Quantitative Investment Management
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Lecture 34
Forward Pricing

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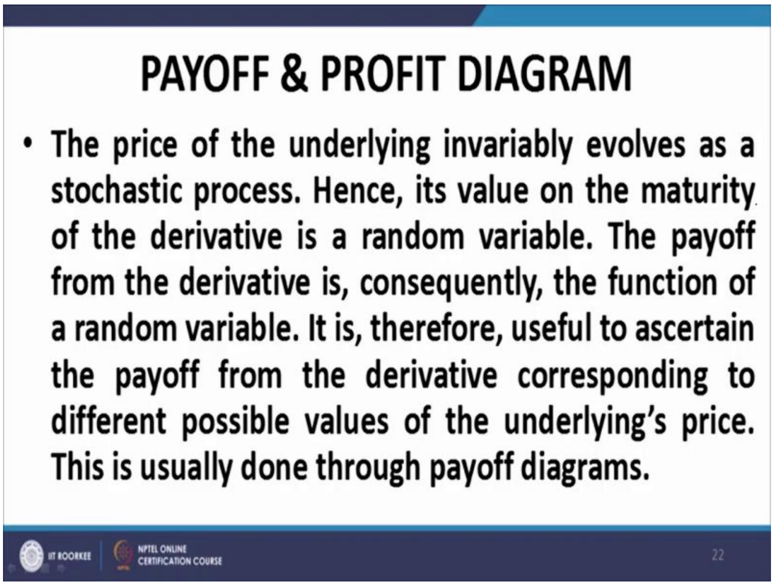

So, let us continue from where we left off payoff and profit diagrams. Now, the price of the underlying which forms the substratum of the forward contract invariably evolves as a stochastic process, what is the stochastic process, stochastic process is a process which evolves with an element of randomness embedded in it.

In other words, it is not possible at any point in time to precisely project, predict the trajectory of the process for all future time, it has an element of randomness, and every at every point of time that randomness manifests itself.

So, we cannot precisely setting at t equal to 0 at any point in arbitrary point in time for that matter, we cannot precisely predict the trajectory of the process henceforth. So, that is a stochastic process and obviously, the underlying assets prices follow a stochastic process.

Because, as I mentioned, if the price process of the underlying is deterministic, that is, if it is perfectly predictable for all future point in time, if it can be given us a differential equation and which has a clear cut preserve trajectory, then there is no positive writing a derivative contract because the derivative contracts value would also be absolutely predictable and then it would make no sense as performing functions of derivative contract.

So, it would not be able to perform functions of derivative contract. So, it is necessary that underlying assets purely those assets, which evolved with a certain element of randomness embedded in them. So, that is what we are talking about here.

Hence, the value of the at the maturity of the derivative is a random variable, the value of the price of the underlying asset, the value of the underlying asset, the price of the underlying asset on the date of maturity of the derivative is a random variable.

So, the payoff from the derivative therefore, depends on the value taken by random variable and it is a function of a random variable. It is therefore useful to ascertain the payoff from the derivative corresponding to different possible values of the underlying's price. This is usually done through payoff and profit diagrams. So, let me explain it, if at a t equal to capital T, which is the maturity of the derivative, the the value that the underlying asset's price could take is a random variable.

Now, corresponding to the value that the price of the underlying asset could take, the payoff of the derivative is determined at the date of maturity. Therefore, the payoff is a function of that random variable, which is the price of the underlying asset.

Now, it is very convenient, it is very useful for the purposes of establishing strategy is setting a t equal to 0, to have a feel about how the payoffs are going to behave given different values, that the price of the underlying asset, which is a random variable could take.

So, that is done through payoff diagrams, we, we take a spectrum of possible prices of the underlying asset because it is a random variable, we cannot predict it exactly, but we can have a spectrum of prices, such that the the price could lie between this particular set of values and then on that basis of corresponding to each such value that that is a possible candidate for the price of the underlying asset, we calculate the payoff then the graphical representation is called the payoff diagram and the mathematical representation is called the payoff function.

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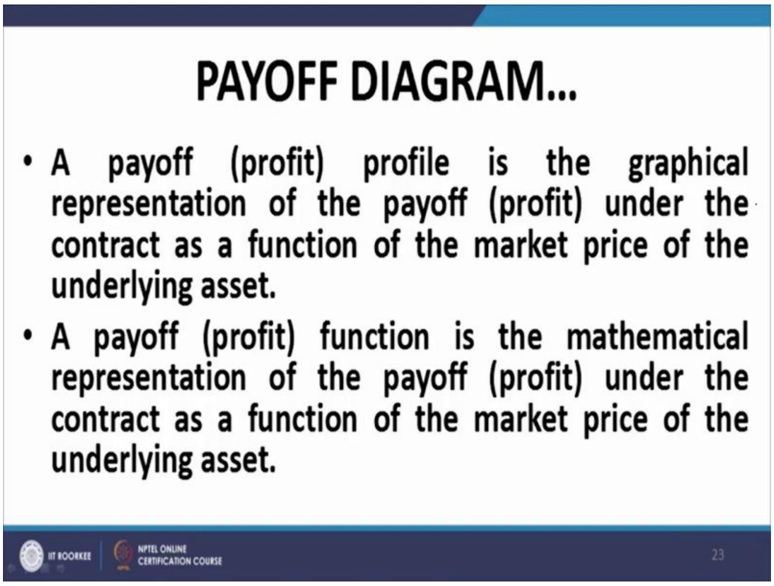

So, a payoff for a profit profile is the graphical representation of the payoff under the contract as a function of the market price of the underlying asset. So, that is what it said this market price is the random variable.

So, we identify certain possible values of this market price. And corresponding to each of these values, we try to work out the payoff of the derivative, we plot the payoff of the derivative corresponding to each of these values. This is called the payoff diagram. A pure function is the mathematical representation.

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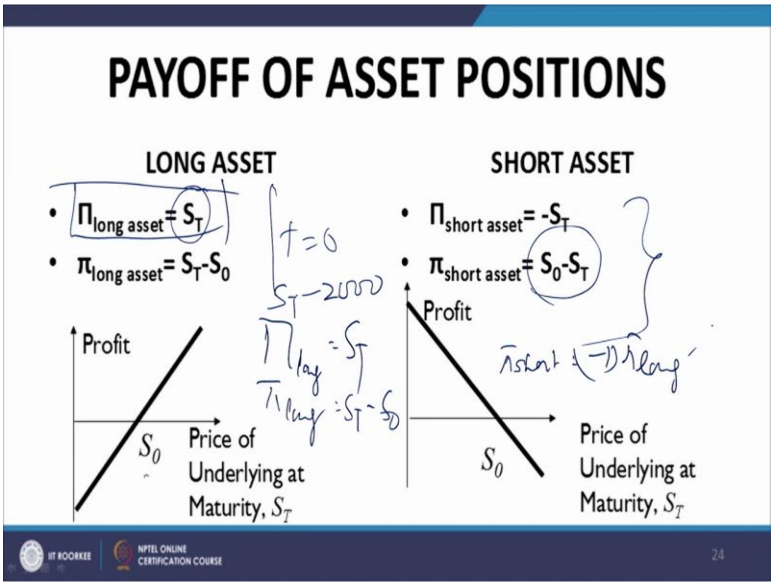

For example, payoff of a long asset position is suppose you have bought an asset, bought a share of let us say a ril at t equal to 0 for a certain price, let us say INR 2000 and you want to work out the profit or the payoff that is going to be generated when you sell the asset 3 months hence, which is t equal to capital T, which is your maturity period or which is your investment horizon.

So, now, these the price of the ril share at t equal to 3 months could is a random variable, let us say it could take any value between 0 and say infinity, the (corre) the corresponding payoff would be what it would be equal to the price at which you are going to sell the asset in the market.

Therefore, the payoff of a long position in a share is equal to the gain prevailing market price of the asset which we call is capital T, capital T represents the maturity S represents the price of the asset. So, S capital T is the price of the asset at maturity a random variable.

Please note, this is the access which is done at t equal to 0. So, we do not know what the exact value as t is going to take, but we have a feel of what possible values for example, 0 to infinity that ST could take and then corresponding to that the that is the payoff from holding one share of ril.

Now, suppose I had bought the share at ril at 2000, then what would be my profit, my profit would be whatever the price is at maturity that is ST, ST minus 2000 because 2000 is my cost, this is called the profit function S pi long is equal to ST this is the payoff and small pi long is equal to ST minus is S0, where S0 is the price that I paid for buying the asset.

So, the net profit is ST minus S0. This is for a long asset position. And the inverse would be for a short asset position, what is the short asset how will you create a short asset position, you borrow the asset, sell it in the market at t equal to 0, you borrow the asset, sell it in the market at t equal to 0, your intention is that at the end of the investment horizon, you will buy the asset from the market in anticipation of the prices having gone gone down. So, you will buy the asset at a cheaper price and replenish the asset to the original owner of the asset. And in that case, obviously, this would be the payoff and the price functions and the profit functions I am sorry.

So, minus ST would be the payoff, because you buy the asset at maturity and deliver it to the original owner so that will entail a cash outflow ST is what is minus ST and the profit would be S0 minus ST the price at which you have sold the asset at t equal to 0 minus the price at which you have bought the asset at t equal to capital T for replenishing to the original owner.

So, the small pi of a short position the profit function of a short position is equal to S0 minus ST and the payoff is equal to minus ST. It is interesting to note here that pi of short is equal to minus 1 into pi of long that shows that this is a zero sum game.

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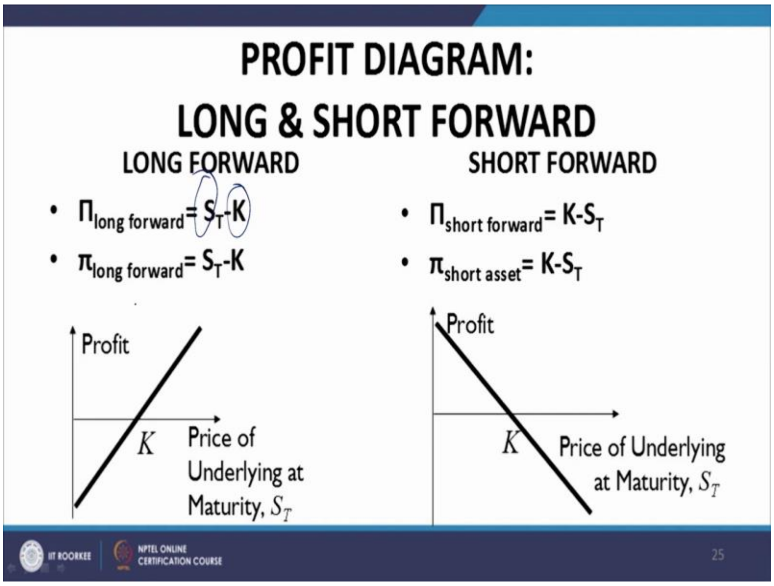

Profit diagram of long and short forward position, what is the long forward position a long forward position is is a position where the holder of the forward contract or the party to the forward contract who has long has the has the right to buy the asset at a predetermined price let us the predetermined price be K.

So, as on date of maturity of the forward contract, the long position will receive the asset and pay a price K and if the whatever is the market price of the asset as on date of the maturity, that is called ST though you can sell that asset in the market at ST and payoff the forward price K.

So, his net payoff it ST minus K which is also its profit then for the part you short in the forward contract who was delivering the asset, he has to buy the asset from the market at ST and deliver it to the long position or the long holder of the forward contract and he will receive a price of K. So, his profit function is K minus ST his payoff function is also K minus ST.

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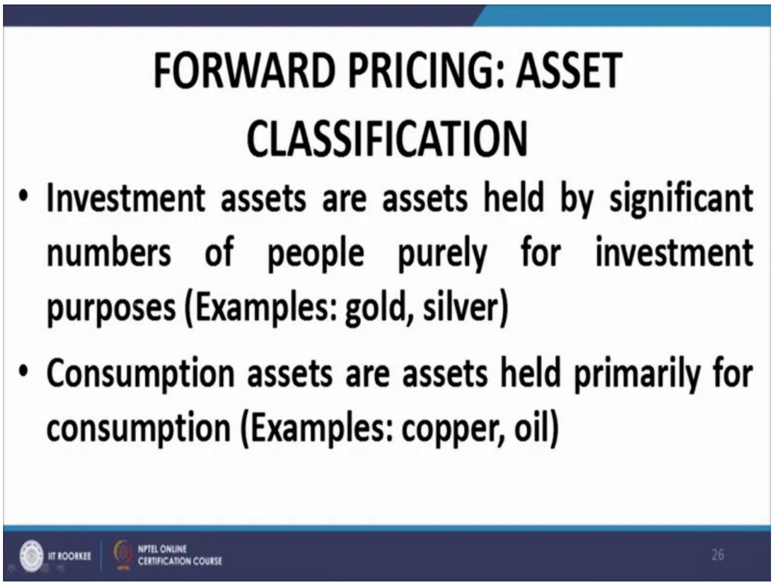

Now, we talk about forward pricing, arbitrage free pricing under a forward contract. For that purpose before we proceed further, we need to classify assets into two types of assets; number one is investment assets number two is consumption assets. Why the rationale behind considering the pricing of our contracts of pricing forward contracts with this discrimination of investment assets and consumption assets will become clear to you as a progress with the pricing.

But basically let us not define the investment assets and consumption assets, investment assets are assets held by significant members of people purely for investment purposes like gold, silver shares and so on. Consumption assets are assets held primarily for consumption like copper, oil, coal, commodities, edibles, oils, and so on.

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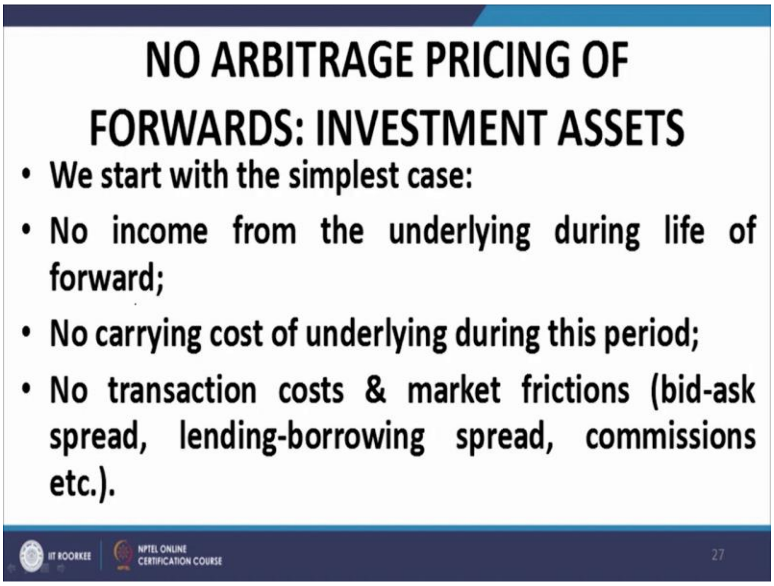

No arbitrage pricing of forward investment assets. First of all, we will consider the arbitrage free pricing of forward contracts arbitrage 3 pricing under forward contracts, or investment assets, we make some fundamental assumptions, some simplifying assumptions, no income from the underlying during the life of the forward contract, no carrying cost of underlying during the life of the forward contract, no transaction costs and market frictions, no bid ask spreads, no lending borrowing spreads and no commissions.

So, these are some simplifying assumptions we now consider the simplest model for the pricing under a forward contract of an investment asset. Let us say it’s a share.

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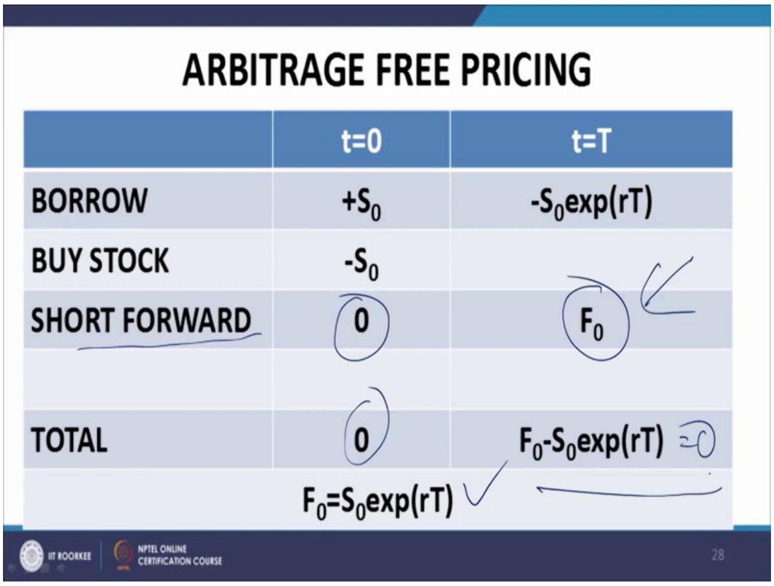

So, let us look at this table here. This is t equal to 0, this is t equal to capital T. At t equal to 0, I undertake some transactions, what are those transactions, I borrow a sum of money S0 from the bank at a certain rate of interest, let us call it r, we shall be more clear on what r represents as we progress along this derivation.

So, S0 is the amount that I borrowed, what is S0, by the way, S0 is the current market price of one unit of the underlying asset S0 is the current market price t equal to 0 market price of the underlying asset of one unit of the underlying asset.

So, we borrow an amount S0 from the bank at r percent per annum, continuously compounded and we use this S0 which we borrowed from the bank to buy one unit of the underlying asset, which we keep with us in safe custody against this position or against this holding of the underlying asset which I have with me in my locker, I write a forward contract and take a short position in our contract.

Remember, what is the short position in a forward contract, a short position in a contract involves the delivery of the substratum delivery of the underlying asset in exchange of the forward price, let K be the or let F0 rather let F0 be the forward price which is agreed at t equal to 0. So, this this short forward envisages what it envisages more (trans) no cash flow the t equal to 0. So, it is 0 here.

But at the maturity that is t equal to capital T, it will involve what it will involve delivery of one unit of the underlying asset and in exchange they are of the party who is long in the forward contract will give me a price which I call F0, which is agreed at t equal to 0 and hence the suffix is 0.

Now, at maturity t equal to capital T what will happen, I will deliver the underlying asset which I have with me in my position in my locker, I will deliver it to the party who is long in the forward contract against that the party who is long in the forward contract will give me the forward price, which is F0 out of this F0 what will I do, I will repay the amount that I have borrowed from the bank together with interest.

So, the amount that I have to repay to the bankers S0 e to the power rT where r is the continuously compounded rate of return rate of interest on the borrowing from the bank. So, the total cash flows if we work it out at t equal to 0 is equal to 0 and t equal to capital T it is equal to F0 minus S0 e to the power rT on the presumption that the entire process is risk free.

What happens if because there is a cash flow of 0 at t is equal to 0 arbitrage free pricing necessitates that the cash flow at the maturity of the derivative contract, maturity of the forward contract must also be 0.

Hence, this must be 0 then that gives us this formula for the forward price. Please remember this is the arbitrage free price under a forward contract, given the assumptions which we have already talked about. And there is another catch here, there is another catch is, there is no default risk of any leg of this transaction.

The entire process is default free, and as a result of which, as a result of which because my initial cash flow is 0 it must necessarily be true that my final cash flow should be 0 otherwise arbitrage process will get invoked and the usual implications would be that the prices will realign themselves so that this equality is satisfied.

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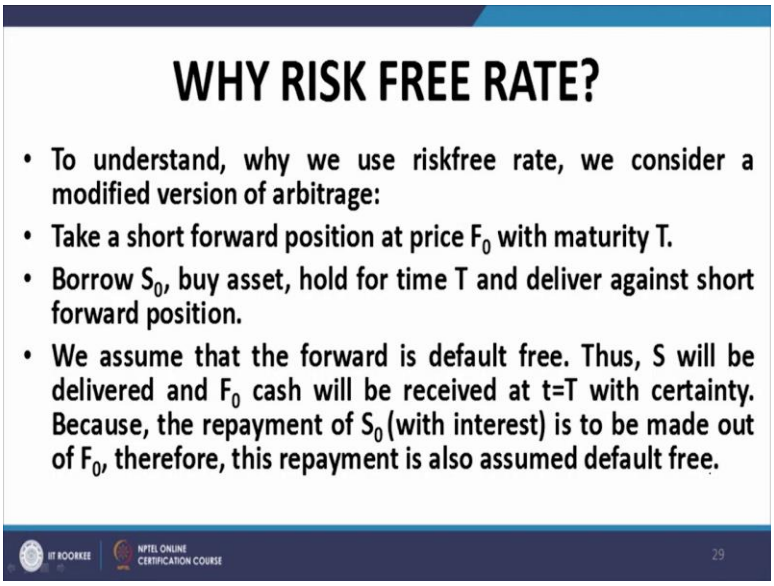

Why the risk free rate? Now, this is interesting, what is r, r is the risk free rate, r is the risk free rate, why r is the risk free rate, you see what is happening. In this case, what was the process we borrowed an amount from the bank, we bought the asset from the market and then we wrote a short forward contract on the asset.

Now, the important thing is we are assuming that the forward contract is default free. If the forward contract is default free, it necessarily implies it necessarily means that we are certain to get the amount of F0 at maturity, we already have the asset with us in our possession, so we can deliver it without default.

So, that leg of the transaction that is as far as the forward is concerned on the premise that the counterparty is not going to default it is entirely default free. And because I have the amount of money F0, and that amount is going to be used for the repayment of my borrowing from the bank, this portion is also default free, and as a result of which the entire process is default free and the rate of interest the bank is going to charge me must therefore be the risk free rate that is the rationale behind using the risk free rate in the context of forward pricing.

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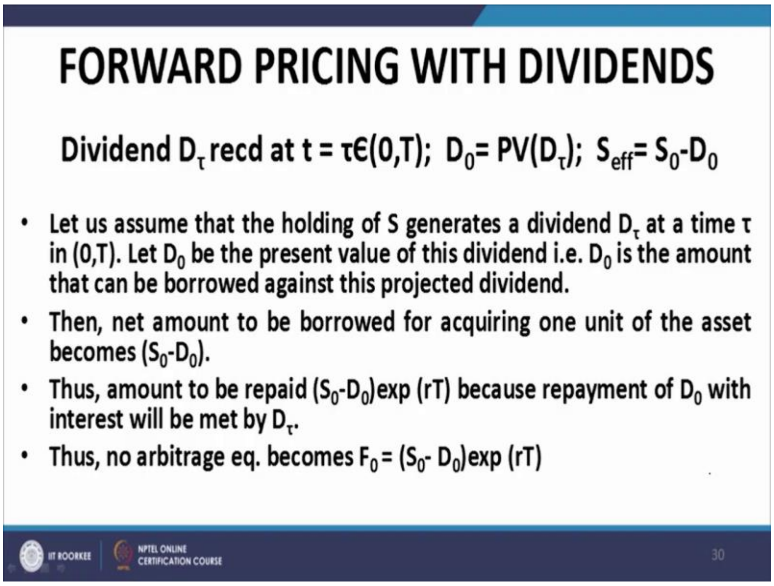

Now, forward pricing with dividends, what happens if during the life of the forward contract, a dividend is envisage a dividend is expected to arise out of the holding of the underlying asset, then what happens, let us see how it impacts the pricing process of the forwards.

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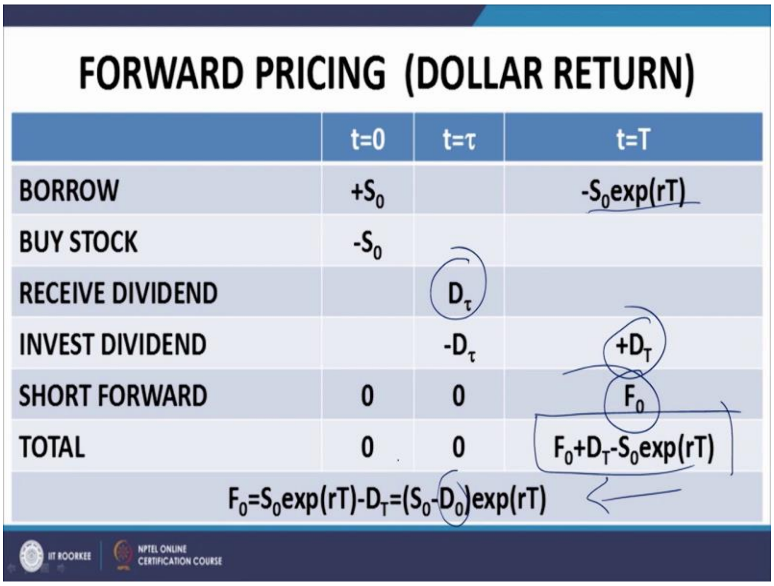

FORWARD PRICING (DOLLAR RETURN)

	$t=0$	$t=\tau$	$t=T$
BORROW	$+S_0$		$-S_0\exp(rT)$
BUY STOCK	$-S_0$		
RECEIVE DIVIDEND		D_τ	
INVEST DIVIDEND		$-D_\tau$	$+D_T$
SHORT FORWARD	0	0	F_0
TOTAL	0	0	$F_0+D_T-S_0\exp(rT)$

$$F_0=S_0\exp(rT)-D_T=(S_0-D_0)\exp(rT)$$

Let us look at this table again. It is pretty much similar to what we had in the previous case, now, again, we borrow we assume that the spot price of the underlying asset per unit is equal to S0. And we borrow an amount of S0 at the rate of return r rate of interest r continuously compounded from the bank and use the same amount S0 to buy one unit of the underlying asset which we keep with ourselves.

So, borrowing S0, using that amount S0 for buying the asset means the cash flow has become 0 at t equal to 0 and at the same time I have one unit of the underlying asset in my custody in my possession in my ownership.

Now, I write a forward contract on this particular one unit of the underlying asset that I have in my ownership, then what happens because I have a holding, I have ownership of this asset I will obviously get it transferred in my name in the books of the company.

Therefore, if the, if and when the company declares dividend during the life of the forward contract, that dividend would accrue to me, that dividend would accrue to the party whose name appears in the register of members of the company and that would be me because I have bought the asset from the market at t equal to 0, and I am holding it in my possession.

So, I will get an amount that is an inflow that will occur at let us say t equal to tau, where tau is any point in time between 0 and capital T capital T is the maturity of the forward contract, 0 is today and tau is some point in between 0 and capital T, what I do is I reinvest this amount

D tau for the remaining life of the forward contract at the same interest rate r and what I will get is DT, what I will get is DT at maturity.

So, let us see what happens at maturity now, at maturity, I have got one unit of the underlying asset with me, I will deliver that asset under my short forward position against that I will receive the forward price F0 please note I repeat again, this F0 is agreed at t equal to 0 then therefore the suffix 0 is is the attached to this F, F represents the forward price at what point in time at t equal to 0 for a maturity of t equal to capital T.

In addition, I will get the amount that I have invested for dividend that is D capital T the future value of the dividend from tau to capital T for an investment from tau to capital T and out of this that is DT plus F0, I have to pay off the borrowing that I have made at t equal to 0 or buying one unit of the asset in the market that together with interest and that will become is S0 exponential rT.

So, again, because the entire set of transaction is default free entire process is default free. Therefore, the and the cash flow at t equal to 0 is 0. There is no intermediate cash flow either whatever intermediate cash flow we had received was invested forthwith and therefore the cash flow at t equal to tau is also 0. There is no other intermediate cash flow. Therefore, the maturity cash flow must also be 0 because it is a set of risk free transactions.

So, that gives us this equation and on simplifying this equation, you get this expression for the future price of an asset, which has which delivers or which is likely to deliver a dividend during the life of the forward contracts.

Please note what is this D0, D0 is the present value of D tau D capital T let me repeat two things D tau is the actual dividend that you receive at t equal to tau D capital T is the future value of that dividend that you receive at t equal to tau calculated at t equal to capital T and D0 is the present value of the dividends that you receive at t equal to tau calculated at t equal to 0.

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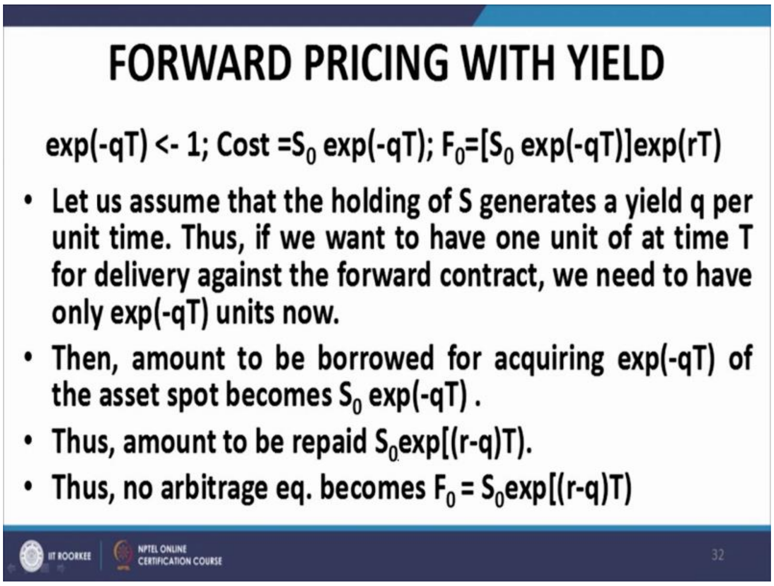

Forward pricing with yield very often it it happens particularly when we talk about (interest rate) sorry, index futures and such other thing that we instead of assigning a particular amount of profit on holding futures contracts, we assign a yield that is going to be likely to occur on the underlying index and even for spot indices even for the let us say BSE index or the NSE nifty and so, on, it is customary that we use the yield exponential continuously compounded yield instead of the value the absolute value for specifying the return on holding or taking a position in this indices.

So, under these circumstances, when you are given a continuously compounded yield on the on your holding of the underlying asset, what happens, what is the forward pricing that is our next exercise. So, let us assume that the holding S generates a yield q over per unit time therefore, if you want to have one unit at time t you have for delivery against the forward contract, you need to have only exponential minus qT units now, or alternatively, if you have one unit of the index or the stock at t equal to 0, the one unit will grow to e to the power qT units at t equal to capital T given the yield as q per unit time. So, that is what we are going to make use in the pricing process.

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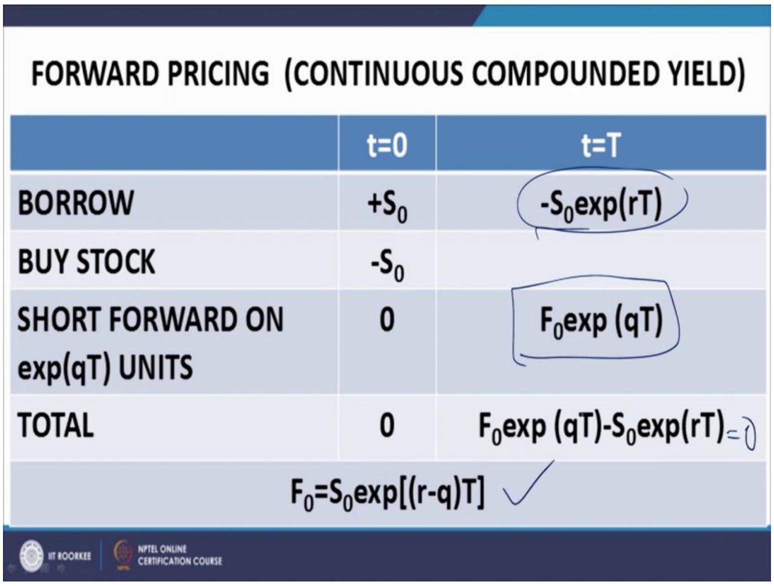

Again, we look at this table, we borrow a certain amount of money S0, which is the current price of the index or the stock let us call it stock for the sake of simplicity, but we are given a continuously compounded yield on the stock instead of an absolute dollar value of dividends. So, instead of amount of borrow at r percent per annum continuously compounded by one unit of the stock. So, the net cash flow at t equal to 0 is 0.

And now is the catch again in this holding of one unit of the underlying assets that you have with you that you have in your position, you write a forward contract you take a short position in the forward contract not for one unit, you take a short position in e to the power qT number of units of the underlying asset.

Why because this one unit that you have bought at t equal to 0, will grow to e to the power qT units at t equal to capital T where q is the rate of growth per unit table the continuously compounded yield as you call per unit time.

So, this t equal to 0, one unit will grow to e to the power qT unit and therefore, you can deliver e to the power qT units against your short forward position. Therefore, you write short forward contract or take a short position in a forward contract not against one unit, but against e to the power qT unit.

So, what will be the majority cash flows because you have a short forward position for e to the power qT units, the payment that you will receive against the short position will be equal to F0 e to the power qT where F0 is the price per unit of the underlying asset in the forward market for maturity at t equal to capital T.

So, instead of F0, you will now get F0 e to the power qT and against this, we have to pay back the loan with interest and that is S0 e to the power rT. Again on the same premise as we have discussed earlier, the cash flow at t equal to 0 as 0, there is no risk embedded in this set of transactions and therefore we assume that the forward is default free and therefore, the maturity cash flow must also necessarily be 0, then that gives us this equation that gives us this equation.

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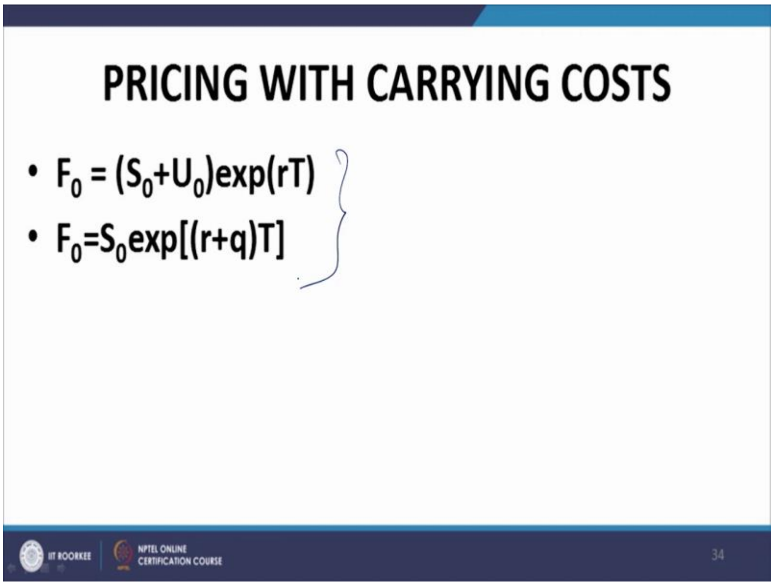

Now, if instead of dividends we have carrying costs, the sign of the D0 will be reversed and the sign of y will be reversed then we get these equations, we get these expressions in the event in the event where instead of cash inflows by virtue of holding the asset, we have to pay a cost, for example, the godown rent, for example, insurance costs, protection against embezzlement, evaporation of the of the underlying liquid or fluid and such other expenses that may arise by holding the underlying asset in your possession in your ownership, then the signs will be reversed, no other change in the in the derivation process.

So, these equations will take over instead of S0 minus T0 exponential rT and S0 exponential r minus qT, we will have S0 plus U0 exponential rT and S0 exponential r plus q into T.

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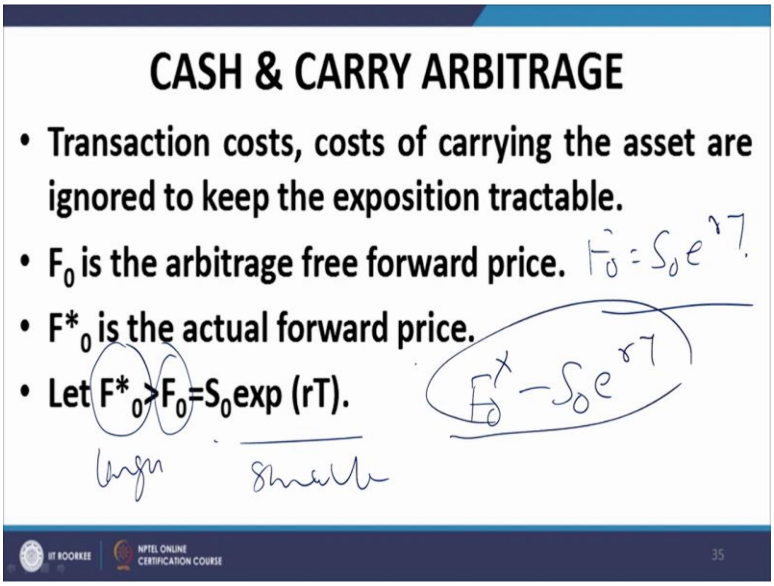

Now, we have worked out certain formula on the basis on the basis of the assumption that there there there will be no arbitrage and if there is a differential then arbitrage will converge the two legs that is a spot leg and the forward leg together so that we in the arbitrage free situation, we have these values, which we would have seen earlier, F0 is equal to S0 e to the power rT. This is the situation which would arise when we are in equilibrium without any arbitrage opportunities.

The next question that I want to address is that if this equality is violated, what would be the arbitrage, how would arbitrage actually activate itself? What would be the steps that an arbitrager will undertake in order to profit from this mispricing? That is the next question.

So, let us assume that the actual forward price which I write by F0 star, which is the actual forward price, which is the forward price prevailing in the market corresponding to a negotiation at t equal to 0 and a delivery maturity at t equal to capital T and F0 is the arbitrage free forward price, which is equal to S0 e to the power rT, let us ignore for the moment dividends and carrying costs and yields and so on. Let us keep that (tract) traction simple.

So, let us assume that for the moment F0 star, the actual market price in the forward market is higher than the no arbitrage or the arbitrage free price corresponding to the same delivery date of the same underlying asset, then how does arbitrage takes place the clearly this is larger and this is smaller.

So, we buy this smaller leg and we sell the larger leg, how do we do it? It is quite simple by this smaller leg means what buy the borrow the money S0, buy the underlying asset and write a forward contract on that for delivery of the underlying asset.

On maturity what happens you deliver the underlying asset you receive F0 star please note this that is the actual forward price and you receive the actual forward price you receive F0 star you pay what you pay S0 e to the power rT that is the amount of S0 that you borrowed together with interest thereon. So, your profit is equal to F0 star minus is S0 e to the power rT and from the assumption that we have made here this is greater than 0.

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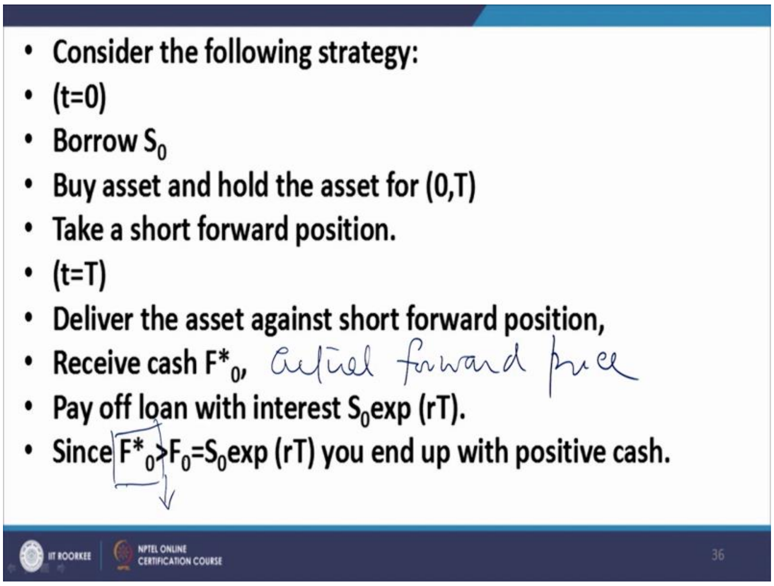

So, let me know go through a step wise, consider the following strategy at t equal to 0 borrow S0 buy asset and hold the asset for 0 to capital T take a short forward position at t equal to capital T what happens deliver the asset against the short forward position receive cash F star 0 the actual forward price pay off the loan with interest S0 e to the power rT and you because F0 star is greater than F0 e to the power rT, you make an arbitrage profit.

So, what will happen everybody will start doing this and this particular quantity F0 is greater will will decrease because everybody will start start writing forward contracts and buying in the spot market and this 0 this will increase and as a result of this differential will gradually be wiped out at equilibrium.

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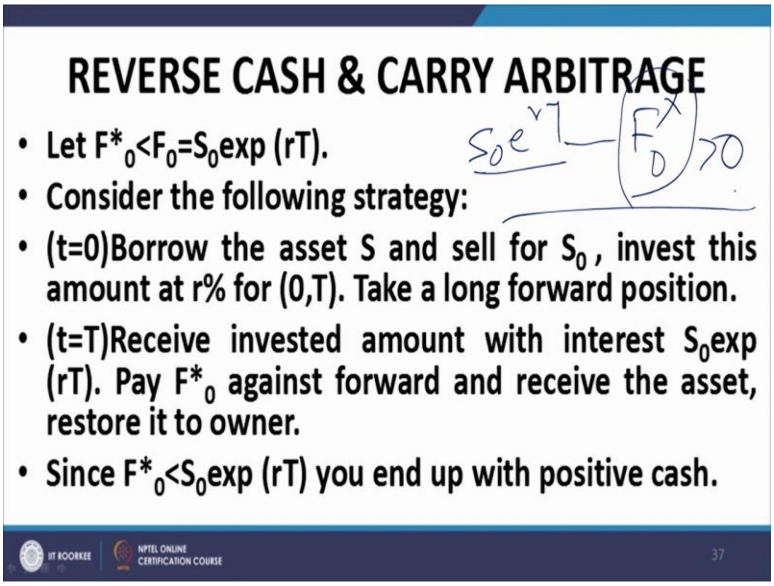

Reverse cash and carry arbitrage. What happens is F0 star that is the forward price actually prevailing in the market is less and the no arbitrage price or the arbitrage free price corresponding to the same underlying same maturity is higher, then we do the reverse strategy, what do we do? We take a long position in the forward contract and against that long position in the forward contract, you borrow the asset at t equal to 0 sell it in the market at t equal to 0 use the proceeds that is S0 to invest at the rate of r for maturity at capital T.

At maturity what will happen, because you are long in the forward contract you will receive the asset from the party and you the asset that you will receive you will replenish it to the party from whom you have borrowed and as far as the investment amount is concerned, you will receive So e to the power rT and against the long forward position, you have to pay a price for receiving the asset of this much, please note the asset that you have received will be replenished to a person from your borrow rate.

So, that cancels out and the price that you have received in the forward market you have paid please note this is the price that you have paid in the for the actual forward market when you took the position at t equal to 0 and this is the proceeds of the investment when you made at which you made at t equal to 0.

So, my assumption this is greater than 0. So, here again you make a arbitrage profit. So, that is as far as cash and carry arbitrage is concerned and reverse Cash and Carry arbitrage is concerned in the reverse Cash and Carry arbitrage please note what we do is we take a long forward position that means we are going to buy the asset in the forward market at the actual for price that is F star 0.

And against this you sell the assets that you have you take a short position in the asset, you borrow the asset from somebody you sell the asset in the market you get an inflow of S0 at t equal to 0, you invest that inflow at t equal to 0 at the rate of r percent so that it becomes S0 e to the power rT at maturity from this, against the long position you have to pay the price of getting the asset which is F0 star.

So, the net proceeds net cash that is available for you is S0 e to the power minus F0 star and the assets that you will receive under the long forward position will be used for replenishing the person from replenishing to the person from whom you had borrowed the asset at t equal to 0. So, that is how reverse Cash and Carry arbitrage operates. So, I will continue from here in the next lecturer. Thank you.