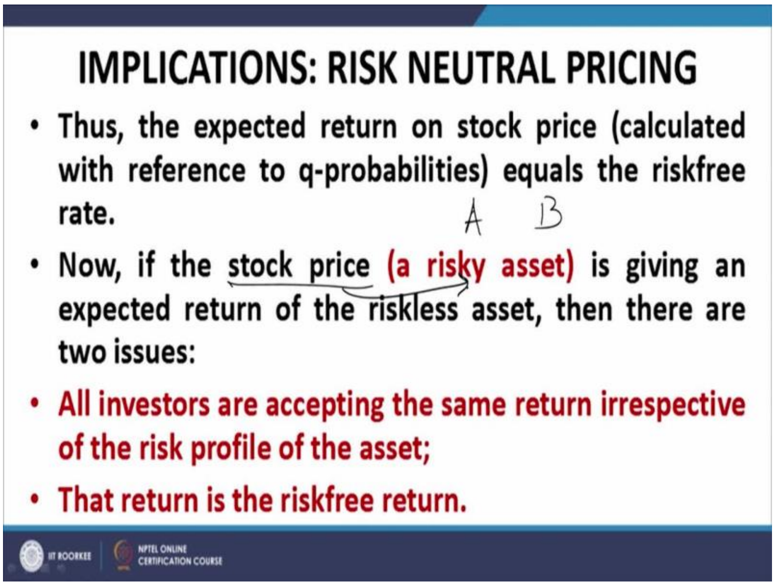

It means that the risk neutral pricing or the risk free pricing that we have talked, that the pricing methodology that we have been talked about, we are talking about the probabilities that we are talking about, the q-probabilities, the synthetic probabilities that we are talking about, are probabilities with reference to a word with reference to an environment, which is risk neutral.

What do we mean by risk neutral? By risk neutral we mean that the, all the investors in that particular environment in that particular are indifferent to risk, they do not care about risk. They are not bothered about risk. Why do I say so? Why do I say that this q-probabilities relate to the risk neutral world? Because a risky asset is giving you a risk free return. And that means what?

That means all the risky assets would give you the same risk free return by virtue of implication. So, that means that the investors that are there in that particular environment, in that particular world are not concerned with risk. They, whatever be the risk, whether the risk of a particular investment is higher or lower or zero, notwithstanding that fact, notwithstanding the riskiness of the asset, they are concerned with only the return, and they would invest in that asset without any reference to without any consideration of risk. That is, what is risk neutrality, indifference to risk.

So, in this situation, obviously, it becomes a one dimensional analysis framework, where the investor would invest in an asset which gives you the higher expected return. That leads to the second point that is salient to this model. The first point, that all the risky assets are providing you the same return, and because all the risky assets are providing you the same return, it follows that the investors who are evaluating those assets are not concerned with risk, not bothered with risk.

The second feature of this model is that all these assets, all these risky assets are giving you the risk free rate of return. Why is that? That is because, let us say you have two assets. Let me explain that by an example, A B. Let us assume that A asset is risky, B asset is not risky. Let us assume that the return on A is higher and the return on B is lower. We are now living in a risk neutral world. Please note we are not transpose, our transpose to a risk neutral world. And in the risk neutral world, risk has no consideration.

What will that manifest itself as? Because A has a higher expected return notwithstanding its higher riskiness, all the people, all the investors in this risk neutral world will invest in A. Nobody will invest in B because nobody is concerned about risk, nobody bothers about evaluating risk, nobody takes the risk into consideration when evaluating any investment. So, because A has a higher expected return, all the investors would go for A. Obviously, this would create a demand supply imbalance.

And as a result of it, the price of A will go up, the price of B will go down, the returns will go, move inversely, the returns on A will decline, the return on B will increase and at equilibrium, we will see that all the, all the investors will get the threshold return and that threshold return will be the return on the risk free asset. That is why in this risk neutral world, number 1, all investors get the same return or assets provide the same return, and number 2, that same return is the risk free rate of return. As we saw by explicit calculation using the q-probabilities.

So let me read it out. Thus, the expected return on the stock price calculated with reference to q-probabilities equals the risk free rate. Now, if the stock price, a risky asset, this is important stock price is a risky asset is giving an expected return of the riskless asset, what does it mean? There are two reasons. Number 1, all investors are accepting the same return irrespective of the risk profile of the asset. And number 2, that return with a risk free return.

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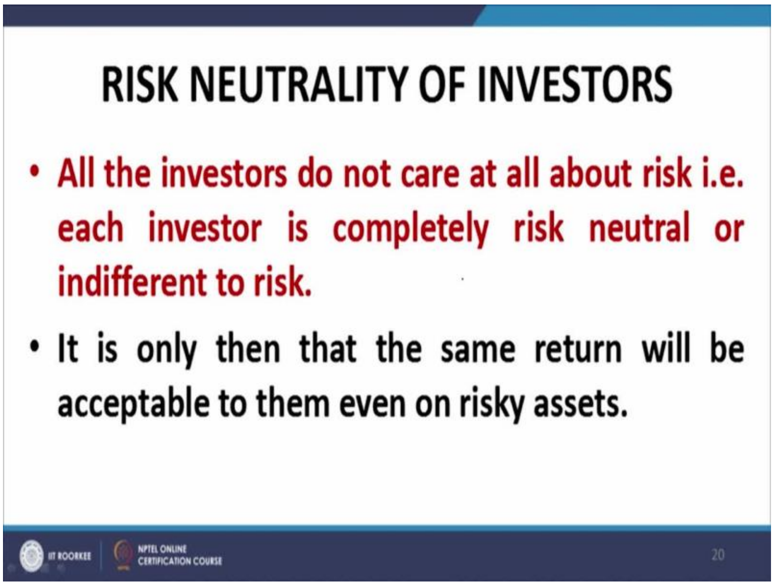

So, risk neutrality of investors, all the investors do not care at all about the risk. Each investor is completely risk neutral, indifferent to risk. So as far as the investment opportunities, investment appraisal is concerned, it is a one dimensional framework depending on the expected return. And that leads to the issue of arbitrage. If one asset is providing you a higher return than others and therefore, all assets must provide the same return and that must be the threshold return which is the risk free rate of return. It is only then that the same return will be acceptable to them even on risky assets.

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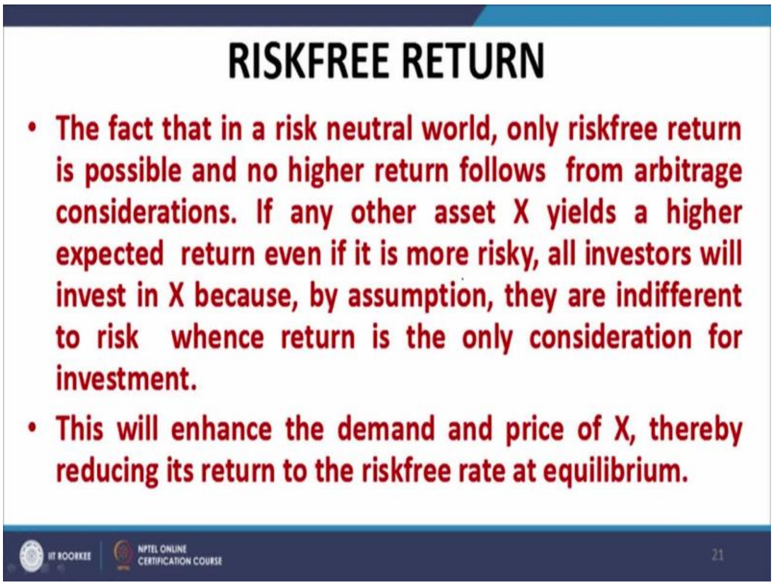

Risk free return. I just explained it. Let me read it out as well. The fact that in a risk neutral world, only risk free return is possible with no higher return follows from arbitrage considerations. If any other asset X yields a higher expected return even if it is more risky, all investors will invest in X because, by assumption, they are indifferent to risk whence return is the only consideration for investment. This will enhance the demand and price of X, thereby reducing its return to the riskfree rate at equilibrium.

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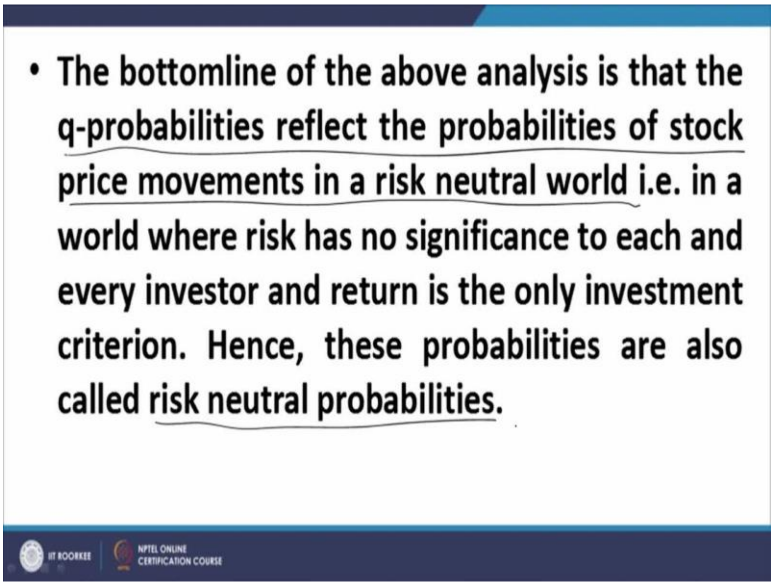

The bottom line of the above analysis is that the q-probabilities reflect the probabilities of stock price movements in a risk neutral world. This is the crux of what I have explained. That is, in a world where risk has no significance to each and every investor and return is the only investment criterion. Hence, these probabilities are also called risk neutral probabilities. So, now we have a formal name, a respectable name for these probabilities, we call them risk neutral probabilities. Instead of addressing them simply as synthetic probabilities or q-probabilities, we have a name for them, we call them risk neutral probabilities.

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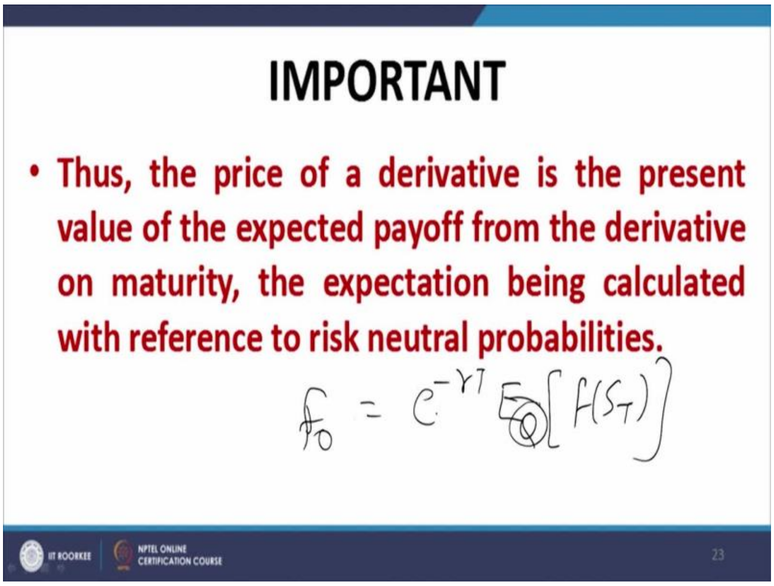

Thus, the price of a derivative is the present value of the expected payoff from the derivative on maturity, the expectation being calculated with reference to risk neutral probabilities. f 0 is equal to e to the power minus r T E Q f of S T. This was the formula that I gave you earlier. And this is what the formula can be interpreted as. This Q is important. We are talking about expectation with reference to risk neutral probabilities.

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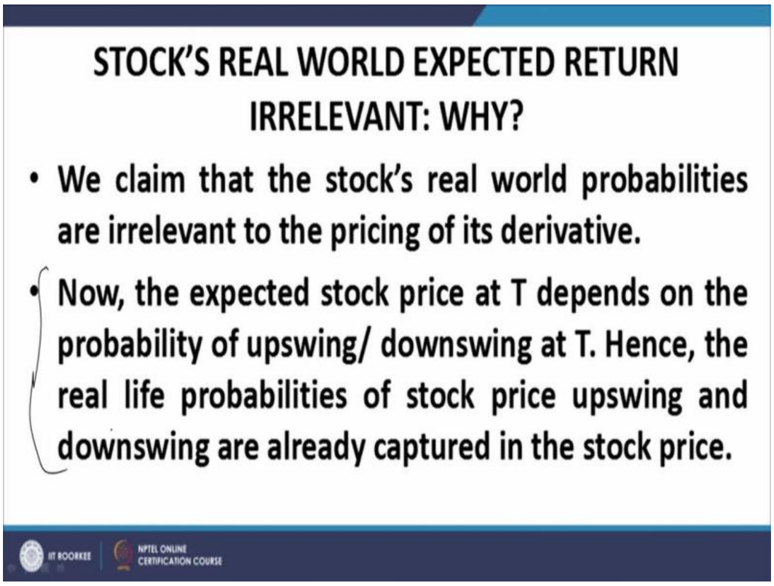

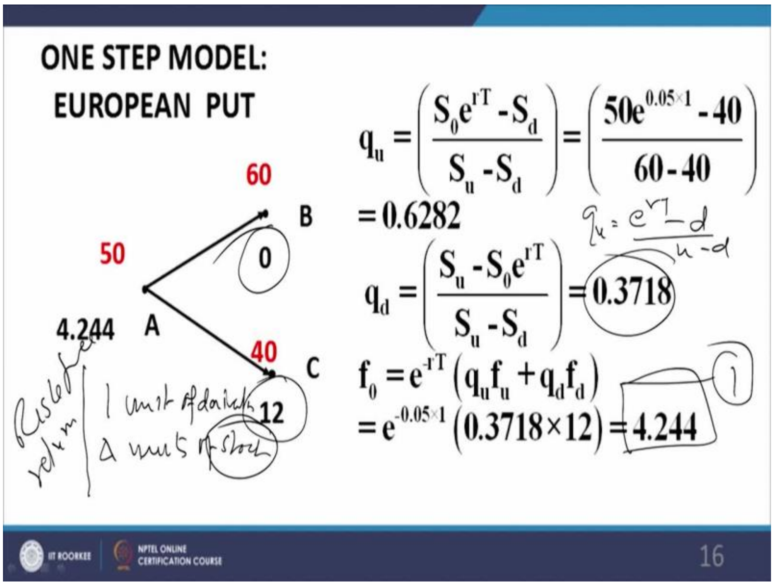

Now, I will come to a very interesting point. The stock's real world expected return is irrelevant. Now, if you look at look at this formula that we got just now, look at the example that we did in the earlier slides. Let us go back and look at this example. There is a very intriguing feature of this particular methodology that we have followed.

And that is, that when calculating the value of the derivative, we have nowhere considered the real world, the actual upstream and downstream probabilities of the stock. You can see here, q u and q d have nothing to do with the real world probabilities of upswing or downswing. They are simply mathematical construct.

Why is that? This is very striking. The reason is, how did we go about valuing this particular derivative? We constructed a portfolio consisting of one derivative, one unit of derivative in delta units of stock. We showed or we worked on the premise that this combined portfolio has no risk and therefore, because it has no risk, it will give you the risk free rate of return. That was the fundamental methodology that we followed.

We constructed a risk free portfolio consisting of the stock and the derivative, we argued that the random, because the model of randomness that goes into the stock price is the same as the model of randomness that goes into the derivative price. Therefore, we, by taking opposite positions, we can annul the randomness of one with the other. And that means what? That means if the randomness is annulled, the asset would behave as a risk free asset and give you the risk free rate of return.

Note the issues, where do the real world probabilities come in? The real world probabilities come in, when we talk about this particular part, the stock component of the hedged portfolio.

The hedge portfolio consists of the derivative and the stock. Now the stock component, obviously, the price of the stock at any instant of time, is dependent on the upswing probability and downswing probability based on the real world data.

So, the bottom line of what I am trying to say is that we have not, actually, we have not ignored the real life probabilities. They have manifested themselves in a latent manner in a hidden manner, when we talk about the stock price in this stock component of the hedged portfolio. So because the hedged portfolio contains the stock, the, and obviously, the price of the, of the derivative is based on, or is situated with the price of the stock as well.

Therefore, what happens is that the upswing and downswing probabilities get captured because of their involvement in the fixation, in the ascertainment or in the market price of the stock. So, we claim that the stocks real world probabilities are irrelevant to the pricing of the derivative.

Now, the expected stock price at T depends on the probability of upswing and downswing of T and hence, the real life probability of stock price, upswing and downswing are already captured in the stock price. So, when we are talking about the stock component of the hedged portfolio, that stock component embeds or incorporates information, data about the probability of upswing and downswing in the real world.

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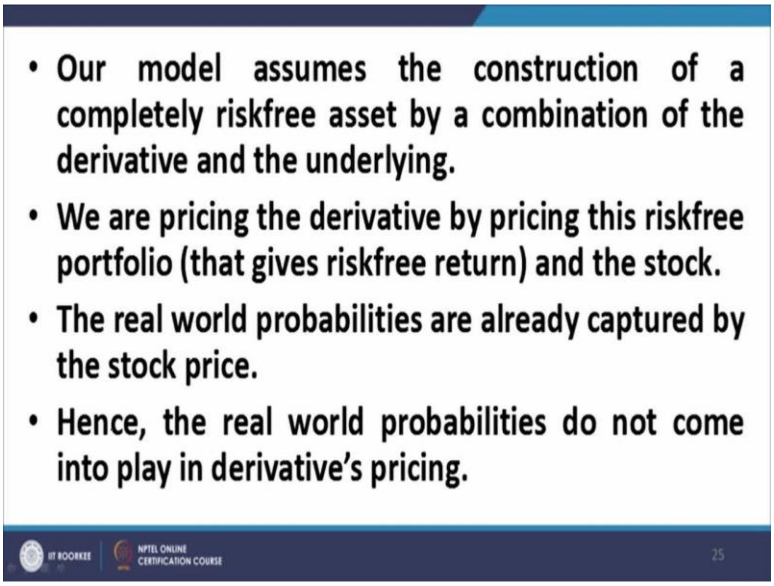

Our model assumes the construction of a completely risk free asset by a combination of the derivative and the underlying. So, we are pricing the derivative by pricing the risk free asset and pricing the stock. The stock contains the data about its own upswing and downswing probabilities in the real world, and the risk free return obviously, we know. So, knowing these things, we are pricing the derivative by implication, rather than explicitly pricing the derivative on the basis of its behavior.

So, that is why, that is how the upstream and downstream probabilities of the real world become involved in this particular model. They are latent, but nevertheless, they are very much there. The real world probabilities are already captured by the stock price. Hence, the real world probabilities do not come into play, explicitly in the derivative price.

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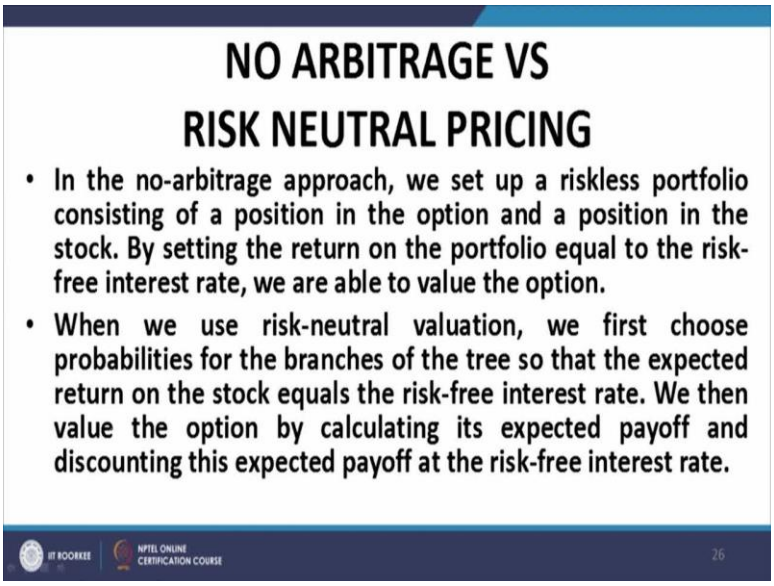

So now we talk about or compare the two approaches. Number 1 approach, now please note, the value that we arrived at for the derivative is the same whether we start with a risk free portfolio, a construction of a risk free portfolio using 1 unit of the derivative, delta units of the stock and then showing that this is a risk free asset and then making it to give the risk free rate of return because it is a risk free portfolio. That is the risk free approach to pricing or we use the q-probabilities that the, where we call there is neutral pricing.

So what is the relationship or what is the difference between them is given on this slide. Let us read it out. In the no arbitrage approach, we set up a riskless portfolio consisting of a position in the option and a position in the stock. By setting the return on this portfolio equal to the risk free rate which is required by arbitrage, it is here that arbitrage comes into play, we are able to value this option, value the derivative. So this is the no arbitrage or arbitrage free pricing.

When the risk neutral valuation, we first choose probabilities such on the branches of the tree, so that the expected return on this stock equals risk free interest rate. That is how we calculate the q probabilities. As you recall, you will, the return on the stock on the risky asset was equal to the risk free rate when calculated with reference to these q-probabilities.

Working backwards, if you have the risk free rate, you can work out the q-probabilities that give you a risk free rate of return. So we work backwards, we calculate the values of q u and q d. And using these values of q u and q d, then we calculate the expected value of the payoff from the derivative corresponding to S u and S d. And then finally, we discount that expected value to give the today's value of the derivative. That is the risk neutral approach to pricing.

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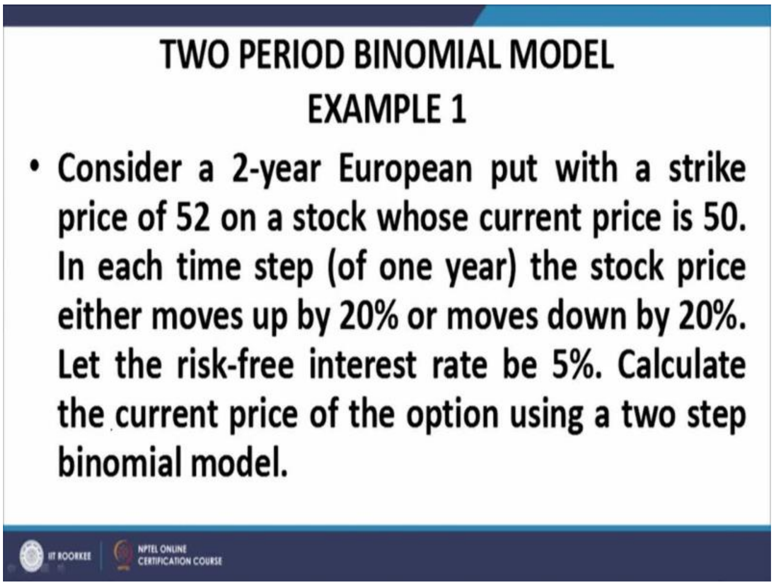

Then we talk about a two period binomial model. We have talked about a one period binomial model. I will not go into the proof of the two period binomial model. Suffice it to say that the model holds not only for one step binomial, it holds for two period or two step, it holds for three steps, it holds for n steps, it holds for as large number of steps as you desire.

In fact, that is the versatility of this approach, that you can, by extending the number of steps in the binomial tree, you can model as many values of the stock price at maturity as you like. Whatever is your feeling, whatever is your estimate about the possible values of the stock, that it could take on the date of maturity, you can value them by extending the number of steps in the binomial model. So let me illustrate this by an example rather than giving you a tedious detailed proof, which is pretty much similar to what we do for the binomial theorem. in senior secondary classes.

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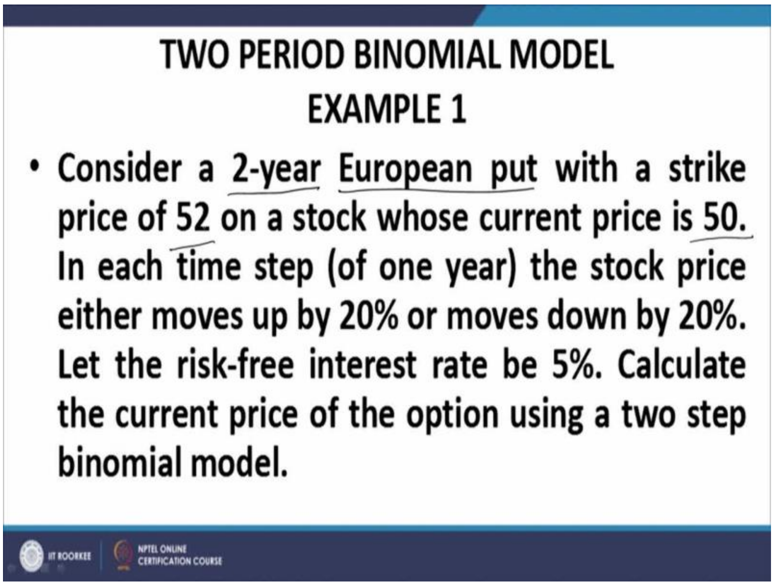

Consider a 2-year European put, 2-year European put with a strike price of 52 on a stock was current prices 50. This is S naught, this is K, and maturity is 2 years, and it is a European put option. In each step of 1 year, so there are two steps here. T equal to 1, that is, at the end of 1 year, and then T equal to 2 at the end of 2 year. At T equal to 1, the stock will make the jump either upwards or downwards of magnitude 20 percent, either 20 percent up or 20 percent down. And then from where it is, at T equal to 1 after the jump, it will again make a jump either upwards or downwards either 20 percent up or 20 percent down.

Let me repeat the pattern of the stock that we are going to model. At T equal to 0, suppose it is at S 0, then a T equal to 1, it will either go to 1.2 S 0 or it will go down to 0.8 S 0 at the end of 1 year. This is the first period. Then from wherever it is at T equal to1 year, whether it is at 1.2 S 0 or 0.8 S 0, it will again make a jump at T equal to 2. That means, it will either go up by another 20 percent or it will go down by 20 percent from the value at which it is at T equal to 1. That is the important part. It will not come back down to 0, it will, down to S 0. From wherever it is at T equal to 1, it will again either jump up or jump down.

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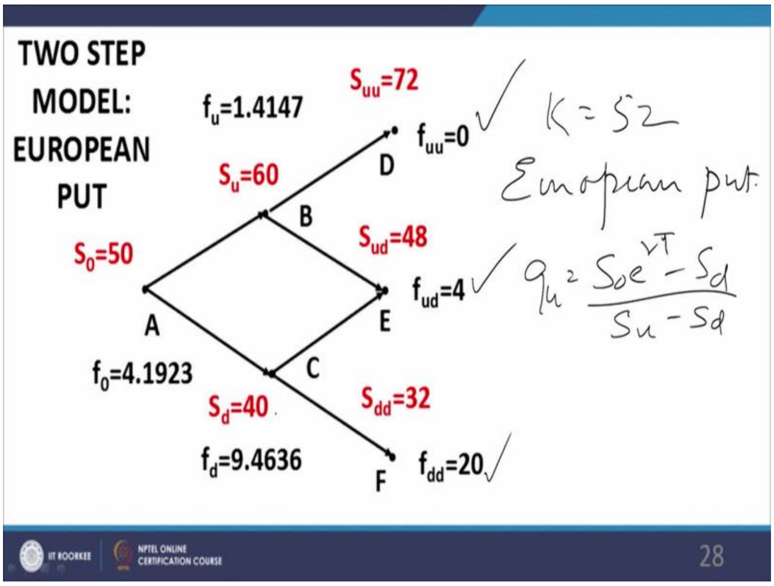

So, this is the model of the stock price. We are at point A. At the point A, S 0 is equal to 50. At T equal to 1 year, we can go either to B where the stock price goes up by 20 percent to 60 or the stock price goes down by 20 percent to 40 at C. And then depending on whether it is at B or C, at T equal to 1 year, it will again make a jump either upwards to 72 or to 48 if it is at 60 at T equal to 1. And if it is at 40 at T equal to 1, again it can either go up to 48 or it can go down to 32.

So, this is the structure of the tree. The claim values at T equal to 2 are easily found out. Recall what is the strike price, the strike price is 52, it is a European put. So, if the stock price is above 52, the payoff is 0. So, that accounts for this 0. If the stock price is 48, then the payoff is equal to 52 minus 48, that is 4. And if it is stock price finishes up at 32, the payoff is equal to 20. That is 52 minus 32. So, these are the payoffs that the three nodes, three extreme nodes, and then we work backwards, we work out the q-probabilities.

How do we work out the q-probabilities? q u is equal to S 0 e to the product r T minus S d upon S u minus Sd. Now, everything is given to us. S 0 is equal to 50, S d is equal to 40, r is equal to, I think r is given as, r is given as 5 percent, T is equal to1 year, u is equal to 60. So, you can find out this value. This value has been found out in the next slide, which is 0.6282.

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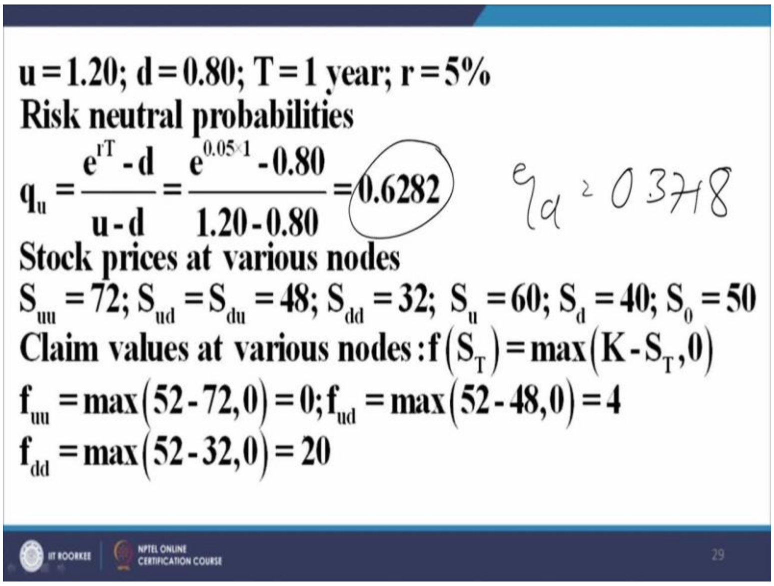

And therefore, q d and q d turns out to be equal to 0.3718. And using these values of the claims, that we have worked out 0, 4 and 20, we can work out the, we can work out the value of the option at B and C. At B turns out to be 1.4147 and at C, it turns out to be 9.4636. And using these values again, we can work out the value at A. And that turns out to be 4.1923.

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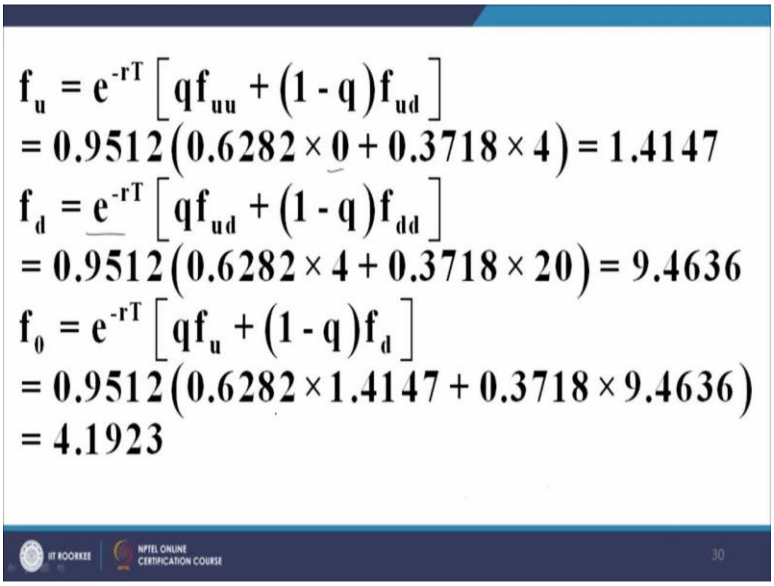

So, the working is given in these two slides, all the working. This is f u’s working, this is f d’s working, and the net result is 4.1923.

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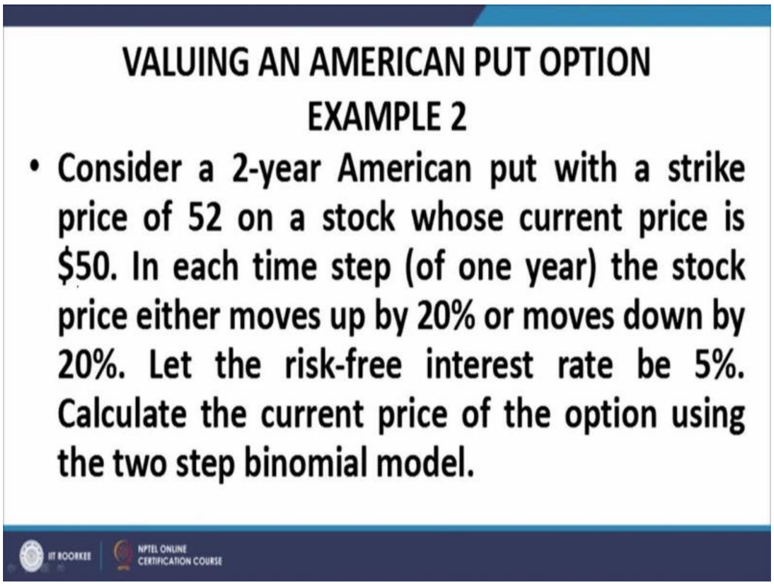

So with, I conlude today's lecture. I will start with how this particular model can be modified to value an American option. That is my next step. Thank you.