Quantitative Investment Management
Professor J. P. Singh
Department of Management Studies
Indian Institute of Technology, Roorkee
Lecture: 04
Arbitrage

So, in before lecture we were talking about the relationship between riskiness and return. And what we found was that if there are two assets, A and B and both have said same future value, then and assuming that A is less risky and B is more risky, the present value, which in a sense represent the prices of the bonds or the intrinsic value of these instruments are of A being less risky would be higher compared to the price of B, which has a higher risk and therefore a lower price.

And this implies that the discount rate used for discounting A which has a lower risk would be lower and the discount rate that is used for discounting the payoff of B which has a higher risk has a higher value. In other words, the discounters carry a component of risk adjusted with respect to risk and higher the risk higher is the discount rate. As we shall see later, these discount rates actually represent the expected returns on the underlying securities in an equilibrium market.

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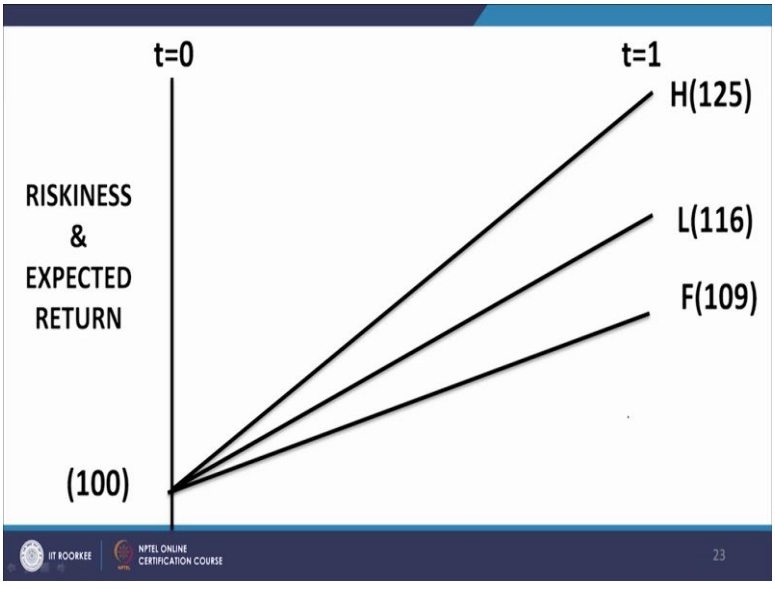

And this is illustrated here in this particular diagram. Again, we have three assets, we have the asset H, which is the highest risky asset, the asset L which is the which is the intermediate risk, and we have the asset F which has the lower risk, all of these three assets that have been created as of today at 100.

Now, what happens to the future values or the value that y equal to 1. We can obviously, because H is the highest risky asset, the investor who invests in H a t equal to 0 would demand a higher rate of return and in order that he gets a higher rate of return, the expected price or the expected value of the asset, at t equal to 1 for asset H must be the highest. This translates to the statement that higher the risk higher should be the expected return on the asset in order that the asset is able to sustain itself in an equilibrium market.

Otherwise, what will happen is if, asset has higher risk and lower expected return compared to any other asset, arbitrage will start, we will talk about arbitrages very soon.

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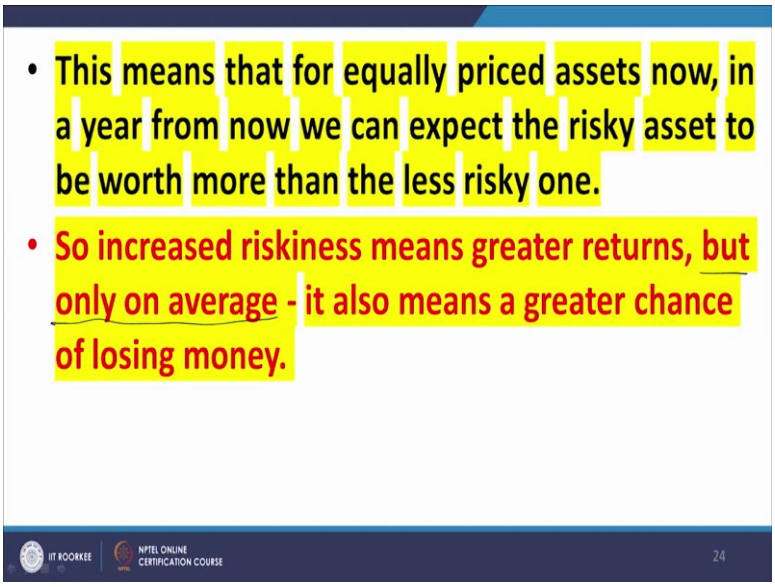

So, briefly summarizing that for equally priced assets now, that if there are two assets A and B, which are equally priced, in a year from now, we can expect a risky asset to be worth more than the risky, riskless asset or the lower risk asset. Again, I repeat, this implies that the expected return on the risky asset needs to be higher than the lower risk asset in equilibrium market.

In other words, increased riskiness means greater returns, but only on the average this is fundamental. So, increased riskiness means greater returns higher the risk higher the returns but which return, expected returns, it is not the actual return. If you are talking about actual returns, there is no question of risk really, because once the actual return is realized, and that is a post factor event so that there is no riskiness attached to that, there is no uncertainty attached to that, and there is no riskiness and that value that we have already obtained is a certain value.

So, there is no question of riskiness there, however, we are talking about future values here, the today's price versus the future values. Now in the case of future values, what we say is that higher the riskiness given a given enrichment value higher has to be the future expected value, but that the important point, the cardinal point is the word expected value, expected return, expected price, expected value is not actual price, is not actual revenue.

What that means is that while the expected price or the expected return has to be higher, in order that the asset sustains itself or even in an equilibrium market, it is also necessary that or it is also followed that the because it is highly risky that asset is higher in terms of risky, it follows that the expected return that we are going to get could be defaulted upon in a greater number of cases or a greater profit with a greater probability compared to an asset which has a low risk.

For example, if an asset is riskless, there is no question of default. And even though if it has a lower value than then that value is absolutely certain and there is no risk. So, that return is certain, but if there is an asset with a high risk, it may give you a higher expected return, but there would also be a higher probability of that asset defaulting in providing you that higher expected return.

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Now, we see about the arbitrage, By making a very brief passing mention about arbitrage. Let us see what arbitrage actually means.

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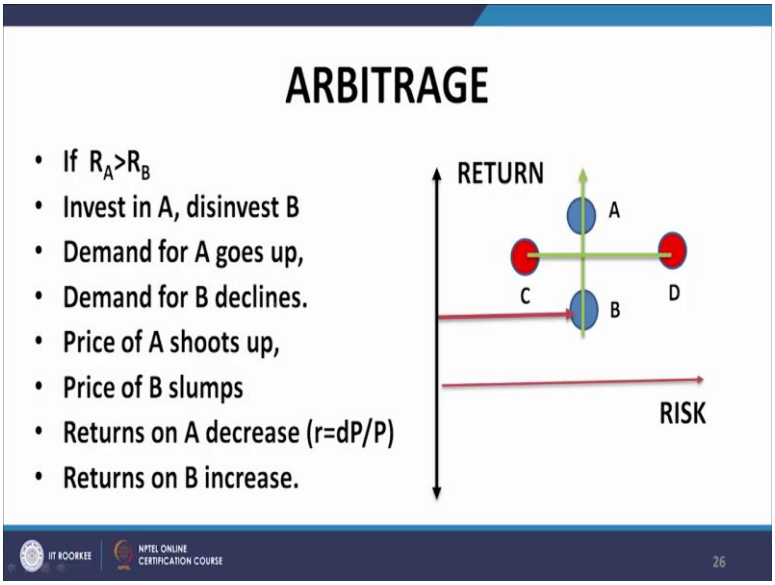

Look at this diagram. As mentioned, whenever we talk about investment, appraisal, investment evaluation, we talk about a two-dimensional framework. It is conventional that the risk is taken along the x axis and the expected return is taking along the y axis. For the moment, we need not delve into what measure of risk we are adopting any measure of risk would be adequate. This the exposition that I am not talking about is pretty much applicable across all measures of risk, So, long as they identify a higher risk asset from a lower risky asset.

Now, let us consider the assets A and B, let us consider the assets A and B in this diagram. Well, as you can see, A has a higher expected return than B for the same level of risk. Now, if the market is to be in equilibrium, these two returns what will happen, because A is giving you a high return for the same level of risk as B the demand for A will increase and the demand for B will decline people who try to get into investment A and people who try to get out of investment B because B is providing you a lower level of return corresponding to the same level of risk as a security A.

There is a security in the market which is security A, which is providing you higher expected returns compared to security B which is providing you lower expected returns but with the same level of risk. So, obviously any rational investor or enter investment A and exit investment B. Now, what would happen as of consequences that the demand for the asset A would increase would escalate whereas the demand for the asset B would decline would fall and with the fall in the demand for B and the increase in the demand for A what happens to the prices, the current prices of A will go up will shoot up and the current price of B will go down will decline will be suppressed.

As a consequence, the returns on security would fall, the expected returns on security would fall, the expected returns on security B would increase then they would tend to converge. In other words, the points A and B as in an equilibrium market would tend to converge. Similarly, if you consider security C and D, you arrive at a similar conclusion C and D both provide you the same level of expected returns, but see has a significantly lower risk compared to the and again because risk is viewed as an unfavorable attribute by investing community it follows that people would exit investment D, people would enter investment C, prices of C would increase, prices of D would decline.

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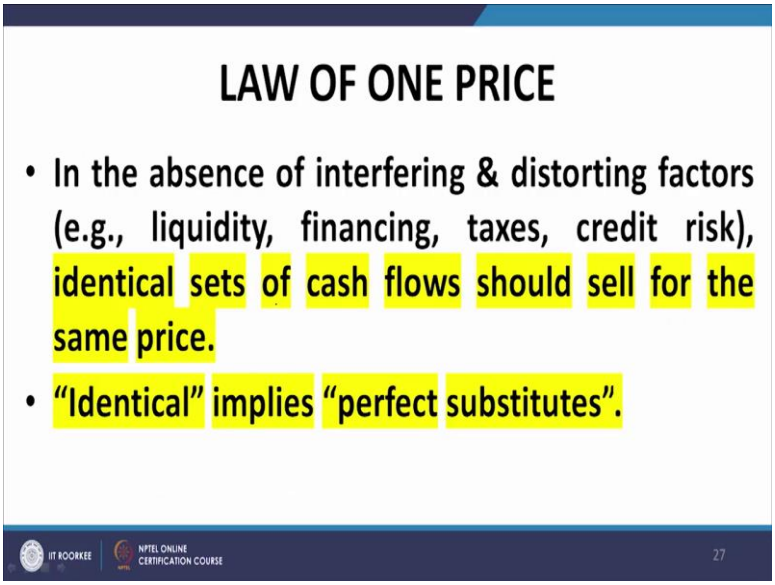

Now, whatever that explained by this diagram can be summarized into a very important theorem which is converted as a law of one price it is so important that is it is termed as the law and it is called the law of one price. In the absence of interfering and distorting factors like liquidity, financing, taxes, credit risk, etc. Identical sets of cash flows sure sell for the same price. So, if you have two assets whose cash flows are identical at any future date, then they must be selling or they must sell at the same price.

If they do not, the phenomenon of arbitrage arbitrage will come into play and gradually the arbitrage profits will be extracted out of the system and consequently, what would happen is that two assets will converge in terms of their returns. What is identical is a very fundamental question, it is not trivial it is by no means trivial, identical implies perfect substitutes. So, when we are talking about identical cash flows, the cash flows should be such that one stream of cash flows can be substituted by the other stream with perfection, with no distortion in the system whatsoever.

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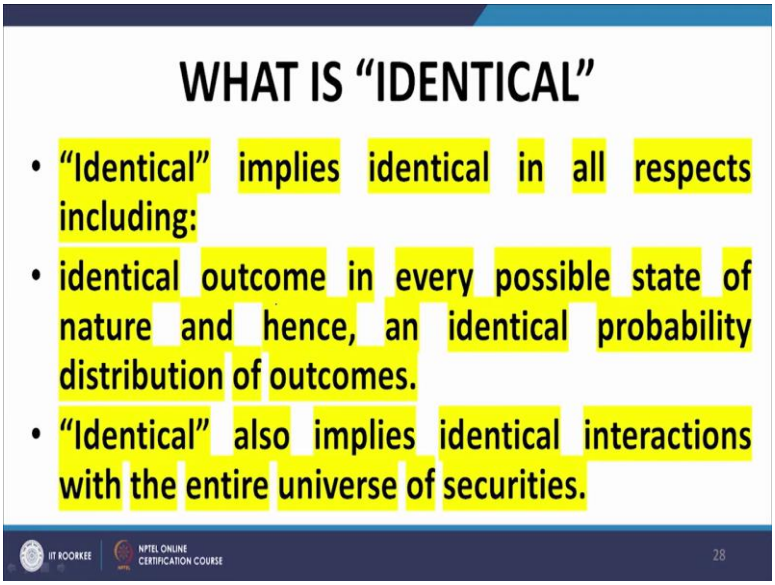

So, identical implies identical in all aspects, this is to reiterate, Its just highlighted, identical implies identical in all respects including identical outcome in every possible state of nature, and hence, an identical probability distribution of outcomes. So, if two cash flows are to be identical, if they are random variables in if there are certain it is a different matter, if they are random variables, that means, the cash flows corresponding to different states are different then the cash flows that arise from asset A and the cash flows that arrives as asset B, in each state of nature, whatever the state of nature may arise, must be identical and it implies that they must have an identical probability distribution.

Secondly, identical also implies identical also necessitates. This is so far, I have not mentioned anything about interaction between securities, but it so happens that there hardly any pair of securities, which in our universe of securities, which have absolutely no interaction with each other to securities, particularly when we talk about equity shares, definitely or usually do have interactions. So, interactions between these two securities that are involved in the arbitrage process must be identical depend the level quantum and quality of interactions between the two securities or between the securities A and the universe and security B and the universe must be absolutely identical.

So, identical also implies that identical interactions with the entire universe of securities, that is, both this parts that, now are captured by saying that the two current flows must necessarily be perfect substitutes of each other. And that means what, that means in an uncertain future value situation, the probability distribution of outcomes must be absolutely identical. And the second important point is that which we have ignored so far is that of interactions, the

interactions of the two securities with our entire universe of securities must be identical with respect to each pair of securities that we have that we consider.

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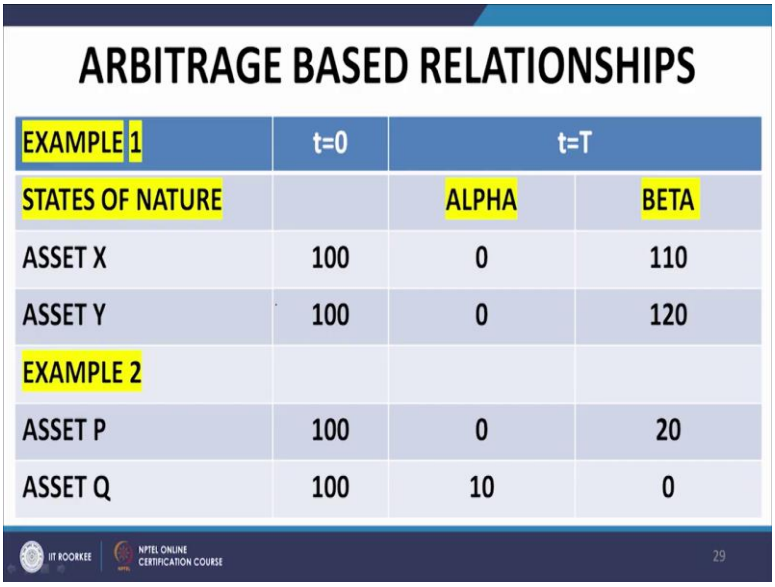

Now let us look at some arbitrage examples. Let us look at this example. We have two assets X and Y, X and Y are two assets. And both are being priced at 100 in the current market, and in the current market price of X is also 100. And the price of y is also 100. The future values of these investments, let us call them investment X and investment Y.

The future value of investment X if the state alpha with future is characterized by only two possible states to keep the exposition simple, the state alpha and the state beta. If the state alpha materializes, the investment takes has a Payoff of zero. And if the state beta materializes on the date of exit of the investment, the outcome is a Payoff of 110.

And the investment Y Payoff of 0, if the state alpha materializes, and if the state alpha becomes materializes, it has a Payoff 20. As you can see it because there are only two states alpha and beta and refined that in the state of alpha, in the state alpha both the payoffs are identical investment X and investment Y. However, in the state beta, the investment X is giving you a payoff of 110, the investment Y it dominates and is giving you a payoff of 120.

And because both these assets are being priced at 100 this surely indicates the possibility of arbitrages because, two identically prized assets are not identical in respect of their future payoffs or the probability distribution of the future payoffs, one is having a higher payoff in one state and both of them are having identical payoffs in the other state in a two state system, and as a result of it, there is definitely a possibility of arbitrage in this example.

Let us look at example two, example two has two assets P and Q both of them again are priced at 100. Again, we are talking about a two-state system, a two-state future where as on the date of exiting of the investment, there are only two possible states alpha and beta. If state alpha materializes, investment P gives you a payoff of 0 and if state beta materializes gives you a payoff of 20. And as far as investment Q is concerned, it gives you a payoff of 10. If state alpha materializes, and if state beta materializes, it gives you a payoff of 0.

Now, what can we say about arbiters, prime of asset, it would indicate to the labor that one would immediately react rather that asset P has a higher payoff in state beta and of 20 whereas asset Q has a payoff of only 10 instead alpha therefore, asset P is definitely superior to asset Q. And as a result of it, because they are trading at the same price and should be arbitrage opportunities, but on a deeper level on a deeper understanding of the problem, we find that it is not so trivial. The relative probabilities of state alpha and beta is not given in the problem.

And because this is not given in the problem, we cannot say that the payoff of asset P is superior to the payoff of asset Q, what happens for example, if the state alpha has a probability of literacy 75 percent and the state beta has a probability of only 25 percent, the absence of a probability distributions of probabilities of the occurrence or the possible occurrence of state alpha and beta not being given to you the information is incomplete, the information is inadequate and we cannot infer that the state Q is superior to state P or vice versa for that matter as well.

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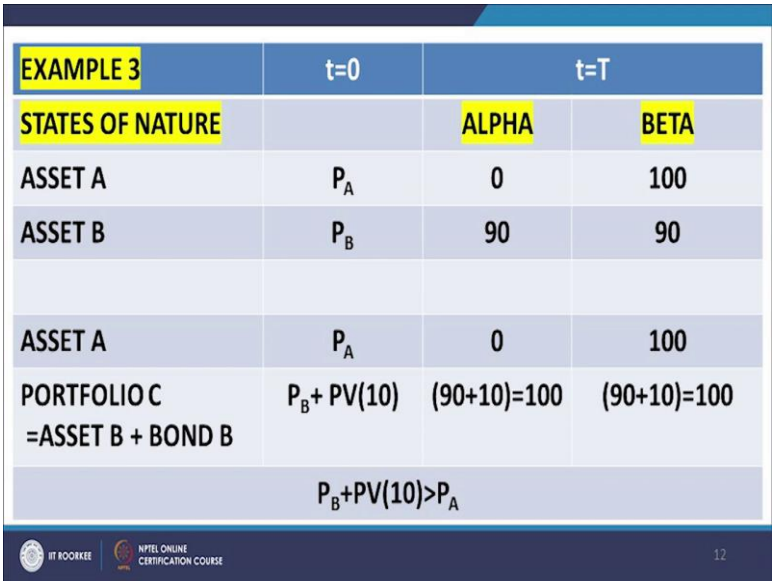

Let us explore issue of arbitrage a little bit further. Now, let us talk about asset A and asset B, let us say the price of asset the price at which at asset A is trading at t equal to zero that is the part of entry of the investment is PA and let us assume that the price of entry into investment B is PB and let us assume again of two state future value and state alpha asset A gives you zero in state beta asset A gives you 100, asset B gives you 19 state alpha and asset B gives you 90 instead beta.

We have neither given the values of PA and PB nor are we given the relative probabilities of state alpha and state beta. So, we cannot work out the expected values of the payoffs of investment A and investment B, but we can arrive at certain bounds here.

Let us look at the situation more carefully examine it more carefully. Suppose, we construct a portfolio. This methodology is very important when we talk about setting a bounce on options and other similar contingent contracts. So, let me explain this in a very simple manner. Let us (())(19:47) form a portfolio C, we live asset A as it is we do not disturb asset A. We form a portfolio C which consists of asset B and we add to it a bond, a bond which has a face value of 10 and which is trading at present value of 10 at the appropriate risk. Let me repeat, I form a portfolio C consisting of asset B as it is and I add to asset B a bond, which pays of 10 in state alpha as well as in state beta so it is a default free bond, it is a certain payment irrespective of what is the evolution of the state of nature.

So, bond B pays of 10 at if state alpha occurs and it pays of 10 if state beta occurs. So, and we because we are adding this bond to a portfolio, we have to pay a price for that and the

present price of the bond is the present value of 10 at the appropriate discount rate, now let us look at this situation. In state beta asset A is giving me 100, asset C or portfolio C is giving me 100.

So, let me repeat in state beta both the portfolio assets, asset A and asset C or portfolio C for that matter. Asset A and Portfolio C, the payoff of both these portfolios is absolutely identical if state beta arises. If state alpha arises, what we find is that as portfolio C is definitely superior to asset A, asset A is having a payoff of 0 and asset C or portfolio C is having a payoff of 100 this total is this 90 plus 10 is 100 and here 90 plus 10 is 100. So, I can write this total here is 100 and I can write this total here is 100.

So, what does it imply, because as portfolios C is definitely superior in either state of nature in state alpha, it is obviously superior instead beta it is identical. So, overall whatever be the relative probabilities of alpha and beta, whatever be the relative probabilities of alpha and beta, except of course alpha having 0 probability which we ignore, we find that the portfolio C is superior to Portfolio A and therefore, the price of portfolio C must be greater than the price of Portfolio A and therefore, we arrive at this relationship, price of portfolio C that is PB plus present value of 10 is greater than PA.

So, by use of arbitrage considerations, we have been able to set upper bound for PA in terms of PB at a lower bound in terms of bounds for PB in terms of PA.

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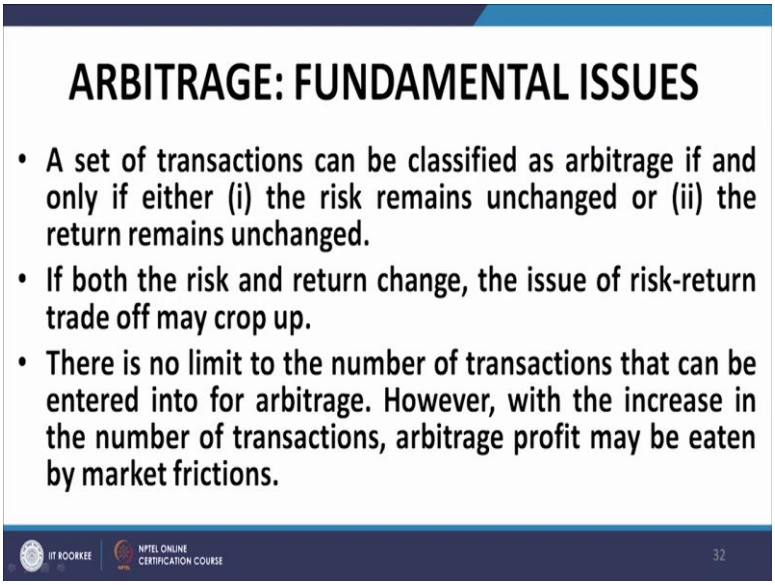

Now, arbitrage some fundamental issues, a set of transactions can be classified as arbitrage if and only if either the risk remains unchanged or the return remains unchanged. In both the risk and return chain the issue of risk return trade off crops up and finally, there is no limit to the number of transactions that can be entered into for arbitrage.

For example, you can do arbitrage by converting rupees to dollars, dollars to Great Britain Pounds, Great Britain Pounds to Swiss francs and from Swiss francs back to Indian rupees, you can indulge in any number of steps appropriate is your own money and you can indulge in any number of steps that you like, if you feel that you are likely to realize an arbitrage profit.

The caveat here is that with the increase in the number of transactions, arbitrage profit may be eaten away by market frictions like the bid ask spread and the lending boring spread and so on. So, the basic thing is arbitrage is generally achieved when there is a two-step approach arbitrage opportunity or at best a three step arbitrage opportunities. If you have a larger number of steps involved in arbitrage, the frictions that are involved in the market in terms of buying and selling spread, bid ask spread that is may eat into the arbitrage possible arbitrage profits.

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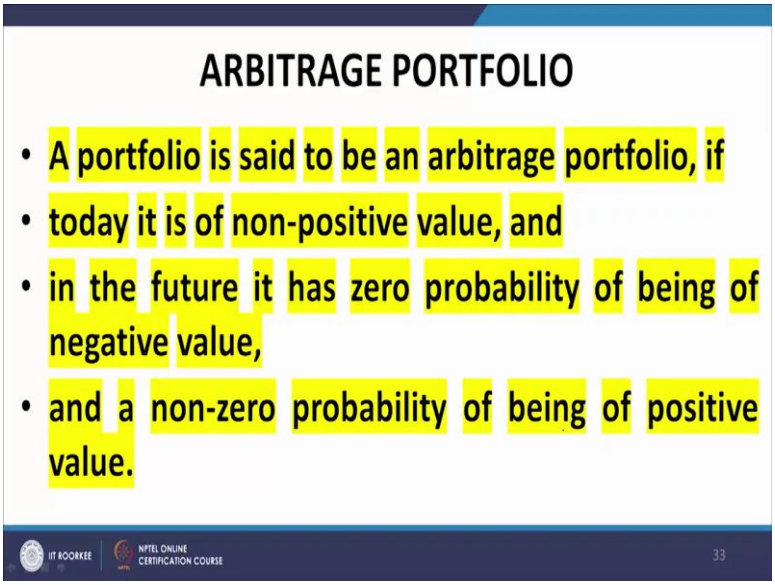

Now we come to a formal discussion or arbitrage. We have talked about arbitrage, the definition of arbitrage, we are discussed arbitrage at the conceptual level, let us not talk about the formal definition of arbitrage what is an arbitrage portfolio. A portfolio is said to be an arbitrage portfolio, if today it is of non-positive value that is, it is of either 0 value or negative value. In the future it has 0 probability of being a negative value and a nonzero probability of being a positive value. That means, it is either the probability of other portfolios certain to return 0 or a positive value. In other words, in the future, the portfolio is certain to return a zero or a positive value and today it has a non-positive value, then it is called an arbitrage portfolio.

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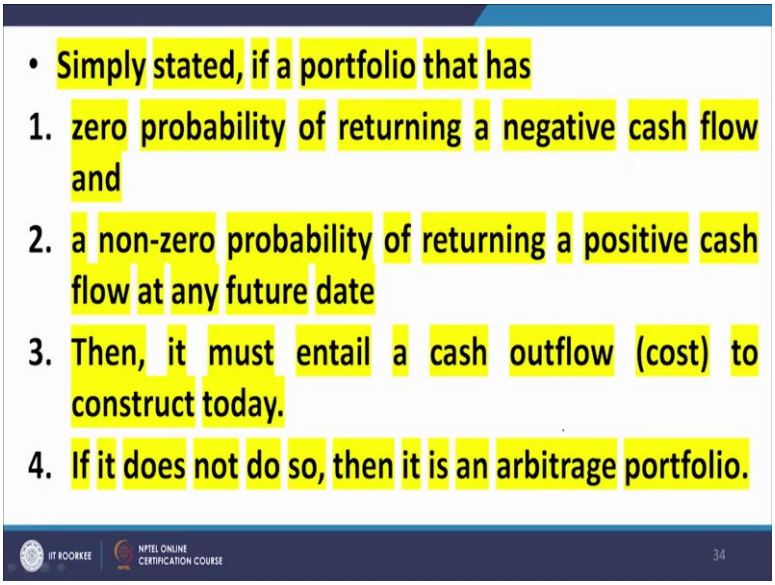

Putting it in other words, putting it more simply in more conversant language, if a portfolio has 0 probability of returning a negative cash flow, that is it is certain to return either a 0 value at maturity or a positive value of maturity or at any future date, then it must entail a cost a cash outflow please remember cost of the cash outflow.

So, it must entail a cash outflow to construct a portfolio t equal to 0. So, let me repeat, if you are constructing a portfolio, if you are able to construct a portfolio that has either the possibility of returning a 0 value or a positive value, but no probability of returning a negative value, then it must entail cost or a cash outflow to construct at t equal to 0.

If it does not do so, if it does not entail a cost to construct this kind of portfolio, which has some probability of returning a 0 value and some probability of returning a positive value, a nonzero probability of returning a positive value please notice that probability of returning a 0 value may be 0, the probability of returning a nonzero positive value must be nonzero.

Then if you are going to construct that portfolio t equal to 0, you must incur a cost, you must incur a cash outflow and if you do not do so, if it so happens that a portfolio is such that it does not entail cash outflow to construct a portfolio, then it is an arbitrage portfolio.

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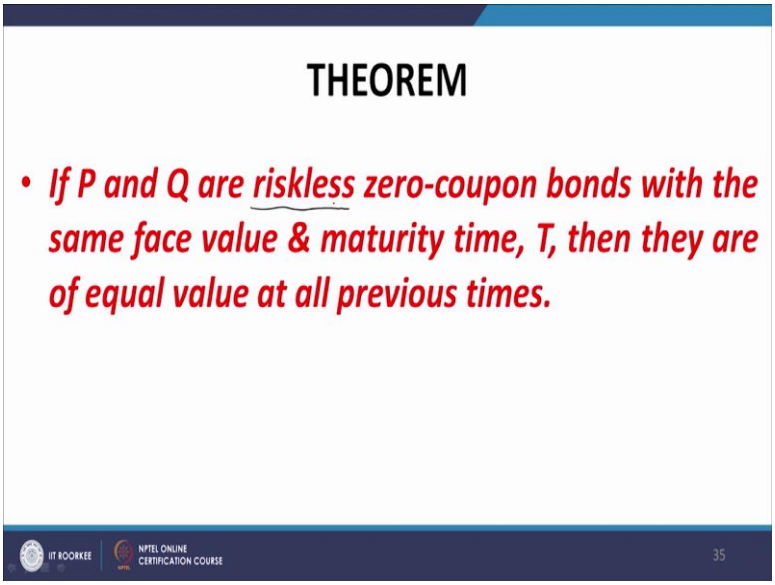

Theorem, if P and Q are riskless zero coupon bonds with the same place value and maturity time t and this is important, let me underline this P and Q are riskless. This riskless word is very important. It will come back when we talk about the proof to this theorem. If P and Q are riskless zero coupon bonds with the same face value and maturity time T, then they are of equal value at all previous times.

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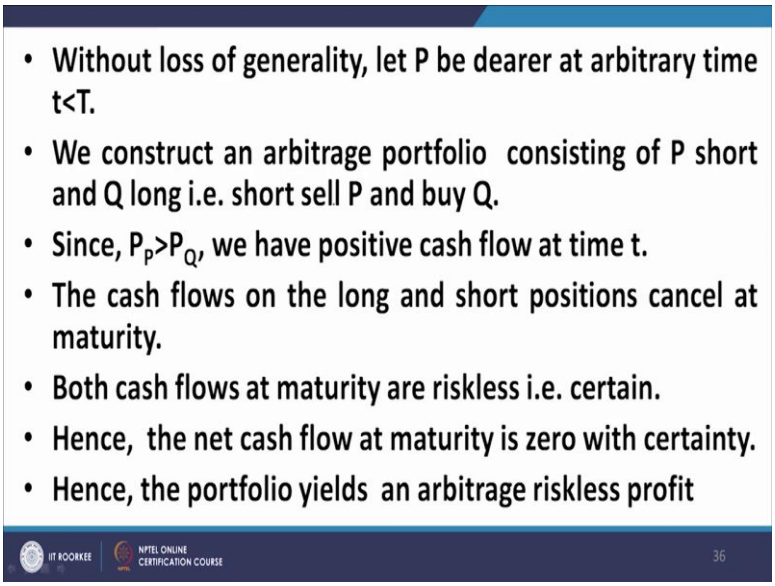

Without loss of generality, let P be dearer, you can either assume P to be dearer or Q to be dearer it does not matter, you end up with the same result. So, let P be dearer at arbitrary time t less than capital T, capital T is the maturity of the bond, at maturity of the bond, the bond is likely to return a certain value and that certain value is with certainty. Please notice what

these are risk free bonds. We construct an arbitrage portfolio consisting of P short, P is more costly, Q is cheaper. So, what we do, we sell P which is more costly, and we buy Q and form a portfolio consisting of Q long P short, remember P is more costly, Q is less costly, P is dearer, Q is cheaper. So, you buy the cheaper one, and you sell the dearer one.

Now, please note, the maturity cash flows of the two bonds are identical. And not only are they identical, they are certain. So, obviously, the maturity value of this portfolio compared to comprising of Q long and P short will be 0. What about its value at t equal to 0? At t equal to 0 because you are sold P you will get and P is dearer. And P is dearer, therefore you will get a higher value for P compared to the amount that you have to invest to get Q because Q is cheaper. So, you get a positive cash flow at t equal to 0 corresponding to a 0 cash flow at t equal to the capital T which is the maturity of the bonds.

Clearly this is an arbitrage portfolio and therefore, arbitrage operations can take place. Due to this possibility of arbitrage the prices will tend to converge and as the outcome is of what has stated that the outcome is that the two prices will tend to converge. I empathize, this for theorem will hold only if the two bonds are riskless because it is only then that we can cancel the cash flow that maturity of these two or at least the bond should be of identical risk and because if you are having bonds of differential risk, then obviously the theorem breaks down.

So, we conclude today and we shall continue from here in the next lecturer, we see about arbitrage free valuation of securities, how does the principle of arbitrage, how does arbitrage enable us to ascribe values to securities that is the next issue that we will talk about in the next lecture. Thank you.