Quantitative Investment Management

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Lecture 43

Option Pricing – Binomial Model – 1

Welcome back. So in the last lecture, towards the conclusion of the last lecture, I was talking about the various strategies that can be implemented using positions and options. We discussed the covered call strategy, the protective put, then the straddles and strangle, strips and straps and we also discussed various types of spread strategies. Let us now move on.

Today, my agenda is to introduce the learners to the concept of option pricing. Now, why we need a special model or a special approach when we talk about the pricing or the valuation of options needs to be understood. What happens is you see in the case of options the payoff from the option is contingent upon the behaviour of the or the payoff of the underlying asset.

How the underlying asset behaves, how the price process of the underlying asset evolves, will finally determine how the payoff is going to be determined or ascertained in the context of the options. And therefore, there is a significant difference when we talk about the pricing of bonds or equity shares and we talk about the pricing of options.

We shall now, now I shall introduce you to the two fundamental approaches that go into option pricing, the first one is called the binomial approach which to some extent you are acquainted with in the context of pricing of bonds. And the second is the Black Scholes, approach the first one is a discrete time discrete variable approach, the second one is a continuous time continuous variable approach.

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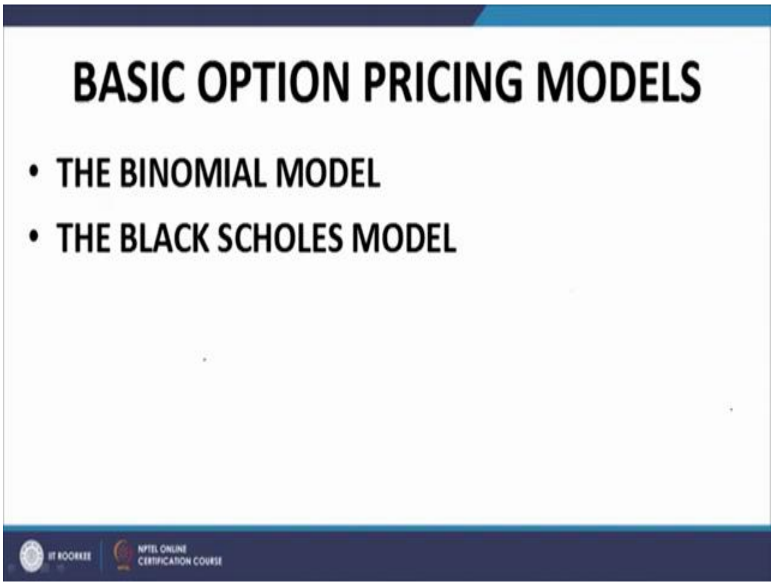

So, let without further I do, let us move into option pricing. These are the two models that I mentioned just now, the binomial model and the Black Scholes model.

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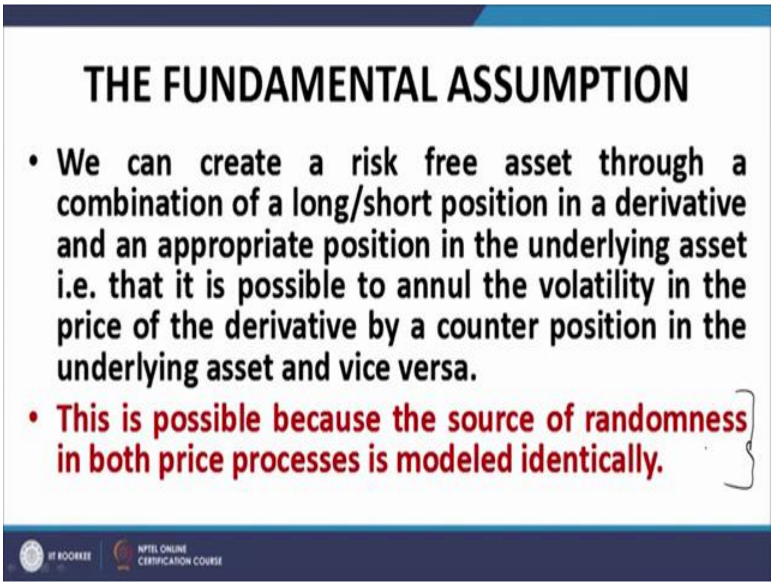

The fundamental assumptions when we talk about both of these models the assumption that goes into these models is that because the behaviour severe of the option or the derivative is a function of the behaviour of the underlying asset, we can model the behaviour of the derivative in terms of the same randomness, let me use this word for the moment, what exactly randomness manifests itself when we talk about the mathematics of it will be clear gradually as we move along this course.

So, let us use the word randomness for the moment, it may sound abstract but it is quite relevant. So, when we talk about the pricing or valuation of a derivative, the behaviour of the price process of a derivative it depends on the behaviour of the price process of the underlying asset. Therefore, we can use the same structure of randomness that we use for pricing or modeling of this underlying assets price for modeling or pricing of the derivative asset.

Let me repeat, this is very fundamental. You see the behaviour of the price process of the derivative by virtue of its very definition of the term derivative, is a function or it is a dependent on the behaviour of the price process of the underlying asset. Now, the underlying ass, the underlying assets behaviour contains a certain element of randomness which you model in a certain way.

Let us call that modeling as randomness. So, because the derivative also has a randomness element which emanates or which arises out of the randomness that is manifest in the behaviour of the underlying price, we can use the same randomness which we use to model the behaviour of the stock price for modeling of the derivative asset.

In other words, the randomness that is used for that manifests itself in the price process of the underlying can be used as a modeling technique or modeling strategy for the modeling of the price or behaviour of the derivative asset. That being the case, if we now work on the premise, if we now work on the premise that the randomness that is there in the derivative and the randomness that is there in the underlying asset is identical, then what should be the implication of this?

The implication of this should be that if I construct a portfolio that is long in one let us say that is long in the derivative and short in the underlying asset then we should be able to annul that randomness because the randomness is the same in both cases. So, it is something like the annulment of two waves which are traveling in opposite directions in the same phase.

So, if you use that analogy, the randomness that is manifest in the derivative can be annulled by the randomness that is there in the underlying asset because it is the same randomness and therefore if you take opposite positions, long in the direction derivative, short in the entire asset, we should be able to create a portfolio that is devoid of randomness.

And if a portfolio is devoid of randomness, it means it is a risk-free portfolio and it should give you that is free rate of return. That is the philosophy, that is the underlying principle and methodology that we use when we talk about pricing of derivatives or whether we talk about the binomial model which is in discrete time, discrete variable model, or we talk about the Black Scholes model which is the continuous time continuous variable model.

So, let me read it out for you. We can create a risk-free asset through a combination of a long oblique short position in a derivative and an appropriate position in the underlying asset, that is, it is possible to annul the volatility in the price of the derivative by a counter position in the underlying asset and vice versa. Why it is so? Because we are modeling both the price processes with this same type of randomness, say, same manifestation of randomness.

This is possible because the source of randomness in both the price processes that is the price process of the derivative and the branch process of the underlying is modeled identically. So, this is what is very important, this is the crux of our assumption. And as a result of this assumption, we are able to construct a risk-free portfolio comprising of a long position in a derivative and a short position in the underlying asset or vice versa.

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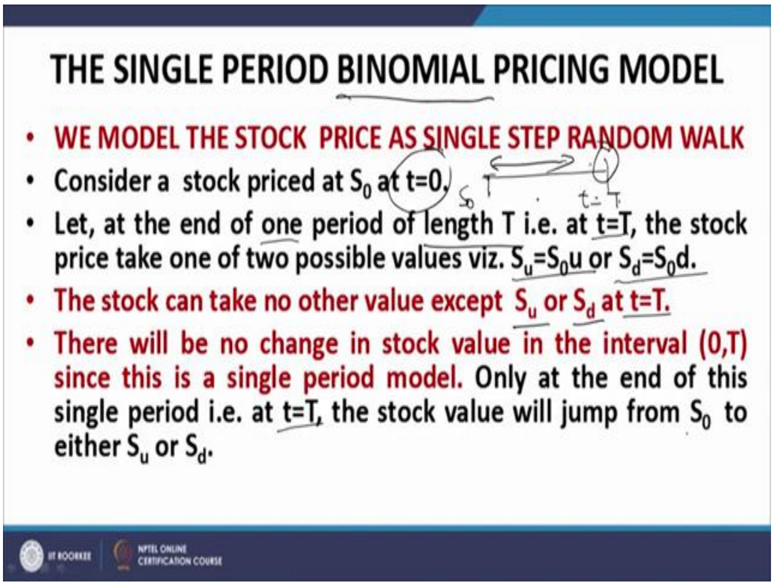

So, now we talk about, now we talk explicitly about the binomial model, that is, the single period binomial pricing model. We model the stock price as a single step random walk. This is the fundamental approach that goes into this particular model that we model the stock price process as a single step random walk. What does it mean is given in the following sentences.

Consider the stock priced at S 0 at t equal to 0, at t equal to 0 the price of the underlying asset is equal to S 0, in line with the usual terminology. Let at the end of one period of length T, now please note we are talking about a single period model, so we start at t equal to 0 and we end at t equal to 1 but this length between 0 and 1 may be any length any unit of time. So we call it at time T, capital T. Although it, there is, the reason why we call it a single period model is because the system will only undergo a transition at t equal to capital T that is, at in terms of the steps, it would be the first step, the only step, the one step.

And in terms of the length of time it would be equal to t equal to capital T. So, please know the difference between the timeline and the time step. The time step is the point at which the system will undergo transition and because it is a single step model, there will be only one point at which the system will undergo a transition. That one point may be at a certain time from the origin from t equal to 0, it may be 1 day, it may be 15 days, it may be 20 days, it may be 1 hour, it may be 1 minute, whatever the unit of time that we take and whatever be that particular time period in terms of which or at the end of which the transition is going to take place.

So, consider a stock priced at S 0 at t equal to 0. Let at the end of one period, this is important, one period, the length of that period may be anything but it has to be one period. This is more explicitly given as I move along the slide, of length T, that is, at t equal to capital T. We started the observation or the process started at t equal to 0, we have, this is, let us say, this is t equal to 0. Then this is t equal to capital T, and it is at this point that the system will undergo a transition.

There would be no movement of the system, no change in the system situation between these two. There will be no change in the system between t equal to 0 and t equal to capital T. Only at t equal to capital T, the system will undergo transition. And the transition may be that the stock price may go up to S u or the stock price may go down to S t.

So, let at the end of one period of length T, that is, at t equal to capital T, the stock price take one of two possible values, that is, S u equal to S 0 u where u is the percentage change in the price or S d where S 0 d where again d is the percentage change in price. If the stock price goes down and u is the percentage change in price if the stock price goes up.

And so that is what I said, t equal to 0, the stock price is S 0. There is no change between t equal to 0 and t equal to capital T. As soon as the clock clicks t equal to capital T whatever the capital T is, please note this point, whatever that capital T is, as soon as the clock click capital T, the system undergoes a transition, it either takes a jump upwards, that is, the stock price jumps upwards or the stock price jumps downwards.

If the stock price jumps upwards, it goes up to S u or, that is, equal to S 0 u or, and if the stock price goes down, it goes to S d, which is equal to S 0 d. So, the stock price can take no other value except S u or S d at t equal to capital T. The stock price can take no other value except S u and S d. This is the reason why we call it binomial.

There will be no change in the stock value in the interval 0 to T. That is why we call it a single period model. We got the transition is occurring at the end of one step, and that is the only transition that is relevant to us. So, there is, there will be no change in the stock value in the interval 0 to T, since this is a single period model, single step model. Only at the end of one thing single period, that is at t equal to capital T the stock value will jump from S 0 to S u or to S d.

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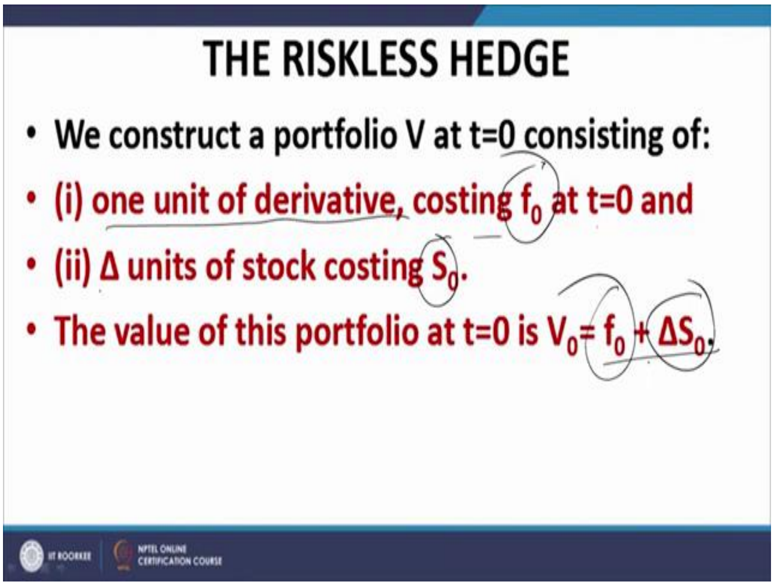

Now, we construct a riskless portfolio in line with the introduction that I gave a few minutes back. What do we do? We take one unit of the derivative, we take a long position in one unit of the derivative that causes f 0 and f 0 is precisely the quantity that we have to calculate. So, the derivative is costing f 0 at t equal to 0. And delta units, what is delta, we will come back to it, we will try to find out by virtue of certain equations.

But delta units of the stock costing S 0 per unit. Please note this is S 0 per unit of the stock so the cost of taking this position, of taking delta units of the stock will be equal to delta into S 0. So, the total cost of setting up the strategy is equal to f 0 plus delta S 0. Please note, I have not used the point that one should be opposite of the other. Let it emanate out of the mathematics.

So, for the moment I construct a portfolio with one unit of the derivative, whatever position it is and delta units of the stock whatever position it is. The derivative cost f 0 per unit and the stock cost S 0 per unit. So the stock component of this portfolio will cost me delta S 0 and the derivative component will cost me f 0. So, the total cost of this portfolio, the total value of the portfolio at t equal to 0 is equal to f 0 plus delta S 0.

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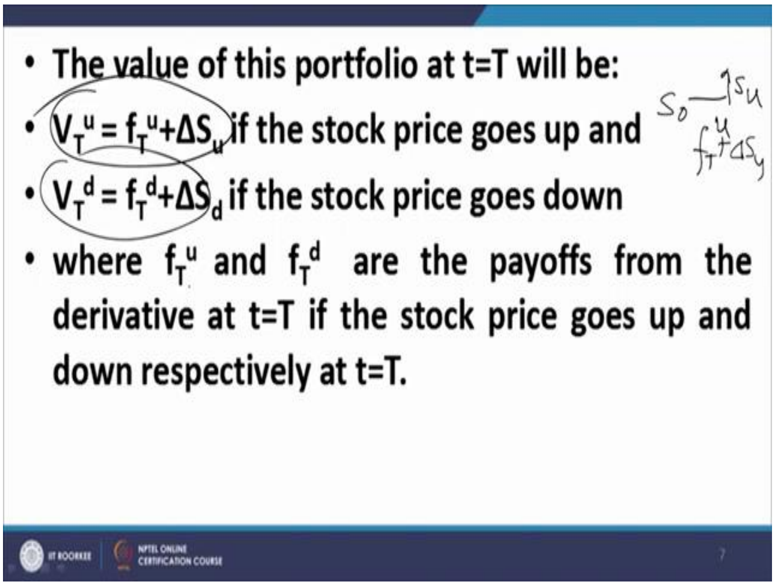

Now, the value of this portfolio at t equal to capital T will obviously depend on whether the stock jumps up or whether the stock jumps down. And not only because of the component of stock that is embedded in this portfolio but also because the derivatives value also depends upon the value of the stock at t equal to capital T.

If the stock price goes up, the derivative will give you a certain payoff, if the stock price goes down the derivative will give you a different payoff. So, depending on whether the stock price goes up, the stock component will obviously change in value but in addition to that the derivative component will also change in value.

Let us assume that the stock price goes up from S 0 to, it goes up to S 0 or S u. And let us assume that the derivatives payoff, if the stock price goes to S u is given as fu T. Then, what is the total value of the portfolio if the stock price goes up? That is equal to fu T plus delta into S u. That is what is given here. fu T is what? It is the value of the derivative if the underlying assets price goes up.

And S u is the value of the stock if the stock price goes up. We have got delta units of the stock so the stock component of our portfolio V will be equal to delta into S u. And the derivative component because, we have only one derivative, it will be worth fu T. Similarly, if the stock price goes down, the value of our portfolio will be equal to this expression Vd T is equal to fd T plus delta into S d.

So, these, now where fu T and fd T, I have already explained, they are the payoffs from the derivative of t equal to capital T if the stock price goes up and if the stock price goes down t equal to capital T.

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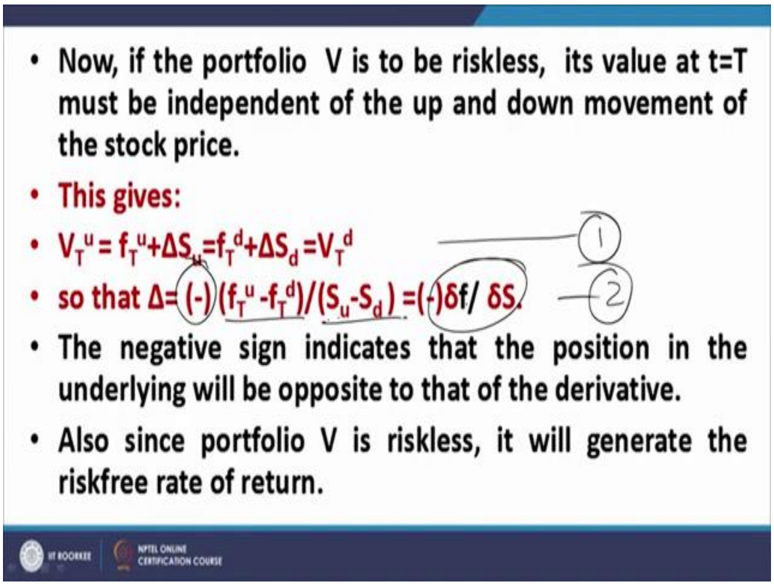

Now, if the portfolio V is to be riskless, what does it mean? It means that irrespective of whether the stock price goes up or the stock price goes down which is a random event, please note this, we are talking about the stock following a random work. So, obviously whether it goes up or down is a random event and it can be modeled as a random variable but the important thing is that it is random.

Now, and this is, the source of risk in the stock price, the, whether the stock price can go up or down, we do not know. Sitting here at t equal to 0, we cannot precisely predict whether the stock price can go up or down, and therefore this is a random event and therefore it can be modeled as a random variable, but the important thing is that this is the source of risk as far as we are concerned.

Now, if we want that this portfolio that we have constructed, using the derivative and the stock to be riskless what does it mean? It means that the value of the portfolio at t equal to capital T, that is the majority date of the portfolio should be independent of whether the stock price goes up or stock price goes down.

So, if the asset is, if the portfolio is to be riskless, it means that the portfolio's value should be independent of whether the asset price or the stock price goes up or the stock price goes down. The portfolio value should not change, if the portfolio value does not change what does it mean?

It means that sitting here at t equal to 0, we are able to precisely predict the value of the portfolio t equal to capital T. And that means that our process is devoid of any randomness and that means that our process is totally risk free. So, if the portfolio V is to be riskless, the value at t equal to capital T must be independent of the up and down movement of the stock price. This gives us this equation. Let us call it equation number 1.

We are simply equating the value of Vu T and Vd T. And by solving this we are able to get the value of delta. I mentioned just a few minutes back that what is delta, we do not know, but we will get it out of the mathematics of the problem which arises or which emanates or which follows from the philosophy of the model.

So, delta is equal to minus fu T minus fd T divided by S u minus S d. This minus sign. Now, this minus sign is coming out of the mathematics. What does it say, what does this minus sign connotes? It connotes that the position in the underlying asset should be opposite to that of the derivatives. If you are long in the derivative, you should be short in the underlying asset and vice versa. So, you can write this as minus of delta f upon delta S. This will be of much use later on. Let us call it equation number 2.

This equation, equation number 2 gives us the value of that delta such that if we hold delta units of the stock and one unit of the derivative in opposite positions one long, the other short, then we have or we have constructed a portfolio that is independent of the movement of the stock price at maturity or at t equal to capital T when the stock price is going to make a random jump upwards or downwards. The stock price is going to change, the derivative value is going to change, but the combination of the stock price and the derivative is not going to change.

The negative sign indicates that the position in the underlying will be opposite to that of the derivative. Also since portfolio V is riskless, it will generate the risk-free rate of return. This is mandated by what? It is mandated by the consideration of arbitrage free pricing. If there are two assets which have identical risk profiles, they must give you identical expected returns. And because this portfolio V that we have constructed is a risk-free portfolio, it is in, it has no risk embedded in it therefore it must also give you the risk-free rate of return.

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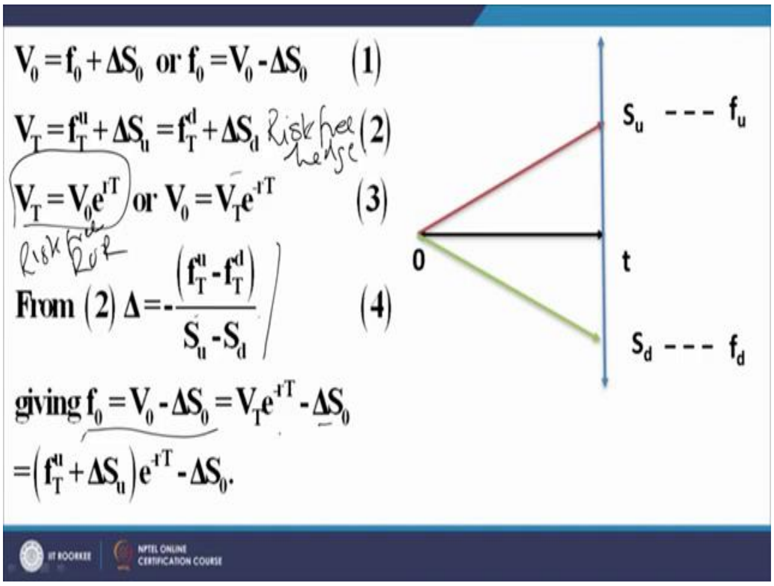

That means what? That means we, from this, we, what we get? We get equation, this equation, this part, V T is equal to V 0 e to the power r T. So, we, I have now, in this particular slide I have set forth the mathematics that is, that we have obtained in the previous few slides. This is, equation number 1 is for the construction of the portfolio and from this, we get f 0 is equal to V 0 minus delta S 0 because V 0 is the total value of the portfolio t equal to 0 which consists of one unit of the derivative worth f 0 and delta units of the underlying asset worth delta S 0.

Then by, this is from the condition of risk-free hedge, it is risk-free hedge and this is the risk-free rate of return as I mentioned just now, risk-free rate of return. So, using equations 1, 2 and 3, in fact from equation number 2, we get this value of delta which I have already explained in the previous slide. And using equation 1 we can write this if this expression and in this expression we substitute V 0 equal to V T e to the power minus r T, this follows from equation number 3. And delta S 0, we keep it as it is and we move to the next slide where we will substitute the value of delta given by equation number 4.

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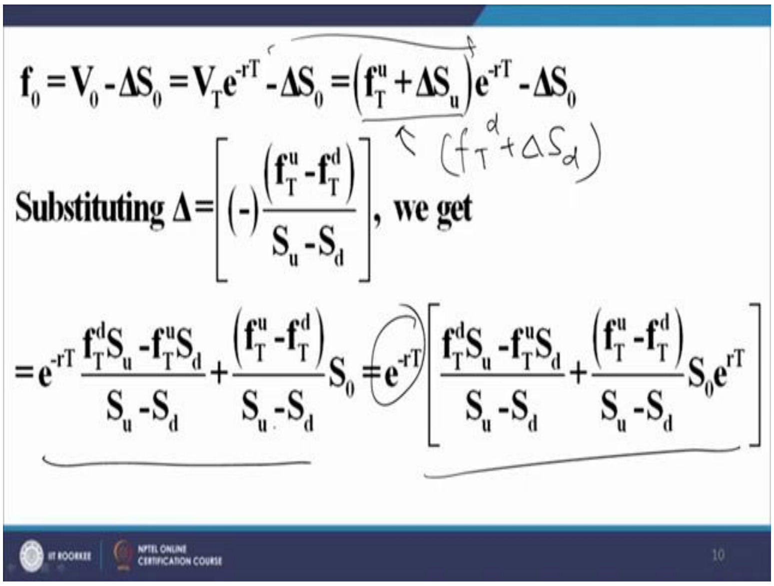

So, f 0 is equal to V 0 minus delta S 0, we have it from the previous slide, is equal to V T e to the power minus r T. We are substituting V 0 in terms of V T. And then we are substituting V T equal to this. We could as well have used fd T plus delta S d instead of this. It would not have made any difference because they are equal. Why they are equal? Because our portfolio is riskless.

So e to the power minus r T is as it is and delta S 0 is as it is. And now we substitute delta S 0 from equation number 4, and we get this rather involved expression which we have here which can be simplified and written in this form, e to the power minus r T is taken common outside the square brackets and the rest of it is rearranged to get this expression.

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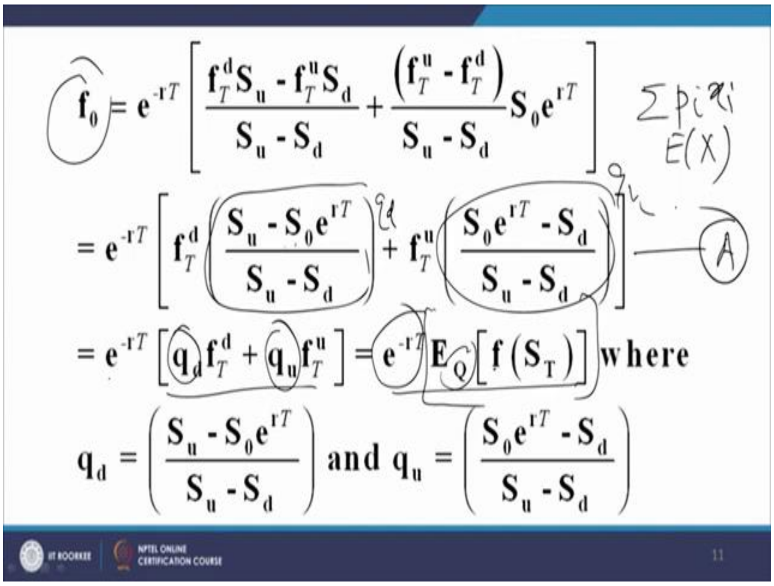

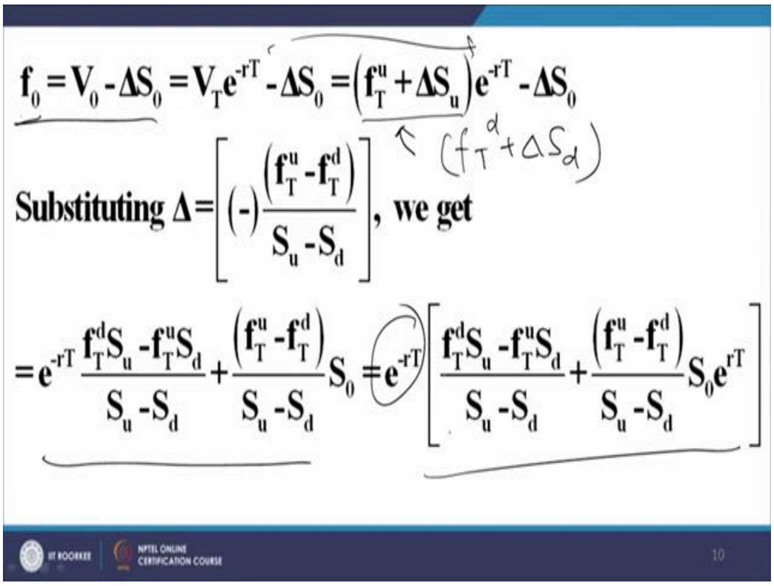

Further rearrangement, you see now what we are doing is simple algebra, we are only rearranging the terms. We, when we rearrange the terms we can write it in this form let me call it equation number say A, this equation number A is nothing but a rearrangement of which equation? The rearrangement of this equation. This, this has been extended, this has been, as we have substituted the value of delta here, after substituting the value of delta here we have taken e to the power minus r T common and then we have rearranged the terms within the square brackets.

And we get this equation number A. Now, we make a substitution. What substitution do we make? We write q d, q d as this expression and we write q u as this expression. That means we get this e to the power minus r T outside and we have, within the square brackets, we have q d fd T plus q u fu T, and which we can write as, it is, you will recall, that is something similar to sigma of p i x i.

And what is sigma p i x i? It is equal to E of X. So, that is precisely what we have done here. We have used the analogy with the expectation value of a statistical or a random variable to write this expression as this expression, the expectation of f of S T. What is f of S T, f of S T is the payoff from the derivative at maturity, and E is the expectation or the expected value of the payoff of the derivative at maturity corresponding to different states that S T could take.

You see S T is a random variable it could take a number of values f S T would be the corresponding value of the derivative in relation to each possible values that S T could take in the spectrum of possible values, so f S T would be different values that the derivative payoff could take E of f S T is what? It is the expected value, it is the mean value of all these possible payoffs that the derivative could give you corresponding to each value that S T could take.

But what does this q represent? q is a measure of probability, q is a special type of probability. Its significance will be clear to you very soon but for the moment, it is a type of probability, we will come back to it in a minute. So, it is nothing, this expression is nothing but the expected value using a certain type of probability of measure. And by using this expression e to the power minus r T out of the bracket, we are simply discounting this expected value.

So, if I put it in words, the value of the derivative f 0, that is here, is equal to the discounted or the present value of the expected payoff from the derivative where the expectation is calculated with respect to a special set of probabilities called risk neutral probabilities. But why it is called risk neutral probabilities, what is the rationale behind using or writing this as q d and writing this as q u, why we can interpret them as probabilities we will come back to it in the following few slides.

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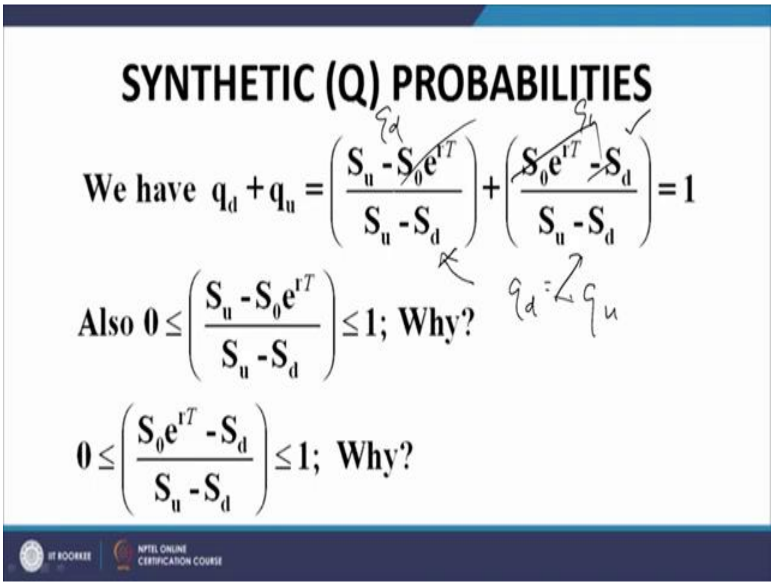

So, this is what the story is so far. Now let us look at, let us investigate the values of q u and q d. What was q d? q d was equal to this expression and q u was equal to this expression. So, the first thing that we observe here is that q d plus q u is equal to 1. It is quite obvious, it is elementary and you can make it, you can see here this term and this term cancel out and we are left with S u minus S d divided by S u minus S d which is equal to 1. So, that is one thing. The second thing which is more subtle, which is more involved is that each of these q u and q d which we define by this term this is q d, this is q u each of these terms q d and q u must lie between 0 and 1. That is our next agenda.

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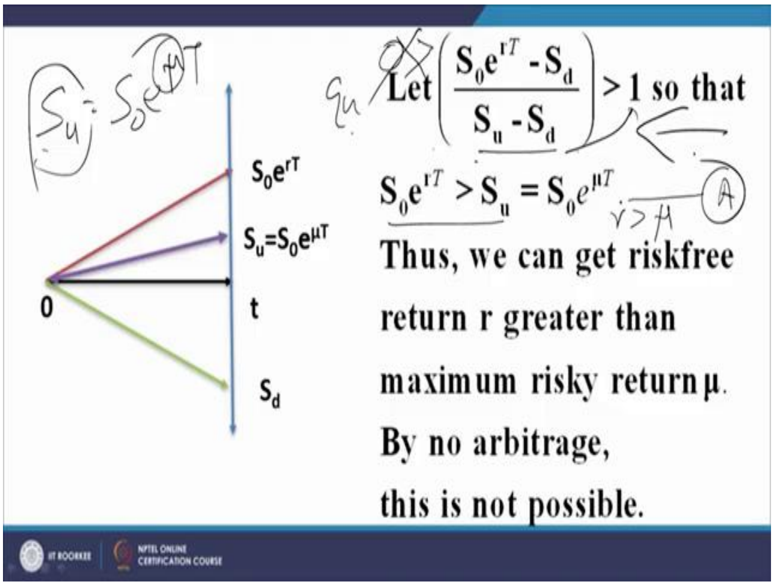

Let us look at this term. This is what? This is equal to q u S 0 e to the power r T minus S d upon S u minus S d, this is equal to q of u. Now, let us assume that this expression is greater than 0. Now, if this expression is greater than, greater than 1, I am sorry, let us assume that this expression is greater than 1.

If this expression is greater than 1, then by cross multiplying what we have is that S 0 e to the power r T is greater than S u because if you cross multiply this to the right hand side, the S d and S d will cancel out and you will get S 0 e to the power r T is greater than S u. What is S u, please note? S u is the upper value or the upside value of the stock price. You see the stock price can take only two values, S u if it goes up as S d, if it goes down. It cannot go more than S u, it cannot go below S d.

It can only take two values. We are talking about a binomial model, and in this binomial model, the stock price can take only two values S u or S d at t equal to capital T when it makes a random jump. Now, what does this give us? Let us assume that if the stock price takes the value S u, then we can write it as S u is equal to S 0 e to the power mu T.

Then obviously this mu, what does mu represents? Mu represents the continuous compounded upward limit of the return on the stock price. I repeat, mu represents what, mu represents a upward limit of the continuously compounded return on the stock price because the stock price cannot go beyond S u. The maximum value that the stock price can take as at t equal to capital T is equal to S u, it cannot take any value more than that.

Therefore, the maximum return that you can get on the stock price is S is mu. Now, by this expression, if you look at this expression, let me call it equation number A again. By this expression, what do we have, we have r is greater than mu. What does it mean? It means that the risk-free return, the certain return, the absolutely certain return that you are getting in this model is higher than the maximum return that would be possible on investment in this stock, if you invest in this stock, the maximum return that you can get is mu.

But if you invest in the risk-free asset, then you could get a return which is higher than mu with certainty. And in the case of the stock you can get a maximum return of u, you may as well end up with a negative return, with a lower return. So obviously, any rational investor would not contemplate investing in this stock. All investors will flock and invest in the risk-free asset because you are getting a certain return which is not only, not only certain but which is also higher.

So, higher and certainty both are incorporated in your investment in the risk-free asset. In the investment in this stock, what happens? Your investment will give you a lower return or a return equal to mu, that is, that, that is, lower than r. And you can even fail to get that lower return as well, you may end up with a negative return as well. On both counts, on the count of return and on the count of risk, your investment in this stock is inferior to your investment in the risk-free asset.

This is not allowed by arbitrage considerations. And as a result of which this condition that we have initiated this discussion on, this particular condition will not hold, will not hold. Similarly, we can prove by introducing arbitrage considerations and so on, that this will be, this cannot be, this cannot be, 0 cannot be greater than this. It cannot be this, this situation cannot hold as well.

In other words, this cannot be negative as well. So, it must necessarily be that this quantity must lie between 0 and positive. This will not hold. 0 is greater than this will not hold. So, similarly, we can prove for q d as well. This is for q u. We can prove the same thing for q d as well. I will continue in the next lecturer. Thank you.