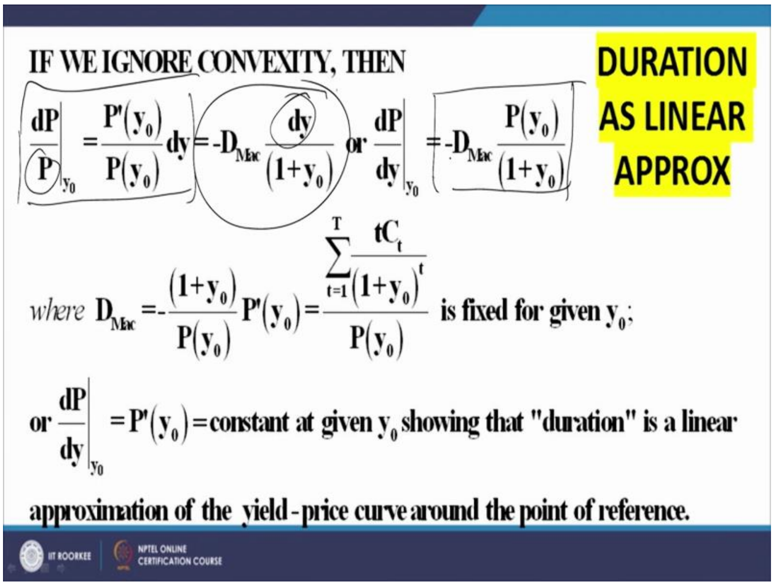
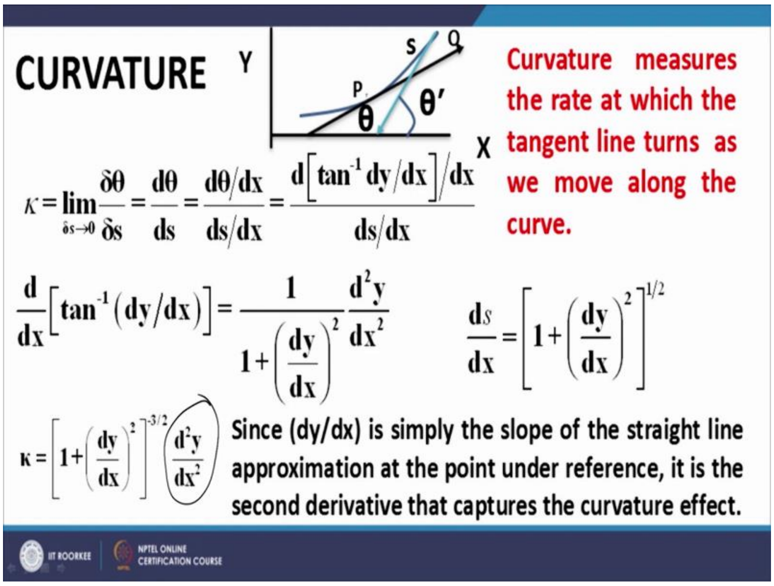
Quantitative Investment Management
Professor. J.P. Singh
Department of Management Studies
Indian Institute of Technology, Roorkee
Lecture No. 18
Duration & Immunization

(Refer Slide Time: 00:35)



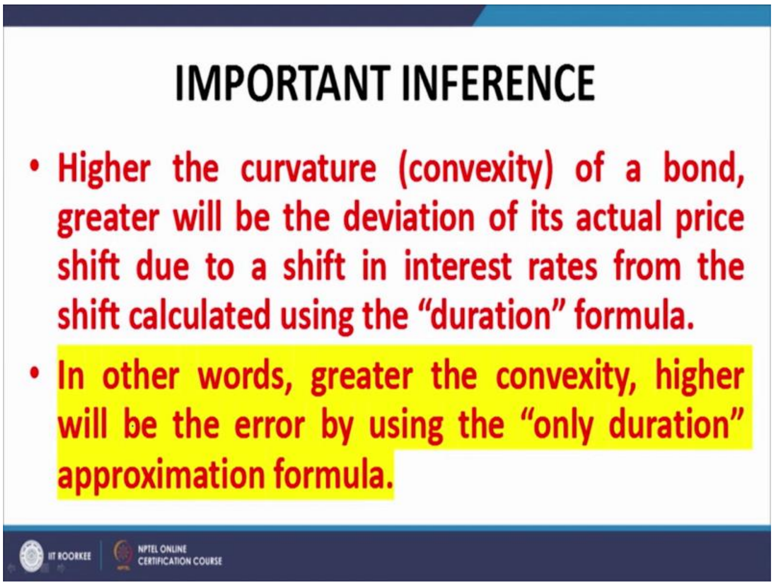
So, before that I was explaining that if we exclude the higher order terms, that if we exclude convexity and other higher order terms in the Taylor series, what we find is that the duration gives us the linear approximation of the yield price curve at the point at which the duration is calculated. I emphasize that that straight-line assumption or the straight-line nature of duration holds only for very small shifts around the given point, infinite decimal shifts around the given point y0.

(Refer Slide Time: 00:59)



Now, in this slide, what I have tried to show is that the it is the second order derivative and not the first order derivative which captures the curvature, which captures the curvature in the yield price curve… in any curve for that matter, the curvature is captured by higher order terms in the Taylor series, the first order term is the linear term.

(Refer Slide Time: 01:21)

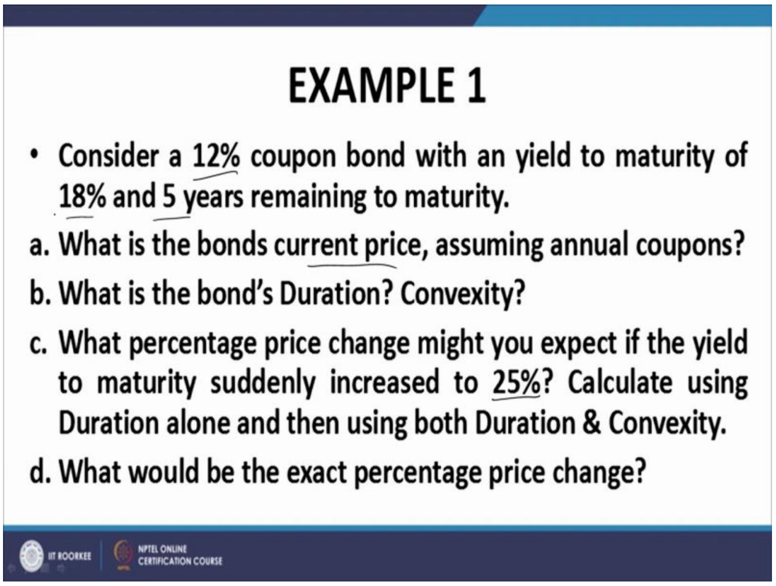


Now, what is the influence of the curvature? What is the impact of curvature on estimation that is very interesting, higher the curvature or convexity the general term is curvature, but, as our curve happens to be convex towards the origin, the curvature is convex towards the origin, we also call it convexity. So, higher the curvature or convexity of a bond, greater will be the deviation of the actual price shift due to a shift in interest rates from the shift calculated using the duration formula.

In other words, if you ignore convexity and you calculate the price in DP upon P percentage price change, the result that you will get will deviate more and more from the actual price change will be different from the actual price change depending on how much is the curvature of the bond, if the curvature is higher, if the convexity is higher, then what happens is that the estimate that we arrived at using the duration using the linear approximation turns out to be of the mark by a greater margin.

If the convexity of the bond is lower, obviously, then the curve is closer to the straight line approximation embedded in the duration and as a result of which the price range that we estimate using the duration for a bond with smaller convexity may be close to its actual value. In other words, greater the convexity higher will be the error by using the only duration formula compared to the complete formula using both duration and convexity.

(Refer Slide Time: 03:08)



Let us do an example, consider a 12 percent coupon bond with a yield to maturity of 18 percent the coupon rate is 12 percent and the YTM is 18 percent the life is 5 years, that has 5 years remaining to maturity. Now, what is the bonds current price? Assuming annual coupons, what is the bonds duration and convexity?

What percentage change might you expect if the yield in to maturity suddenly increased to 25 percent from 18 percent change of 7 percent plus 7 percent. Calculate using duration alone and then using both duration and convexity and compare it with the projected price change using the cash flow formula, it is a comprehensive example, the solution is here.

(Refer Slide Time: 04:01)

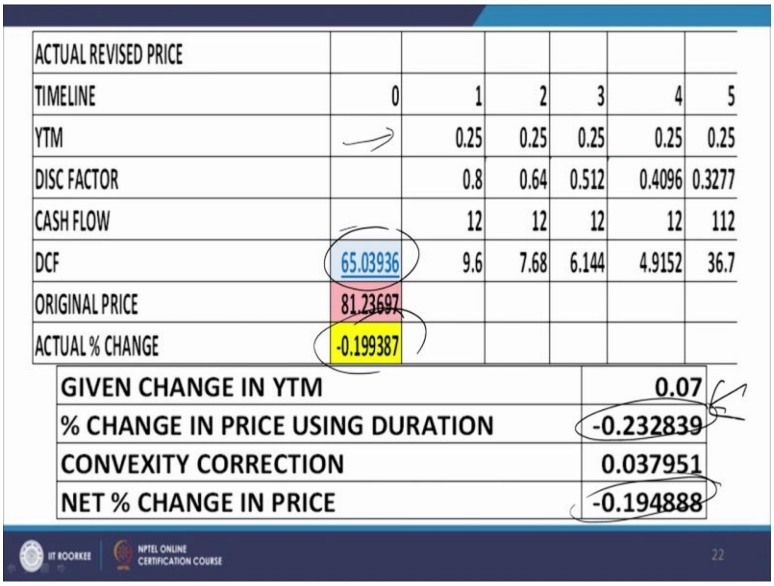
TIMELINE	0	1	2	3	4	5
YTM		0.18	0.18	0.18	0.18	0.18
DISC FACTOR		0.84745763	0.71818	0.60863	0.5157889	0.4371
CASH FLOW		12	12	12	12	112
DCF	81.23697	10.1694915	8.61821	7.30357	6.1894665	48.956
tC(t)		12	24	36	48	560
DISC tC(t)	318.8557	10.1694915	17.2364	21.9107	24.757866	244.78
DURATION	3.925007					
t(t+1)C(t)		24	72	144	240	3360
DISC t(t+1)C(t)	1752.167	20.3389831	51.7093	87.6428	123.78933	1468.7
CONVEXITY	10.7843					

When you discount all the future coupon payments and the base value at maturity. So, you can see here, this is the YTM. So, that constitutes the discount rate, these are the DCF values or discount factor values at 18 percent the second line this is the discount factor values at 18 percent. This is the coupon payment, this is the coupon payment and this is the coupon payment plus the redemption of principal at maturity.

Then you compute the discounted future cash flows from the bond using these discount values. What you arrive at is 81.23 this is the current market price of the bond that corresponds to the YTM of 18 percent.

Then we work out the duration, to work out the duration we need to find TCT the product of the time at which the cash flows takes place there together with the magnitude of this cash flows, which is given in this row, and then we discount each of these cash flows or tC t is rather time weighted cash flows or cash flow weighted time at the discount rate of 18 percent which is the YTM and we arrived at this expression. Dividing it by the… dividing this expression by the price we get this expression for the duration 3.925 years. Similarly, we work out the convexity and we find that the convexity is 10.78 years squared.

(Refer Slide Time: 05:41)



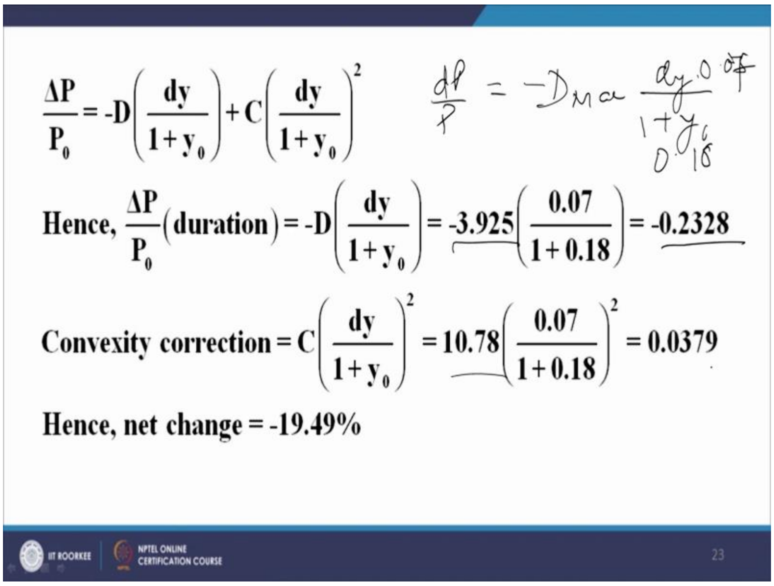
ACTUAL REVISED PRICE						
TIMELINE	0	1	2	3	4	5
YTM		0.25	0.25	0.25	0.25	0.25
DISC FACTOR		0.8	0.64	0.512	0.4096	0.3277
CASH FLOW		12	12	12	12	112
DCF	65.03936	9.6	7.68	6.144	4.9152	36.7
ORIGINAL PRICE	81.23697					
ACTUAL % CHANGE	-0.199387					

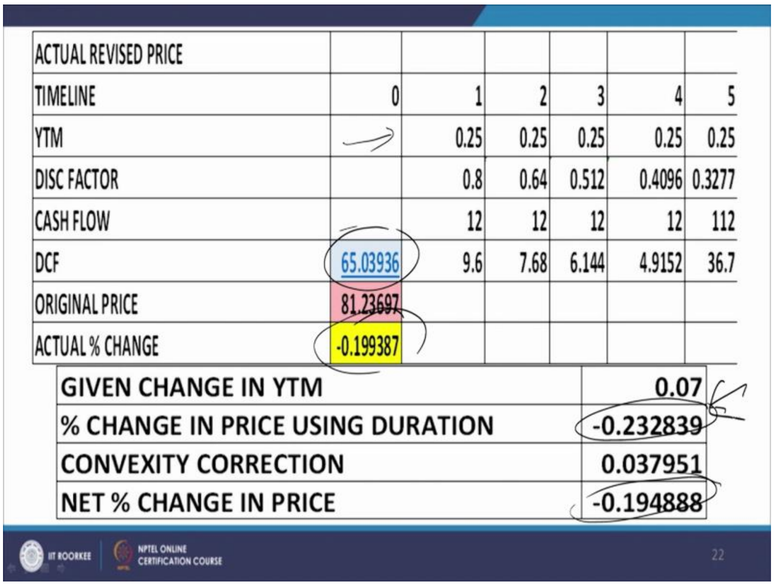
GIVEN CHANGE IN YTM	0.07
% CHANGE IN PRICE USING DURATION	-0.232839
CONVEXITY CORRECTION	0.037951
NET % CHANGE IN PRICE	-0.194888

Now, what is the actual price change to work out the actual price change, we use the new YTM and we again work out the DCF at the new YTM of 25 percent and what we get is the actual prices 65.039. So, original price was 81.23 the new price is 65.03. So, the actual change is 19.94 percent. If we use only duration and calculate the change in price, it turns out to be 23.28 percent.

And if we introduce the convexity correction, we end up with 19.48 percent which is pretty much close to our actual pricing of 19.94 percent, please note here the YTM is changing by as large as 7 percent and not withstanding that, the if we use both direction and convexity we are arriving at a pretty close estimate of the percentage price change. So, this is how this example can be done. And I recommend that all the learners work it out for themselves also.

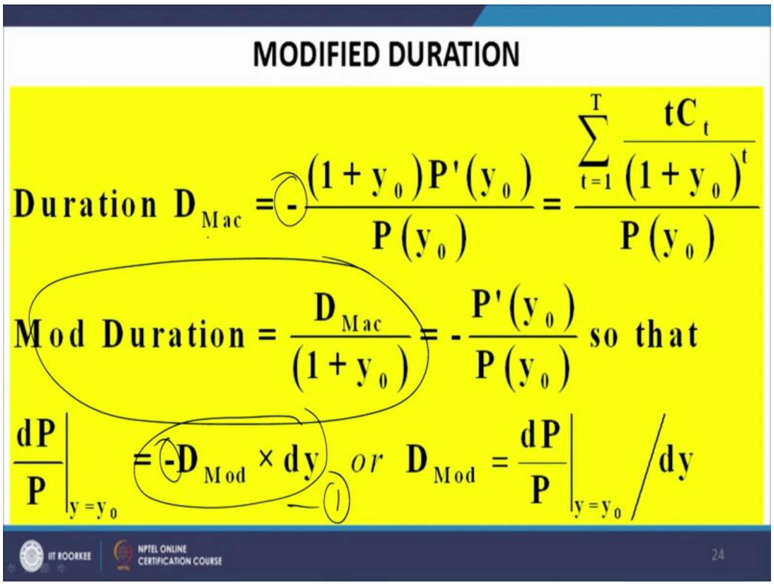
(Refer Slide Time: 06:57)





This is the summary of how the calculations have been made. This is the duration figure, this is the convexity figure and how do we work out the price change dP upon P this is the percentage price change is equal to minus DMac into dy 0 dy rather upon 1 plus y zero y0, y0 here is 18 percent, dy is 7 percent so, that is report 07, when you plug in these values what you get 23.28 which has the figure that we have here.

(Refer Slide Time: 07:44)

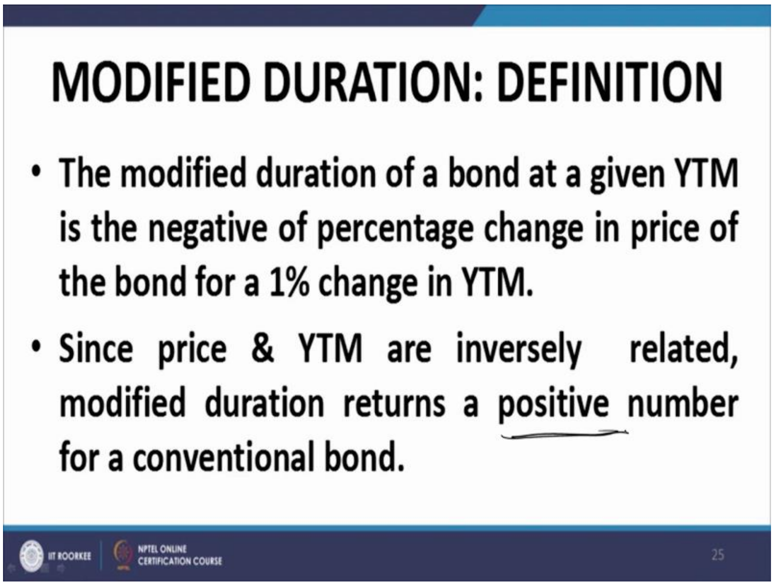


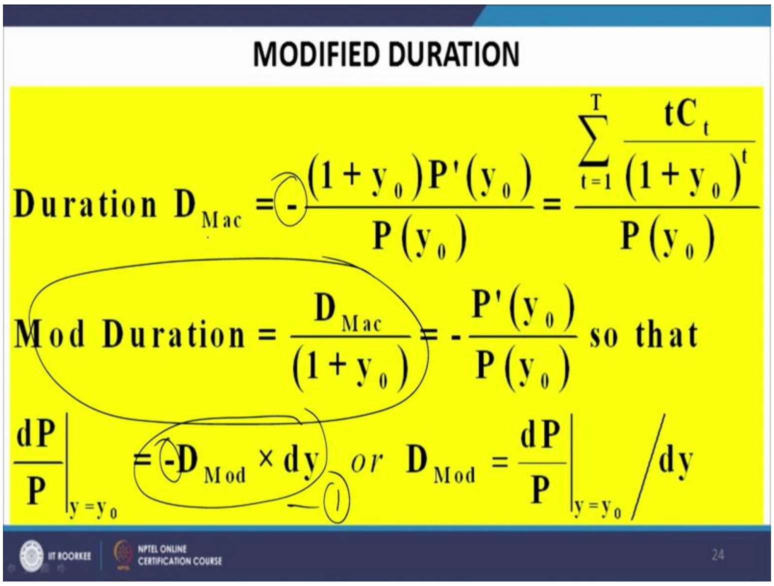
So, let us continue, then we talk about modified duration which we will come be coming back to duration, but in a different context. Let us talk about modified duration, modified duration is a modified version of Macaulay duration, we define modifies duration in terms, in relation to Macaulay duration by this expression.

So, D modified is equal to DMac divided by 1 plus y0, where y0 is the YTM at which the Macaulay duration is calculated. And when you use this expression for the modified duration, what we find is that the percentage change in price is given by this rather simple expression, the percentage change in price is given by this rather simple expression.

Why? Because the factor of 1 plus y0 is absorbed into the definition of duration in the case of modified duration and we end up with a simple expression for the modified, therefore, in fact, this expression this formula let us call it 1 can be used to give out to provide a definition for modified duration as it is the percentage change in price corresponding to a unit change in the YTM.

(Refer Slide Time: 09:02)

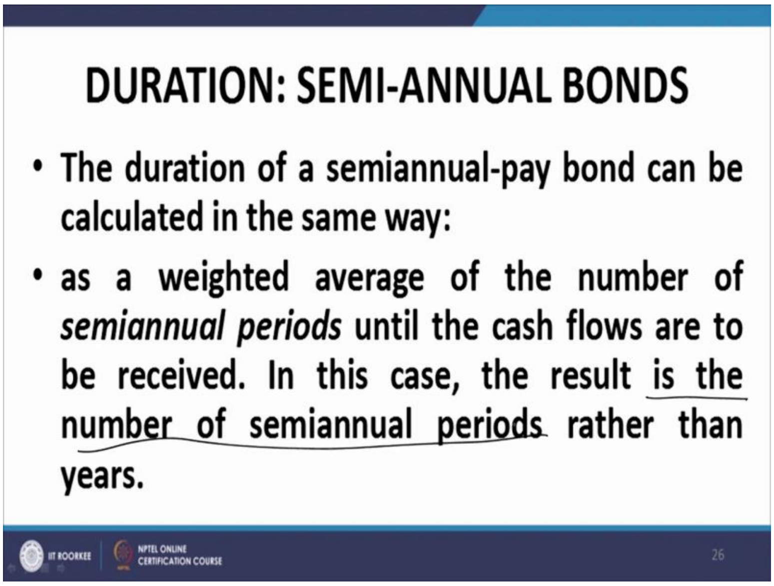




So, that is the definition, the modified duration of a bond at a given YTM is a negative of the percentage change in price of the bond for a 1 percent change in YTM. The price and YTM are inversely related modified duration returns a positive figure please note as in the case of duration, we also add a minus sign when we talk about modified duration.

So, that the figure that we get at the output is positive it is more convenient to deal with positive figures and that is the reason that we introduce a negative sign. Whenever we talk about duration or modified duration. Here as you can see, here, we have a minus sign here and you can see here also when we talk about Macaulay duration, you also have a minus sign here.

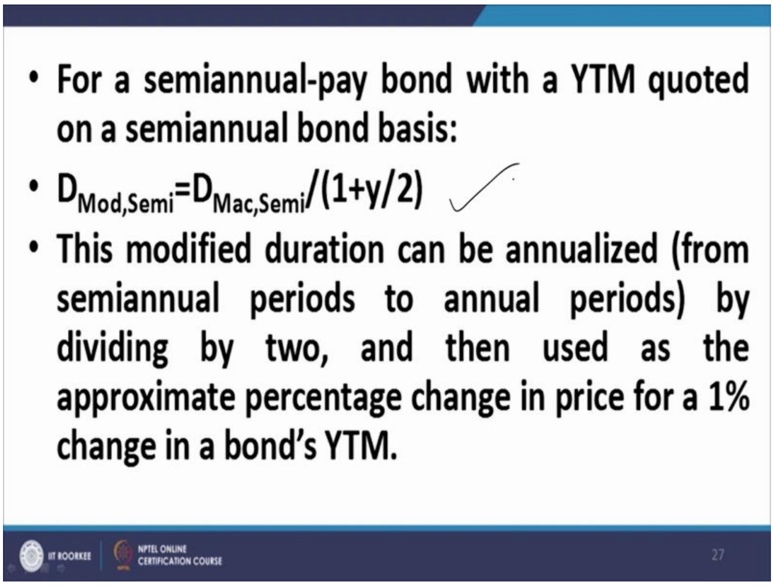
(Refer Slide Time: 09:57)



Duration of semi-annual bonds, the duration of a semi-annual pay bond can be calculated in the same way as a weighted average of the number of semi-annual periods until the cash flows are to be received. For each cash flow, instead of using… you see it is quite straightforward. Instead of using 1 year as the measure of time, we use 6 months or half year as the measure of time, because the cash flows are occurring at intervals of half year.

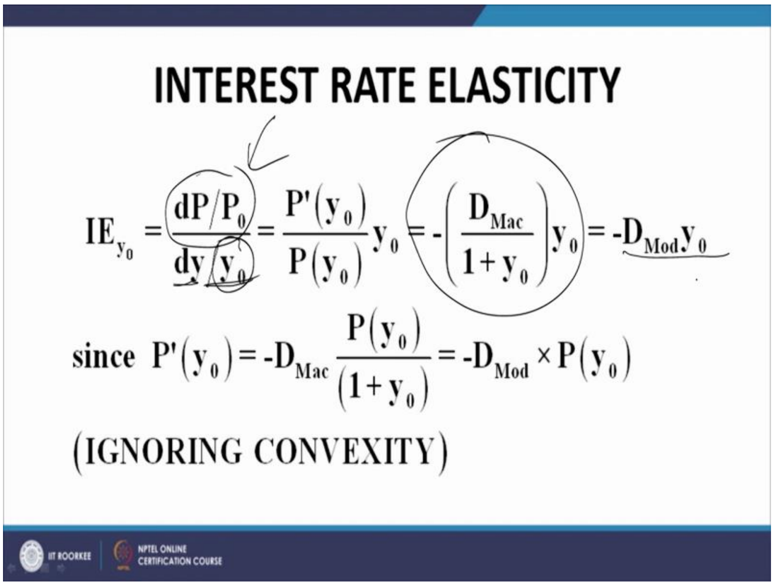
So, at the end of each half year, we use that cash flow and we use the semi-annual period as a measure of time and we calculated the cash flow or cash flow weighted time that will be worked out in terms of the semi-annual period and using that semi-annual period we can use work out the annual duration of a bond. So, the duration of a semi-annual pay bond can be calculated in the same way as a weighted average of the number of semi-annual periods until the cash flows are to be received. In this case, the result is the number of semi-annual periods rather than years.

(Refer Slide Time: 11:14)



And for a semi-annual pay bond with a YTM quoted on a semi-annual bond basis, we work out the duration as per this formula, a modified duration as per this formula, the modified duration can be annualized from semi-annual periods to annual periods by dividing by 2 and then used as the approximate percentage change in price for a 1 percent change in the bonds YTM.

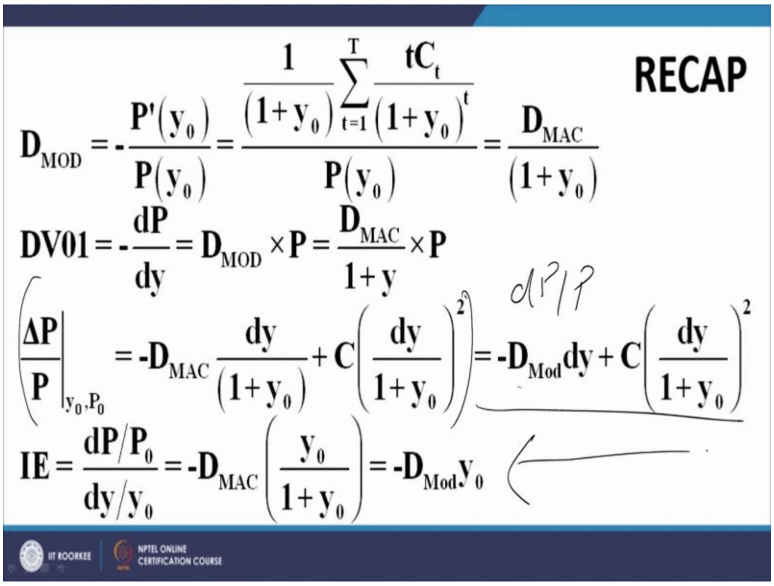
(Refer Slide Time: 11:44)



So, we talk about interest rate elasticity, interest rate elasticity is given by the percentage change in the price of a bond divided by the percentage change in the YTM as a percent of the original YTM. The change in the YTM as a percentage of the original YTM please note this, the numerator is the percentage change in price there is no issue about this, there is no confusion about this.

But, as far as the denominator is concerned, it is the change in YTM expressed as a percentage of the original YTM and this is what we call the interest rate elasticity in terms of duration, it turns out to be this expression minus DMac upon 1 plus y0 into y0 or minus D modified into y0, here y0 is the point or is the YTM at which we are working out the interest rate elasticity and importantly we are ignoring the convexity of the yield price relationship.

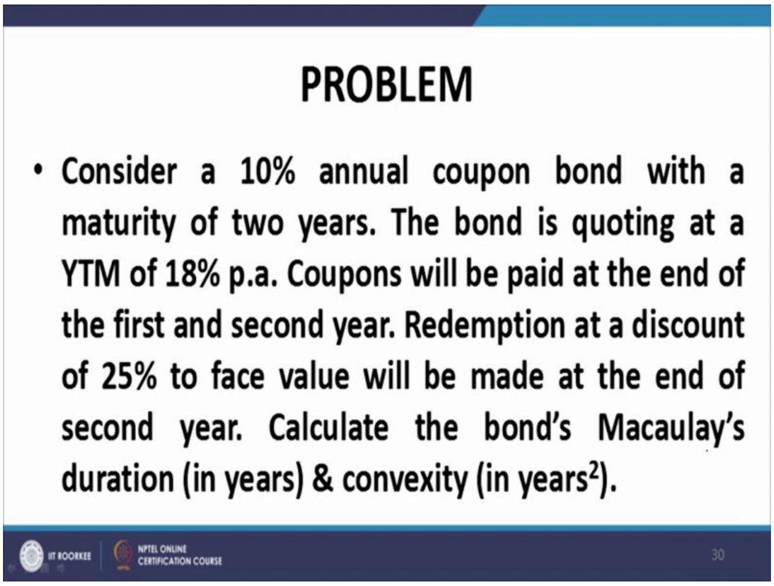
(Refer Slide Time: 12:59)



So, a quick recap here. We started with DV01, which is the change in not the percentage change in price change the absolute change in price corresponding to a given change in YTM or a 1 percent change in YTM or a unit change in YTM with a negative sign of course, to return a positive figure because YTM and price are inversely related.

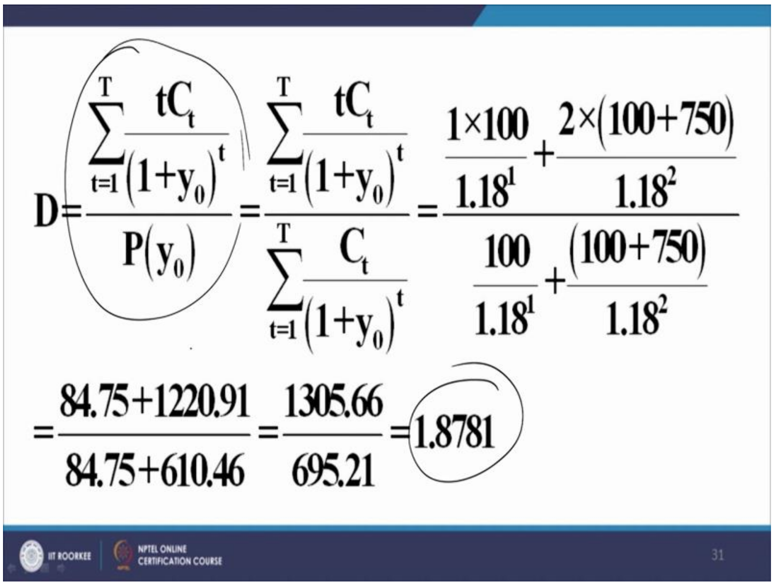
Then we talked about the duration and convexity of the bond which is captured by this expression. I also introduced modified duration in terms of which dP upon P can be written as this expression. And then, we talked about interest rate elasticity, which is given by this expression.

(Refer Slide Time: 13:45)



As another problem consider a 10 percent annual coupon bond with a maturity of 2 years, the bond is quoting at a YTM of 18 percent, coupons will be paired at the end of the first and second year. Redemption at a discount of 25 percent of face value will be made at the end of the second year. Calculate the bolts Macaulay duration and convexity. This is a rather simple problem much simpler to what we did earlier.

(Refer Slide Time: 14:12)



This is the solution to this problem, we end up with a duration of 1.8781 years using the formula for duration, this is the formula for duration that we have discussed earlier as well.

(Refer Slide Time: 14:28)

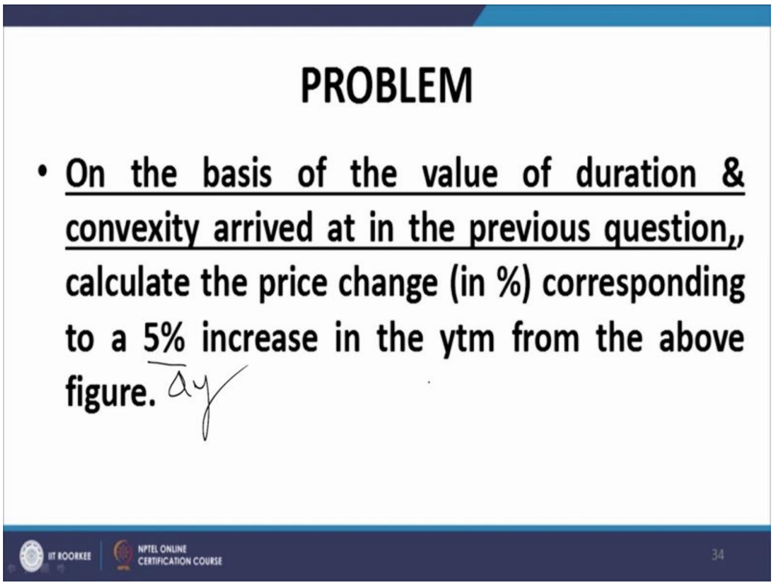
$$C = \frac{\sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y_0)^t}}{2P(y_0)} = \frac{\frac{1\times 2\times 100}{1.18^1} + \frac{2\times 3\times (100+750)}{1.18^2}}{2\times\left(\frac{100}{1.18^1} + \frac{(100+750)}{1.18^2}\right)}$$

$$= \frac{169.50+3662.73}{84.75+610.46} = \frac{3932.23}{2\times 695.21} = 2.7562 \text{ years}$$

And this is the convexity, the convexity turns out to be 2.7562 years.

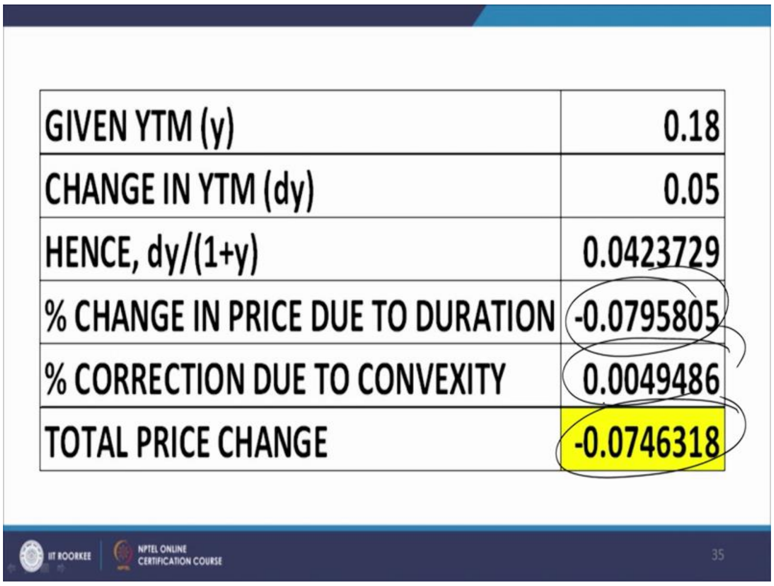
(Refer Slide Time: 14:38)

LET FACE VALUE BE	1000		
YEAR		1	2
COUPON PAYMENT	0.1	100	100
REDEMPTION VALUE	0.75		750
TOTAL CASH FLOWS		100	850
YTM	0.18	0.18	0.18
DISCOUNT RATE		0.847457627	0.7181844
CURRENT PRICE	695.20253	84.74576271	610.45677
TIME WEIGHTED CASHFLOWS	1305.6593	84.74576271	1220.9135
DURATION	1.8780992		
t(t+1) WEIGHTED CASHFLOWS	3832.2321	169.4915254	3662.7406
CONVEXITY	2.7561983		

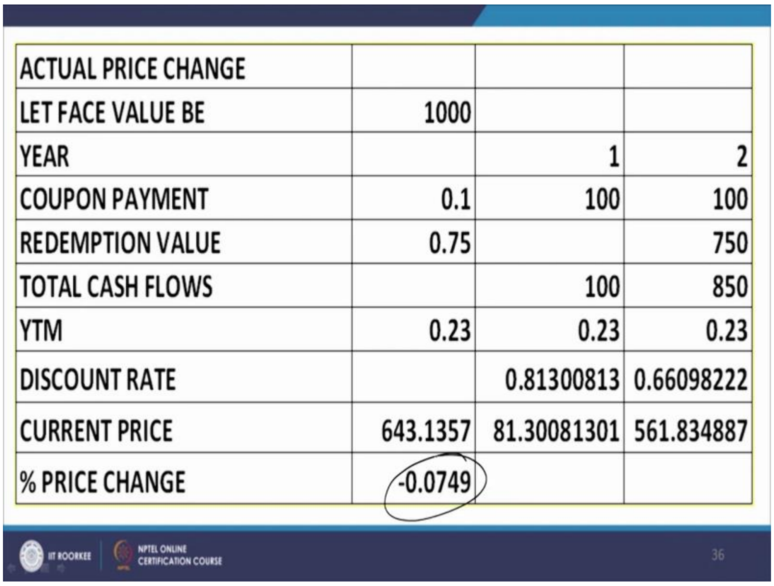


This is the Excel working of this formula of the question, on the basis of the value of duration and convexity arrived at in the previous question. Calculate the price change in percentage corresponding to a 5 percent this is delta y. Increase in the YTM from the above figure. So, the original YTM was how much the original bottom was 18 percent, the original YTM was 18 percent, the YTM is increased to 23 percent.

(Refer Slide Time: 15:09)



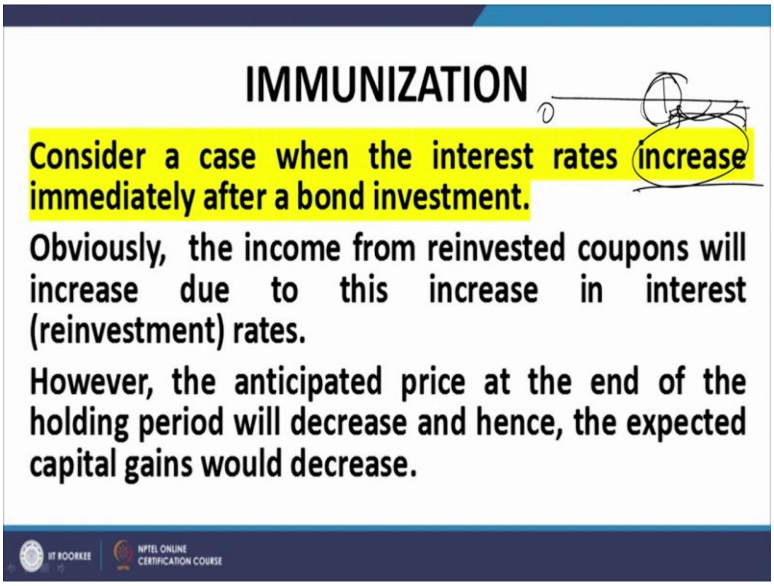
GIVEN YTM (y)	0.18
CHANGE IN YTM (dy)	0.05
HENCE, dy/(1+y)	0.0423729
% CHANGE IN PRICE DUE TO DURATION	-0.0795805
% CORRECTION DUE TO CONVEXITY	0.0049486
TOTAL PRICE CHANGE	-0.0746318



ACTUAL PRICE CHANGE			
LET FACE VALUE BE	1000		
YEAR		1	2
COUPON PAYMENT	0.1	100	100
REDEMPTION VALUE	0.75		750
TOTAL CASH FLOWS		100	850
YTM	0.23	0.23	0.23
DISCOUNT RATE		0.81300813	0.66098222
CURRENT PRICE	643.1357	81.30081301	561.834887
% PRICE CHANGE	-0.0749		

And this is the change in change in the price of the bond using duration alone. This is the convexity correction, this gives you the change in price with up to second order correction that is correction together with the duration and the convexity and this is the actual change in price 7.49 percent and you have 7.46 percent as the best approximation up to second order.

(Refer Slide Time: 15:48)



Now, we talk about immunization. I had given a feel of what immunization is right at the beginning of the last class when I talked about investment in a bond. I mentioned that the if there is a change in the interest rates, obviously, there would be a change in the reinvestment income and the capital gains due to an investment in a bond.

Now the important thing as I also mentioned the second very important facet, and that was that this these two forces operate in opposite directions. Nevertheless, notwithstanding the fact that they operate in opposite direction and would analyse each other to some extent, the extent of an enrolment depends on the holding period of the investor. If the holding period is shorter, the impact on reinvestment income is less, the impact on capital gains is much more and therefore, again the enrolment will not be perfect.

Conversely, if the holding period is very close to maturity, then what happens reinvestment income effect is more the price effect or the capital gains effect is less and again there would be a mismatch and the enrolment will not be accurate. However, there is exist one such point in this line, this is 0, this is T let us call this H.

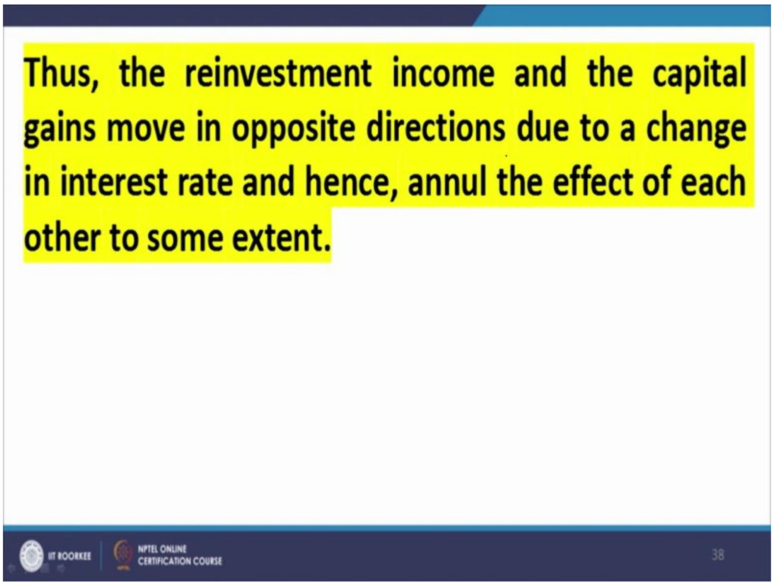
H is my holding period, there is one special holding period, such that if you hold the bond up to this holding period H, then what happens. Then, if there is a change in interest rate, during the life of the investment, the change in the reinvestment income, a chain corresponding to or consequential to the coupons reinvestment exactly matches the change due to the change in the price. And as a result of which these two effects exactly cancel each other.

And because these two effects exactly cancel each other, what happens is, the investor becomes immunized against the change in the interest rate, if there is an increase in interest rate, the increase in investment income is exactly neutralized by the decline in price. And if there is a decrease in interest rates, that decrease in investment in reinvestment income, again is neutralized by the increase in the price of the bond.

So, consider a case when the interest rates increase immediately after a bond investment. Obviously, the income from reinvested a coupon increase please note this word. Obviously, the income from reinvested coupons will increase due to the increase in interest rates, reinvestment rates, because the interest rates have increased. So, all the coupons will be reinvested at the higher rates.

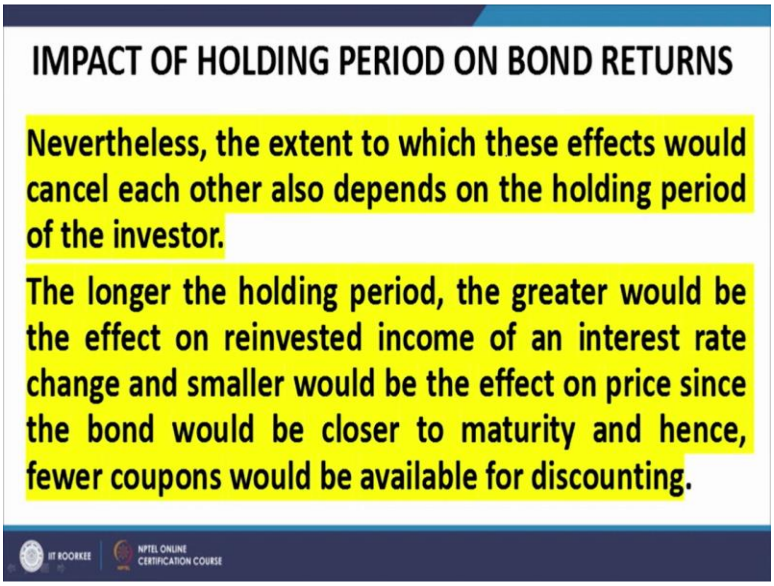
However, the anticipated price at the end of the holding period, please note this is not the maturity value, this is not the maturity value, this is the value at this point, when the bond will have to be sold in the market to realize or to end the investment horizon. The anticipated price at the end of the holding period, which is the investment horizon will decrease why? Because the rate, the coupons and the final value or maturity value will be discounted at a higher rate, the interest rates have increased. So, enhanced the expected capital gain would decline.

(Refer Slide Time: 19:21)



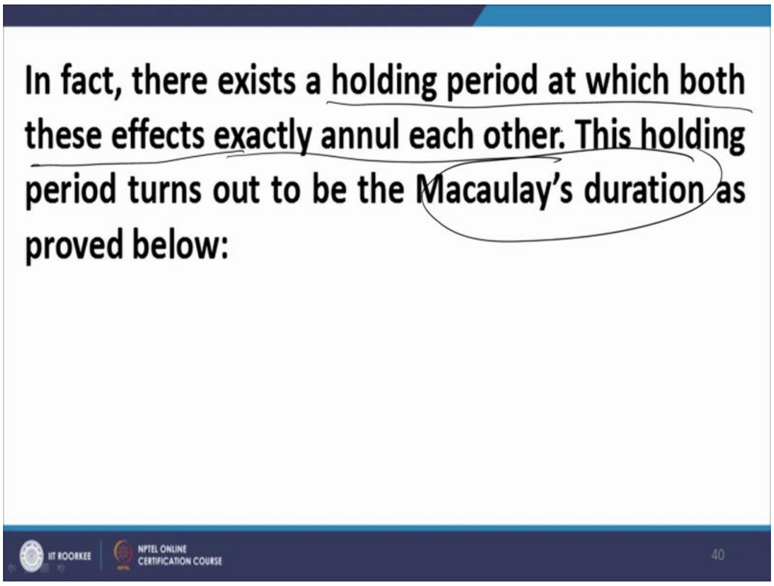
Thus, the reinvestment income and the capital gains move in opposite directions due to a change in interest rates and hence, annual the effect of each other to some extent.

(Refer Slide Time: 19:33)



I have dealt with it in a lot of detail. Nevertheless, the extent to which these effects would cancel each other would also depend on the holding period of the investor. The longer the holding period, the greater would be the effect on reinvested income of a interest rate change and smaller would be the effect on the price since the bond would be closer to maturity and hence fewer coupons would be available for discounting, as well as the annual cash flow would also be very close to the holding period and therefore, the discounting periods would be the period of discounting or the terminal cash flow will also be lower.

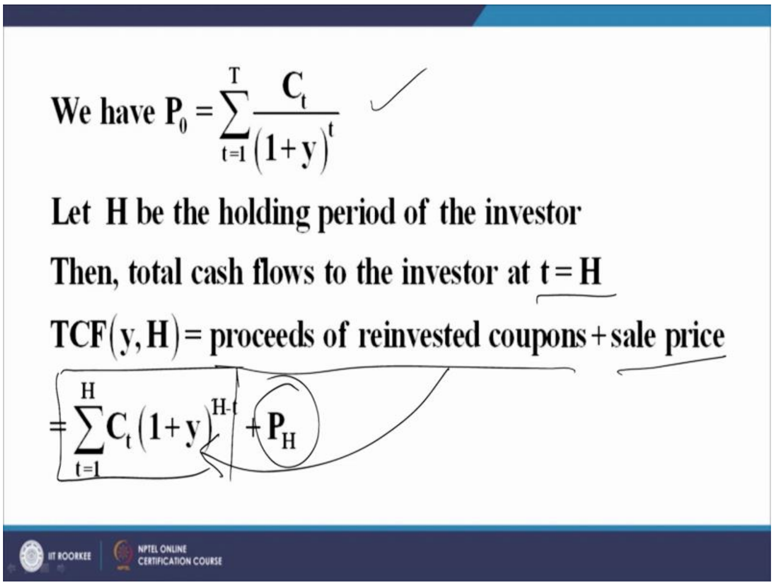
(Refer Slide Time: 20:14)



In fact, there exists a holding period at which this is the important part, holding period at which both these effects exactly annul each other this holding period turns out to be the Macaulay duration. So, Macaulay duration in addition to being a measure of interest rate risk is also has a very important property that if the holder, if the investor holds the bond up to its Macaulay duration, if the investment horizon coincides with the Macaulay duration, and there is a small shift in interest rates during his holding period, his return does not change, does not change.

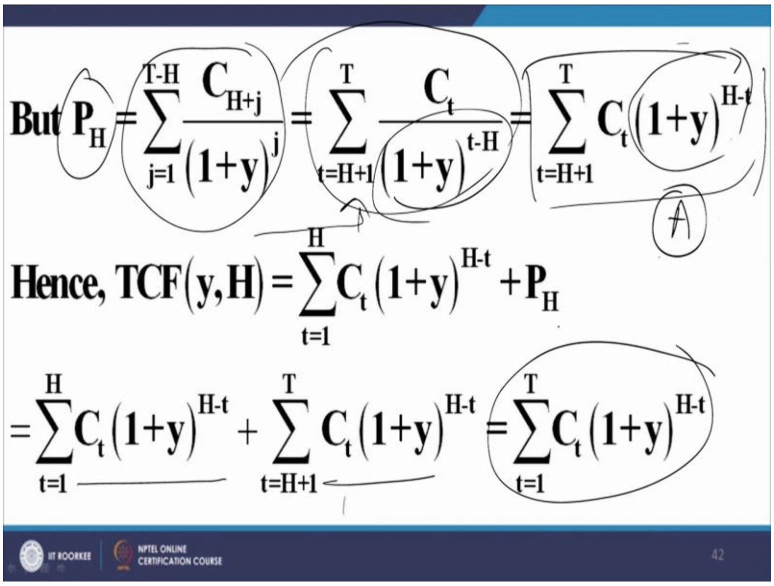
Because the reinvested income and the capital gains that move in opposite directions and annul each other for that particular holding period. We will take it up with an example also later on. So, let us see how we arrive at this expression that Macaulay duration constituents and that holding period for which the return is independent of small changes in the yield shift.

(Refer Slide Time: 21:26)



This is the formula for the price of a bond or formula for the YTM of the bond you will look at it either way, price in YTM they are closely related quantities, let H be a holding period of the investor then the total cash flows for the investor at T equal to H will consist of two parts, it will consist of proceeds of reinvested income plus sale price, the proceeds of reinvested income can be written in this form. And this is the price at which the bond will be liquidated in the market at the end of the holding period.

(Refer Slide Time: 22:10)



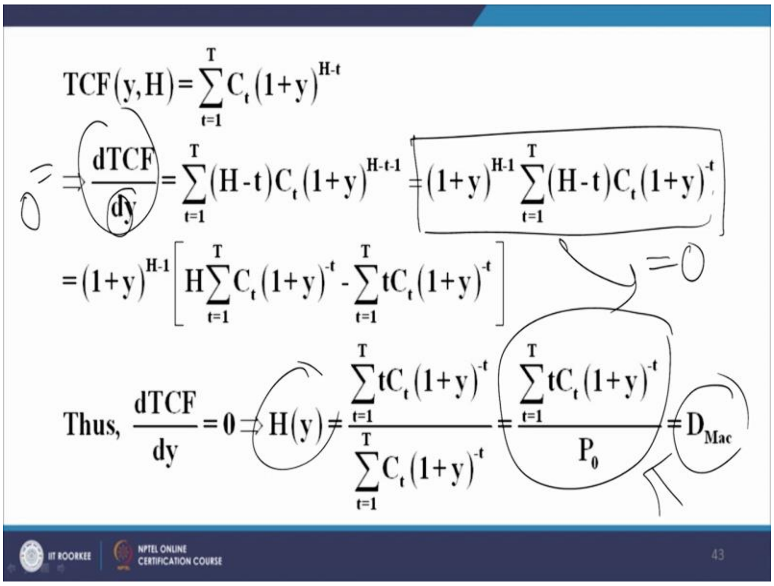
Now, what is PH? PH is obtained by discounting all future cash flows from T equal to H onwards all future cash flows arising from the bond from T equal to H onwards up to its maturity and discounted back to T equal to H. I repeat, how do we get PH, we get PH by discounting all future cash flows from the bond from T equal to H onwards at the appropriate market rates, up to or bring them back to T equal to H. So, the formula turns out to be this expression, we start with j equal to 1 that is it starts with a CH plus 1 then CH plus 2, CH plus 3.

So, we write it as CH plus j, where j is sum from 1 to T minus H, there will be T minus H coupons, H plus 1, H plus 2, H plus 3 up to T, T minus H coupons and they will be discounted backwards, the coupon at t equal to H plus 1 will be discounted for 1 period the coupon because you are discounting up to the point T equal to H. So, the coupon at T equal to H plus 2 will be discounted for two periods and T equal to H plus 3 will be discounted for 3 periods and t equal to capital T would be discounted for T minus H periods.

The objective is to bring everything back to T equal to H not T equal to 0, because we are working out the price PH, we are working out the price at T equal to H by a revision of the indices, we can write this expression in this form by a renaming of the index from substituting the index from j equal to j to t where t takes the value from H plus 1 to capital T, where capital T is the maturity H is the holding period. Let me remind you and in terms of the index t, I can write this expression in the form of this expression.

And which can be further written is this expression. Very simple. We simply take this to the numerator, we take this denominator to the numerator and we arrive at this expression. So, the total… when we substitute the values of PH given by this equation, let us call it A in this equation, let us call it B substituting from A in B substituting the value of PH from A and B, what do we get? We get this expression which can be combined together and we get this result. So, this is the result that we have this is a total cash flows that will arise at T equal to H, which is the end of our investment horizon, when we liquidate the bond at this point.

(Refer Slide Time: 25:14)



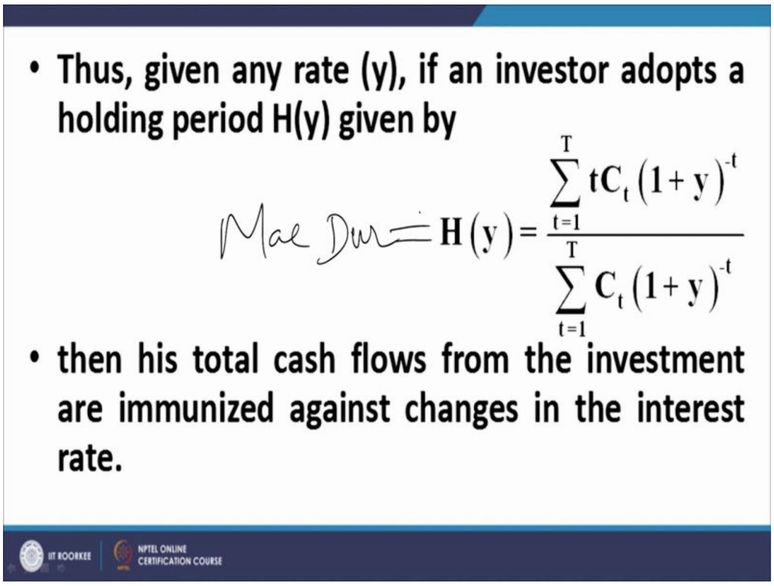
So, what we do now is, please note this this expression the total cash flows is a function of obviously, the interest rate that is why the YTM of the bond and the holding period as I mentioned, holding period is everything to do with the cash flow that we are going to realize, at the end of that holding period.

So, we take the first derivative with respect to y by taking the first derivative what we get, we get this expression. Our next objective is to equate this first derivative to 0. Why are we doing this, we are doing this why, because we are looking for that holding period, we are looking for that value of H such that if I hold the ball for that value for that holding period for that investment horizon, then the change in the total cash flows consequent to a small change in y that is dy turns out to be 0, that in other words, the slope of the curve at this point turns out to be very flat is almost 0, so that if there is a small change in YTM, either to the left or to the right, that does not impact our total cash flows to any significant extent.

So, when we do that, when we equate this to 0, when we could this expression to 0. And we simplify a little bit, what we get is H of y, H as a function of y is given by this expression. And believe it or not, when you see this expression very carefully, we find that it is nothing but DMac it is nothing but the formula that we arrived at for DMac.

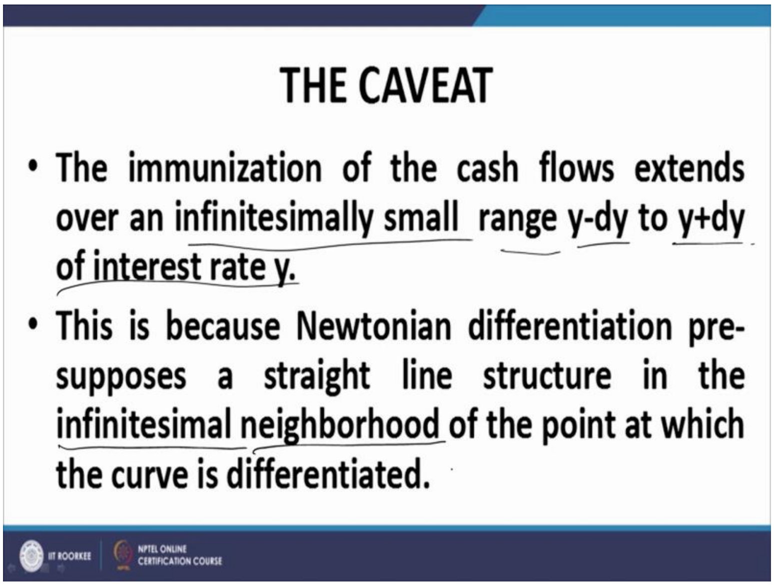
So, what is the outcome of all this stuff, the outcome of all this stuff is at if you hold the bond for a period, if your investment horizon for holding the bond coincides with a bonds Macaulay duration, then if there are small shifts in the interest rate, the small shifts in the interest rate will not affect your overall return.

(Refer Slide Time: 27:23)



That is given any rate why if an investor adopts a holding period Hy given by this expression, which is equal to the duration Mac duration you end up with the total cash flows from the investment or immunized against changes in interest rates.

(Refer Slide Time: 27:44)

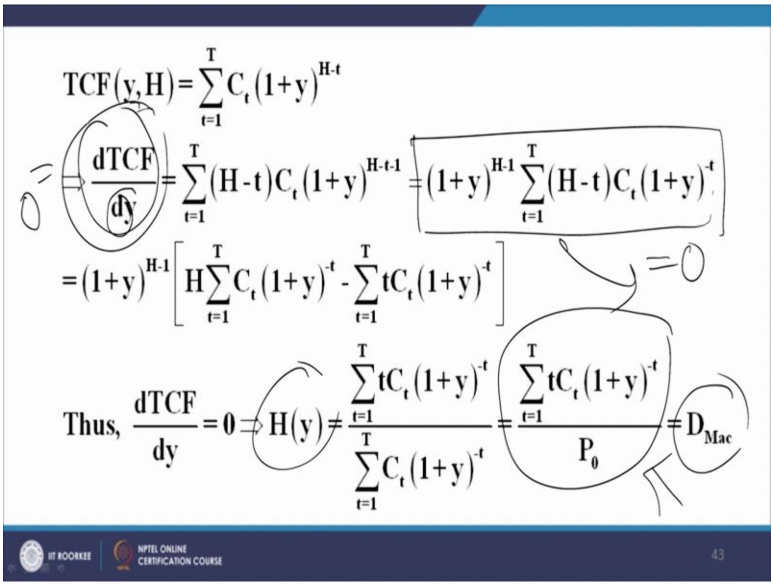


The caveat to this formula looks very attractive, very exciting, very innovative, but there is always a catch to such a formula, the catch is that the immunization of the cash flows extends over infinitesimally small range y minus dy to y plus dy of interest rate y. If there is a significant or a massive change in interest rates, this formula breaks down, it is only it only holds if the change in y is of infinitesimal nature is a very small nature either to the left or to the right whatever the case will be, but the change the magnitude of the change has to be very small.

If the magnitude of the change is small, then what happens what happens is that the investor gets immunized against any change in return any change in the total cash flows at the end of his holding period obviously, his investment is fixed. So, the return that he gets over his holding period is also unchanged due to or notwithstanding that there is a minor change in interest rates to the left or to the right.

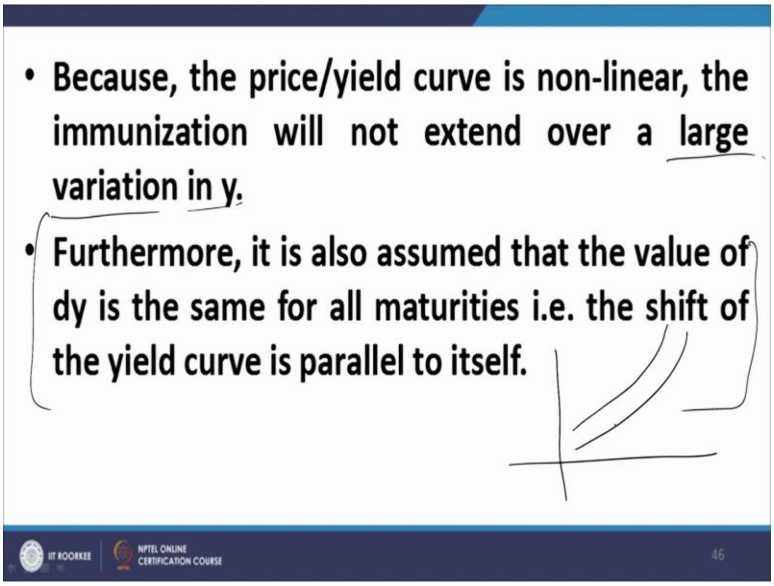
This is because Newtonian differentiation presupposes a straight-line structure in the infinitesimal neighbourhood, this line is or this is a very important infinitesimal neighbourhood, whenever you work out a derivative under the Newtonian concept, you assume that around that point at which the derivative is being calculated, the curve can be approximated well by a straight line. So, that is precisely what we are doing here.

(Refer Slide Time: 29:33)



Please note, we have worked out the Newtonian derivative here, this is the Newtonian derivative, and when we talk about the Newtonian derivative, we assume that the relationship between TCF and y at that point is almost a straight line.

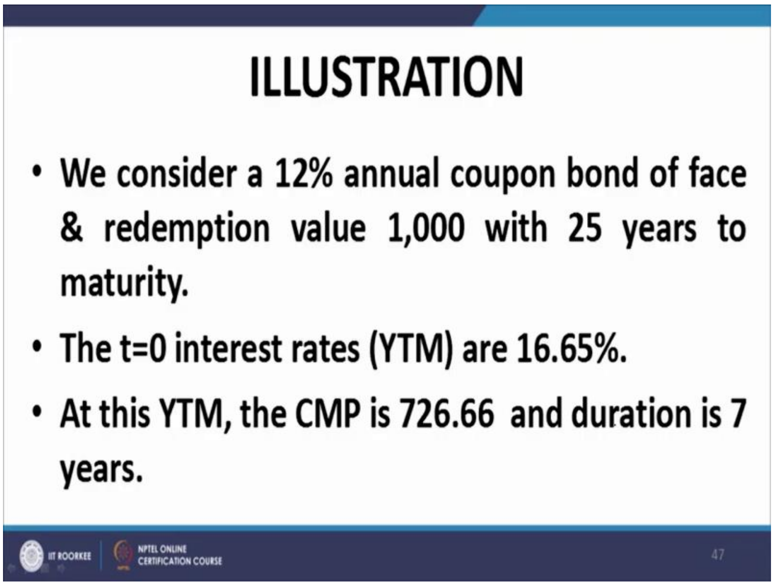
(Refer Slide Time: 29:45)



Because the price yield curve is nonlinear, the immunization will not extend over a large variation in y. So, the immunization or the protection that we have consequent to this derivation, consequent to our holding the bond upto its duration, will not hold. If there is a significant change in the YTM than this immunization breaks down.

Furthermore, it is also assumed that the value dy is the same for all maturities, that is, the shift of the yield curve is parallel to itself. This is another very important assumption, but I will defer for it for detailed discussion to a later class, when I talk about portfolio… fixed income portfolio management. What we are basically saying is that the yield curve is shifting parallel to itself, the yield curve is shifting parallel to itself and that is the rates, the interest rates corresponding to different maturities are changing by the same amount dy.

(Refer Slide Time: 30:55)



Then there is an example, which I shall take up in the next lecture. Thank you.