Quantitative Investment Management
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Lecture 38
American Option - 1

So, let us continue. We will now talk about American options. What are American options? American options are options that have the special right, a special privilege that they may be accessed at any point up to the date of maturity of the option. In contrast to European options which can be accessed only on the access date, American options can be accessed at any point during the life of the option. There is a very interesting paradox to start with when we talk about American options. If you have an American call on a non-dividend paying stock, it is never optimal to early exercise rather to early exercise the American option.

In other words, it is never justified. It is never justified on rational considerations that you early exercise the American call on a non-dividend paying stock, that is, you do not or you should not at least it is not optimal that you exercise the American call prior to its maturity on an underlying which does not entail any dividend payout during the life of the option. This is very interesting.

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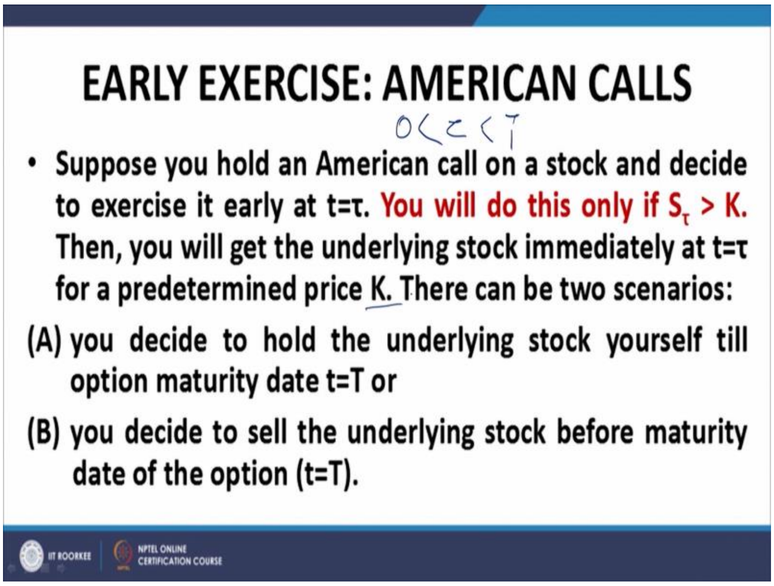

Let us take it up. Suppose you hold an American call on a stock and please note whenever we use this stock, we are talking about non-dividend paying stock for the purposes of this article. So, suppose you hold an American call on a stock and decide to accept it at t equal to tau, where tau lies between tau lies between 0 and capital T. That is, you are exercising it prior to maturity. You are exercising the option prior to maturity at any let tau be any arbitrary point in time between 0 and capital T.

You will do so only if S tau is greater than K because if S tau is less than K, you will not get any pay off from the option. There is no point in exercising the option. There is no rationality behind exercising the option of S tau less than K. If S tau is greater than K, what will you do? You will exercise the option, you will buy the asset at K and you will sell it in the market at S tau and you will make a profit of S tau minus K. Of course, for the moment ignoring the premium on the option.

Then you will get the underlying stock immediately and at a predetermined price K and obviously, if you choose, if you want, you can sell the underlying asset as I mentioned just now at S tau and make a profit of S tau minus K, make an instantaneous profit of S tau minus K or you can also hold on to the stock up to the date of maturity of the options. There are two possibilities in fact.

Number one, you decide to hold on to the stock after exercising the option at t equal to tau up to t equal to capital T, that is the date of maturity of the option or you exercise the option at t equal to tau and within the period tau to capital T you decide to dispose of the stock at the then prevailing market price. We will examine each of these two options separately.

Let us look at the first option. The first option is you decide to hold the underlying assets. You decide to hold the stock yourself up to option maturity, up to date t equal to tau. Let me repeat what has happened. You have exercised the option at t equal to tau where tau is some intermediate point between zero and capital T.

In other words, you have early exercised the option, American option and after exercising the option which would imply that you buy the stock, you receive the stock and you keep it with you up to t equal to capital T and you make the payment for the stock at t equal to tau for the excess price.

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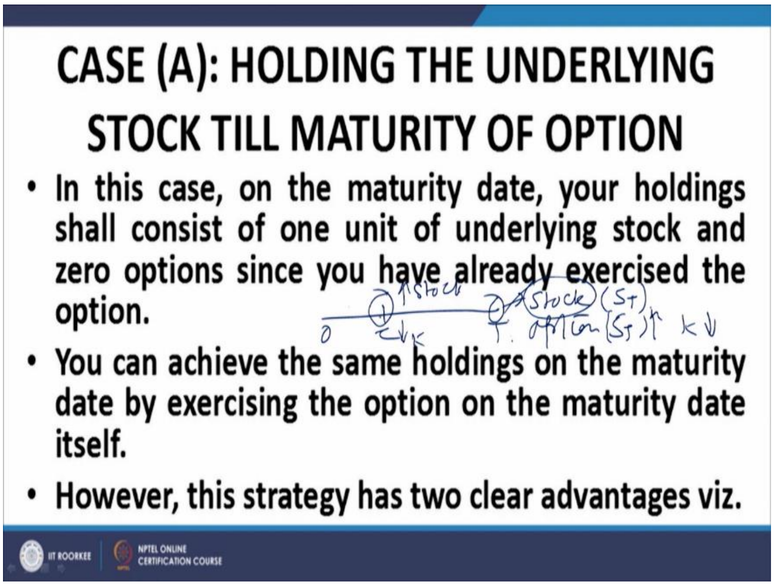

So, let us assume that you hold the stock up to t equal to capital T. Now the argument is the argument must be understood very carefully that now if you have exercised the option at t equal to tau, what is your holding at t equal to capital T? Say this is 0. This is somewhere here, let us say tau and here is capital T.

Now, at this point you have exercised the option. So, what has happened? You have paid the cash equal to K and you have received the stock. So, at this point when you reach this point, what is your holding in this portfolio? Your holding is only equal to the stock and the stock will be worth what the stock will be worth St.

And please note you have made the payment at t equal to tau. Now, the argument is that suppose you do not exercise the option. Then what happens? Then as on this date, t equal to capital T, your holding is what? Your holding is an option, call option that is American call on the same stock. So, at this point t equal to capital T what happens if you exercise the option? If you exercise the option at t equal to capital T that is the maturity of the option, not early access, not early access. Then what happens? Then you get a stock which is worth St and you pay the price which is K. So, this is an outflow, this is an inflow.

Now, please note one very obvious factor in the first strategy when you had early exercised the option, you had paid the price at t equal to tau. In the second case, when you have not earlier exercised the option when you have exercise the option and maturity, you have paid the price at t equal to capital T. Obviously, capital T is after tau. Tau is here, capital T is here. That means by early exercising you have made the payment earlier and because you have made the payment earlier, you lose interest for the period from t equal to tau to t equal to capital T on the amount of exercise price. This is one fundamental drawback. One fundamental issue against the early exercise of the option.

Now, the second point is even more intriguing. When you reach this point, t equal to capital T and you still have the option with you and suppose the stock price plummets down, stock price goes down rapidly is in a downswing and is well below K, then you can always think about it. You can always use the discretion not to exercise the option if at the last minute you feel that you have made a mistake, you do not want to hold the stock, you can always let the option lapse.

And by letting the option lapse, your loss is confined to the amount of premium that you have paid on the option because you have not bought the stock the loss on the stock is not to your account. Only the amount of premium that you have incurred for buying the option is the loss that you would encounter if the stock price goes down and you decide not to invest in the stock.

On the other hand, if you have already exercised the option, the stock is already with you and if the stock price goes down significantly, the entire loss is to your account because you are the owner of the stock. In the second case, you are the owner of the options, not the stock. So, you can get out of it. You can review, you can revisit your decision to invest in this stock. Okay, the stock is going down.

Let us avoid investing in it and in that case, if you let the option lapse, the only cost that you incur is the amount of premium that you have paid. So, this is the protection. This is the kind of insurance against the stock fall if there is a massive stock fall, then you are better placed if you have not exercised the option because the loss on the stock will not be to your account, you have not bought the stock as yet.

And if you have that right in that case to revisit the decision to invest in the stock, let the option lapse and get away with a loss equal to the amount of premium. This is not available if you have already exercised the option at t equal to tau because you have the stock with you, the option is gone, you have the stock with you and whatever price occurs in the stock is to your account.

So, these are two fundamental reasons which motivate or which mandate that early exercise of American calls on a non-dividend paying stock is never optimal. Let me repeat early exercise of American calls on non-dividend quipping paying stock is never optimal. Number one, the issue of early payment of exercise price. Number two, insurance against fall in the stock price, enabling you to revisit the decision of investing in the stock.

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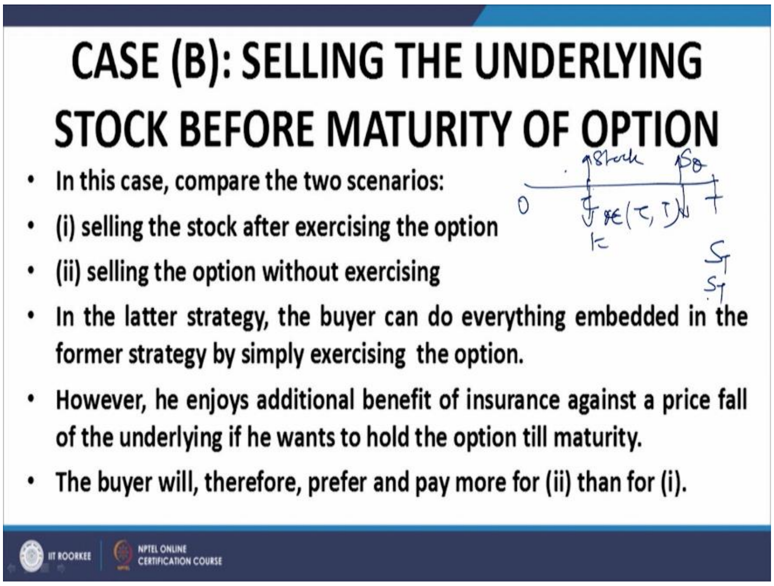

Now, let us look at the second situation. Second situation is if you sell the underlying asset between tau and capital T you have exercised the option. What happens? This is let me draw the timeline, 0 tau and capital T. So, at this point you have exercised the option, you have paid K, you have bought stock and within this period tau and capital T, tau comma capital T at any point you sell the stock. You sell the stock and you receive as theta, let us call it theta at any point between theta is contained in tau and t any arbitrary point between tau and t.

Now, let us compare the two situations from the perspective not of your perspective, from the perspective of the person who is going to buy the stock at t equal to theta. The question is would he be better off buying the stock at t equal to theta or would he be better off buying the option unexercised at t equal to theta? That is the big question.

Let me repeat. Buying the stock, you have exercised the option, the stock is with you and Mr. X buys the stock from you at t equal to theta for the price S theta. That is one possibility. The second is we compare this possibility with the second one which is that you sell the option unexercised, option not exercised to somebody to let us say to Mr. X at a certain price.

The question is which is better from the perspective of X? Let us assume that it is now X who will hold the stock up to maturity. Then what happens? Then the holding of X becomes worth S capital T because he has bought the stock from you at theta and he is holding up the stock and this holding is worth now is capital T and if he buys the option from you, he keeps the option with him and as on the date of maturity, that is S capital T, he decides to exercise the option.

Then again, he has one stock with him which is worth S capital T. So, as far as he is concerned, the stock that he will be holding or the portfolio that he will be holding whether he buys the stock at t equal to theta or buys the option at t equal to theta and exercises that option at t equal to capital T will be equal to St.

But again, as in the previous case, Mr. X will have that opportunity that if the stock goes down, if the stock goes rapidly plummets, then he could get out of the investment in the stock because he has not bought the stock, he has not exercised the option. So, because of that prerogative, that additional benefit, that additional insurance, that additional protection against a fall in price and the right to revisit his decision whether to invest in the stock, he would be happier that he has got the option and he has not invested in the stock at t equal to theta. And therefore, when he buys the option unexercised, he would be willing to pay a higher price than the price of buying the stock less the exercise price.

In other words, if the option is excess at t equal to theta, the pay-off is S theta minus K but if the option is unexercised, the pay-off would be slightly higher. You would be able to sell the option at a slightly higher price. So, there again, you find that if you do not decide to hold the stock to maturity and somebody else decides to hold the stock to maturity, howsoever long the chain may be, somebody would be there who would be willing to hold the stock to maturity because otherwise the stock price would not be sustained.

Somebody has to be there who is willing to hold the stock to maturity. This process or this derivation would hold in the context of that person as well and as a result of which it would not be optimal that he buys the stock after you exercise the option, he would be better off buying the option instead.

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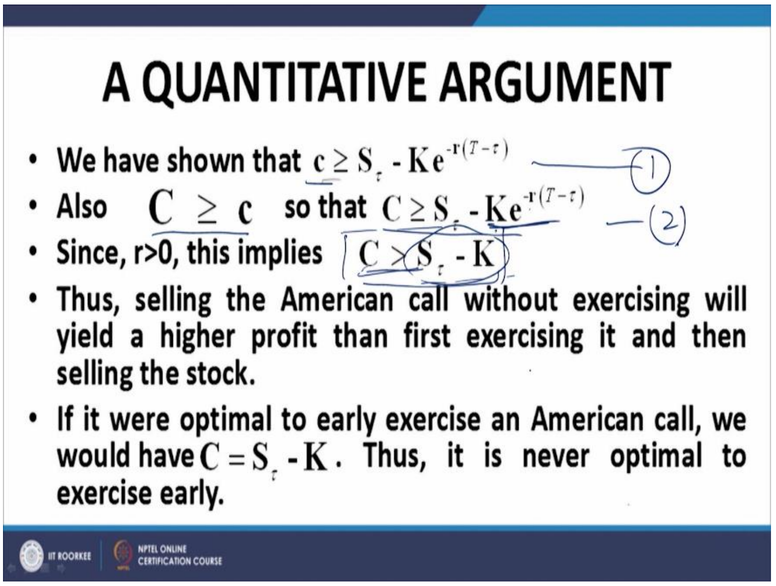

There is a quantitative argument in favour of this as well. We have already established that the lower bound of a European call is given by equation number 1. We have already established that the lower bound of a European call, small c means European call is given by equation number 1. Also, we know that because the American call gives you a certain additional right over whatever you can do in the context of the European call. So, therefore the price or the value of the American call must be greater than at worst case, it should be equal to that of the European call.

European call can be exercised only at maturity. American call can be accessed anytime up to maturity. So, whatever the European call holder can do, American call holder can also do. So, obviously, the price of the American call has to be either greater than and in the worst-case scenario, equal to the price of the European call.

That means this therefore, the American calls lower bound is also given by this expression. This is equation number 2. Because capital C is greater than equal to small c, the lower bound for small c must also be a lower bound for capital C. And that is precisely what this is saying. Since r is greater than 0, what is rr is that a free rate and r is in practical circumstances, r would be greater than 0.

If r is greater than 0 this implies that C is greater than S tau minus K. Because the right-hand side this expression where if r is greater than 0 obviously, the present value of K is less than the actual value of K and therefore, this is the present value of K. This is the present value of K. When we are deducting present value of K from S tau I will get a larger number compared to when we are deducting K from S tau. Why? Because present value of K is smaller than the value of K. Therefore, C has to be greater than now. please not equal to is missing S tau minus K.

What is S tau minus K? S tau minus K, the instantaneous profit, instantaneous pay off that you derive from exercising the American call. If you exercise the American call at any arbitrary point in time between t equal to zero and t equal to capital T, let us say t equal to tau the profit that you make, you buy the asset at K and sell it in the market at S tau, the instantaneous profit or the instantaneous pay off the rather, if you ignore the price of call is S tau minus K. That means what? That means if you immediately exercise the option, you get a profit of S tau minus K and if you sell the option, the worst-case scenario, even in the worst-case scenario, you get something better than S tau minus K. That is precisely what I explained a few minutes back in a descriptive manner.

So, what is C? C is the price that you get if you sell the option unexercised, unexercised. What is S tau minus K. S tau minus K is what you get if you exercise the option and then sell the stock that is S tau minus K. You exercise the option, buy the stock at K, sell it in the market, you get S tau minus K. You sell the option without exercise, you get capital C and we know that C capital is greater than S tau minus K. That immediately implies that it is not optimal to early exercise an American call or a non-dividend paying stock.

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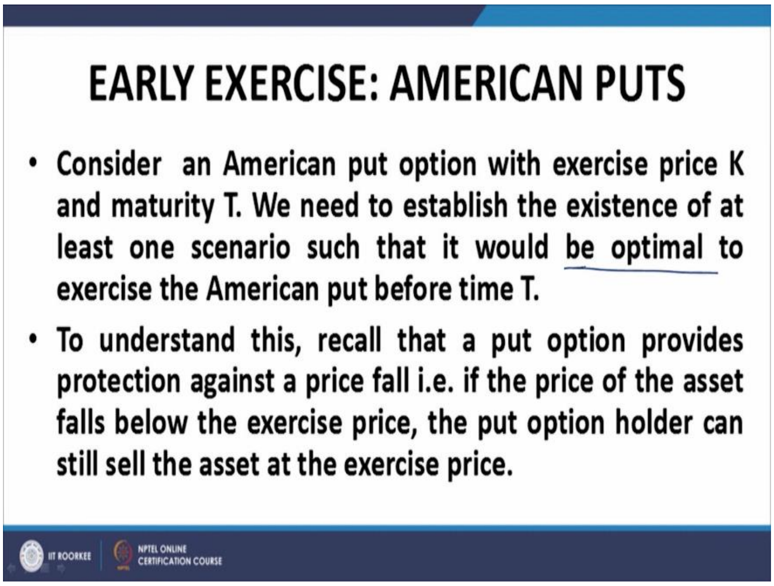

Early exercise of American puts. This is even more interesting. What is a put option? A put option is the right to sell and what is American put? An American put is the right to sell the underlying asset at any point between zero and capital T, where capital T is the majority of the option. Consider an American put option with excess price K and maturity capital T.

We need to establish the existence of at least one scenario. At least one scenario because we are trying to contradict the contention that early access of American put is also not optimal. Is it true? If it is true, it has to be true for all scenarios. If we can identify or even if we can identify one scenario in which it is not true, then the statement breaks down.

The statement that early exercise of American puts is not optimal breaks down if we can identify even one single scenario in which it is optimal to early exercise in which it is optimal in which it is optimal to early exercise the American put. If you can establish one scenario in which it is optimal to exercise the American put, the contention, the argument, the statement that early access of American puts is not optimal will break down. That is what we will try to do.

We need to establish the existence of at least one scenario such that it would be optimal to exercise the American put, it would be optimal. Please note this. Please note this fundamental thing. What we are trying to do is we are trying to contradict this statement that it is never optimal by identifying a situation where it may be optimal. To understand this, recall that a put option provides protection against a price fall. That is, if the asset price falls below the excise price, the put option holder can sell the asset at the excess price. That is what is the characteristic of a put option.

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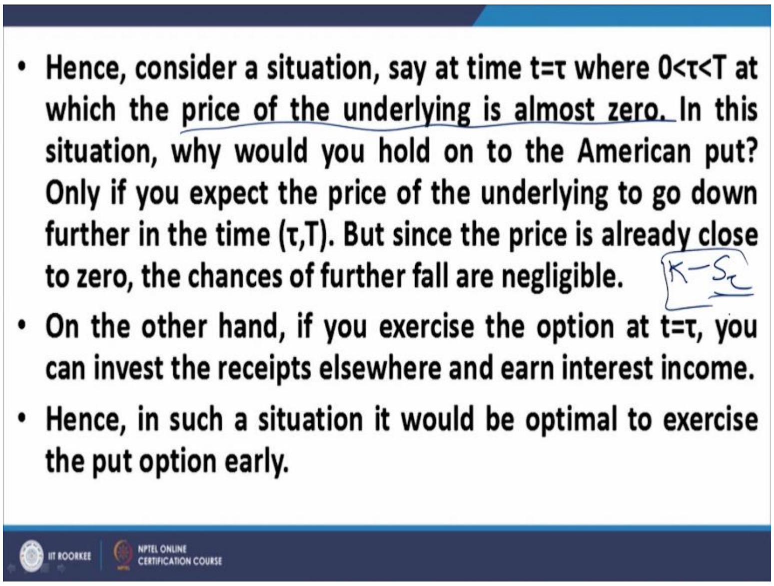

Hence, consider a situation, say at time t equal to tau, where 0 less than t that is, tau is some intermediate point between 0 and capital T at which the price of the underlying asset is almost 0. Price of the underlying asset is almost 0. In this situation, why would you hold on to the American put only? Why would you hold you see, the temptation is that you exercise the option, get the exercise price and deliver the asset and lower the price of the asset, lower the market price of the asset greater is the benefit to you. Greater is the temptation that you exercise the put because the put will give you the right to sell the asset at a higher price relative to what the market and the market price for the moment, we are assuming is very low.

So, the question is why would you not exercise the put option? Obviously, you will not exercise the put option in only one circumstance that is, you expect the price of the underlying asset to fall even further because if the price of the underlying asset falls even further, then what happens? Then your price your profit increases, your payoff increases. What is the payoff? The payoff is given by K minus S tau at any point in time t equal to tau if S tau is less than K, K is greater than S tau.

Now, the lesser the value of S tau, the closer S tau is to 0, the smaller S tau is the greater is this profit. So, if you are not exercising the put option at a given point in time as tau you are holding on to it for the remaining period tau to capital T, you would only do so if you have a perception that the stock price could fall further. And therefore, at that point in time when the stock price falls further, you exercise the option and you get a greater profit K minus S theta let us say where S theta is less than S tau.

But the question is, if the stock price at tau is already very small, negligible very small, then what is the probability that it would go even below, more or less or would fall down further? The probability is very low. So, the rationale or the motivation to hold on to the option decreases when the price is already very low, massively low, almost close to zero the temptation is to exercise the option.

The rational strategy would be to exercise the option rather than waiting further because waiting further is unlikely to result in further decline in price if the price already at tau is very low here. So, that means what? That means it is better that you exercise the option at t equal to tau. Then there is an added advantage. Contrast the situation with the situation that we encountered in the case of American call. American call was the right to buy. So, you are paying in cash, you are paying the exercise price.

Here it is a right to sell, so you are receiving the cash. The earlier you receive the cash, the better it is. You can reinvest the cash into some other profitable adventure and that means what? That means the earlier you exercise the put option, the better it is. That is the rational that if the stock price is small enough, the stock price is low enough, there would be little motivation to not exercise the option and it may be true that the exercise of the American put at this earlier point in time be optimal because at the end of the day you are getting the exercise price. You can reinvest the exercise price. The later you exercise the option, the later is the receipt from the exercise price and the later is your investment, the loss and the consequential loss of interest.

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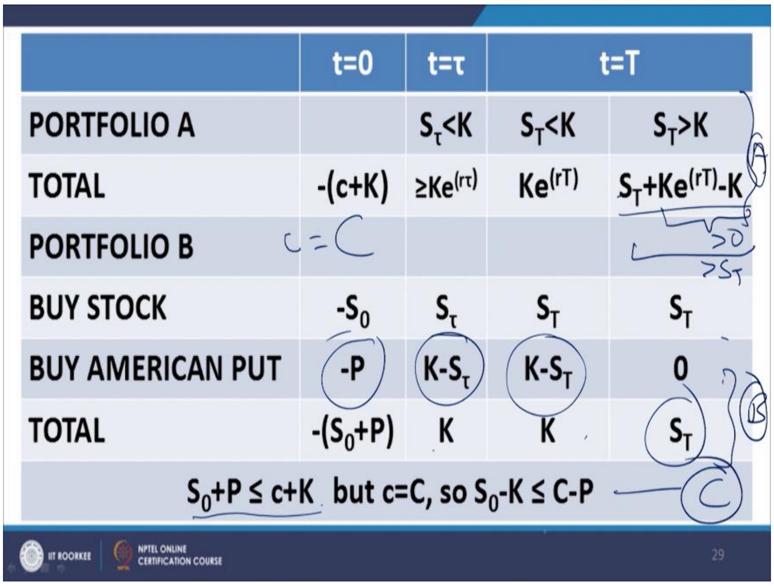

	$t=0$	$t=\tau$	$t=T$	
PORTFOLIO A		$S_\tau<K$	$S_T<K$	$S_T>K$
TOTAL	$-(c+K)$	$\geq Ke^{(r\tau)}$	$Ke^{(rT)}$	$S_T+Ke^{(rT)}-K$
PORTFOLIO B				
BUY STOCK	$-S_0$	S_τ	S_T	S_T
BUY AMERICAN PUT	$-P$	$K-S_\tau$	$K-S_T$	0
TOTAL	$-(S_0+P)$	K	K	S_T

$S_0+P \leq c+K$ but $c=C$, so $S_0-K \leq C-P$

Now, put call parity for American options. It is not a parity relationship. Please note this. It is something similar to a parity relationship. It is not an equality relationship but let us do it. The process is pretty much identical to what we had in the case of European options. We have portfolio A, we have a European call here we shall be talking about it later on.

But for the moment, European call and because the Americanness manifests itself as a right to exercise the option at any point between 0 and capital T, let the arbitrary time be termed as tau, be called tau and then we also have to consider the payoffs at t equal to tau. So, this is an intermediate time t equal to tau.

And then we have the maturity at t equal to capital T as usual. As far as the European call is concerned, it will not be exercised at any point earlier but it may have a non-zero value. Please note that at t equal to tau it may have a nonzero value, but invariably a non-negative value.

Obviously, a call option cannot have a negative value. A long position in a call option can never have a negative value. Why is that? Because it is a right. Right cannot have a negative value. So, because it is a right, what happens that the value of the European call is either 0 at the worst case scenario it is 0, but it would be greater than 0 or worst case scenario, zero.

And if St is less than K, the pay-off is zero, if St is greater than K, the pay off is St minus K and we invest. Now, please note in the case of the earlier put call parity, we were investing present value of K. Here we are investing K. Please note this reference because you are investing K at t equal to tau, the value of this investment will be K e to the power r tau, where r is the relevant rate. And at t equal to capital T, the investment would be worth K e to the power rT. And please note that K e to the power rT is independent of the state of the stock price whether it is less than K or it is greater than K.

So, this is as far as portfolio A is concerned, the cash outflow at t equal to 0 is equal to c plus K. The value of the portfolio at t equal to tau is greater than or equal to depending on the value of the call. If the value of the call is 0, the value of the portfolio is K e to the power r tau and if it is greater than 0, the value of the portfolio is greater than K e to the power r tau. And as far as t equal to capital T is concerned, if St is less than K, the portfolio is worth K e to the power rT and if St is greater than K, the portfolio is worth St plus K to the power rT minus K. Now, please note this part. This part is greater than 0. So, this whole thing is greater than St.

Now, we look at portfolio B. Portfolio B, we buy one unit of the stock cost us S 0. Its value at t equal to tau is S tau. Its value at t equal to capital T is St, and St is obviously, whether St is less than K or greater than K does not matter it will always remain St. American put the price that you pay is capital P. Please note this.

So, it is minus capital P. Now, at t equal to tau, you will obviously exercise it only if S tau is less than K. If S tau is greater than K, there is no point in exercising the American put. You will not get any pay off. So, if a S tau is less than K, the payoff will be equal to K minus S tau. And in the event that you hold on to the American put up to maturity, you get K minus St. If St is less than K and 0 otherwise.

The total pay off in this case is at t equal to tau, it is K. At t equal to capital T, St less than K, it is K. At t equal to capital T, St greater than K, it is St. You compare this pay offs. Let us call this pay offs A. Let us call this payoffs B. You compare payoffs A and B, what you find is that in each case t equal to tau, t is the to capital T St less than K. t equal to capital T St greater than K.

Portfolio A outperforms portfolio B, the payoffs of A are higher than portfolio B. Pay offs at every point because tau is an arbitrary point. Please note that we have not identified any specific tau at any point between zero and capital T. So, the payoff at any arbitrary intermediate point for portfolio A is greater than B and the portfolio and the pay off at maturity of portfolio is also greater than that of portfolio B or worst-case scenario, it can be equal when r is equal to 0.

The theoretical possibility of r being zero means that in that situation, when r is equal to zero, the payoffs at maturity of the two portfolios can be identical. So, that gives us this. That means the cost of setting up the portfolio A has to be greater than the cost of setting up a portfolio B and that gives us this relationship.

And now because of the early exercise issue because early exercise of American calls is never optimal, we get small c is equal to capital C or capital C is equal to small c. You may look at it either way but why it is so? Because capital C, that is the Americanness does not result in any incremental advantage because it is never optimal to use the Americanness property. It is never optimal to early access the American option and that is the Americanness property. In other words, because it is not optimal, it is not worth anything and therefore capital C will be equal to small c, substituting capital C for small C what we get is expression number C which is one part of the put-call parity for American options.

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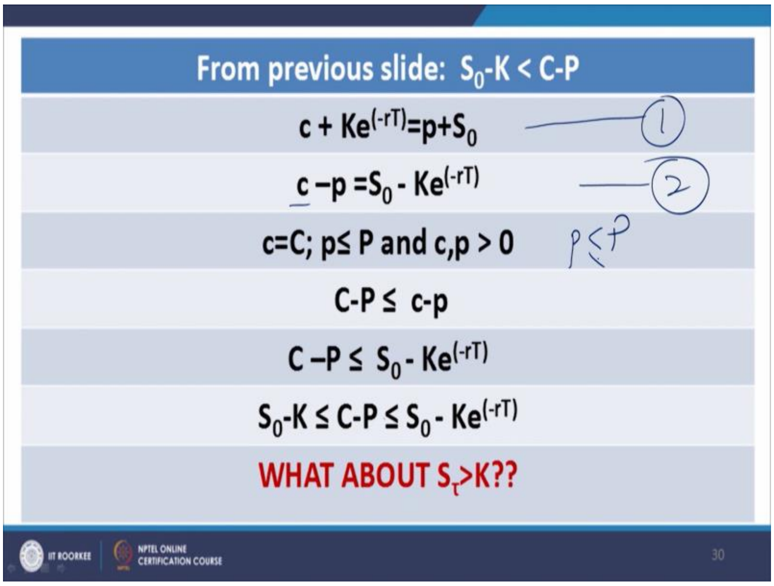

Let us look at the other side. Now, from the previous slide, what we have is S 0 minus K is less than C minus P. Now, from put-call parity for European options, we have this equation number 1. This is straightforward. Rearranging this equation. Equation number 1 we get equation number 2. Now, this small c can be replaced by capital C for the reason I mentioned just now but because early excess of American puts may be optimal in extreme situations, we must have small p is less than capital P or less than equal to rather in capital P.

Worst case scenario it can be equal. So, small p is less than equal to capital P. Why? Because there may exist situations where the early exercise gives you a certain value addition. You can early exercise with optimality and therefore, small p is less than equal to capital P and therefore, it means capital C minus capital P. Look at equation number 2. From equation number 2, we get equation number 3.

This is equation number 3. You see small c and capital C are same. Small P is smaller than capital P. Smaller or equal to that is therefore capital C minus capital P must be smaller than or equal to small c minus small p and therefore capital C minus capital P using equation number 2 now, capital C minus capital P must be less than equal to S 0 minus K e to the power minus rT present value of K. Therefore, combining the equation that we derived in the previous slide and this equation, equation number 4 we get equation number 5 which represents the equivalence of put-call parity for American options. We will continue from here in the next lecturer. Thank you.