Quantitative Investment Management
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Lecture 37
Put Call Parity and Arbitrage

Welcome back. So, let us continue from where we left off towards the end of the last lecture I was talking about Put Call Parity. Put Call Parity is a relationship between the no Arbitrage prices of a Put Option and a Call Option both European both having the same underlying, both having same maturity and both of the same time of course European options so and of the same maturity, same underlying as well and the same exercise price too. So, if these two options are being traded in the market what kind of a relationship should prevail, should prevail in the market. So, that no Arbitrage profits can be siphoned off is given by the Put Call Parity relationship.

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	t=0	t=T	
PORTFOLIO A		$S_T<K$	$S_T>K$
TOTAL	$-c-Ke^{(-rT)}$	K	S_T
PORTFOLIO B			
BUY STOCK	$-S_0$	S_T	S_T
BUY PUT	$-p$	$(K-S_T)$	0
TOTAL	$-S_0-p$	K	S_T
$c+Ke^{(-rT)} = S_0+p$			

The derivation I had discussed in the last lecture I will derive the same thing in a different approach. So, that you can appreciate the nuances of where certain quantities that came into that seemed abstract and so far, as coming into the picture when we derived the Put Call Parity in the last lecture

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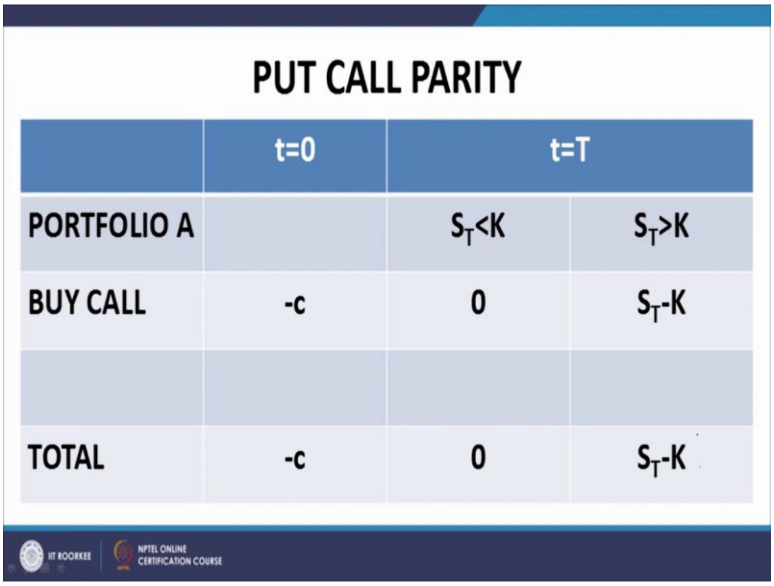

For that purpose let us start with this again we have a Portfolio A, Portfolio A comprises only of a Long Call Option since you have bought the call the cash there is a cash outflow at t equal to 0 and therefore we have it as minus c the payoff at maturity depends on the stock price if the stock price is below the exercise price the payoff will be 0 and if the stock price is above the exercise price the payoff will be equal to S T minus K where S T is the prevailing stock price on the date of maturity of the option it is a random variable.

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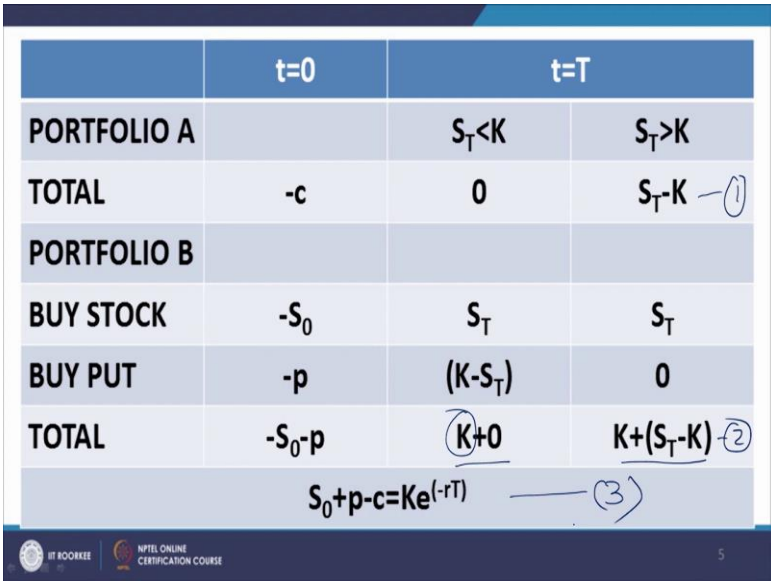

This is Portfolio A, Portfolio B we buy one unit of the underlying asset that is costing us S 0 at t equal to 0 and the value of this holding of one unit of the underlying asset will be S T at irrespective of whether S T is less than K or S T is greater than K at t equal to capital T that is the maturity rate of the co of the option, we buy a Put Option.

So, we have a cash outflow at t equal to 0 of minus p and the payoff from the Put Option at the date of maturity is going to be K minus S T if S T is less than K and 0 otherwise if we total out the constituents or the payoffs and the cost of Portfolio B the cash outflow at t equal to 0 is minus of S 0 minus of p the cash inflow at t equal to capital T is equal to K if S T is less than the first is less than K and if S T is greater than K it turns out to be S T.

Now, we can write this payoff in the form of this expression which we have here and if I compare the payoff from Portfolio 1 or Portfolio A and Portfolio B represented respectively by 1 and 2 what I find that there is a difference of an amount K the payoff from the Portfolio B exceeds the payoff from the Portfolio A by an amount K you can see if S T is less than K the payoff from Portfolio A is 0, the payoff from Portfolio B is K and if S T is greater than K the payoff from Portfolio A is S T minus K, the payoff from Portfolio B is S T.

So, the differential again turning out to be K in other words irrespective of the state of nature at the maturity rate of the option the two Portfolios A and B differ by an amount K and so far as the payoff is concerned the payoff from B will exceed the payoff from A notwithstanding how the stock price emerges, how the stock price evolves at maturity by an amount K therefore the cost of the two Portfolio should differ by an amount equal to the present value of K the cost of Portfolio B should be K present value of K greater than the cost of Portfolio A and that gives us this expression that I write as equation number 3 which is nothing but a realignment of the Put Call Parity relationship that we derived earlier.

So, either way we arrive at the same conclusion except for the realignment of the constituents of Portfolio A and Portfolio B. Now, let us look at a situation where this equality relationship that is the Put Call Parity is violated in other words the two options that are being traded Call Option price and the Put Option price do are not conforming to the Put Call Parity that we arrived at the two options being of the same type European on the same underlying same maturity and same exercise price. Please note these are fundamental requirements for the validity of the Put Call Parity same underlying same maturity same exercise price and same type that is European.

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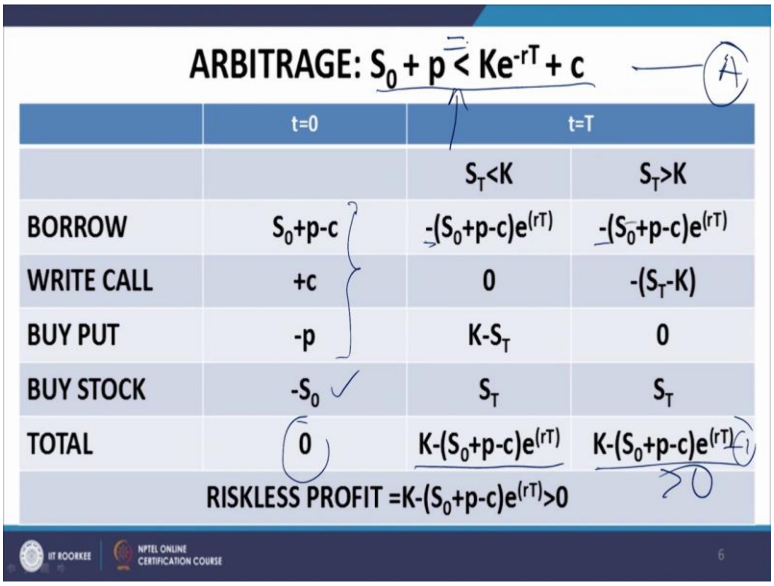

So, let us assume that the Put Call Parity relationship is violated in this form that we have written here at the top S 0 plus p is less than you see the parity requires that S 0 plus p equal to we assume that at a certain instant of time S 0 plus p is less than K e to the power minus r T that is the present value of K plus c then how would the Arbitrage process manifest itself such that this inequality is reduced to an equality the market forces align themselves operate in such a way that this inequality disappears and it is replaced by an equality.

Let us see how or what are the pro steps in other words let us see what are the steps that if you know at a particular time, point in time this situation arises in the market the situation exists in the market an arbitrator who recognizes this inequality can extract profits out of this and thereby push the inequality towards the equality for that purpose what we what we do is we borrow an amount S 0 plus p minus c where the symbols have their usual meaning I will not repeat them again and again. We write a Call Option since we are writing a Call Option will get a cash inflow at t equal to 0 that is plus c and we buy a Put Option we are long in a Put Option.

So, we have to pay the premium of p that is represented by minus p. So, the aggregate of the cash outflows or cash flows rather at t equal to 0 turns out to be S 0 and we use this amount as 0 for buying one unit of the underlying asset in the spot market at t equal to 0 that in that is represented by this minus S 0 factor here. So, the aggregate cash flow at t equal to 0 is 0 as you can see here. Let us look up the payoffs at maturity again we identify the payoffs with respect to the states of nature in which the stock price can evolve we have 2 situations S T less than K where K is the exercise price and S T greater than K where K is exercise price and S T is the price of the underlying asset at the date of maturity of the European options.

So, the amount borrowed S 0 plus p minus c will grow to S 0 plus p minus c, e to the power r T, the future value that is and this will have to be paid back to the bank and therefore this there is a minus sign here and the amount is independent of what state the system is at t equal to capital T whether S T is less than K or S T is greater than K this payoff this or this payment or repayment to the bank is independent of the state of evolution of the stock price then we have written a Call Option.

So, the payoff from the Call Option if S T is less than K 0 the holder of the call will not exercise the call and it will be minus S T minus K if the holder of the call decides to access the Call Option which he would do as a proven investor if S T is greater than K then the Put Option payoff is given by K minus S T and if S T is less than K and 0 otherwise, as far as the stock is concerned the price of the stock will be S capital T and this S capital T is again independent of the state of evolution in the sense that whether S T is less than K or S T is greater than K the stock would be worth S T in any case.

So, when we work out the aggregate payoff what we get is K minus S 0 this expression K minus bracket S 0 plus p minus c, e to the power r T and you can see that this payoff is independent of the state of nature it is independent of whether S T is less than K or S T is greater than K the payoff that you get is K minus S 0 plus p minus e to the power r T and that is shown here let us call it 1.

And what about the sign of 1 in view of what we have here let us call it A for that matter in view of A you can clearly see that the quantity that we have as 1 turns out to be positive by a slight realignment we can slight realignment followed by multiplication by the factor of e r T which is invariably positive.

What we find is that this expression will always be greater than 0 and therefore what is happening without any investment at t equal to 0 you are earning a risk less profit at t equal to capital T which is independent of the state of evolution of the system and that means that this profit is a certain profit it is a riskless profit and that means Arbitrage forces will come into play and gradually the operate in such a way that this siphoning of profits will cause this inequality that we have here to disappear or the prices of the various items that are appearing in this equation will realign themselves such that this inequality is replaced by an equality.

So, similarly if the inverse inequality holds if this less than is replaced by greater than all the process the all the steps would be reversed and again we will have a situation where we and the arbitrator can make risk-free profit with a 0 investment and again the forces will realign themselves such that the greater than would be replaced by equality.

So, that is the that is how the Arbitrage pricing mechanism operates in practice when you have a situation where there is a possibility of stifling of profits without making any investment risk less profits certain profits without making any initial investments all the market players who are usually vigilant about this kind of opportunities jump into the fray and gradually extract the profits out until the prices realign themselves.

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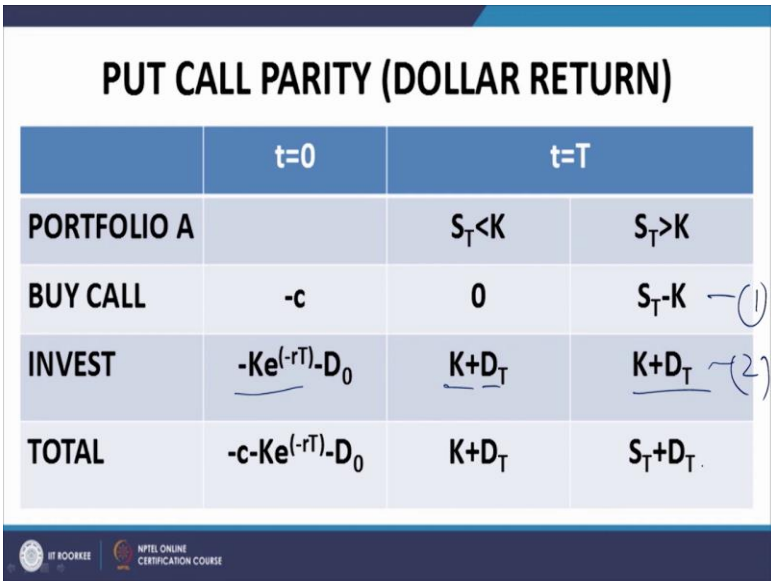

PUT CALL PARITY (DOLLAR RETURN)

	t=0	t=T	
PORTFOLIO A		$S_T<K$	$S_T>K$
BUY CALL	-c	0	S_T-K (1)
INVEST	$-Ke^{(-rT)}-D_0$	$K+D_T$	$K+D_T$ (2)
TOTAL	$-c-Ke^{(-rT)}-D_0$	$K+D_T$	S_T+D_T

Now, we talk about Put Call Parity in the presence of a Dollar Return please note that in the derivation that we did just now we assume that during the life of the two options that is the Put Option and the Call Option there was no cash inflow or cash outflow arising from the holding of the asset from the holding of the asset S that, that is there on one side of the parity relationship or in one of the one of the Arbitrage Portfolios, this asset does not yield any income or any NTL any cash or flow during the life of the option this was the assumption this was the premise on which the Put Call Parity was derived right now and also in the last lecture.

Now, we look at a situation where the holding of the asset results in a cash inflow by virtue of dividends by way of dividends during the tenure during the term during the life of the two options the Put Option and the Call Option. So, again we construct Portfolios A and B and then arrive at an equality relationship by considering Arbitrage issues and so the first

Portfolio here consists of a Long Call and that is given in this equation number 1 or which has been explained earlier so let us not waste time on this.

The second part is instead of investing an amount equal to the present value of K we now invest an amount equal to the present value of K plus the present value of the dividend that is going to be received on the holding of the underlying asset in Portfolio B. So, let me repeat instead of investing K e to the power r minus r T that is the present value of K we are now investing the present value of K plus the present value of dividend that is due to be received by holding the asset in Portfolio B that we shall come to right now.

So, the maturity value of this investment will be equal to K m plus the value the future value of the dividend at t equal to capital T that is D T. So, and this is independent of the state of nature let us call it number 2 the aggregate payoffs at maturity are K plus D T if S T is less than K and S T plus D T if S T is greater than K.

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	t=0	t=T	
PORTFOLIO A		$S_T<K$	$S_T>K$
TOTAL	$-c-Ke^{(-rT)}-D_0$	$K+D_T$	S_T+D_T
PORTFOLIO B			
BUY STOCK	$-S_0$	S_T+D_T	S_T+D_T (3)
BUY PUT	$-p$	$K-S_T$	0 (4)
TOTAL	$-S_0-p$	$K+D_T$	S_T+D_T
	$c+Ke^{(-rT)}=S_0-D_0+p$ (5)		

Now, let us look at Portfolio B when we look at Portfolio B we buy one unit of the stock and because you are buying one unit of the stock and you are holding the stock with you, you will get dividend on this and that dividend will be in terms of its future value will be equal to D T D capital T therefore the total payoff merely by holding the stock merely by investing an amount of S naught and buying this stock in the spot market will be equal to S capital T plus D capital T because the dividend that you will receive at any point.

So, let us say t equal to tau you can reinvest the dividend and when you reinvest the dividend you get an amount D capital T at maturity. So, the total cash flow at maturity by virtue of buying one unit of the stock and holding it up to the maturity of the options is S capital T plus D capital T and again this is independent of whether S T is less than K or S T is greater than K this is equation number 3 and the we buy a Put Option we take a long position in a Put Option and that is gives you what is number 4.

The payoff is K minus S, S T if S T is less than K and 0 otherwise the cost is p when you aggregate the total payoffs in Portfolio B we find that if they are equal to the payoffs and Portfolio A and because the intermediate dividend that was received on Portfolio B was reinvested forth with therefore there are no in no cash inflows or outflows from the system in the po in the period 0 to capital T and number 2 we are not considering any default risk whatsoever.

So, all the cash flows are certain in that sense and as a result of which because the payoffs are equal for Portfolio A and Portfolio B the cash flows at t equal to 0 must also be equal and that gives us equation number 5 and which is our Put Call Parity in the presence of a dividend payment you can see here that basically it is the same situation that we encountered when we did the pricing of forward contracts on an underlying asset with payoffs of dividend.

The spot price gets adjusted by the present value of dividend that is precisely what is happening here if you look at this carefully instead of S 0 we are having S 0 minus t 0 in other words the spot price of the stock is adjusted by the amount of the present value of dividend and the otherwise the Put Call Parity remains unchanged

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PUT CALL PARITY (YIELD)

	t=0	t=T	
PORTFOLIO A		$S_T<K$	$S_T>K$
BUY CALL	-c	0	S_T-K
INVEST	$-Ke^{(-rT)}$	K	K
TOTAL	$-c-Ke^{(-rT)}$	K	S_T

When we have then we have another situation where we consider the Put Call Parity when the holding of the stock gives the yield or results in a continuously compounded yield like continuously compounded interest rate on a foreign currency or index stock indices if you have the possession of your hold as the equivalent of stock indices you may get a yield on the average yield in terms of a continuously compounded percentage or the return on the holding of a index of stocks may be represented in terms of some kind of an average rate continuously compounded rate that is what we call the yield and in that case the Put Call Parity gets modified slightly.

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	t=0	t=T	
PORTFOLIO A		$S_T<K$	$S_T>K$
TOTAL	$-c-Ke^{(-rT)}$	K	S_T
PORTFOLIO B			
BUY ASSET	$-S_0e^{(-qT)}$	S_T	S_T
BUY PUT	-p	$K-S_T$	0
TOTAL	$-S_0e^{(-qT)}-p$	K	S_T
$c+Ke^{(-rT)} = S_0e^{(-qT)}+p$			

As far as Portfolio A is concerned there is no change. So, I will not spend time on it. As far as Portfolio B is concerned instead of buying one unit of the underlying asset instead of taking a long position in one unit of the underlying asset by investing an amount of S naught we now take long position in e to the power minus q T units of the underlying asset e to the power minus q T units of the underlying asset.

And therefore the cash flow occurring at t equal to 0 for acquisition of these many units of the underlying asset is equal to S 0 e to the power minus q T and this e to the power minus q T units will grow to one unit as in the date of maturity of the options which will command a price of S capital T in the spot market on the date of maturity of the two options otherwise the rest of it is again similar to what we had in the case of the previous cases and as a result of which we arrive at this equation.

The Put Call Parity when the holding of the underlying asset entails a continuously compounded yield I repeat the Put Call Parity relationship when the holding of an asset results in a continuously compounded yield given in terms of percentage the previous case was relevant when the holding of the underlying asset resulted in a Dollar return, a Rupee return, a return in terms of money value during the life of the options here the return is represented as a continuously compounded rate per in terms of percentages per annum.

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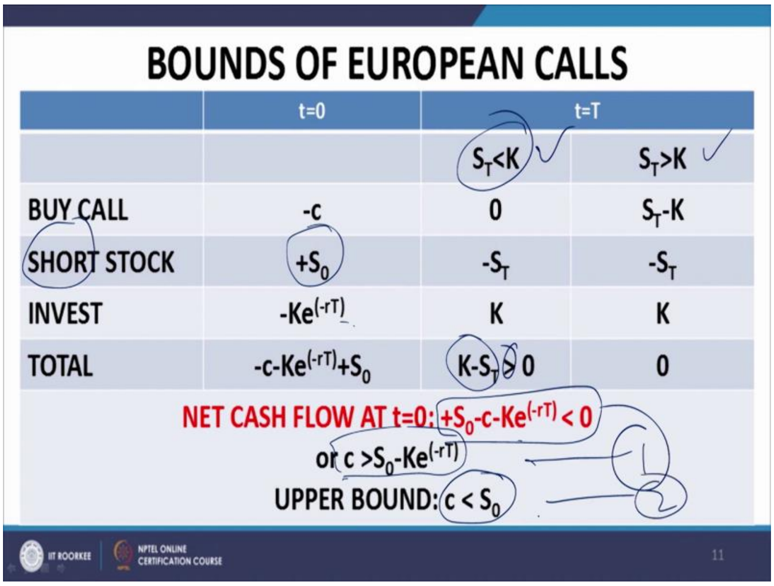

Bounds of European Calls what should be the theoretical Arbitrage free, Bounds on European Calls and European inputs that is our next exercise quite simple really one can derive it straight away from the Put Call Parity as well but here is the Portfolio approach we buy one call we get a payoff of 0 if S T is less than K, payoff of S T minus K if S T is greater than K we shot one unit of the stock that would that would result in a cash inflow.

Please note because we are shorting the stock it would result in a cash inflow at t equal to 0 of S 0 and a cash outflow of S T at t equal to capital T the maturity of the option and this S T outflow will be independent of whether S T is less than K or S T is greater than K we invest an amount equal to the present value of K and that gives us K at the date of maturity of the options irrespective of again whether S T is less than K or S T is greater than K.

Then what we find is that the aggregate payoff at maturity if S T is less than K is K minus S T and because if S T is less than K here therefore this has to be positive and in the other case it is 0. So, we are having a situation where in one state of nature we have a positive cash flow and in the other state of nature we are having a 0 cash flow there is no situation where we are having a negative cash flow.

So, irrespective of what is the probability, irrespective of what is the probability of occurrence of this state or this state, the result would be that the cost of this Portfolio has to be positive in other words at t equal to 0 there has to be a cash outflow and therefore this expression that we have here must be less than 0 and when we use this expression here as less than 0 we can write or we arrive at c is greater than S 0 minus K e to the power minus r T.

So, this is in a sense this is the lower bound of the Call Option this is the Arbitrage free based lower bound of the price of a European Call Option right. Now, talking about the upper bound, what is the upfront what is the what is the Call Option? A Call Option is a right to buy the asset. So, it is quite natural that the price of the Call Option can never exceed the price or the value of the underlying asset and therefore this is this is quite elementary this is quite straightforward.

Let me repeat, what is the Call Option? A Call Option is a right to buy the underlying asset therefore its value the value of the Call Option cannot exceed the instantaneous value or inter instantaneous price of the underlying asset because you if you hold the underlying asset you are pretty much doing what the Call Option could do for you.

So, in no way can the Call Option do better than holding the underlying asset and therefore the price of the Call Option cannot exceed the price of the underlying asset and that is represented by this equation. So, equation number 1 gives us the, gives us the lower bound and equation number 2 gives us the upper bound on Arbitrage considerations of European call.

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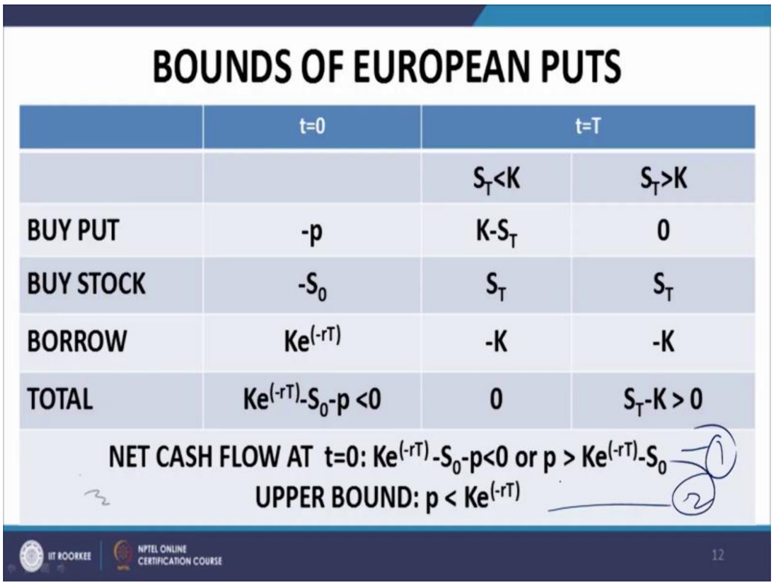

BOUNDS OF EUROPEAN PUTS

	t=0	t=T	
		$S_T<K$	$S_T>K$
BUY PUT	$-p$	$K-S_T$	0
BUY STOCK	$-S_0$	S_T	S_T
BORROW	$Ke^{(-rT)}$	$-K$	$-K$
TOTAL	$Ke^{(-rT)}-S_0-p<0$	0	$S_T-K>0$

NET CASH FLOW AT t=0: $Ke^{(-rT)}-S_0-p<0$ or $p > Ke^{(-rT)}-S_0$ (1)

UPPER BOUND: $p < Ke^{(-rT)}$ (2)

Similarly, we can derive the lower and upper bounds on European puts and the lower bound turns out to be this K e to the power minus r T that is present value of K minus S 0 this is the lower bound and the upper bound in this case is a bit interesting, what is the Put Option? Put Option is the right to sell the asset at the exercise price therefore the value of the Put Option can never be greater than the exercise.

So, it has to be less than the exercise the second thing is that the excise price will be received we are talking about the European options please note. So, the exercise will be received only on the date of maturity of the option and therefore the current value of the Put Option can never be more than the present value of exercise and that is precisely what is obtained or what is given in equation number 2 here.

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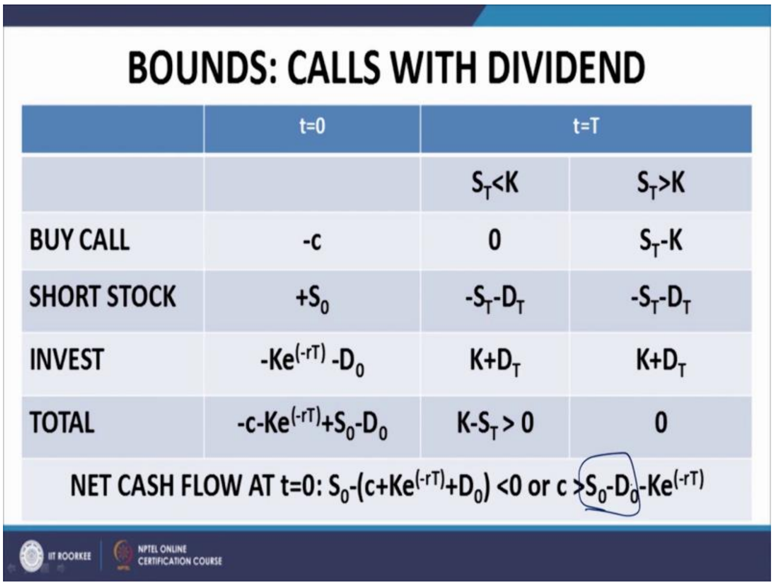

I leave this as an exercise bounds on calls with dividends you will see that this factor comes into play as has been in the case when we talked about forwards as well as when we talked about Put Call Parity. The stock price this spot stock price the price of the stock at t equal to 0 gets adjusted for the present value of dividend the present value of dividend is deducted from the stock price at t equal to 0 when we use the put call when we use the or when we work out the bounds on the Call Options European calls.

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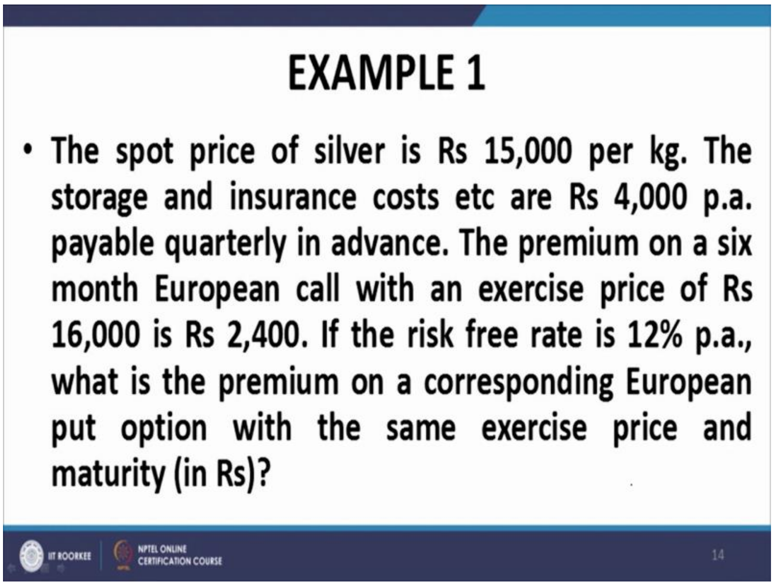

Let us do an example the spot price of silver is Rupees 15000 per kg. The storage and insurance costs are Rupees 4000 per annum payable quarterly in advance. The premium on a 6-month European call with an excise price of Rupees 16 000 is Rupees 2400 if the risk free rate is 12 percent per annum what is the premium on a corresponding European Put Option with the same exercise price and maturity it is a simple application of the Put Call Parity relationship.

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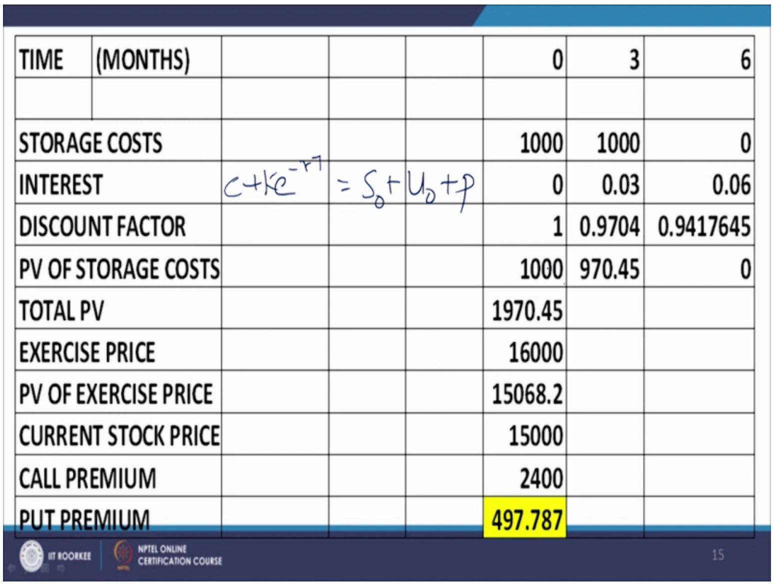

TIME	(MONTHS)				0	3	6
STORAGE COSTS					1000	1000	0
INTEREST		$C+Ke^{-rT}$	$= S_0+U_0+p$		0	0.03	0.06
DISCOUNT FACTOR					1	0.9704	0.9417645
PV OF STORAGE COSTS					1000	970.45	0
TOTAL PV					1970.45		
EXERCISE PRICE					16000		
PV OF EXERCISE PRICE					15068.2		
CURRENT STOCK PRICE					15000		
CALL PREMIUM					2400		
PUT PREMIUM					497.787		

What is the Put Call Parity relationship but please note here in this case the holding of the underlying does not give you any return in the form of income or yield on the contrary it entails incurrence of cost for the carrying and insurance of the set insurance of silver that is the underlying asset therefore is although the model that we will use is the dividend model is the Put Call Parity with Dollar dividend but the sign will change instead of a Dollar dividend being deducted you will add the present value of the of the carrying cost. So, the Put Call Parity that we shall use is c plus K e to the power minus r T is equal to S 0 plus U 0 plus p where what is U 0, U 0 is the present value of the carrying cost that will be incurred for holding the asset.

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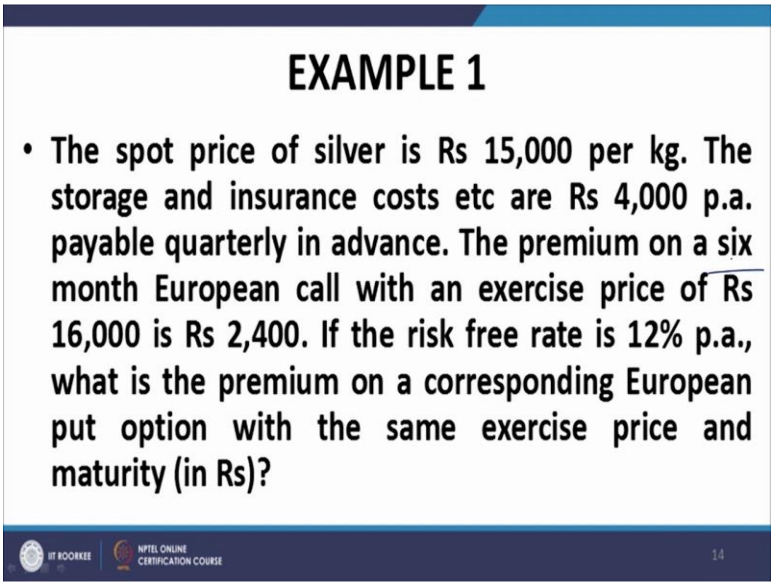

So, here this is the working out of the of the problem the total present value because please note what is the important thing here the pattern of the payment of carrying costs is important and you can look at it here again the storage and insurance costs are Rupees 4000 per annum payable quarterly in advance.

So, you will be paying 1000 at t equal to 0, 1000 at t equal to 3 months there will be only 2 payments because the life of the option is life of the option is 6 month. So, because the life of the option is 6 months first payment will be at t equal to 0 which will cover the period from 0 to three months and then the second payment will be equal to t will be paid at t equal to 3 months which will cover the period from t equal to three to t equal to six months.

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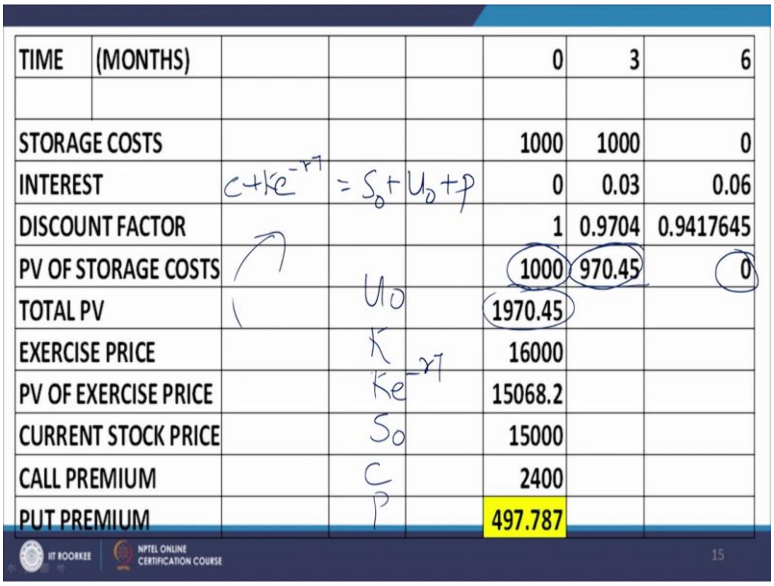

TIME	(MONTHS)				0	3	6
STORAGE COSTS					1000	1000	0
INTEREST		$c+Ke^{-rT}$	$= S_0+U_0+p$		0	0.03	0.06
DISCOUNT FACTOR					1	0.9704	0.9417645
PV OF STORAGE COSTS					1000	970.45	0
TOTAL PV			U_0		1970.45		
EXERCISE PRICE			K		16000		
PV OF EXERCISE PRICE			Ke^{-rT}		15068.2		
CURRENT STOCK PRICE			S_0		15000		
CALL PREMIUM			C		2400		
PUT PREMIUM			P		497.787		

So, the total there will be only two installments of carrying cost that would be paid not three installments and the present value of these two carrying cost installments that is 1000 and 970.45 turns out to be 1970.45 this is my U 0 the rest is simple this is the exercise plus K this is the present value of the exercise price K e to the power minus r T where T is 6 months, r is the risk free rate and the current stock price is S naught. So, we have got everything here the current call price is also given the put price on using this equation here it turns out to be 497.787.

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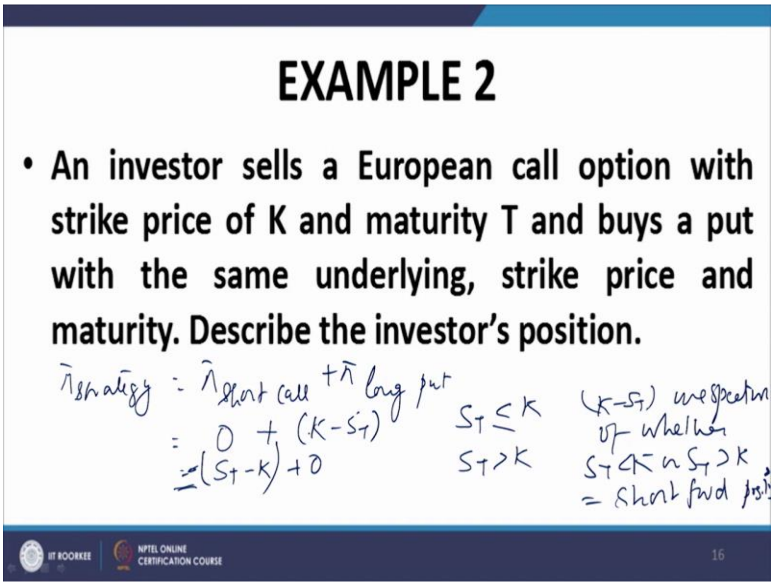

This is this is another example an investor sells a European Call Option with a strike price of K and maturity T and buys a Put Option with the same underlying, strike price and maturity.

Describes the investor's position let us try to understand this we are short in the course. So, pi of strategy, pi of strategy is equal to pi of short call plus pi of and long put, that is equal to 0 plus 0 plus K minus S T for S T less than K or less than equal to K you may say and short call. So, there should be a minus sign here as well.

So, it is K minus S T irrespective of whether S T less than K or S T greater than K and this is nothing but a short forward position. So, this is how this problem is done this is a short call. So, there is a minus sign here please note this and there is a Long Put here. So, there is no negative sign here it is a positive sign.

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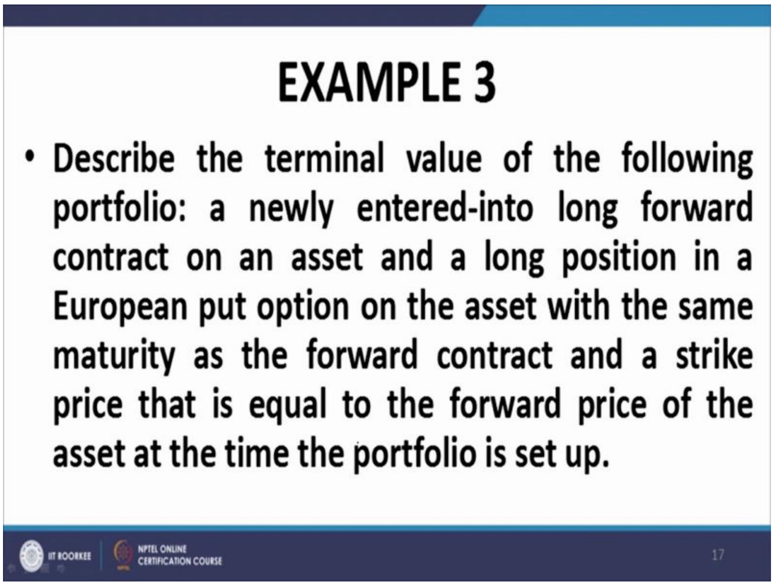

Describe the terminal value of the following Portfolio and newly entered into long forward contract on an asset and a long position in a European put option on the asset with the same. So, what we want what we need to find is pi long forward plus pi long put.

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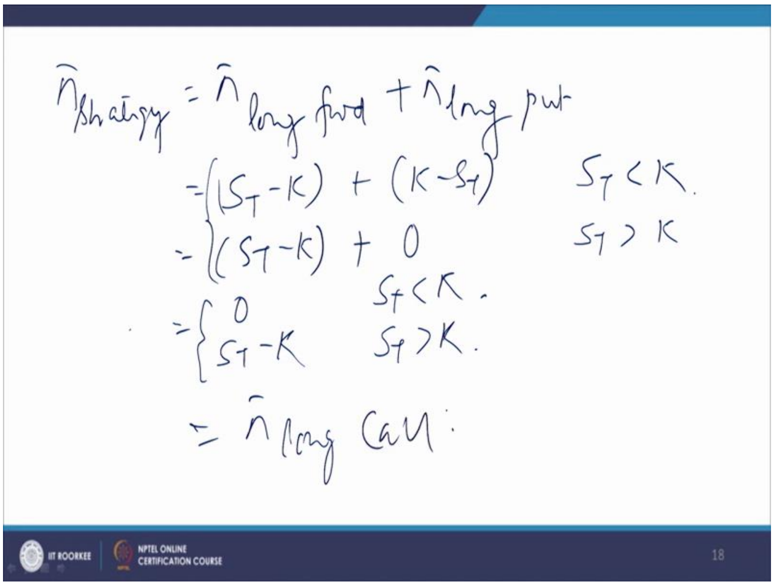

So, let us work it out pi strategy is equal to pi long forward plus pi long. So, that is equal to S T minus K plus K minus S T for S T less than 0 and S T minus K plus 0 for S T greater than K this is equal to 0, S T less than K and S T minus K, S T greater than K and this is nothing but pi of a Long Call. So, that is how this problem is to be done then we come to American Options which we shall take on the next lecturer thank you.