Quantitative Investment Management
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Lecture No. 20
Duration Properties

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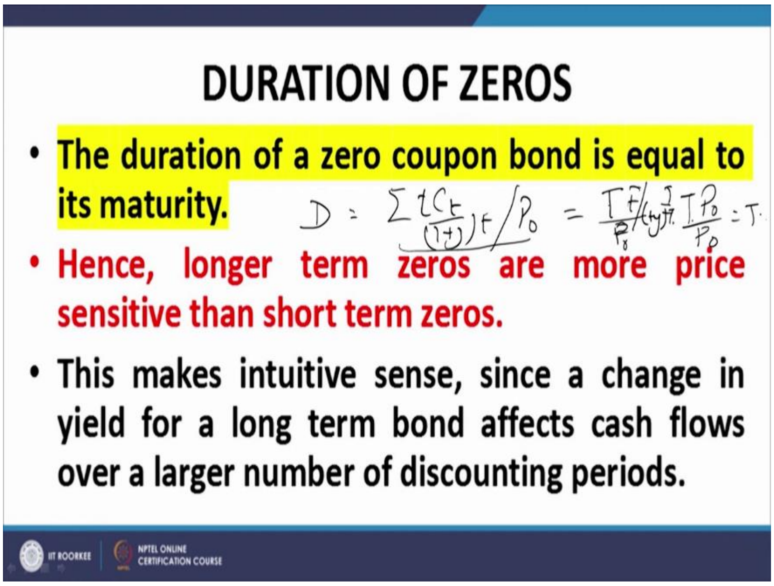

So, let us continue from where we left off. As far as zero-coupon bonds are concerned the duration of the zero-coupon bond is equal to its maturity is quite elementary to establish this particular point, how do we do it? The duration is given by D is equal to sigma tCt divided by 1 plus y to the power t and the whole is divided by P0.

Now, if you look at the zero-coupon bonds, all intermediate cash flows are 0 and we have only one cash flow which is at maturity t equal to capital T and which equals the redemption value of the bond this will be P and this will be discounted at 1 plus y to the power T. Now, f upon 1 plus y to the power t is nothing but P therefore, P and P will cancel out and we are left with t let me repeat the this is the formula for the duration that we have here.

Now, for in the case of zero-coupon bond because all the intermediate cash flows are 0, we have this t and the final calculus f and this will be discounted for t periods capital T period. So, we will divide it by 1 plus y to the power t and when you divide it by 1 plus y to the power t that is nothing but today's price and that that is equal to P0.

So, we have this is equal to TP0 upon P0 and that is equal to T as longer-term bonds or longer-term zeros are more price sensitive than short term zero that is natural because the duration of a longer term 0 will be equal to its maturity which is longer and therefore, the longer-term zeros will be more sensitive to changes in interest rate compared to short term zeros. This makes intuitive sense since the change in the yield curve long term bond affects cash flows over a longer period of discounting periods.

So, that is the financial rationale behind the mathematics, the mathematics says that longer the maturity, longer the duration, longer the duration, longer is the sensitivity of the bond to changes in interest rates. And as far as the finance is concerned, longer maturity means longer number of periods larger number of periods of discounting. So, if the interest rate changes, the impact is magnified.

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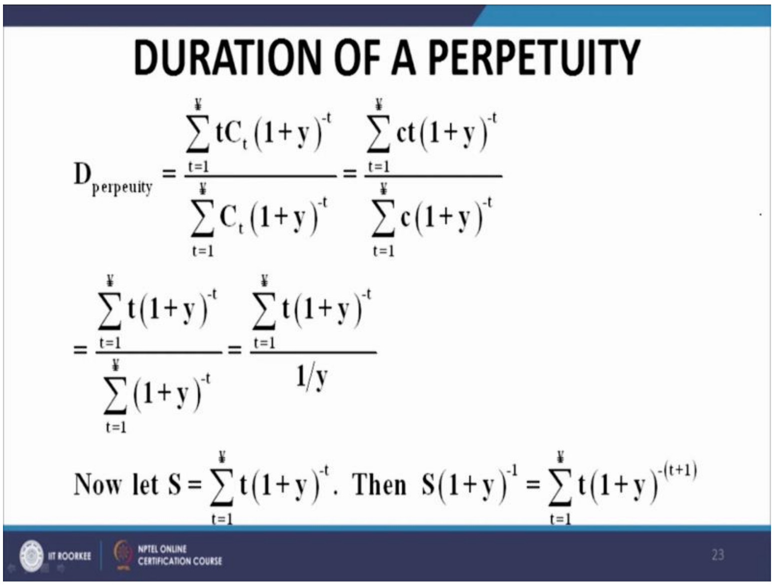

Duration of a perpetuity. This is again quite simple. It is essentially a summation exercise of an infinite geometric progression, I will leave it as an exercise to the mathematically oriented learners. But as I repeat, it is simply a summation of an infrared geometric progression.

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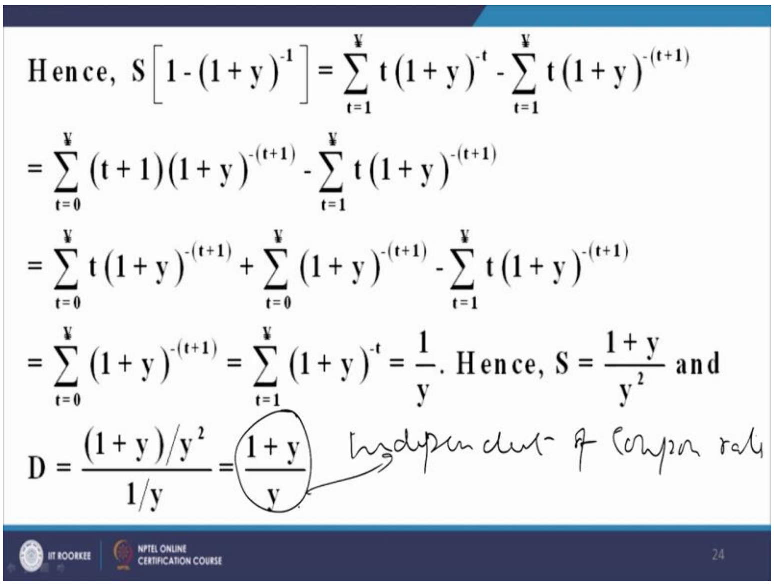

And the results that we get there the result is meets some mentioned, the result is very interesting in the sense that there is no coupon rate involved in that duration and the duration of perpetuity is independent of the coupon rate, independent of coupon rate. As you can see in this formula, it depends solely and wholly on the current YTM of the bond at the YTM that the board has been created, that is the only determining factor of the duration of a perpetuity.

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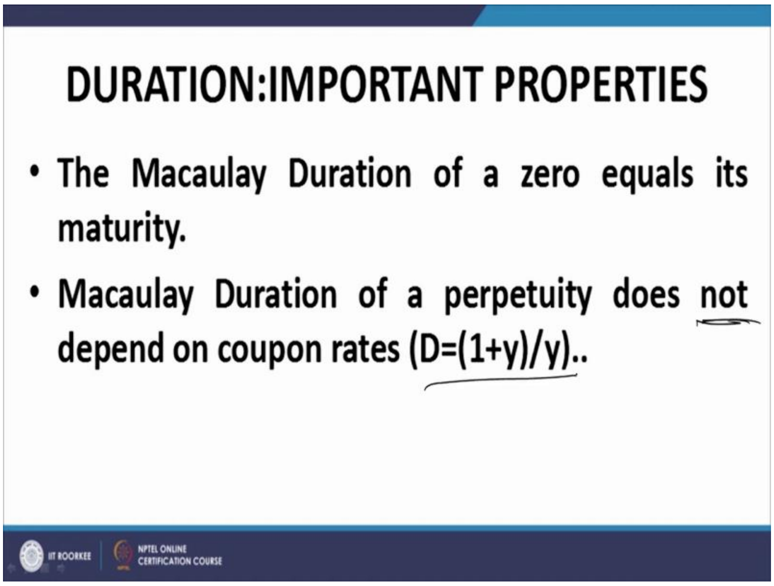

So, duration important properties the Macaulay duration of a zero bond equals zero coupon bond equals its maturity. Explain that Macaulay duration of a perpetuity does not depend on the coupon rate does not depend on the coupon rate and it is given by this expression.

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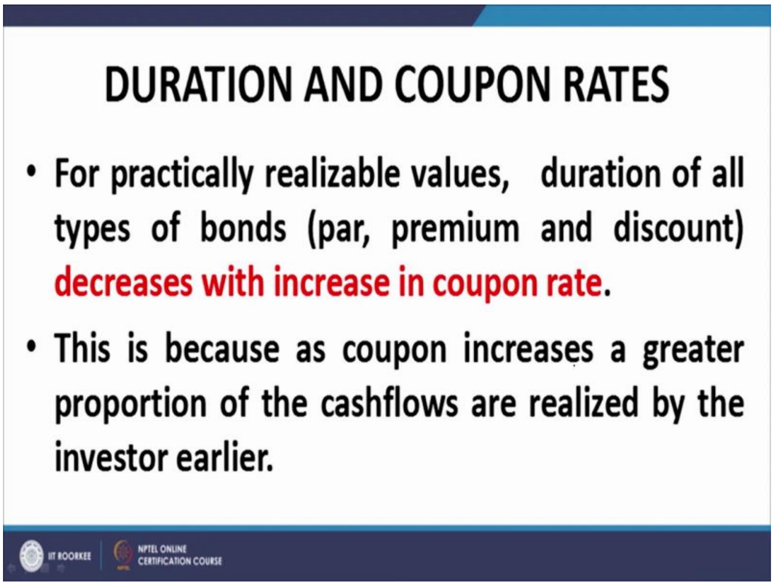

Duration and coupon rates, for practically realizable values, duration of all types of bonds par, premium and discount decreases with increase in coupon rates, higher the coupon rate lower is the duration, why is that? That is because as coupon rate increases, a greater proportion of the cash flows are realized by the investor earlier. So, because higher the coupon, higher is the cash outflow during the intermediate periods on account of coupon payments and as a result of which greater is the cash outflow during the earlier years.

And therefore, the duration which in a sense is the weighted time with the weights being the proportion of cash flows corresponding to each time is point in time at which the cash flow is occurring weights are the given cash flow divided by the sum total of all cash flows present value of all given present value of given cash flow divided by the sum total of the present values of all cash flows, these are the weights with the corresponding time at which these cash flows are occurring. So, as the coupon rate increases, this time becomes shorter and therefore, the duration decreases.

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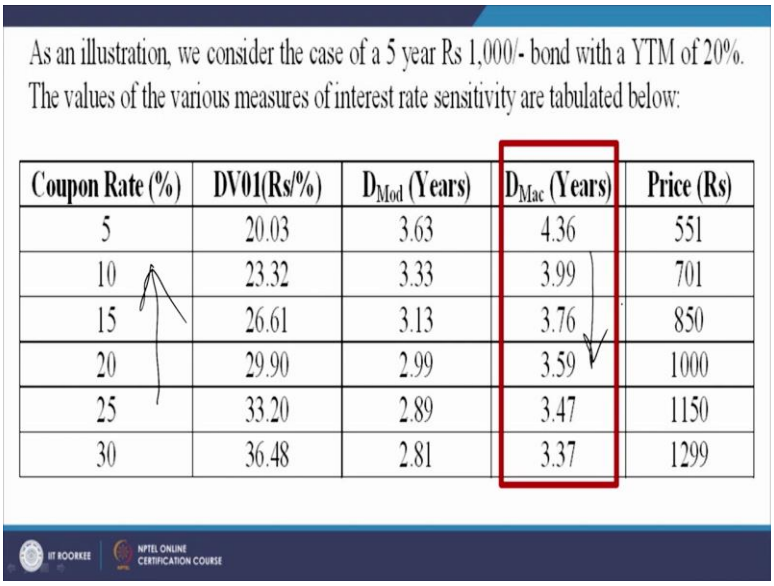

As an illustration, we consider the case of a 5 year Rs 1,000/- bond with a YTM of 20%. The values of the various measures of interest rate sensitivity are tabulated below:

Coupon Rate (%)	DV01(Rs/%)	D_{Mod} (Years)	D_{Mac} (Years)	Price (Rs)
5	20.03	3.63	4.36	551
10	23.32	3.33	3.99	701
15	26.61	3.13	3.76	850
20	29.90	2.99	3.59	1000
25	33.20	2.89	3.47	1150
30	36.48	2.81	3.37	1299

This can be explicitly seen in this example, in this example, we consider the case of a 5-year rupees 1000 bond with a YTM of 20 percent. The values of the various measures of interest rate sensitivity are tabulated below given coupon rate, if it is 5 percent DMac turns out to be 4.36 years, if the coupon rate is 10 percent DMac turns out to be 3.99 years and if it is 15 percent 3.76 years, and if it is 30 percent if the coupon rate is 30 percent DMac turns out to be 3.37 years. So, it is clearly seen that the as the coupon rate increases, as the coupon rate increases, the duration decreases.

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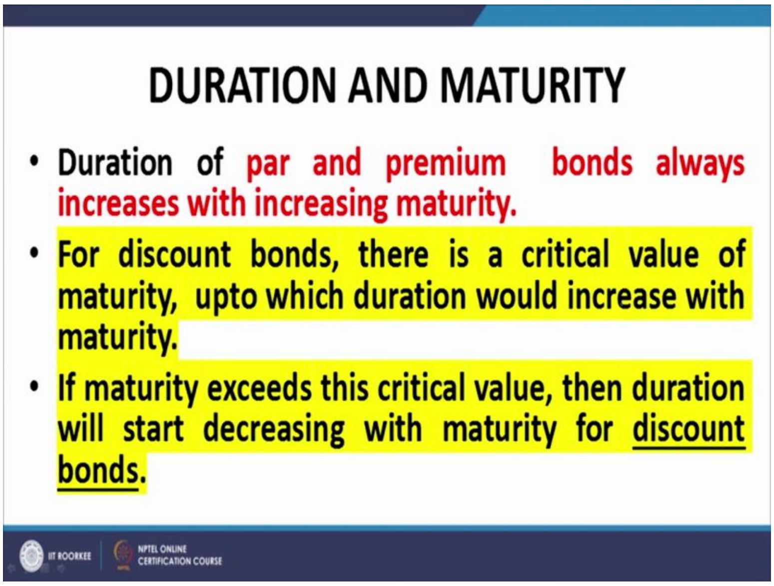

Duration and maturity, duration of par and premium bonds, bonds that are quoted at par and bonds that are quoted at premium always increases with increasing maturity. Why is that, for this law, because, as the maturity increases, obviously, the length of time increases the number of discounting periods also increases. And therefore, the duration of par and premium bonds tens to increase.

But there is a slight catch here, the catch relates to the discount bonds for discount bonds as a critical value of maturity up to which duration would increase with maturity and thereafter, if maturity exceeds this critical value, then duration will start decreasing with maturity for discount bound.

So, in the behaviour of this duration, insofar as discount bonds are concerned in relation to maturity is slightly anomalous, in the sense that initially the duration increases until a critical value is reached. And if maturity then exceeds, maturity then exceeds, the maturity corresponding to that critical value of duration, then the duration starts decreasing. However, as far as par and premium bonds are concerned, duration invariably increases with increasing maturity.

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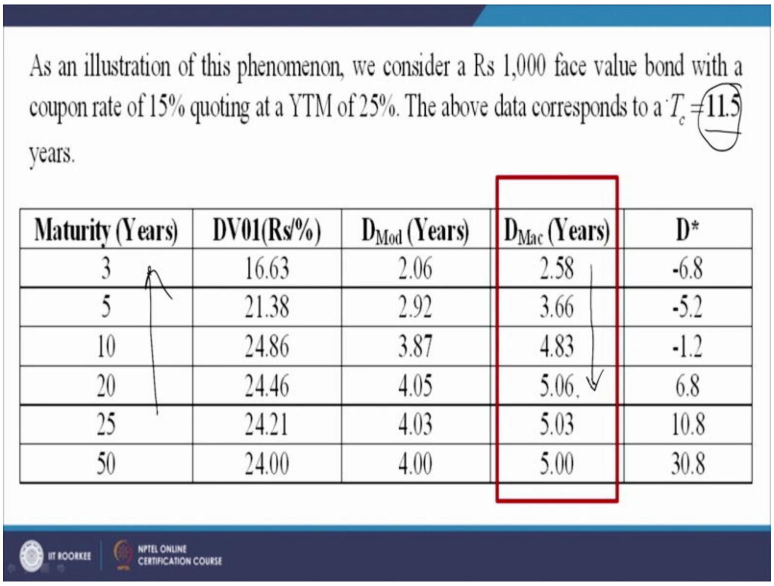

Maturity (Years)	DV01(Rs/%)	D_{Mod} (Years)	D_{Mac} (Years)	D*
3	16.63	2.06	2.58	-6.8
5	21.38	2.92	3.66	-5.2
10	24.86	3.87	4.83	-1.2
20	24.46	4.05	5.06	6.8
25	24.21	4.03	5.03	10.8
50	24.00	4.00	5.00	30.8

This is again shown in this table. This table relates to 1000 face value bond with a coupon rate of 15 percent quoting at a YTM of 25 percent and the maturity is shown in table 1 as maturity increases from 3 years to 5 years, 10 years 20, 25 and 30 years define the behaviour of DMac you can see that initially the DMac is increasing, this is increasing up to 5.06 and then it starts decreasing 5.03 and 5.00.

So, the interesting part is that in this case, the critical maturity or the maturity that corresponds to the critical duration turns out to be 11.5 years. So, up to 11.5 years that duration increases and beyond 11.5 year maturity, that duration starts decreasing. You may note that, this is a discount bond, the YTM is 25 percent and the coupon rate is 15 percent. So, clearly this is a discount bond.

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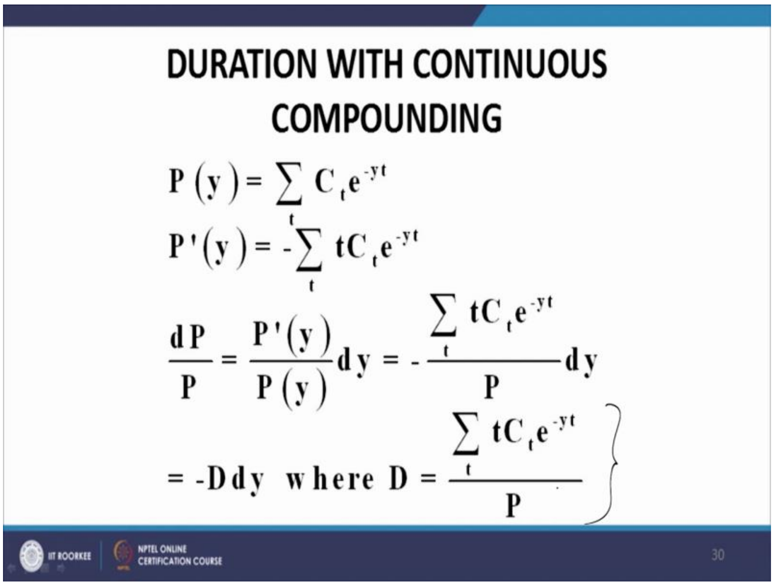

Duration is continuous compounding I shall not delve into this in much detail, but the formula gets slightly modified here. It is self-explanatory, it is very simple.

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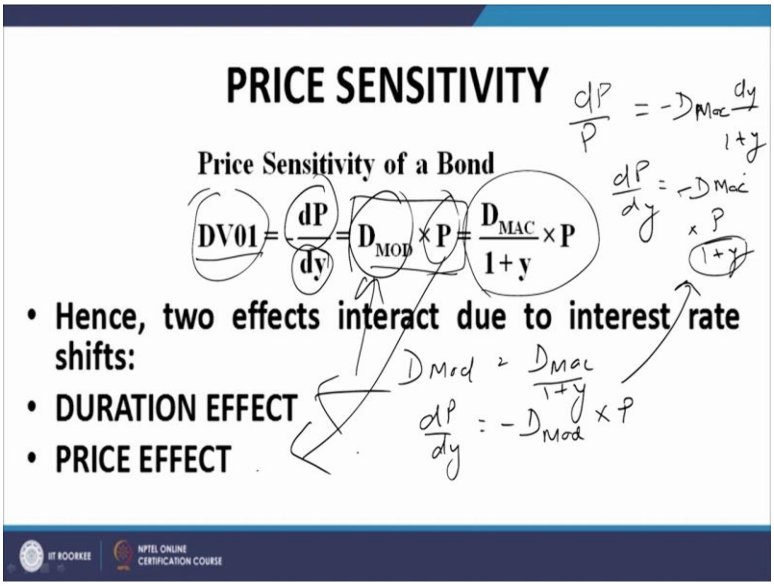

So, let us move on price sensitivity, price sensitivity is a synonym or is another name given to what we referred to earlier as DV01, DV01 which was the negative of what? Negative of the absolute change in price to the corresponding change in yield negative of the absolute change in price that is dP to the corresponding change in yield that is dy, why negative sign because we want to return a positive figure for the sake of convenience.

Now, by using the formula for modified duration, or Macaulay duration, we can write DV01 either in this form or in this form, either of the two are good enough or identical. In fact, as you remember, how did we define DMac? DMac was defined in terms of dP upon P is equal to minus DMac dy upon 1 plus y, what does this give you four dP upon dy that gives us dP upon dy is equal to minus DMac into P divided by 1 plus y.

And if we recall that D mod is nothing but DMac divided by 1 plus y what do we get? Is by substituting in this equation, what we get is dP upon dy is equal to minus d mod that is modified duration into P this quantity gets absorbed into DMac with this and we have DP upon dy is equal to minus D mod into…. So, if you look at this formula carefully, particularly this formula, if you look at this formula carefully, what we find is that, there are two factors that are contributing to DV 01 or for that matter, DP upon dy first factor relates to the duration of the bond and second factor relates to the price of the bond.

In other words, the two effects that interact to give you a value for the sensitivity of a bond corresponding to interest rates with the duration effect, this is the duration effect this particular and then the price effect, which is this. So, depending on how the duration and the price behave for various types of bonds, we can arrive at some expressions or some generalizations for the value of DV01.

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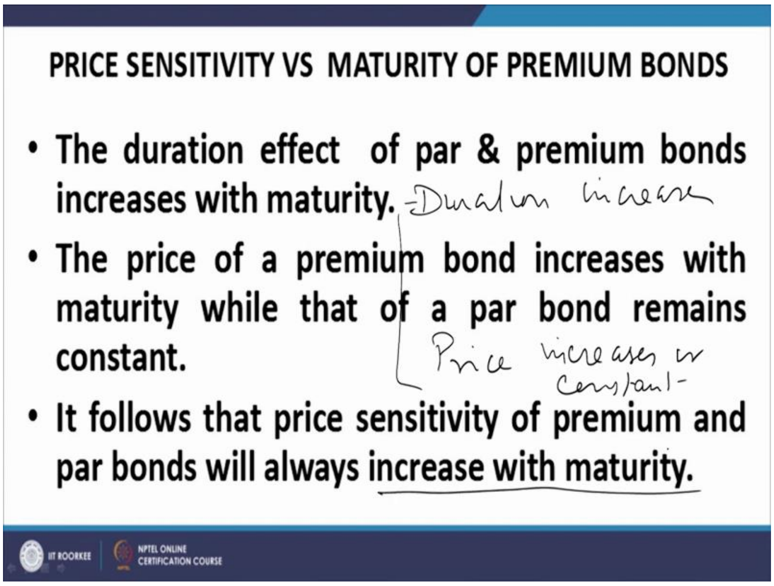

The duration effect for par and premium bonds increases with maturity. Let us examine the price sensitivity with reference to maturity, price sensitivity with reference to maturity. The duration effect of par and premium bonds increases with maturity, the price of a premium bond increases with maturity and the price of a par bond remains unchanged with maturity. In other words, if there is a bond coating at par and the first maturity is 5 years, and if its maturity is increased to 20 years, it will still be sold at par.

So, that is the important thing. In the case of a premium bond, the premium increases with maturity in the case of a par bond, price remains unchanged with respect to maturity, the duration effect increases or duration increases price either increases or constant, increases for premium bonds and is constant for par bonds.

Therefore, when you have the combined effect the combination of these two what will happen to the combination of these two? The combination of these two will increase. In other words, what do we arrive at? We arrive at the (())(13:39) that the price sensitivity of premium and par bonds will always increase with maturity. Why? Because both these effects are either increasing or remaining constant.

So, for premium bonds both the effects are increasing, the duration increases with maturity, the price increases with maturity and as a result of it, both of them are operating in the same direction and will add on to each other and in the case of the par bonds by the duration will increase with maturity the price will remain constant and as a result of it the combined effect will increase will be an increase and hence the price sensitivity of par and premium bonds both increases with maturity.

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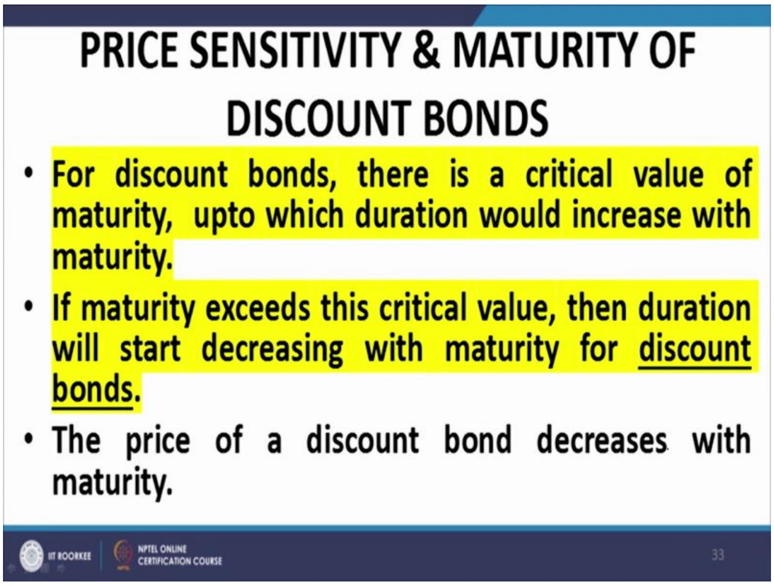

Price sensitivity and maturity of discount bonds. Now, here we have some anomalous behaviour. Why do we have the anomalous behaviour? If you recall a few minutes back I mentioned that the duration of discount bonds initially increases up to a critical value of maturity and thereafter, the duration starts decreasing. I repeat, the duration initially increases up to a critical value of maturity for discount bonds, for discount bonds that is important not for par and premium bonds. And thereafter, the duration starts decreasing.

As far as the price effect is concerned, as the price increases, as the maturity increases of our discount bond, the price decreases as the maturity increases for discount bond, the price decreases, therefore, what happened?

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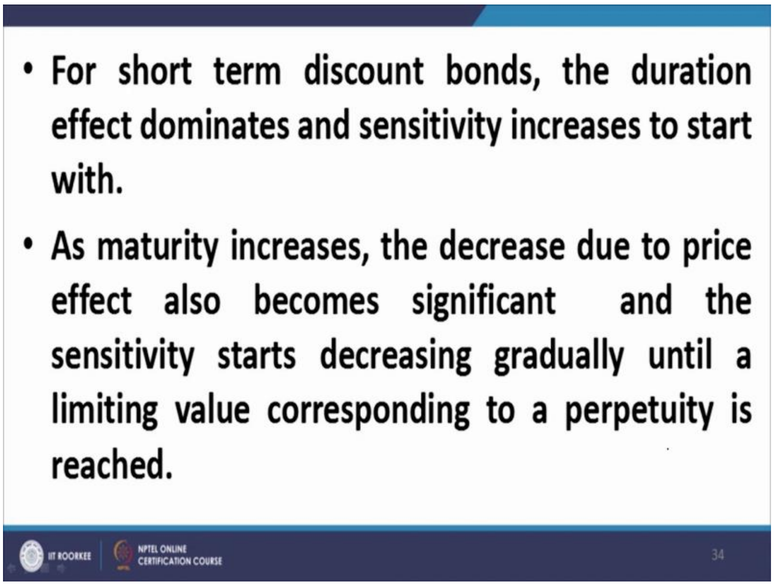

The price of the for short term discount bonds what will happen? The increase in duration will manifest itself will dominate itself over the price effect and as a result of which the price sensitivity will increase. However, gradually as you increase the maturity, the price effect becomes more prominent and later on the duration also decreases. Therefore, duration effect results in a decrease the price effect results in a decrease and the consequences that price sensitivity of discount bonds subsequent to a certain value of maturity starts to decrease.

Initially, it increases for short term bonds, which are quoting at a discount what happens the duration increases price effect is not so, significant duration effect overwrites and we have an increase in price sensitivity. But as the maturity of the bond increases, the duration also starts decreasing the price obviously, for discount bonds decreases with maturity both tend to operate in the same direction and as a result of which the sensitivity decreases.

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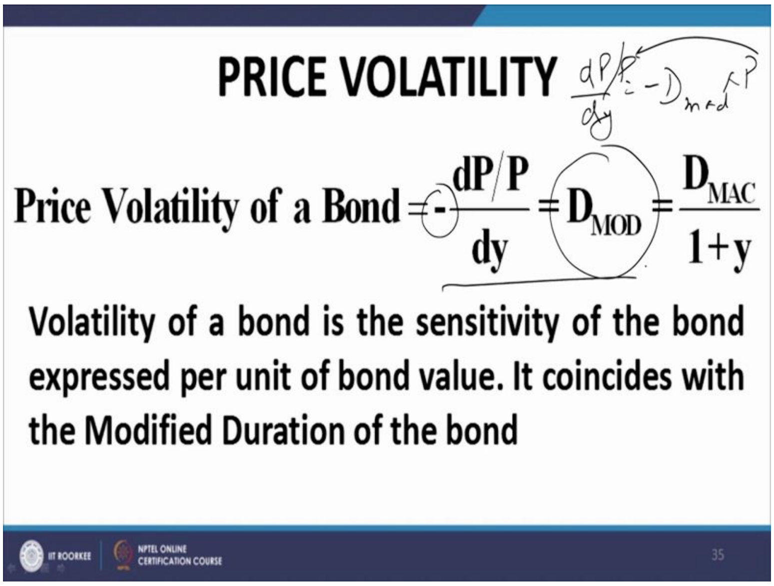

Price volatility. Price volatility is again analogous to a quantity that we have already studied and that is modified duration. Modified duration is also called price volatility. What do we… what is the definition of price volatility? The negative of the percentage change in price for a unit change in yield the negative of the percentage change in price.

Please note the difference between price sensitivity and price volatility; price sensitivity was the absolute change in price corresponding to a unit changing… unit change in YTM? Here, we are talking about the percentage change in price corresponding to a unit change in YTM. So, the difference lies in the numerator in the numerator in the case of sensitivity to the absolute price change the numerator in the case of price volatility as a percentage price change, of course, we have a negative sign here in both cases, in the case of sensitivity and in the case of volatility, and this equals the D mod or DMac upon 1 plus y as you can easily work out.

Because what did we have? We had dP upon dy is equal to minus D mod into p. So, if we bring this P over is what we get is Dp upon p divided by dy is equal to minus D mod that is precisely this formula. So, volatility of a bond is the sensitivity of the bond expressed per unit upon value it coincides with the modified duration of the bond.

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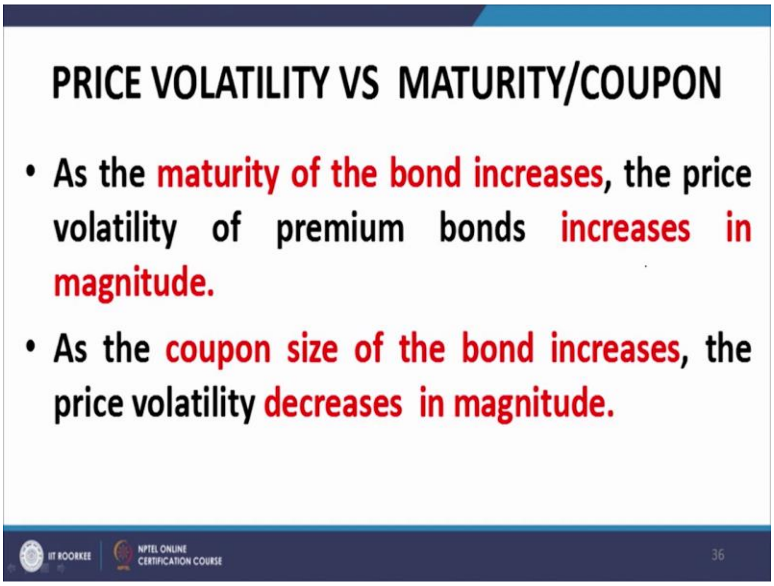

Price volatility versus maturity. As the maturity of the bond increases, the price volatility of premium bonds increases in magnitude. As the coupon size of the bond increases, the price volatility decreases.

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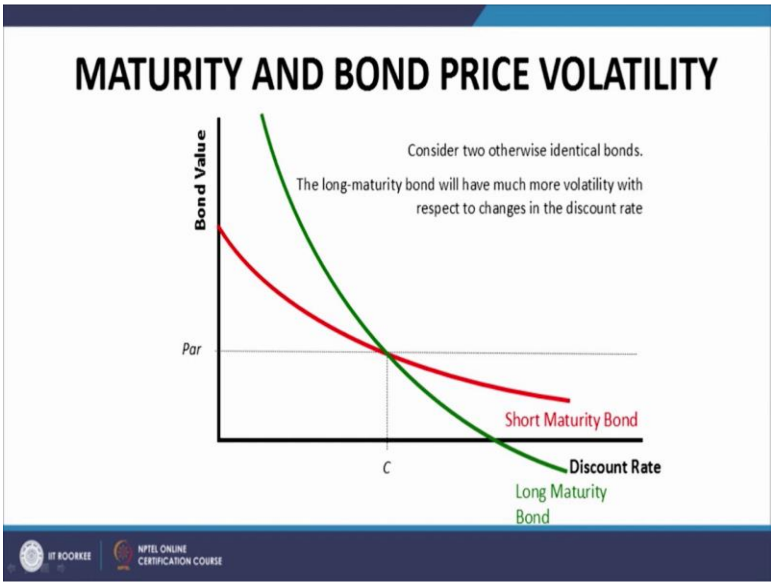

That is seen in this figure here, you have the short maturity bond in the red and the long maturity bond in the black, you can clearly see that the steepness of the red curve is much less than the other one. And as a result, what we infer is that a long maturity bond has greater price volatility compared to the price volatility of a shorter maturity bond.

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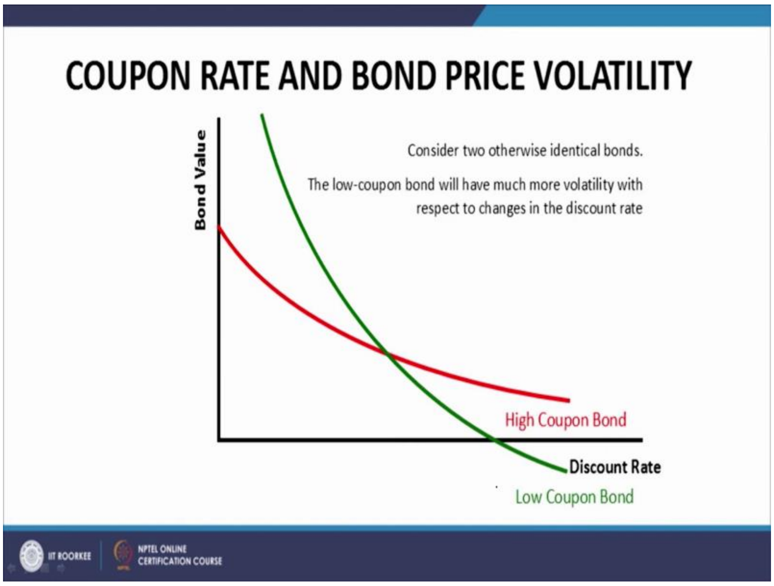

And here is the relationship between the coupon rate and the price volatility. You can see there the red is the high coupon bond and the other one is the low coupon bond and the low coupon bond has a higher price volatility compared to the high coupon bond. High coupon bonds, this is the cash flows that are going to occur earlier. And as the cash flows are going to occur earlier, the impact of (())(19:20) will be less, the volatility would be less if the coupon rate is higher, and the volatility would be more if the coupon rate is lower. If it is a zero-coupon bond, it will have the maximum volatility.

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YTM	MATURITY	PRICE	PRICE SENSITIVITY	PRICE VOLATILITY
40	01	88.57		
41	01	87.94	63	0.71
40	02	71.84		
41	02	70.88	96	1.33
40	10	32.83		
41	10	31.93	90	2.75
40	100	30.00		
41	100	29.27	73	2.43

The issues that are explained just now are illustrated in this example, let me bring to you this example in some detail. The YTM I have taken at 40 percent and 41 percent for computing the changes in price and the changes in the percentage price. So, I have taken different maturity bonds, 1 year maturity, 2 year maturity, 10 year maturity and 100 year maturity. I have taken four different bonds with maturities of 1, 2, 10 and 100 years, the YTM is assumed to change from 40 percent to 41 percent.

And the price changes are given, the absolute price changes are given in this column and the percentage price changes are given in this column corresponding to this one percent change in YTM. And you can see here, as far as price sensitivity is concerned DB01 is concerned as the maturity increases that DB01 or the price sensitivity increases initially then then it starts going down 63, 96, 90 for 10-year bond and 73 for 100 year bond.

So, it initially increases from 63 to 96 for 1 to 2 year bonds and then it decreases from 96 to 90 and then to 73. A similar pattern is observed in the case of price volatility initially 0.71 for 1 year bond 2.75 for 10 year bonds, but then it starts decreasing and 2.43 is the volatility for a 100 year maturity bond.

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MATURITY AND BOND PRICE VOLATILITY: EXAMPLE

- Two bonds X and Y are both 12% annual coupon bonds of the face value of 1,000. They are redeemed at par after two years and ten years respectively. Calculate the percentage change in price of each bond when the market interest rates change from 5% to 6% and from 25% to 26%.

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For Bond X

$$P = P(12\%, 5\%, 2) = 120PVIFA(5\%, 2)$$

$$+PVIF(5\%, 2) = 1130$$

$$P^* = P(12\%, 6\%, 2) = 120PVIFA(6\%, 2)$$

$$+PVIF(6\%, 2) = 1110$$

$$\text{Price Volatility} = -\frac{(P^* - P)}{P} = 1.78\%$$

Similarly, Price Volatility of Bond Y $= 6.42\%$

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RESULTS

- $P_X(12\%, 5\%, 2) = 1130$
- $P_X(12\%, 6\%, 2) = 1110$
- PRICE VOLATILITY OF X =1.78%

- $P_Y(12\%, 5\%, 10) = 1541$
- $P_Y(12\%, 6\%, 10) = 1441$
- PRICE VOLATILITY OF Y = 6.42%

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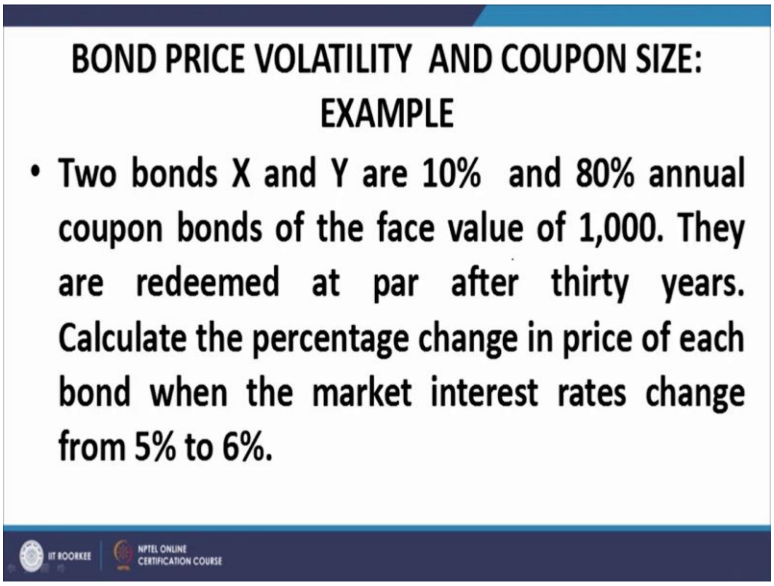

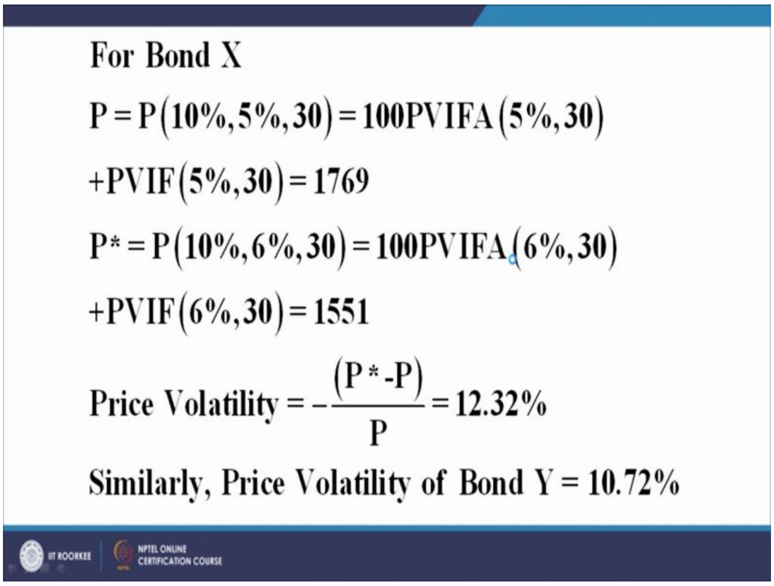

These are some examples which are quite straightforward. So, I leave it as an exercise for the learners, the solutions are there in this presentation. So, I will not go into these examples are quite straightforward. Any issues can be discussed in the discussion forum when this course is launched.

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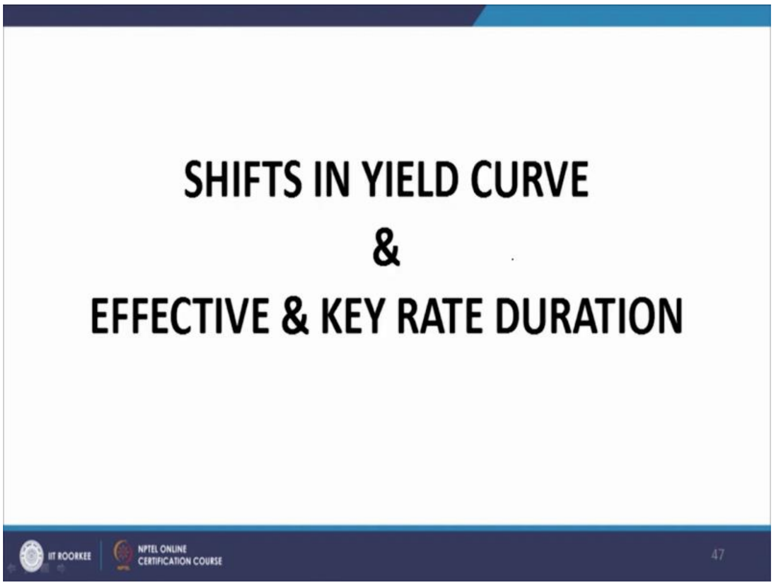

Now, we talked about shifts in yield curve. And the issue of duration, how does the duration relate to the shifts in yield curve that is our next…

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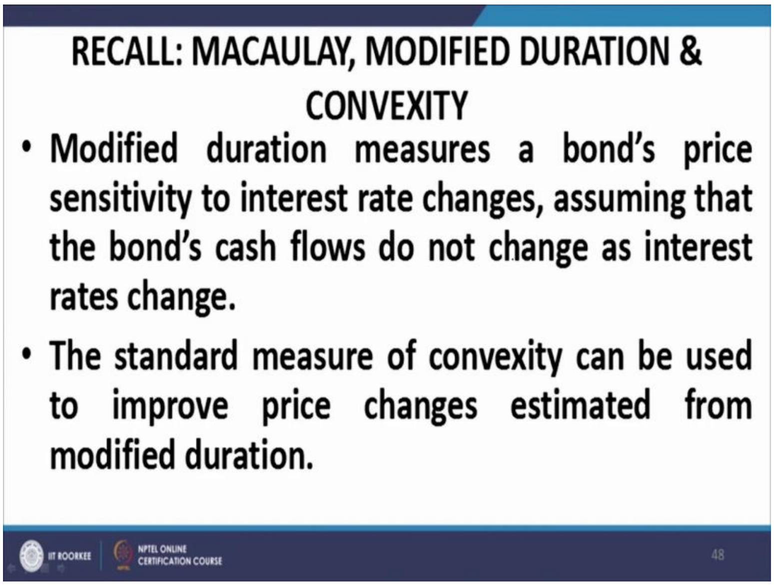

Now, this is the recalling of the Macaulay and modified duration and convexity modified duration measures a bonds price, sensitivity to interest rate changes, assuming that the bonds cash flows do not change as interest rate since, that is usually the case because the cash flows are more or less fixed by the contract of issue of the bond and as a result with the cash flows do not change, but the market return changes which will manifest itself as a change in price of the bond. The standard measure of convexity can be used to improve price changes estimated from modified duration, because convexity is actually the second order correction.

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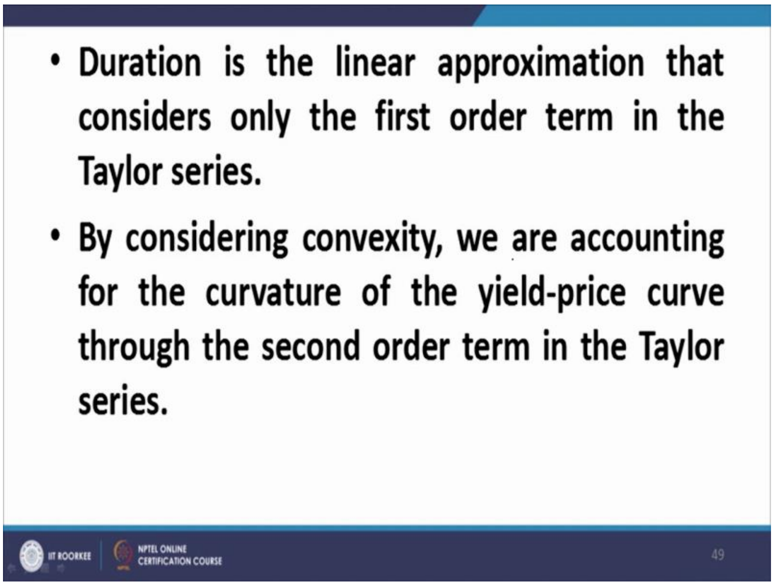

See as I mentioned in the last lecture duration is a linear approximation that considers only the first order term in the Taylor series. And that assumes that the price… yield price curve is linear is a straight line around in the infinitesimal neighbourhood of the point at which we are working at with the duration is calculated or the price changes being worked out.

By considering convexity, we are accounting for the curvature of the yield price curve through the second order term in the Taylor series. So, this is an add on this is an improvement on the rough estimate that we get by using the duration, duration assume the straight-line structure convexity is an improvement on the straight-line structure because the actual structure is convex.

And in fact, if you can use the further corrections in the Taylor series, you will arrive at a more accurate result. But it is believed by market players that such corrections are not required do not carry any weight in actual realization of figures. So, usually we have corrections of the second order that is the convexity, duration as the linear approximation and convexity as the second order correction.

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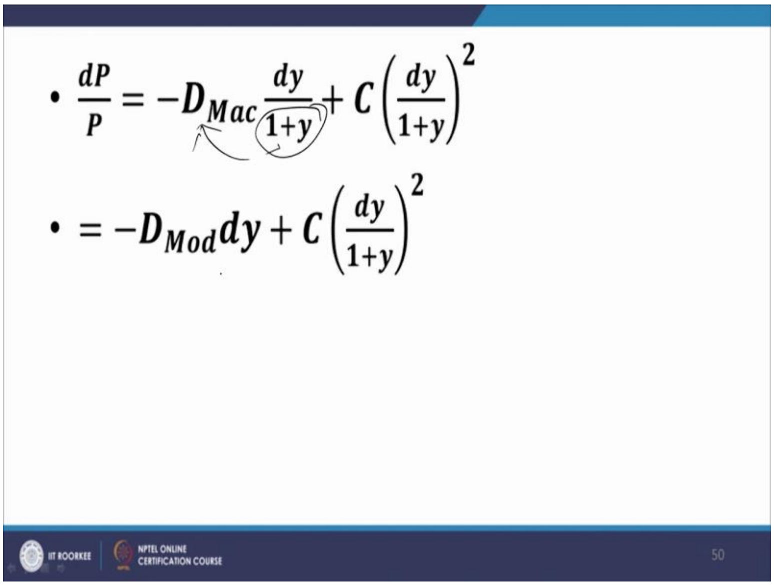

These are the formula that I have talked about many a times. So, the percentage price change is given by minus DMac into dy upon 1 plus y plus C into dy upon 1 plus y squared in terms of modified duration, this 1 plus y factor is captured in this definition. And what we ended up as D Mod into dy or the first term.

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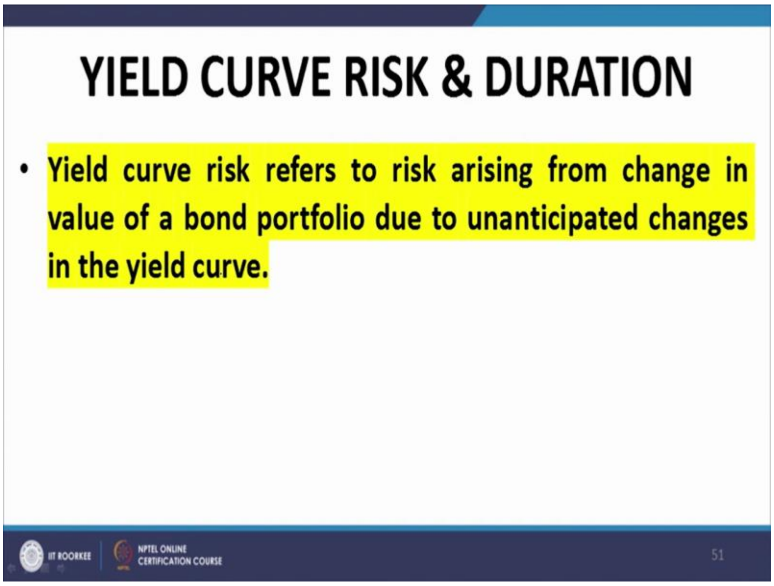

Yield curve risk and duration, yield curve risk refers to the risk arising from changes in the value of a bond portfolio due to unanticipated changes in the yield curve. It will be more explicit this definition would be more explicit as we move on. Basically, you see, the yield curve is not a static quantity. It is a dynamic depiction of the interest rates.

In other words, what I am trying to say is that interest rates are not constant quantities, they keep on fluctuating from time to time, very frequently, just like equity prices, maybe not so rapidly, maybe not with such frequency but all the same. They are not constant quantities. And as a result of it, there could occur different types of shifts in the yield curve.

For example, the yield curve as a whole may move upwards or downwards parallel to itself. In other words, all the spot rates corresponding to different maturities either increase or decrease by the same magnitude that can happen, it can also happen that the yield curve becomes more steep or less steep.

In other words, the short-term interest rates tend to decrease and the long-term interest rates tend to increase that is called steepening of the curve, then there could be another situation where the yield curve changes its curvature or in other words, the short-term interest rates increase the long-term interest rates increase, our intermediate term interest rates do not change.

So, in essence, we can decompose the changes in the yield curve into three typical types, level shifts, which involve a parallel shift to itself, then we have the steepness shift, we have the short term interest rates decrease the long term interest rates increase or vice versa of course, the steepness may increase the steepness may decrease as well, and we have the changing curvature increasing curvature, short term and long term interest rates increase whereas the intermediate interest rates do not change or change marginally.

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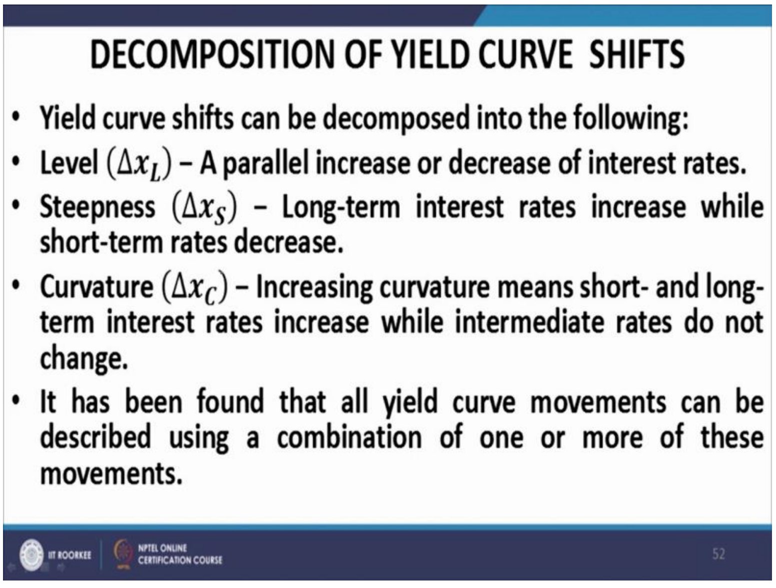

So, this changes in interest rates manifest themselves as changes in the value of the portfolio. And when these changes in interest rates are unanticipated, I repeat when they are unanticipated the corresponding risk is called the yield curve risk.

So, as I mentioned, just no yield curve shifts can be decomposed into the following level shift a parallel increase or decrease of interest rates, steepness, long term interest rates increase, while short term interest rates decrease curvature, increasing curvature means short and long-term interest rates increase, while intermediate rates do not change. It has been found that all yield curve movements can be described using a combination of one or more of these movements.

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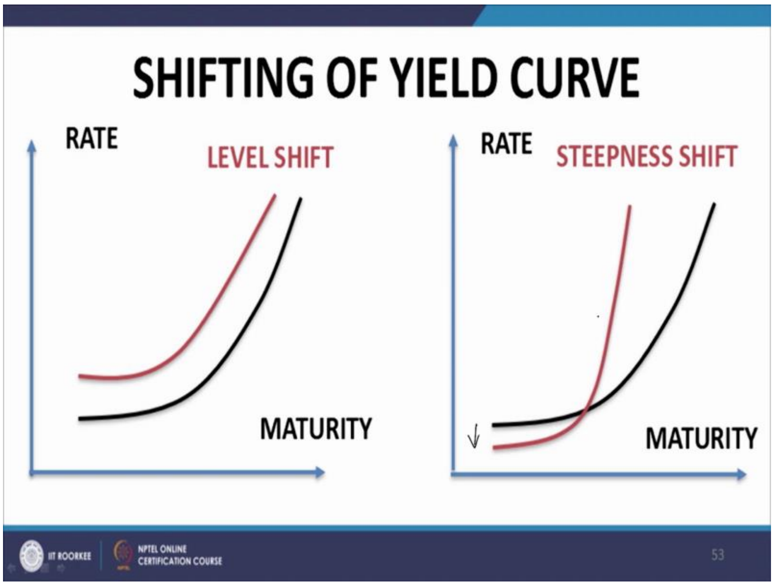

So, this is the these are examples of what I mentioned just now, here you can see that the yield curve is shifting paler to itself suppose that the black one is the original yield curve and the brown one is the change yield curve the shifted yield curve, you can clearly see that all the interest rates of all maturities, spot rates of all maturities have shifted by the same amount, the curve has shifted parallel to itself.

Here you can see that the change in the steepness of the curve, the short-term interest rates have declined you can see here the brown curve with the change curve or the shifted curve. And you can see that the long-term interest rates have increased and that is resulting increase in the steepness of the yield curve. Of course, there can be a situation where there is a decrease in the steepness of the yield curve as well.

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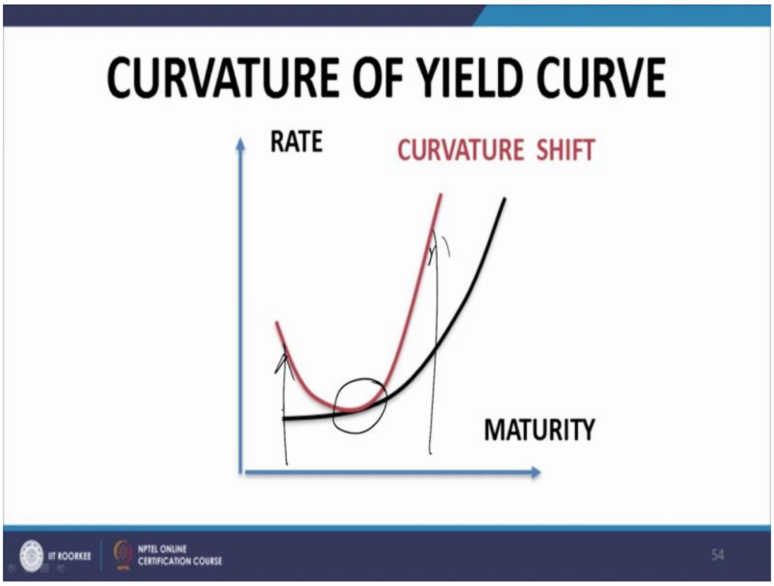

Here you can see there is a change in the curvature of the yield curve, the short term rates and the long term rates both are increasing, whereas the intermediate term rates are approximately unchanged.

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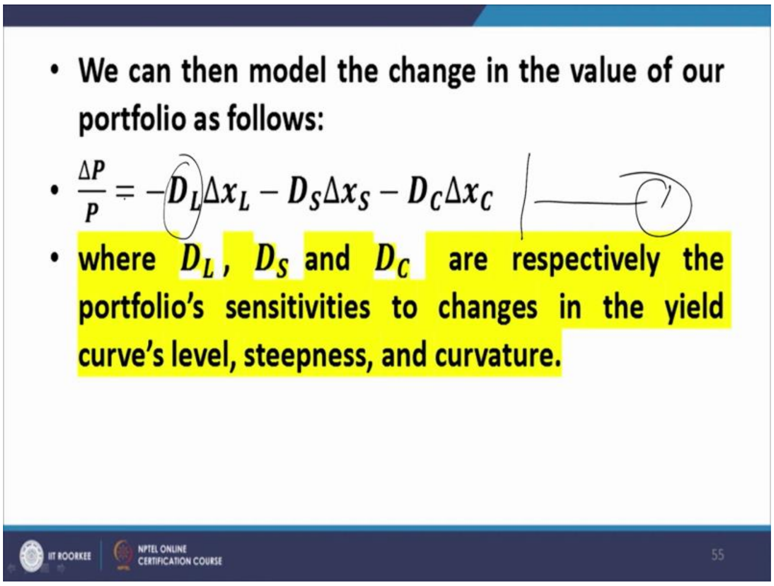

We can then model if suppose we agree or it is accepted that the yield curve shifts can be captured by these three types of shifts, that is level shift, steepness shift and curvature shift, then it is logical that we model the percentage price change in our security in terms of what is given as equation number 1. DL, DS and DC are the respective portfolios sensitivities to changes in the yield curve level, steepness and curvature.

Basically, what we are trying to say is we are trying to model the entire price change or the percentage price change in terms of these three shifts, how much do each of these three shifts contribute to the overall percentage price change that is captured by this formula, and DL, DS and DC are respectively the prices corresponding to unit changes in all each of these shifts.

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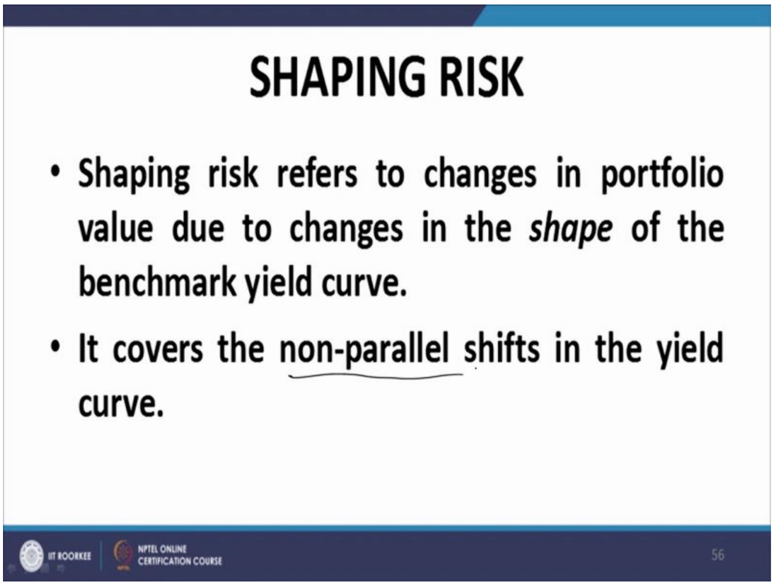

Shaping risk refers to the change in portfolio value due to change in the shape of the benchmark yield curve. That means it excludes the level shift part it covers the non-parallel shift in the yield curve. So, shaping shifts include what? Include the change in steepness, and including changing curvature but they do not include the level shift or the parallel shift in the yield curve.

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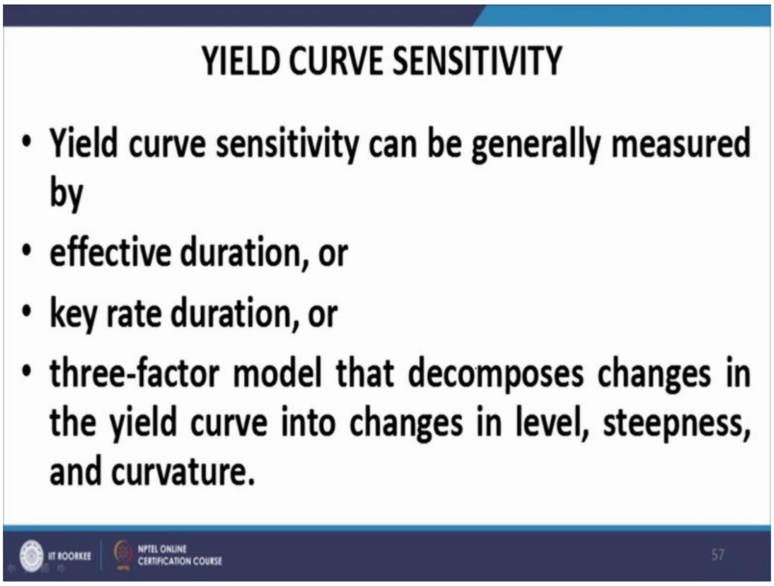

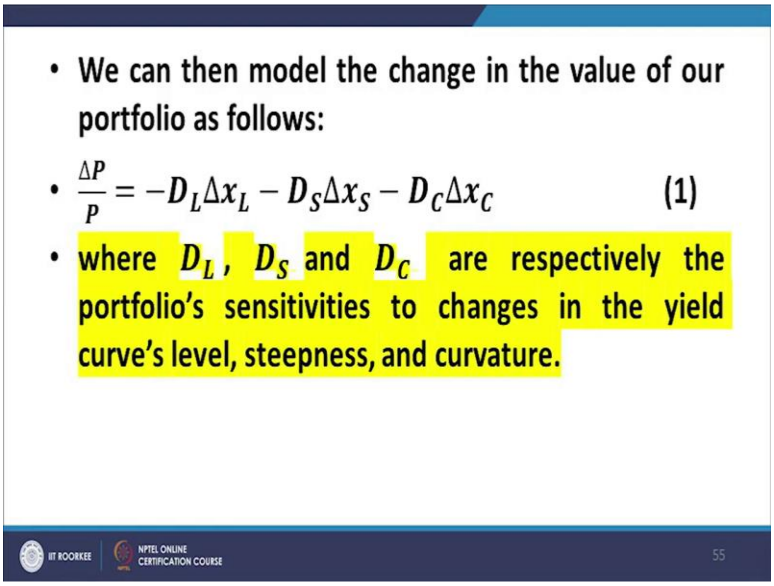

Yield curve sensitivity can be measured by effective duration, it can be measured by key rate duration, and it can also be measured by the three factor model that briefly I touched upon just a few minutes back this model equation number 1 on the slide. This is where we conclude today.

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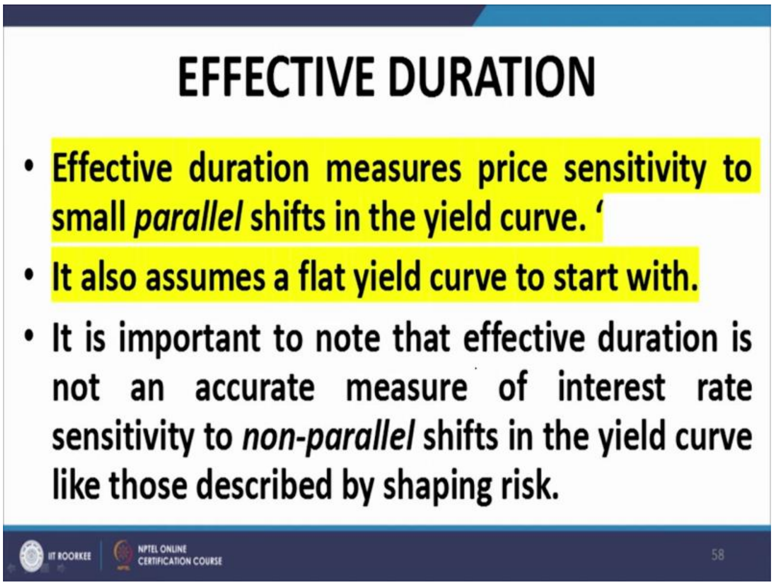

In the next lecture, we will start with effective duration and then take a key rate duration. Thank you.