Quantitative Investment Management
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Lecture 21
Effective Duration

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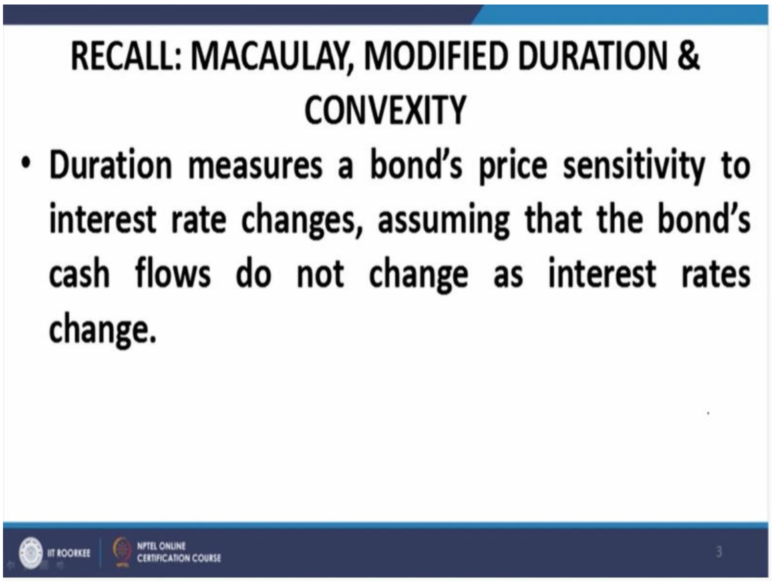

Welcome back. So, let us continue. We start by doing a brief recap of the important points that we have done, discussed so far in the context of interest rate risk. Duration, which is the most important measure of interest rate risk, it measures the bond's price sensitivity to interest rate changes, assuming that the bonds cash flows do not change, as interest rates change.

In other words, we are not talking about floating rate bonds, we are talking about fixed rate bonds, where the cash flows are given by the contract of issue and they do not depend on the changes in interest rate, they of course depend on the contracted interest rate, the interest rate that is specified in the offer document. And duration measures the percentage change in price, corresponding to the given change in the interest rates or the YTM for that matter.

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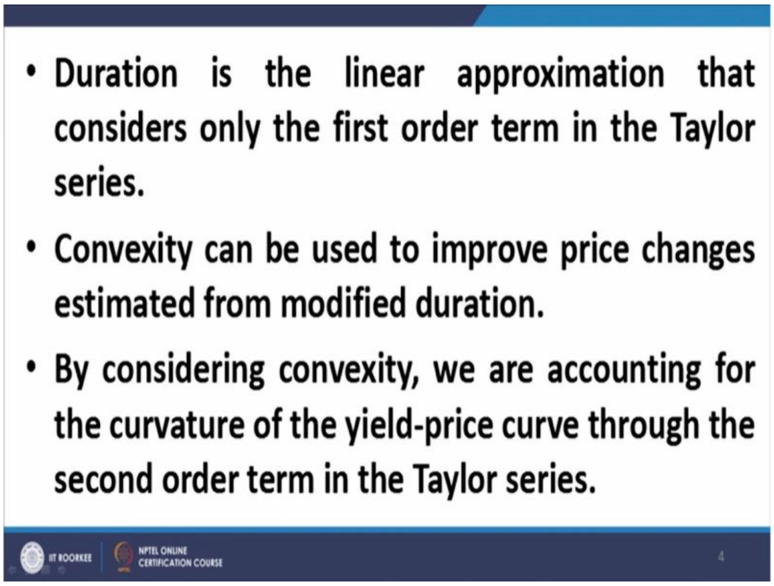

Duration is the linear approximation; I explained this point very carefully in the last couple of lectures. Duration is a linear approximation that considers only the first order term in the Taylor series. When you expand the price of a bond around its YTM and you retain the first order term, you get what is the definition of the duration of the bond.

Convexity is the second order correction. It is used to improve the results that we obtain by using duration as a measure of the percentage price change corresponding to a given change in YTM. The important thing here is that both the duration and convexity relate to the curvature of the yield price curve. Duration assumes that the yield price curve is a straight line in the neighborhood, in the close nexus of the point at which the YTM is originally given and infinite decimals or small changes in YTM are being considered.

Convexity improves upon this by capturing some portion of the convex nature of the yield price curve. I need to emphasize that the yield price curve is not a straight line, it is a convex curve and the convexity is captured by the second order term in the Taylor series. And that gives rise to the definition of convexity. By considering convexity we are accounting for the curvature of the yield price curve to the second order term in the Taylor series.

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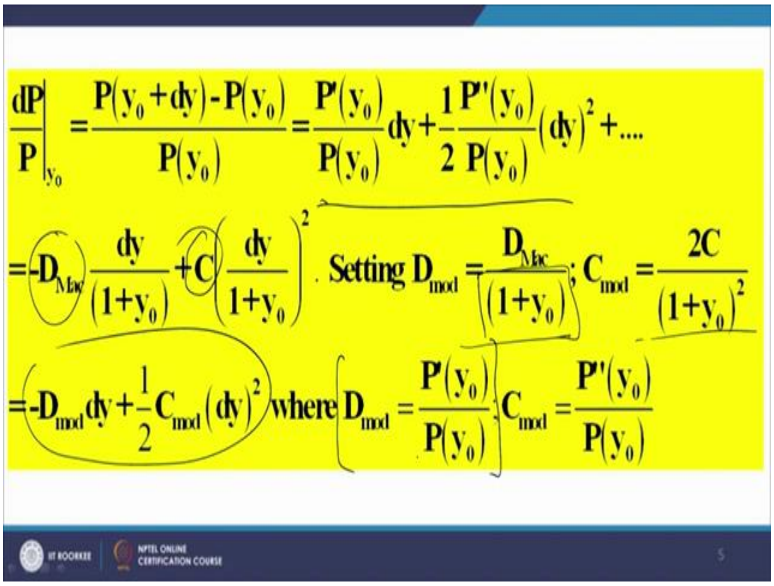

This is what we have, what I have mentioned in the last couple of minutes. If you expand the price of the bond around y0 as a Taylor series and divide throughout by P of y0, what you get is the right-hand side here, this is what you get. The first term enables us to define that duration and the second term enables us to define the convexity. D mac refers to the first derivative of the price with respect to y0 or with respect to y. And the C, that is the convexity refers to the second order derivative or a second derivative of the price with respect to Y.

The definition of duration are given by these expressions. When we talk about the modified duration and the modified convexity, modified duration absorbs the factor of 1 plus y0 in the definition of duration itself. And similarly modified convexity absorbs the factor of 1 plus y0 square in the definition of convexity itself.

In terms of the modified duration and the modified convexity, we can add the percentage price change in the form of this expression. This is a very important expression. Please note the prefactor of 1 by 2. This is the convention adopted by some of the writers, some part of the literature segregates of this 1 by 2 from the definition of the modified convexity and shows it as a prefactor.

So, I have used that convention, while defining modified convexity. However, when you define convexity, you include this factor of 2 within the definition of convexity, this has reference to the notation that is used in most of the textbooks, in particular, the textbook by Elton and Gruber, on security analysis and portfolio management. So, these are the basic definitions of modified duration.

Again, as I mentioned, they capture the first, duration captures the first order derivative of price with respect to Y and convexity captures the second order derivative. Convexity captures the curvature of the yield price curve; duration assumes that the yield price curve is a straight line in the immediate neighborhood of the point at which the interest rate shift is being considered.

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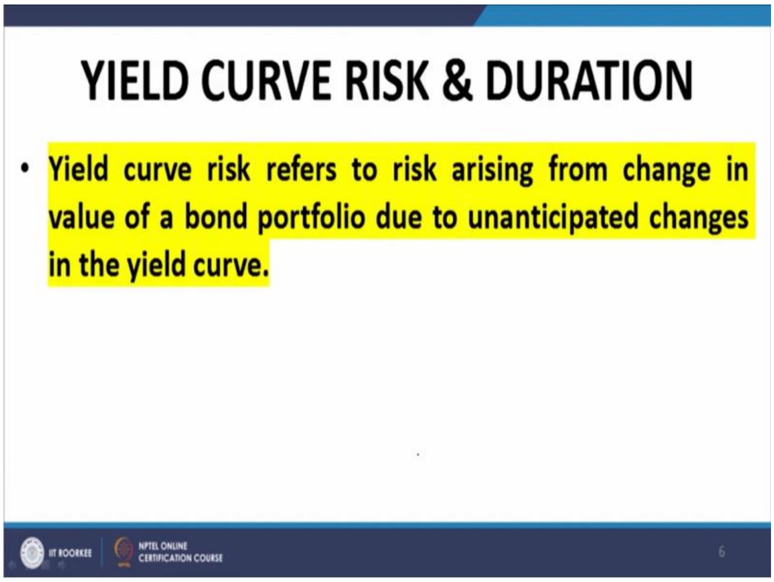

Then I moved over to yield curve risk and duration. Yield curve risk, refers to the risk arising from change in the value of a bond portfolio due to unanticipated changes in the yield curve. Please note, this word unanticipated as a special significance here. We use the term unanticipated because if there is an anticipated change that would be recognized by the market, that would be absorbed by the market in the pricing process and as a result of it that loses its relevance. However, if the changes is unexpected, if the change is unanticipated, then of course, that is what contributes to the risk in essence.

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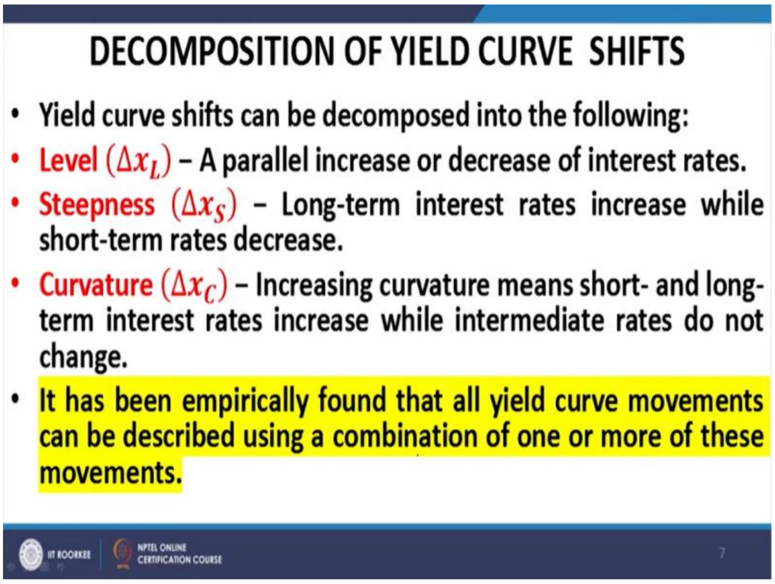

Now, based on empirical observations, the yield price shifts or the yield price movements can be broadly classified into three categories. We have the level shift which has a parallel movement of the yield curve with respect to its original position.

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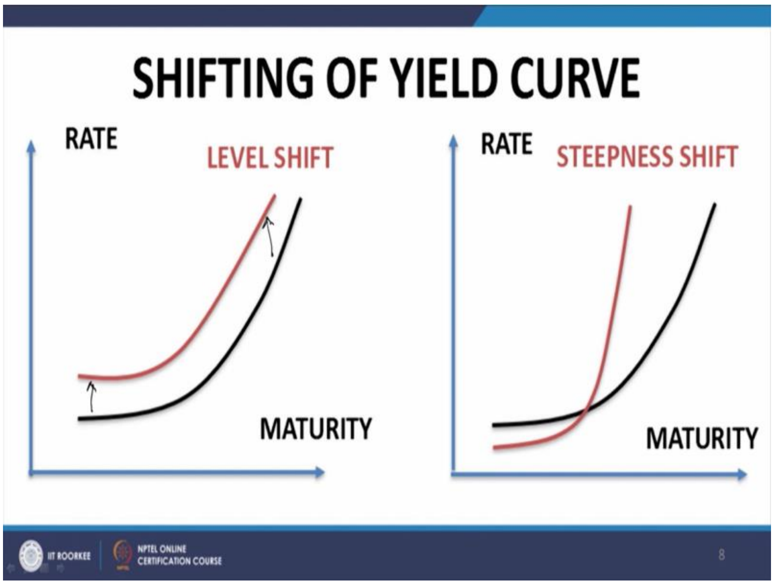

As you can see here, in the first diagram, in the left-hand panel, this is a level shift, the brown curve is the shifted curve from the other curve. And you can see that the shift has taken place parallel to itself. You can see here; this is the magnitude by which the interest rates have changed across the entire shift process. So, this is a parallel shift. This is called a level shift.

Then the second shift is called the steepness shift, where the short term or the short end interest rates decrease. The long end interest rates tend to increase so, that the yield price curve increases in steepness as you can see here, the brown curve versus the other curve which the other covers is much less steep, then the brown curve which is the more, which is the changed version or the shifted version of the yield price curve.

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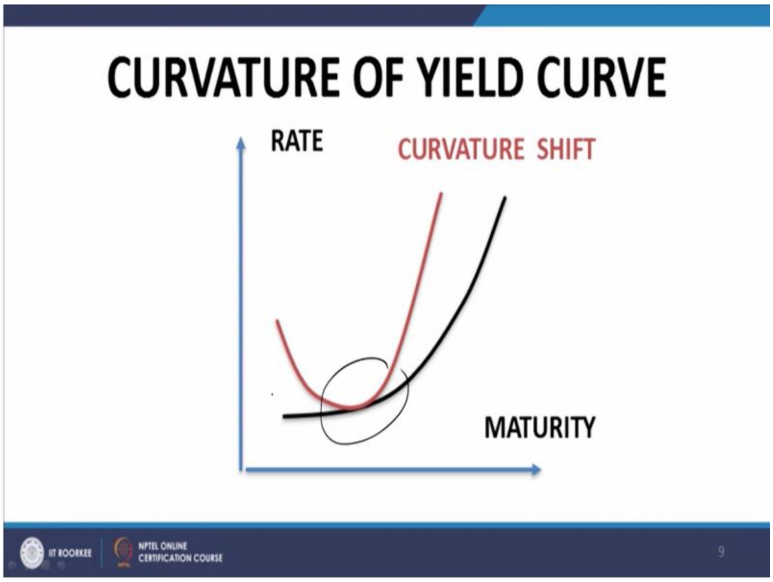

Then finally, we have the curvature shift where the short end as well as the long end interest rates increase whereas, the intermediate interest rates, interest rates around this portion remained more or less close to what they were originally in the original yield curve.

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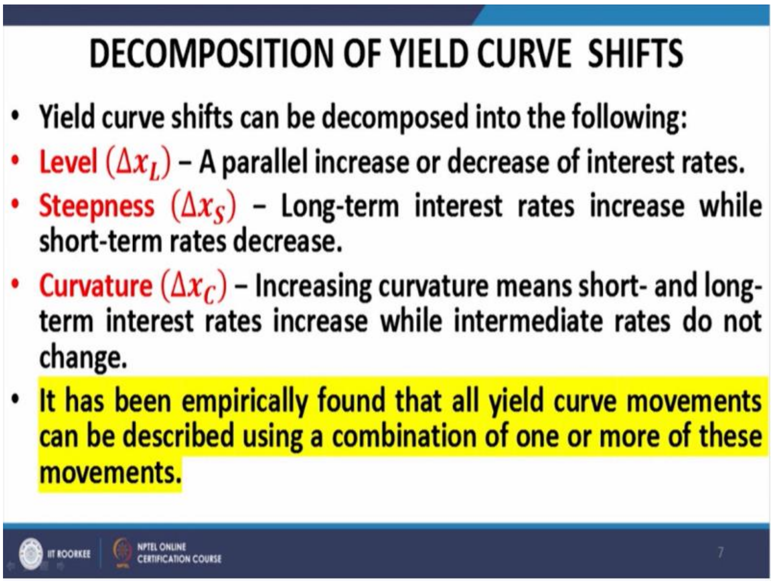

So, it is empirically found that all yield curve movements can be described using a combination of one or more of these movements. So, that is what I mentioned, empirical data indicates that the movements in the yield curve, the shifting of the yield curve or the evolution of the yield curve it takes the form of either one of these three change in level, changes in steepness or change in curvature or a combination of one, either the other two or one of them, or the two or three of them rather.

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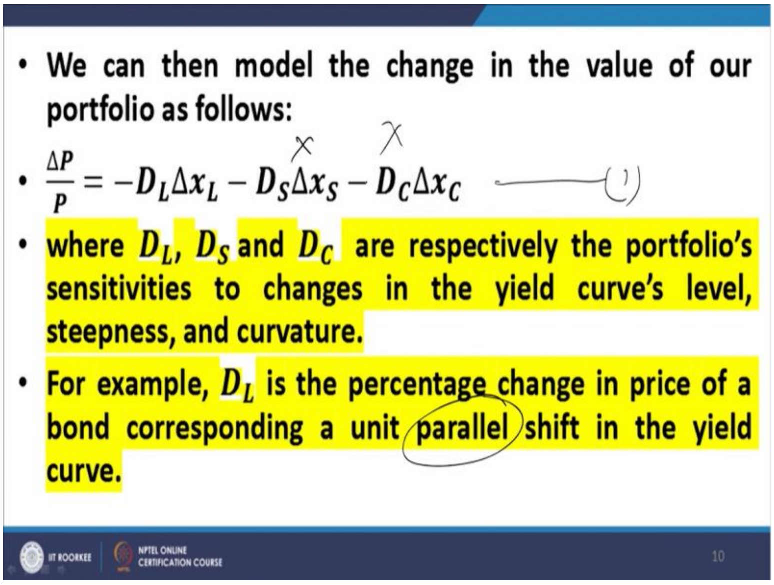

So, we can model the change in the value of our portfolio as follows. The percentage price change can be expressed in terms of this equation, it is a very important equation, it is called the three-factor model, where the three factors are the level shifts and the steepness shifts and the curvature shift.

The shift coefficients or the sensitivities as you may call them, captured by DL, DS and DC. So, these are respectively the portfolio sensitivities to changes in the yield curves level, steepness and curvature. For example, you may define DL as the percentage change in price of a bond corresponding to a unit parallel shift. This parallel word is very important that is what is called a level shift, the parallel shift in the yield curve. If you ignore these two terms, you get what is the definition of the modified duration of the bond.

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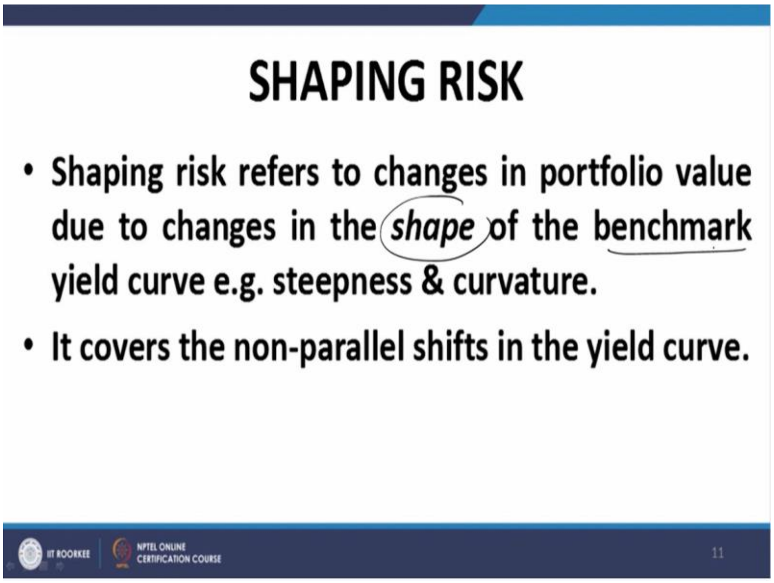

Shaping risk refers to the changes in portfolio value due to changes in the shape of the benchmark yield curve. For example, the steepness or the curvature. So, the changes in the portfolio value with reference to changes in the shape, shape of the benchmark give you what is called the shaping risk, it covers non-parallel shifts in the yield curve.

So, we are not talking about level shifts here, we are talking about non-parallel shifts. Non-parallel shifts are the benchmark of obviously contribute to changes in the percentage price of the bond and that is what gives rise to the shaping risk or that is what is called the shaping risk.

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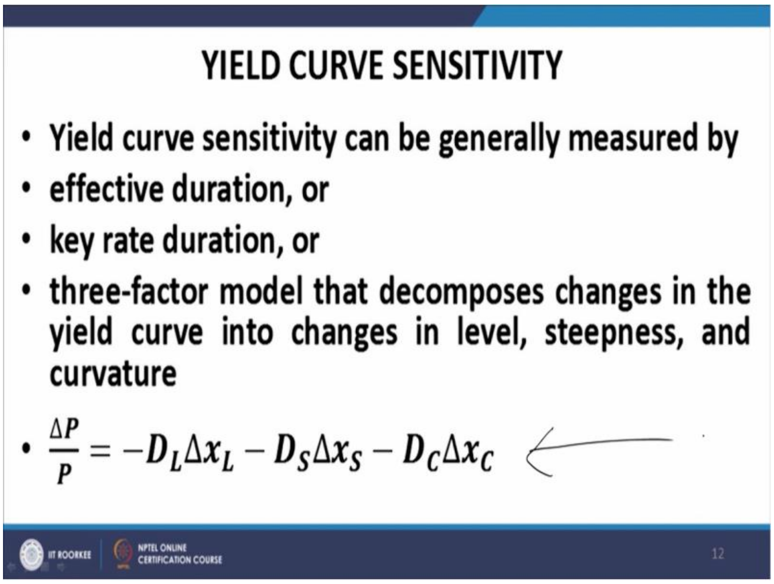

Then we have the yield price sensitivity. It can be measured in terms of the effective duration, which is analogous to modified duration. We have a new term which is called the key rate duration which I will explain later. And then, we have the three-factor model which I have introduced just now, you can recall this equation shown a couple of slides earlier. So, the three-factor model decomposes changes in the yield curve into changes in the level or steepness or curvature of that curve.

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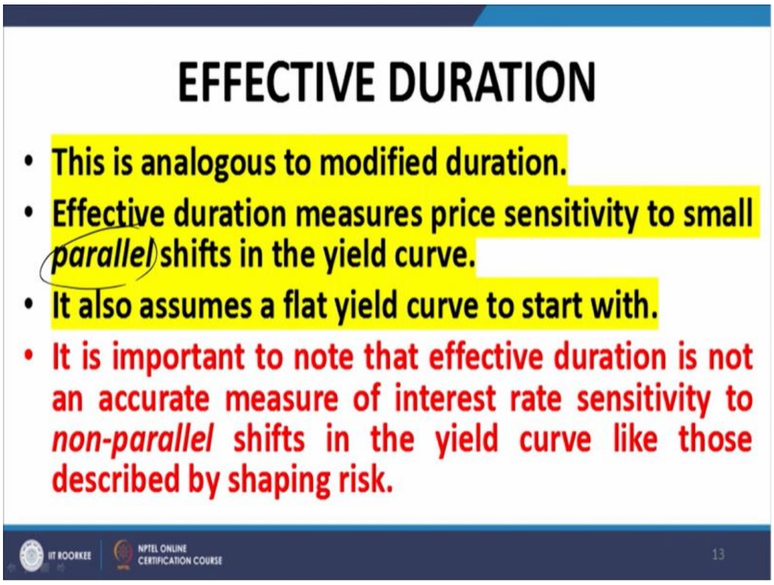

Effective duration is analogous to the modified duration. Effective duration measures the price sensitivity to small parallel shifts. This is what is important. It is a level shifts or parallel shifts that we are talking about here. So, effective duration measures the price sensitivity to small parallel shifts in the yield curve.

It also assumes a flat yield curve to start with, these issues will become more clear as we progress along today's lecture. But for the moment effective duration or modified duration, which are almost similar terms to capture the impact of level shifts in the yield curve on the price or the percentage price of the bond.

Now, it is important here to note that effective duration is not an accurate measure of interest rate sensitivity to non-parallel shifts in the yield curve, like those described by the shaping risk. So, for that purpose we have the concept of key rate duration or the three-factor model, I shall be talking about key rate duration in today's lecture. So, stay with me for the moment.

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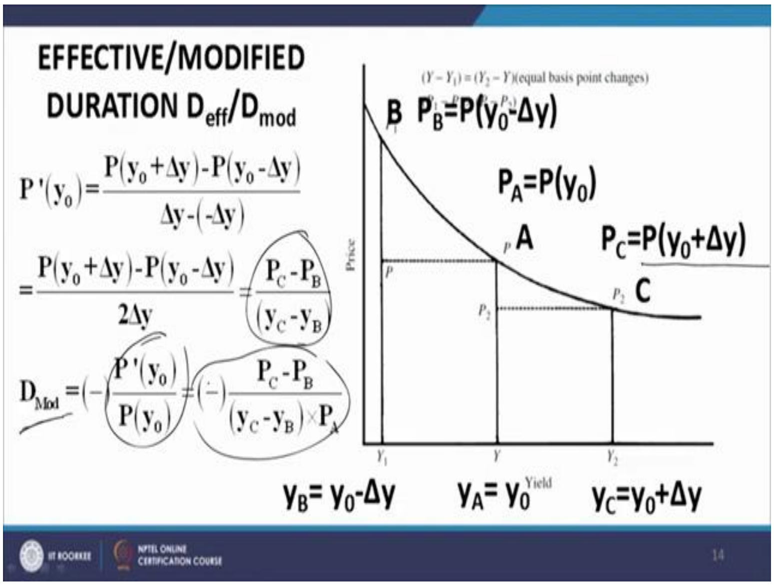

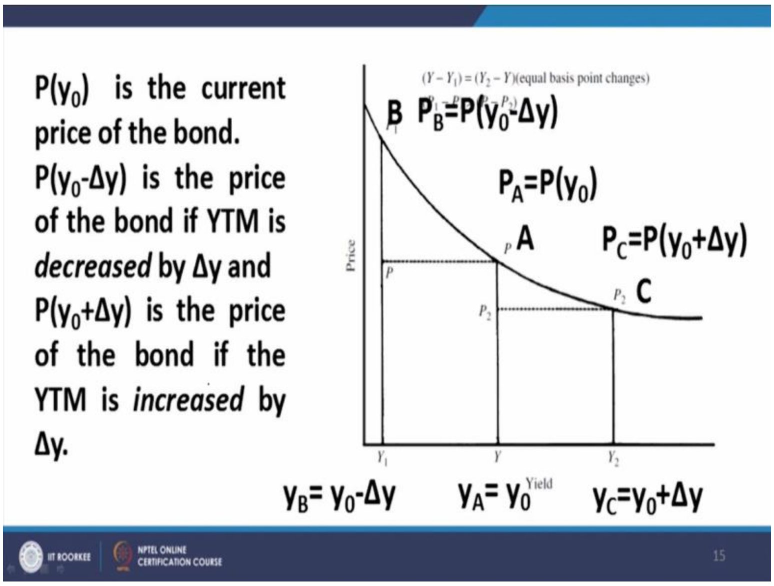

Now, this is the approximate calculation, I emphasized the word approximate calculation of effective duration or modified duration which are more mathematically identical. Now, the important thing here is the effective duration is given by P dash of y upon Py a negative sign as a prefactor and this quantity being evaluated at y0.

So, let me repeat D mod is given by minus P dash of y divided by P of y evaluated at the point y equal to y0 which is the point of reference at which we are computing the impact of the shift in interest rates. The point that needs to be emphasized with reference to this derivation is that we move in both directions, we take the average of movements of the yield in both directions, we first consider the movement of the yield in the positive direction by a small amount delta y.

So, y0 becomes y0 plus y0 plus delta y and the corresponding price is given by P of y0 plus delta y and we call it PC the point C represents the price at y0 plus delta y. The point A represents the price at y0 and the point B represents the point at y0 minus delta y. So, PC is equal to p of y0 plus delta y, PA is equal to P of y0 and PB is equal to P of y0 minus delta y.

So, to calculate P dash of y0, what do we do? We subtract PC from PB and we divide it by the total change in yield from the point C to the point B that is equal to delta y minus minus delta y. So that is 2 delta y. So that gives us this expression here and on substituting this expression of P dash of y into the expression for modified duration that we have here, we get the expression for the modified duration at this quantity. So, actually the derivation is quite simple, then let us go through the singular features of this derivation.

P y0, as I mentioned is the current price of the bond Py0 minus delta y is the price of the bond if the YTM decreases by delta y, and P y0 plus delta y is the price of the bond if the YTM increases by delta y. You can see this all this in the favor also, then it does not warrant much of explanation.

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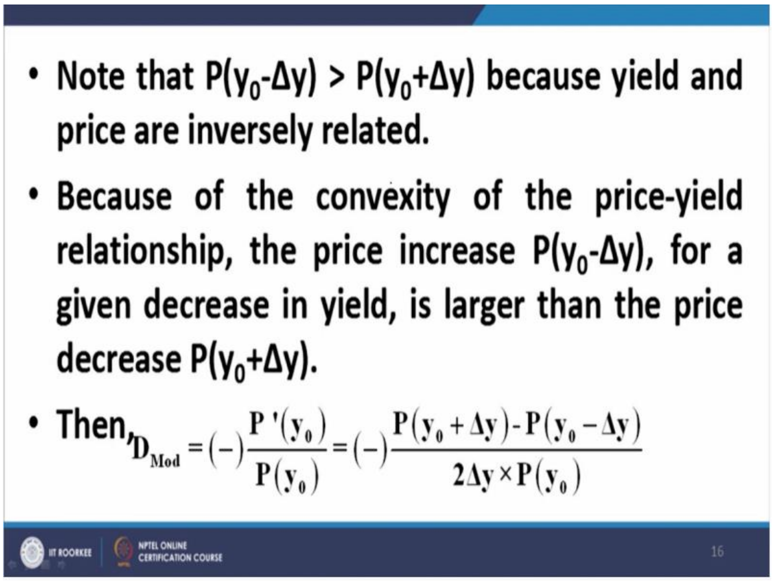

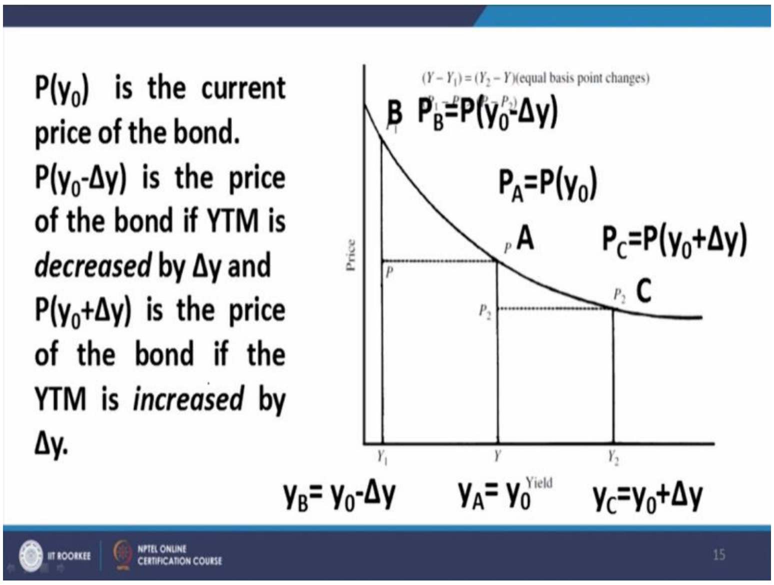

Now because of the inverse relationship between y that is the YTM of the bond and the price, we have P of y0 minus delta y is greater than P of y0. You can see here the slope is negative and you can see here that PB is greater than PC, because of the convexity of the price yield curve or the yield price curve, the price increase y0 minus, P of y0 minus delta y for a given decrease in yield is larger than the price decrease to P of y0 plus delta y corresponding to an increase in yield by delta y.

Again, you can see this in the slide. The, if when the price -- when the yield decreases from y0 to y0 minus delta y, the change in price is given by PB minus PA. And when the yield increases from y0 to y0 plus delta y, the change in price is given by PC minus PA. Obviously, the magnitude of PB minus PA is greater than that of PC minus PA. So, this is the convexity effect.

This means that the price change corresponding to a decline in yield is more or the increase in price corresponding to a decline in yield is more corresponding to, compared to the decrease in price corresponding to an equivalent increase in yield.

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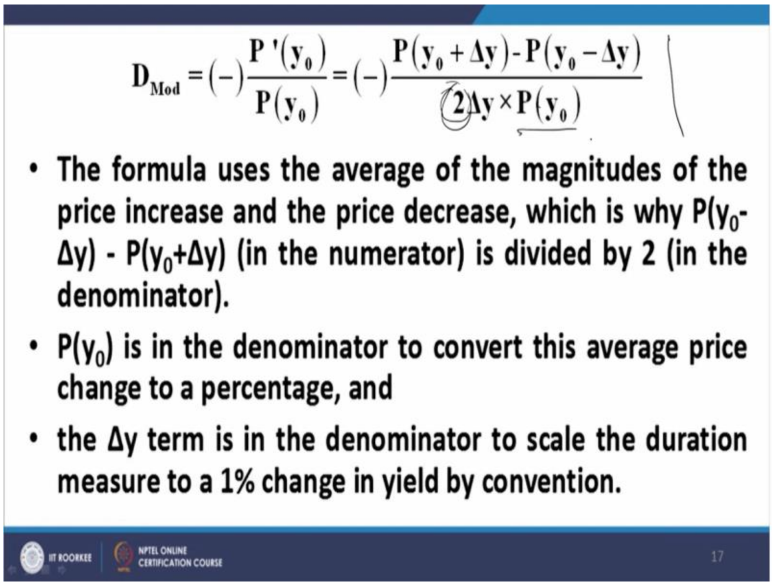

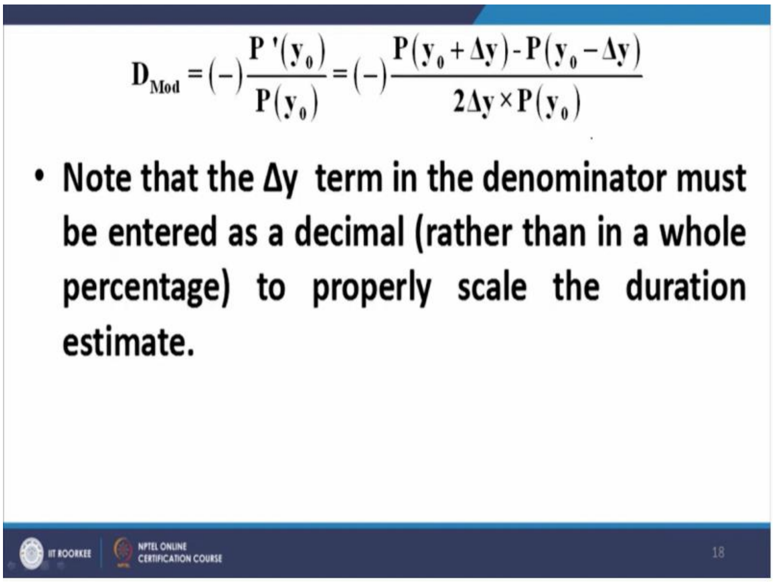

The formula uses the average of the magnitudes of the price increase and price decrease. You can see here in this expression; we are using the average, this 2 is the average in quantity. And the prices that are being considered are P of y0, P of y0 plus delta y and p of y0 minus delta y. So, it uses the average of the magnitudes of the price increase and the price decrease, which is why P of y0 minus delta y minus P y0 plus delta y in the numerator is divided by this 2 quantity. P of y0 in the denominator then this quantity then contributes to is retained or is introduced to convert the average price change to a percentage price change.

And finally, the delta y term in the denominator is used to scale the duration measure to a 1 percent change in yield which is the conventional definition of duration, modified duration. Note that that delta y term in the denominator must be entered as a decimal rather than as a whole number. For example, if the yield change is 1 percent, it should be 0.01 that you are introducing as the value of delta y and it should not be 1 percent to properly scale the duration estimate.

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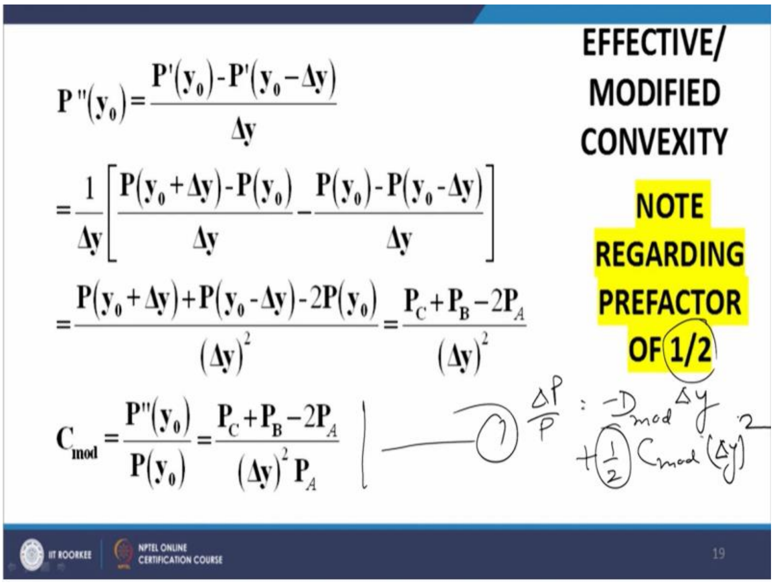

Similar to modify duration, similar to effective duration, we have modified or effective convexity, and the derivation again is absolutely straightforward, it is high school calculus, so, I will not devote time to this but the result that we get here is given an equation number 1 here and please note that issue regarding this prefactor of 1 by 2, which I talked about earlier, we exclude this 1 by 2, when we work out the modified convexity that is the convention for it in some of the textbooks, particularly the textbooks that refer, that are recommended by CFA or that are prepared or that are in reference to the syllabus of the CFA examinations.

So, in those textbooks, this factor of 1 by 2 is not incorporated in the definition of convexity. As a result of which when we try to work out delta P upon P what we use is minus Dmod into delta y. There is no problem regarding this. But as far as convexity is concerned, we introduced this factor. So that when we write the convexity on modified convexity, that this 1 by 2 factor is excluded when we define modified convexity and that is what is being used here. So, into delta y squared. This is a repetition of what we talked about at the beginning of today's lecture.

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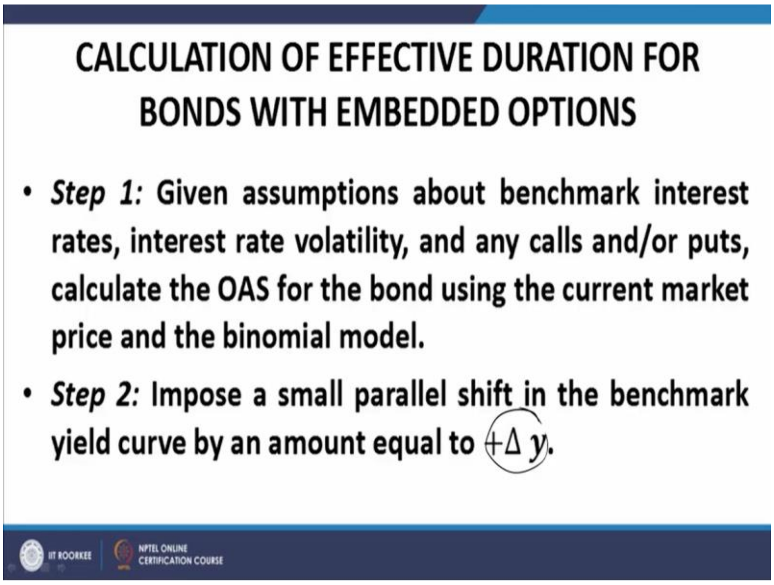

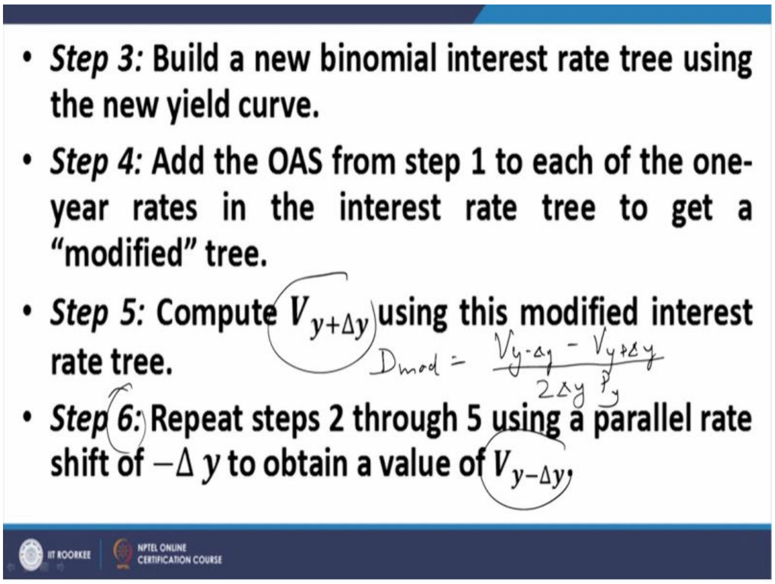

Now, calculation of the effective duration for bonds with embedded options. This is very intriguing. I will explain the process stepwise. The first step is that given assumptions about the benchmark interest rates, interest rate volatility and any calls or puts calculate the option adjusted spread for the bond using the current market price and the binomial model.

So, you have got the benchmark rates, you got the benchmark tree, you have got the interest rate volatility on the basis of it that benchmark rates would be calibrated to give you the binomial interest rate tree. Then once you have the calibrated binomial interested rate tree, you look at the bond that you have with you and you work out the payoffs of the bond or the prices of the bond or the values of the bond, computed values of the bond at each of the nodes that constitute the tree taking into account, the option payoff.

If at any point in time, if it is a callable bond, and if at any particular node you find that the computed price of the bond is more than the strike price of the, of the option embedded in the bond, then you should replace the computed price by the strike price. And on that basis, you arrive at a particular figure let us say you arrive at the figure X.

Now what you do is you add a certain spread to the, to all the interest rates that are involved in the interest rate, same spread, let us call it delta the same spread I reiterate this it must be the same spread, no change, across all the interest rates that are being, that are in the interest rate tree, you add the same spread, you do the same exercise again. And please note, after incorporating this delta, let us say you had a spread of 100 basis points.

Then after incorporating 100 basis points at each of those interest rates which constitute the tree, you should again examine with the new interest rates the option status that is whether an option is in the money or out of the money again, as per the new interest rates, if you find that at a particular node, the option becomes exercisable, the computer price is more, the strike price is less you should replace the computer price by the strike price.

Now, by the process of iteration, you should arrive at that value of delta which I mentioned was 100 basis point, that is not necessary, you start with 100 maybe, you arrive at a certain value for the bond which does not correspond to the current market price you again, say you change it to 90, you change delta to 90. You do the entire exercise again and again you find that you are closer to the price but you are still not there.

Then you take the 84 and using 84 basis points added to each of the tree, you find that when you work out the option payoffs and everything you find that you end up with the current market present, that 84 basis points will give you the option adjusted spread, that is the option adjuster spread.

So let me read it out once again, given assumptions about benchmark interest rates, interest rate volatility and calls or puts, calculate the option adjusted spread for the bond using the current market price and the binomial model, impose a small parallel shift in the benchmark yield curve by an amount equal to delta y.

Now please note this shift is different from the option adjusted spread that I just talked about. That option adjusted spread, the delta that I was talking about when I explained step number one is to be added, it is not, it is changing the benchmark rate, it is an add on to the benchmark. It is a spread over and above the benchmark rate. So, you are adding it to the all the benchmark rates, you are not changing the benchmark rate to step number one, you are changing this benchmark rate to step number two, you are retaining the spread as it is, but you are changing these benchmark rates by small quantities delta y, you are increasing it by delta y.

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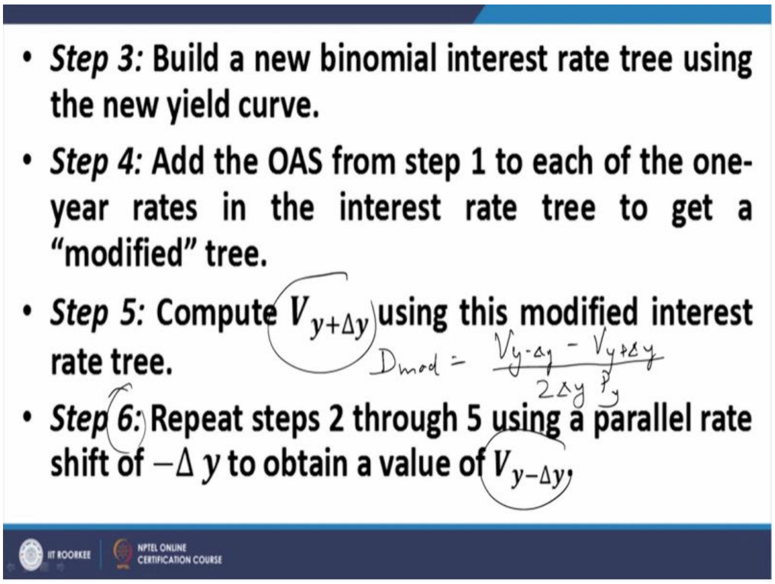

Then what we get is you build a new binomial interest tree because your benchmark rates have changed throughout by delta y. So, you construct a new binomial interest rate tree using the new yield curve. Add that is what I mentioned, now you add the same delta that you had calculated in step one, you add that same delta of step one to the new benchmark rates, which are the increased benchmark rates by a figure of delta y over the original or the initial benchmark rates.

So, you add those delta y to the initial benchmark rates. And then, you add the option adjusted spread and you calculate the price. So that will give you Vy plus delta y. You repeat the entire process by not adding but subtracting the same delta y from the benchmark rates, the rest of the process is the same.

So, in this step that is in step number six, what you are going to do is you are going to reduce the benchmark rates by the same amount that you increased in step number four and five, add delta which is the option adjusted spread and then, work out using the payoffs on the options wherever the options turn out to be the money and then arriving at a certain price that is, that will give you Vy minus delta y, that will give you Vy minus delta y.

So, you have Vy minus delta y, you have Vy plus delta y that will enable you to work out the duration, it is quite simple. It will be what, it will be Vy minus delta y minus Vy plus delta y divided by 2 delta y into P of or V of y. Let me write it down for you. You will get duration, modified duration that is D mod or D effective is equal to V y minus delta y minus V y plus delta y.

Please note y minus delta y is on the left-hand side, why? Because we have a minus sign when we define the modified duration. So, this is to be divided by 2 delta y into P of y. That is how you work out the modified duration of option embedded bond.

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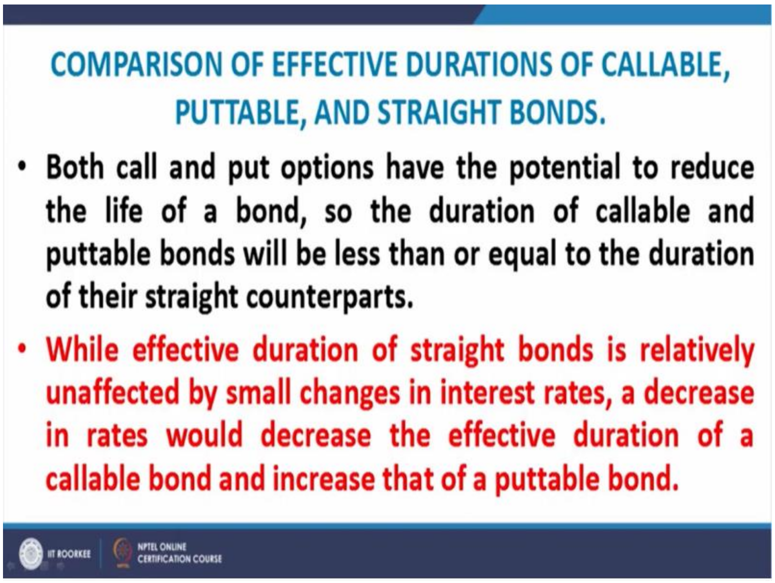

Comparison of effective durations of callable, puttable, and straight bonds. Both call and put options have the potential to reduce the life of a bond, because if the option is going to be accessed early and naturally the life of the bond is going to be reduced. So, both call and put option in the event that either of the options turns out to be in the money, the assumption is the presumption is that the option holder, the person who is long in the option will exercise the option and whenever he exercise the option the bond closes out.

Therefore, the life of the bond with the option, could be either equal to or less than the life of the corresponding straight bond. So, they have the potential to reduce the life of the bond. The options, embedded options in the bond have the potential to reduce the life of a bond and so the duration of the call or puttable bonds will be less than or equal to if the options turn out to be out of the money at all nodes. Then naturally they become redundant, they become useless and we end up with the duration as in the case of a straight bond, corresponding straight bond.

So, both call and put options are the potential to reduce the life of a bond, so the duration of callable and puttable bonds will be less than or equal to the duration of the straight counterparts, while effective duration of straight bonds is relatively unaffected with small changes in interest rates, decrease in interest rates could decrease the effective duration of a callable bond and increase that of a puttable bond. Why is that?

Please note, this point that whenever there is a decrease in interest rates, what happens to the price of the bond? The price of the bond increases and if the price of the bond increases, the option which was hitherto out of the money may become in the money and if it becomes in the money, the option holder whosoever it is, in this case, the issuer of course if it is a call option, then the issuer will exercise the option and callback the debt, callback the bond, and therefore in such a situation what happens, there is a potential where there is a potential of the exercise of option where there is a strong probability of the exercise of the option, the bond duration will decrease.

The greater is the probability of the exercise of the option, the greater is the sensitivity of the bond to the interest rates. A small increase, a small decrease in interest rates could result in a significant increase in the price of the bond. And significant increase in the price of the bond could result in a significant increase in the probability of exercise of the bond, exercise of the option embedded in the bond.

And what happens? If the option gets exercised, we know that the bond contract emanates, issuer calls back the bonds and therefore, their duration will decrease. So, let me read it out once again. While effective duration of straight bonds is relatively unaffected by small changes in interest rates, decrease in rates, would decrease the effective duration of a callable bond and increase that of a puttable bond.

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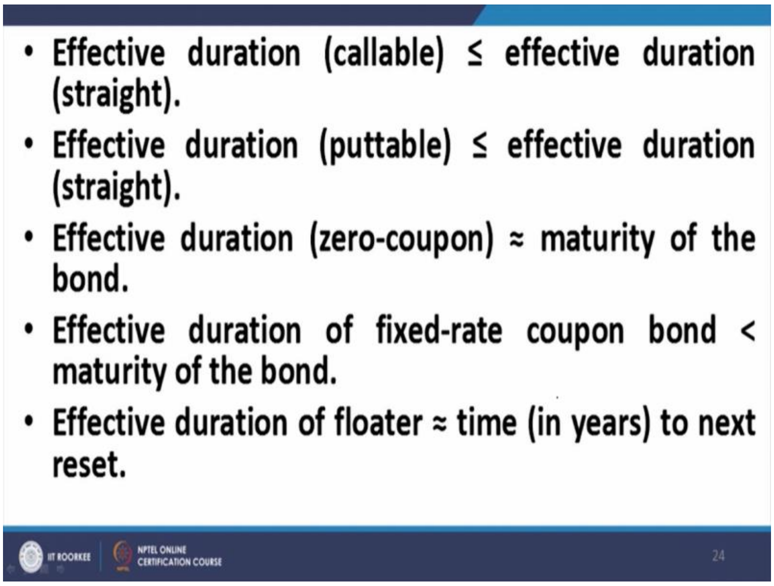

So here is the table for you, effective duration of a callable bond is less than equal to effective duration of straight bond, effective duration of a puttable bond is also less than equal to effective duration of a straight bond because I mentioned a couple of minutes back the embedded options have the impact of reducing the life of the bond in the event that they turn out to be in the money during the life of the bond.

Effective duration of a zero-coupon bond is equal to maturity of the zero-coupon bond, effective duration of fixed rate bond is equal to maturity of the bond is less than I am sorry, effective duration of fixed rate coupon bond is less than the maturity of the bond, effective duration of a floater is equal to or approximately equal to time in years to the next reset.

We shall be talking later on about floating rate bonds. So, you can leave it for the moment, but let me reiterate, effective duration of callable and puttable bonds could be less than or equal to the corresponding straight bonds. Effective duration of zero-coupon bonds equals the maturity of the bond and effective duration of fixed rate coupon bonds is less than the majority of the bonds. We will talk about key rate duration in the next lecturer. Thank you.