Decision Making With Spreadsheet
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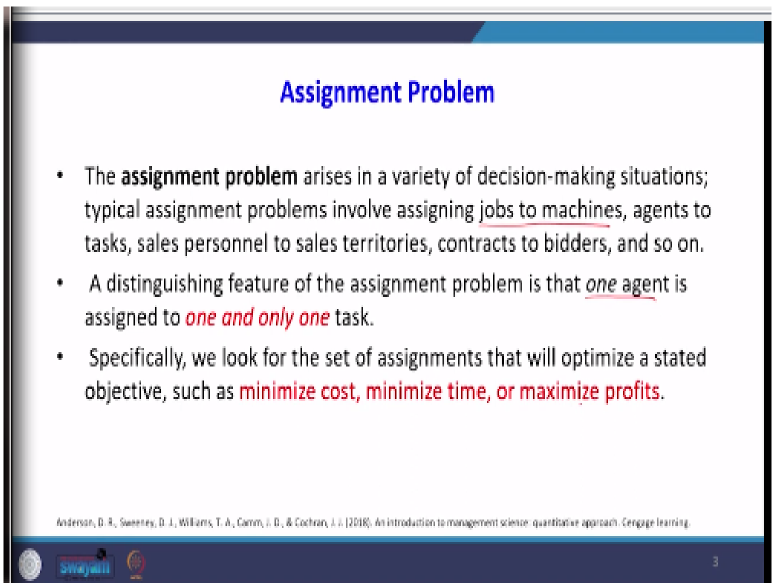
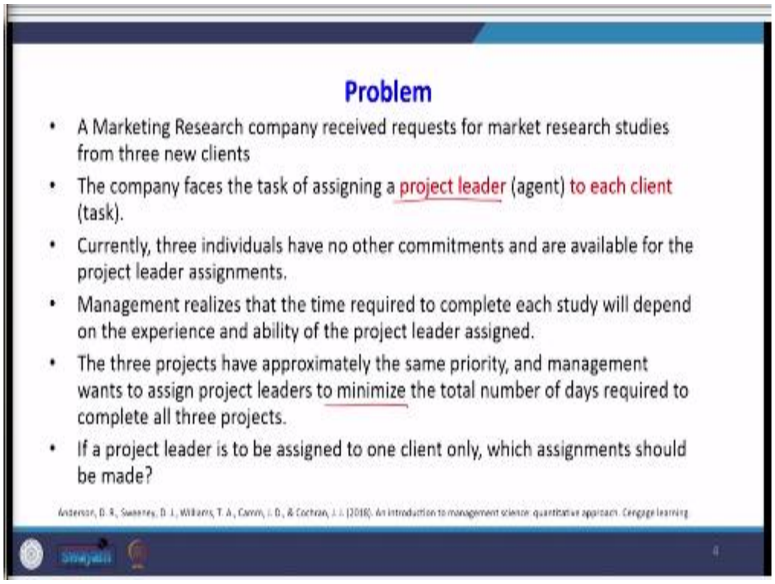
Lecture-20
Assignment and Shortest Path Problem

Dear students, in the previous class we have discussed transportation and transshipment problems. In this lecture, I am going to explain the assignment and shortest path problem. We can say the assignment problem is a special case of our transportation problem. So, the agenda for this lecture is the assignment problem and shortest path problem. So, what is this assignment problem?



Assignment problem arises in a variety of decision-making situations. So, the typical assignment problems involve assigning jobs to machines. There are some jobs that have to be assigned to machines. Agents to task, there is some agents, there are tasks, so the agent has to be assigned to the task, sales personnel to sales territories, there is some sales personnel these people have to be a lot of different sales territories, then contract to bidders.

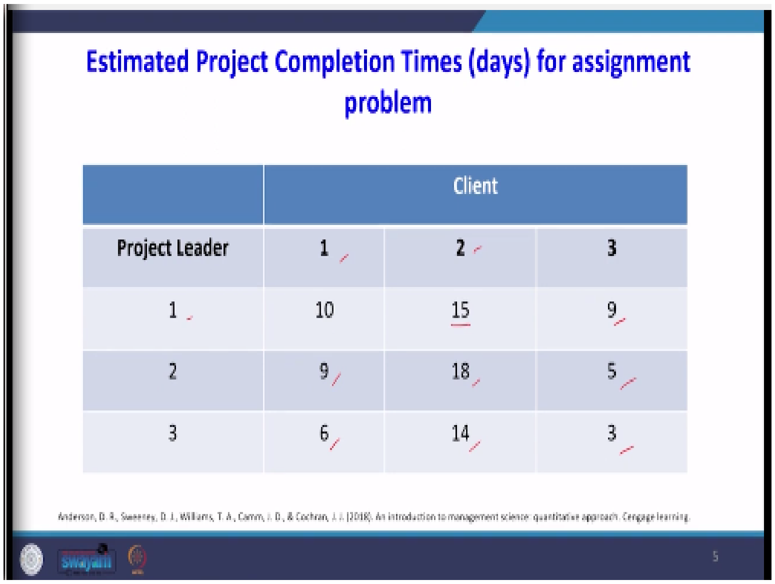
So, like this, there are so many applications of assignment problems. A distinguishing feature of this assignment problem is that one agent is assigned to one and only one task, which is the assumption. Specifically, we look for the set of assignments that will optimize the stated objective, such as minimizing the cost, minimizing the time, or maximizing the profit, so this is the objective function of our assignment problem.



So, I have taken a sample problem from this Anderson et al., book. The problem is like this: a marketing research company received a request for market research studies from 3 new clients. The problem the company faces is the task of assigning a project leader, you call an agent, to each client task. There are some project leaders who are there; there are some clients who are there, so this project leader has to be assigned to different clients.

Currently, 3 individuals and 3 project leaders have no other commitments and are available for the project leader assignment. So, the management realizes that the time required to complete each study will depend on the experience and ability of the project leader assigned, three projects have approximately the same priority. Management wants to assign project leaders to minimize the total number of days required to complete all three projects.

Here objective function is to minimize the total number of days required to complete the projects. If your project leader is to be assigned to 1 client only, which assignment should be made, that is a problem. There are some project leaders; there are some clients, so we have to allocate these project leaders to the clients, so that the total project duration is minimized.



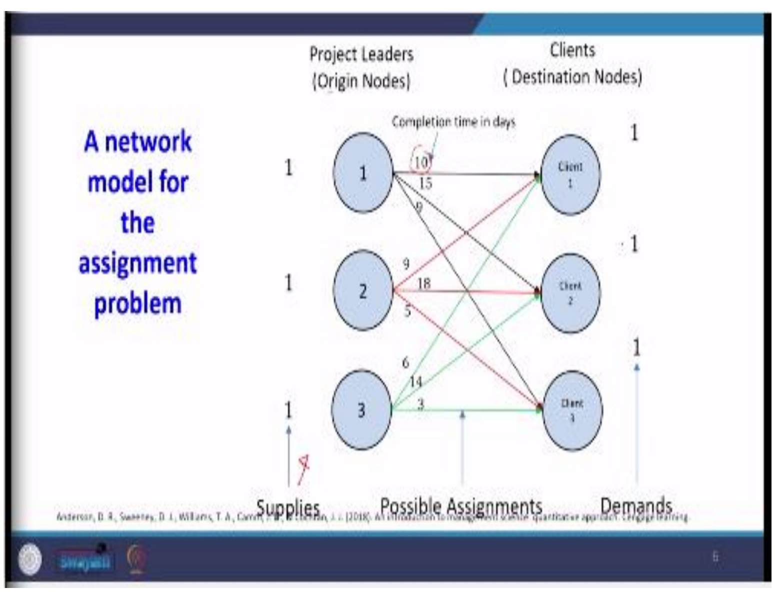
Estimated Project Completion Times (days) for assignment problem

	Client		
Project Leader	1	2	3
1	10	15	9
2	9	18	5
3	6	14	3

Anderson, D. R., Sweeney, D. J., Williams, T. A., Camm, J. D., & Cochran, J. J. (2018). An introduction to management science: quantitative approach. Cengage learning.

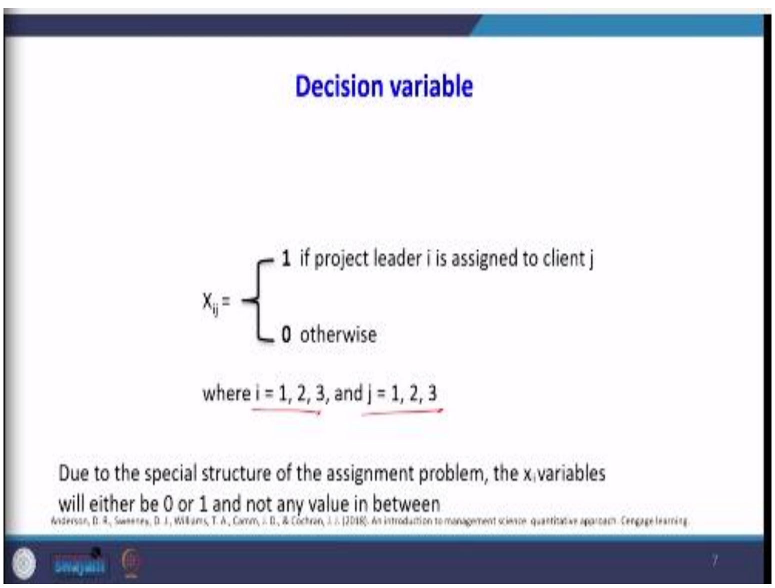
This table shows the estimated project completion time; the unit is days for assignment problems. For example, how to read this? Leader 1, if he is assigned to client 1 he may take 10 days to complete it. If Project Leader 1 is assigned to Client 2, he may take 15 days, so if Project Leader 1 is assigned to Client 3, it will take 9 days. Maybe the specialization of the project leader may be on the client 3 projects so that he is taking less time.

Project leader 2, for client 1, will take 9 days and will consume 18 days for client 2 and 5 days for client 3. Similarly, for project leader 3, he will take 6 days for client 1, 14 days he will take for client 2, and 3 days for client 3. So, here, the objective is we have to assign these project leaders to different clients so that the total completion time is minimized.

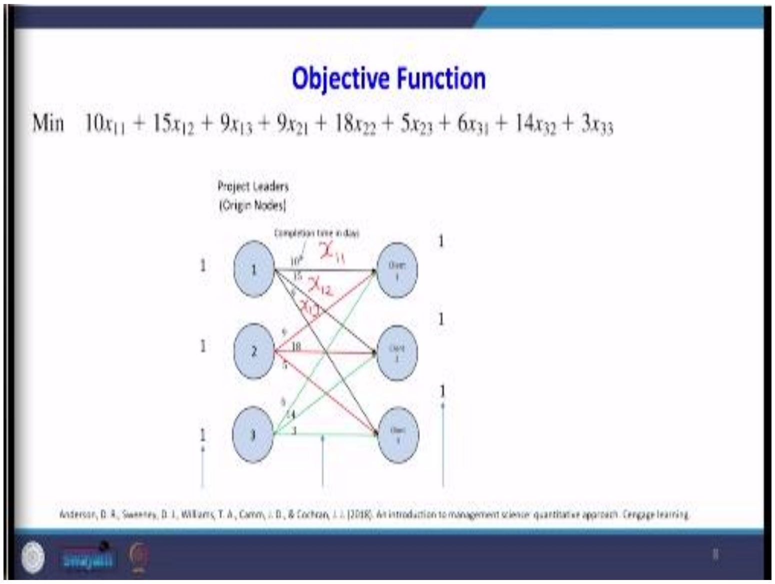


Now, the assignment problem I am going to present is in the form of a network. What is the network? It is like our transportation problem. Here the way we have represented the transportation problem in the form of a network in the same way we can represent the assignment problem also in the form of a network. On the left-hand side, you see there are project leaders equivalent to origin nodes, and on the right-hand side, there is a destination node, clients, and only one project leader, so 1, 1, 1 on the left-hand side, the right-hand side also 1, 1, 1.

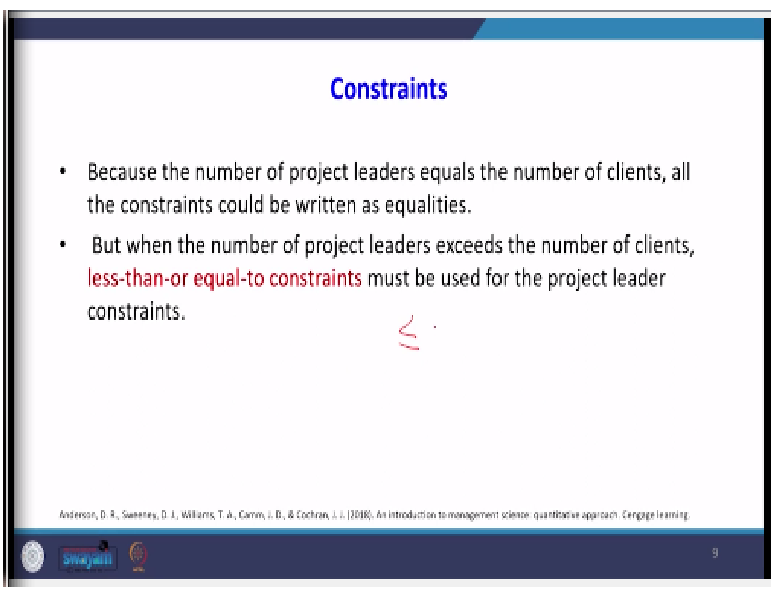
So, for example, this 10 represents completion time in days; what it says is if project leader 1 is assigned to client 1, he may take 10 days. What does the 15 represent? Project leader 1 is assigned to client 2, which may take 15 days, similarly 9 days. So, this is equivalent to our transportation problem context, equivalent to supplies. The right-hand side is equivalent to demand; you see only one person so it is 1. The right-hand side only has one client, so the right hands are also 1, 1.



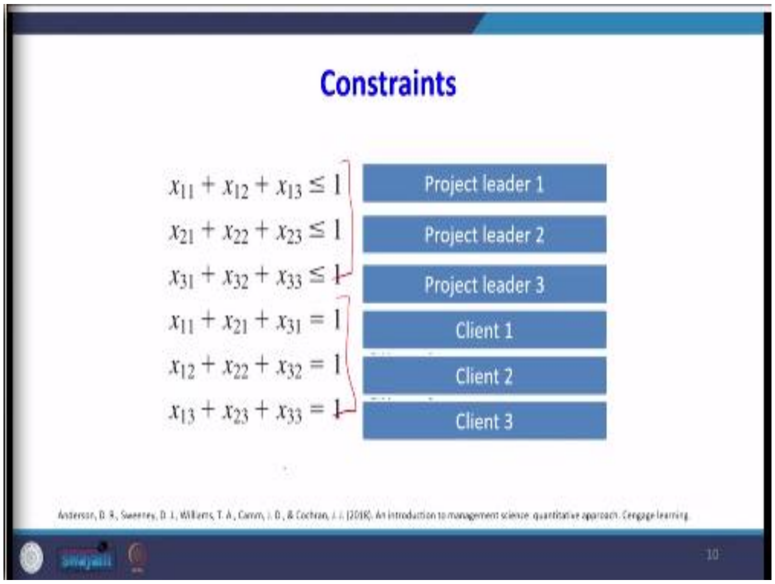
Now, here, the decision variable is binary. So, Xij can take only 2 possibilities; the value of Xij = 1 if the project leader i is assigned to client j, 0 otherwise. There are three project leaders, i = 1, 2, 3, and 3 clients, j = 1, 2, 3. Due to the unique structure of this assignment problem, the Xij variables will be either 0 or 1, not any value in between; that is the assumption for our decision variables; it is a binary decision variable.



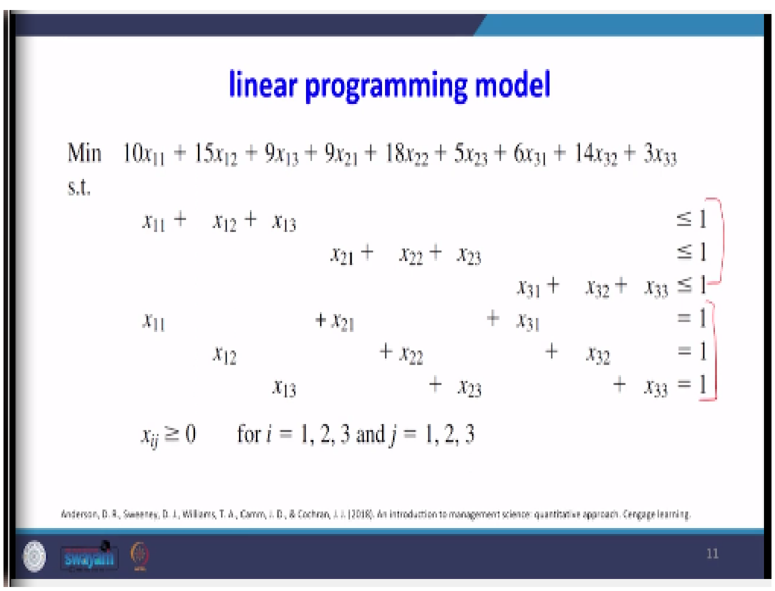
Now, the first task is to write the objective function; what is the objective function? Total number of completion days that have to be minimized. So, if I say this is x 11, this is one decision variable, leader one is assigned to client 1, so how many days will he take? 10 days, so 10 x11, this is x12, 15x12, this is x13, then 9x13 similarly 9x21, 18x22 and 5x23. For the third project leader, 6x31 + 14x32 + 3x33, this is the value of the objective function that has to be minimized.



What are the constraints? Because the number of project leaders is equal to the number of clients, all the constraints should be written as equalities. But when the number of project leaders exceeds the number of clients less than or equal to constraint must be used for project leader constraint. So, this type of constraint has to be used where when the number of project leaders is higher than the number of clients.



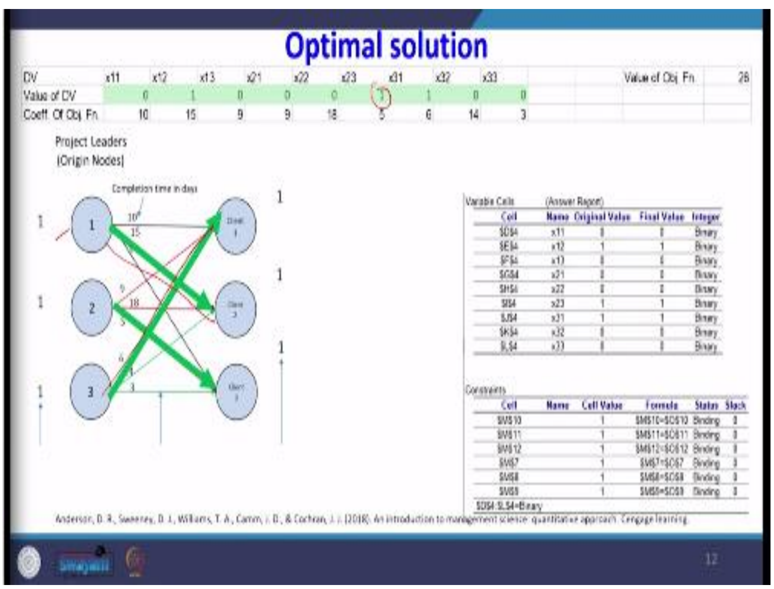
Here, first, we will write about supply constraints that are for the project leader. So, x11, x12 + x13, you can write equal to 1 or less than or equal to 1; there will not be any problem. That means any one project leader can be assigned to any one client. So, these 3 are supply constraints; these 3 are demand constraints. You see that the demand constraints are equal to the sign.



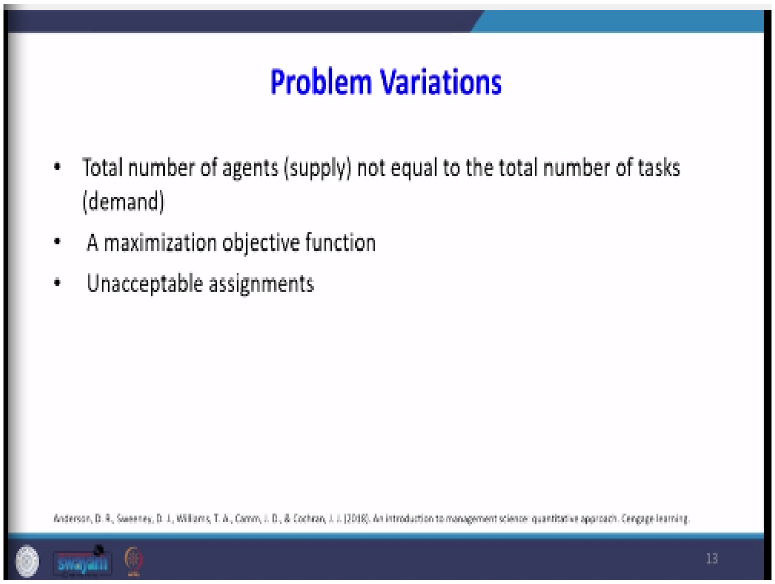
Now, this is a complete linear programming model for our assignment problem. The objective function has three constraints for supply and three constraints for demand, and this value of Xij is binary. Now, I am going to solve this assignment problem with the help of a solver. I have brought that assignment problem to solve in the Excel solver. So, as usual, I have written decision variables, the value of decision variables, the coefficient of the objective function, I have written all the constraints, I have written the value of our objective function also.

Then I go to data, go to the solver, look at this, it is a minimization problem, changing cell is D4 to L4. You see that I have added these changings D4 to L4. It is binary, and their constraint is required. Then I have written all are equal to constraint; you can write less than or equal to also, but equal to constraint also will work, and there will not be any problem. Now, when we solve it, the value of the objective function is 26; we get the values of x12 = 1, x23 = 1, and x31 = 1. What does it say?

Leader 1 is assigned to Client 2, leader two is assigned to Client 3, and Project Leader 3 is assigned to Client 1. So, at present, what is the total completion time? 26. If you go for some other combinations, your 26, maybe you may get some other bigger than this number; the minimum value is 26. So, now we will go to the presentation.

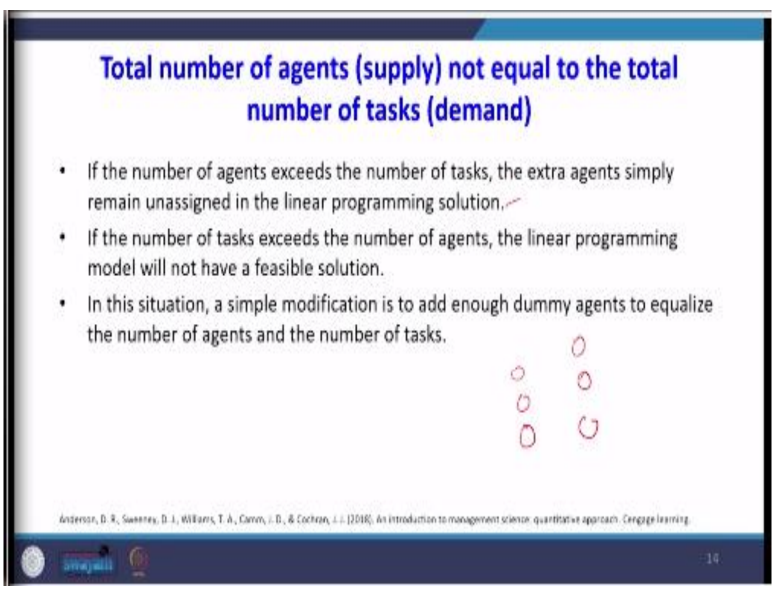


Now, I have brought the solution here. So, what solution are we giving? See, project leader one is assigned to client 1; what is the cost of that one? How many days will he take? He will take 15 days to complete Leader 2 because why the Leader 2 assigned to Client 3? Our x 23 value is 1; how much time will he take? 5 days then x3 = 1, so this 3 is assigned to 1, how many days will it take 6, so 6 + 5 11, 11 + 15 26. So, the total time to complete all the client's projects is 26 days.



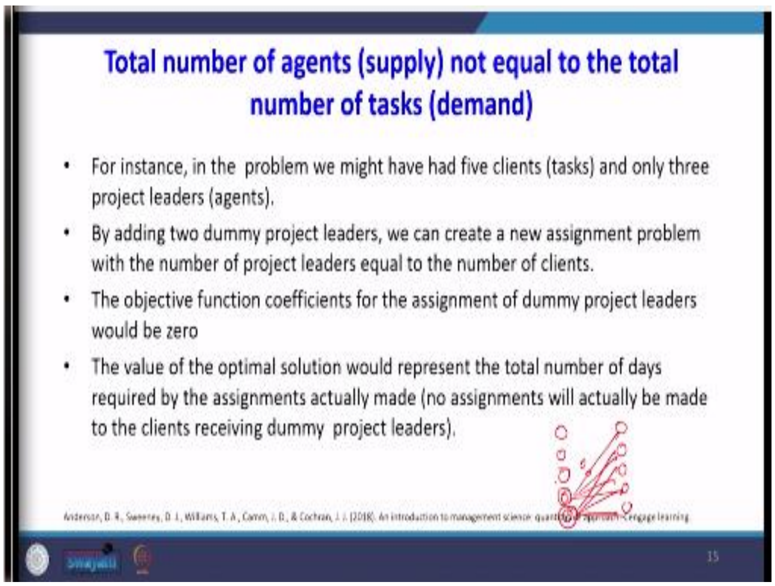
Now we can look at different variations of this assignment problem, and there are 3 variations possible. The first variation is the total number of agents is not equal to the total number of tasks; here, we had three agents and three tasks; this need not be equal. So, if it is not equal, how can we handle this? I will explain. Another one, the problem which you have written is a minimization problem, in case if it is a maximization problem, what to do?

Sometimes, there may be unacceptable assignments, and some project leaders may not be capable of doing certain clients' jobs, so there may be a restriction. So, how do we bring this restriction into our traditional assignment problem?



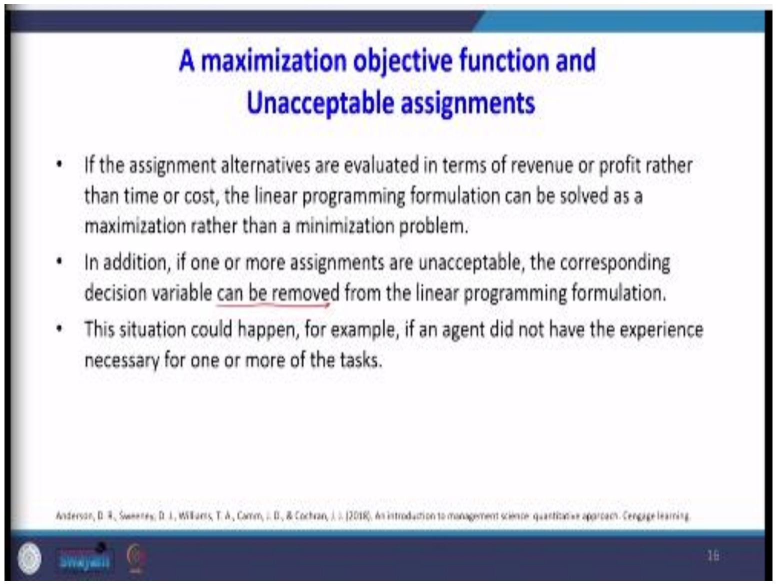
Now we will see the first case; what is the first case? The total number of agents is not equal to the total number of tasks. If the number of agents exceeds the number of tasks, then extra agents remain unassigned in the LP problem. Then there is no problem if there are more agents than the task, but if the number of tasks exceeds the number of agents, the LP model will not have a feasible solution; what is the meaning?

There are fewer people, but there are more projects; in this case, your LP will not have a feasible solution. In this situation, a simple modification is to add enough dummy agents, so we have to add one more dummy agent to equalize the number of agents and the number of tasks. After adding that, what do you have to do?



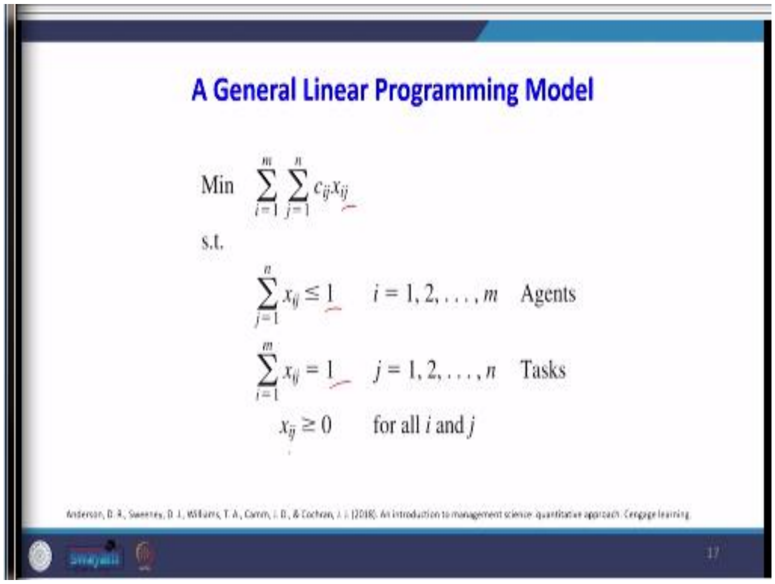
We are discussing the total number of agents not equal to the total number of tasks. For instance, the problem is that we might have had five clients; for example, see that I have five clients: 1, 2, 3, 4, 5, and only 3 project leaders: 1, 2, 3. So, what do we have to do? By adding two dummy project leaders, I am giving some two lines here, two dummy project leaders we can create a new assignment problem with the number of project leaders equal to the number of clients.

When both the numbers are equal, it is possible to get a feasible solution. Here is the objective function coefficient in case, for example, you write this way. The cost will be 0, so the objective function coefficient for the assignment of a dummy project leader would be 0. The value of the optimal solution would represent the total number of days required by the assignment actually made. So, what the meaning is that no assignment will actually be made to the clients receiving dummy project leaders; that is the meaning of the value of your dummy project leaders.

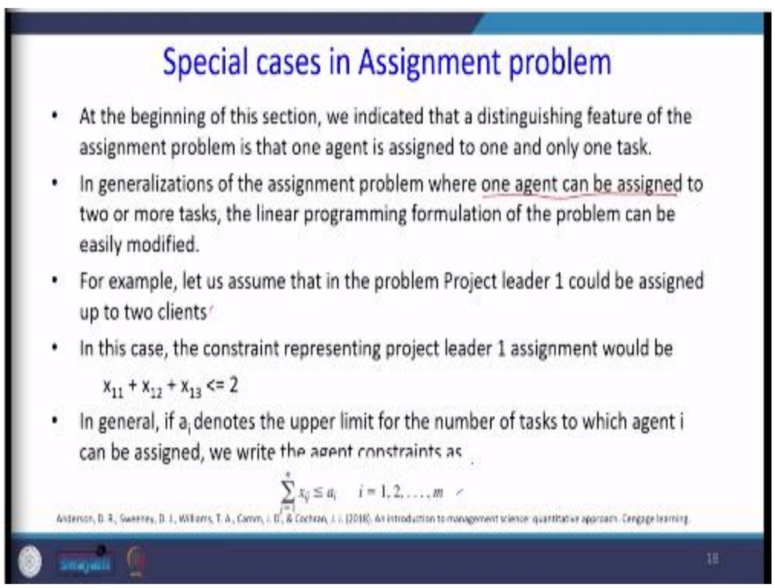


Another variation is maximization, the objective function is maximization, or sometimes there may be unacceptable assignments. If the assignment alternatives are evaluated in terms of revenue or profit rather than time or cost, the LP formulation can be solved as a maximization problem rather than a minimization problem. Simply, instead of selecting minimization, just select maximization of our solver, and you will get the solution.

In addition, if one or more assignments are unacceptable, so some leaders may not have the expertise to do certain tasks. So, the corresponding decision variables can be removed from the LP formulation. This situation could happen, for example, if an agent did not have the experience necessary for one or more of the tasks.

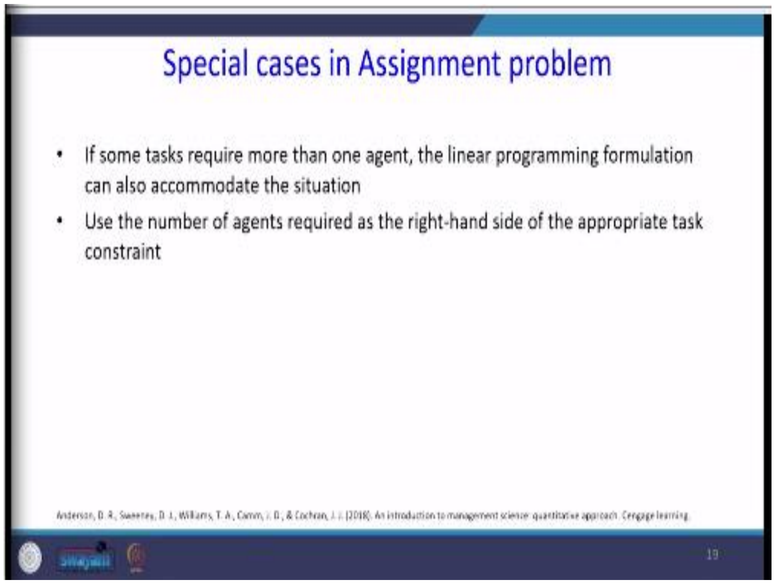


So, this is our general linear programming model for our assignment problem. So, this also is a minimization; for agents less than or equal to 1, the task should be equal to 1;, here the Xij is the binary value.

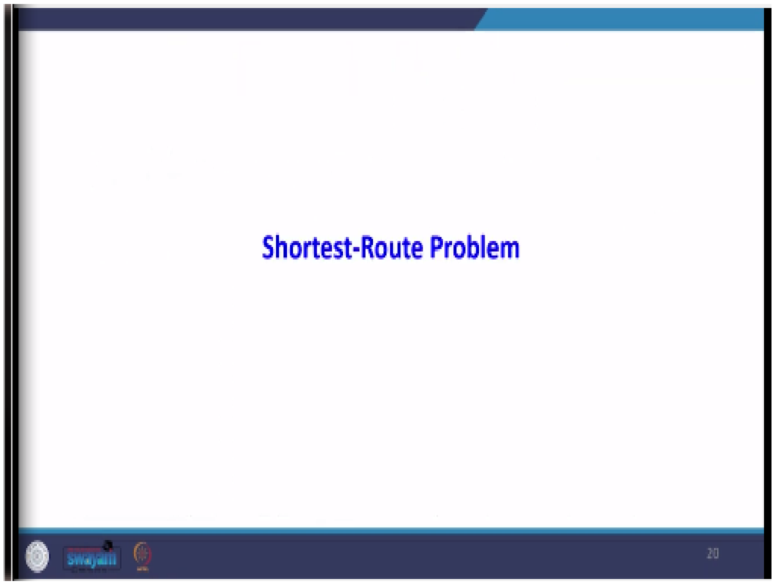


Sometimes, there are more possibilities of special cases in assignment problems. You can remember at the beginning of this section; we indicated that the distinguishing feature of the assignment problem is that 1 agent is assigned to one and only one task. In a generalization of assignment problem where 1 agent can be assigned, that means 1 project leader, for example, can be assigned 2 or more tasks. So, in that situation, your LP formulation of the problem can be easily modified. For example, let us assume that project leader 1 could be assigned up to 2 clients.

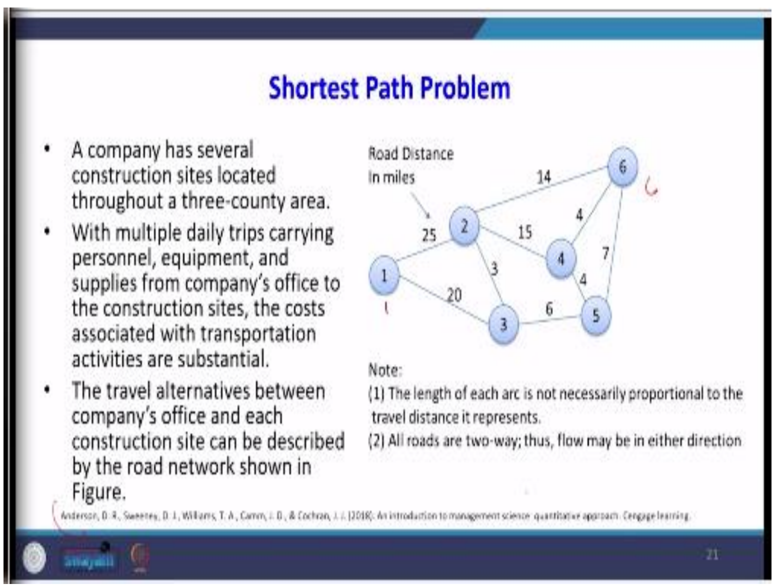
In this case, the constraint representing project leader 1 assignment would be X11, X12, X13 less than or equal to 2. That means that the project leader can be assigned any 2 agents, in general, if I denote the upper limit for the number of tasks to which agent I can be assigned. So, we can write sigma of j = 1 to n Xij is less than equal to a. Because here a project leader can be applied for 2 projects we put less than or equal to 2, that is the modification in the assignment problem.



Another one is if some task requires more than one agent, which is the opposite of what you have discussed, then the LP formulation can also accommodate the situation. What do we have to do? We have to use the number of agents required as the right-hand side of the appropriate task constraint. The right-hand side of the task may be put equal to 1, instead of that, so that task may require maybe 2 leaders. instead of 1, you have to put 2; that is another special case.

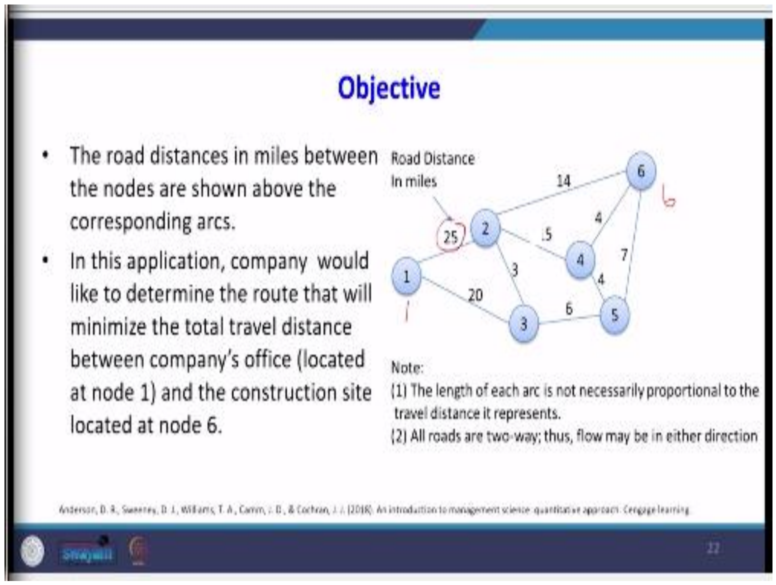


Now, we will go to the next type of problem called the shortest-route problem.

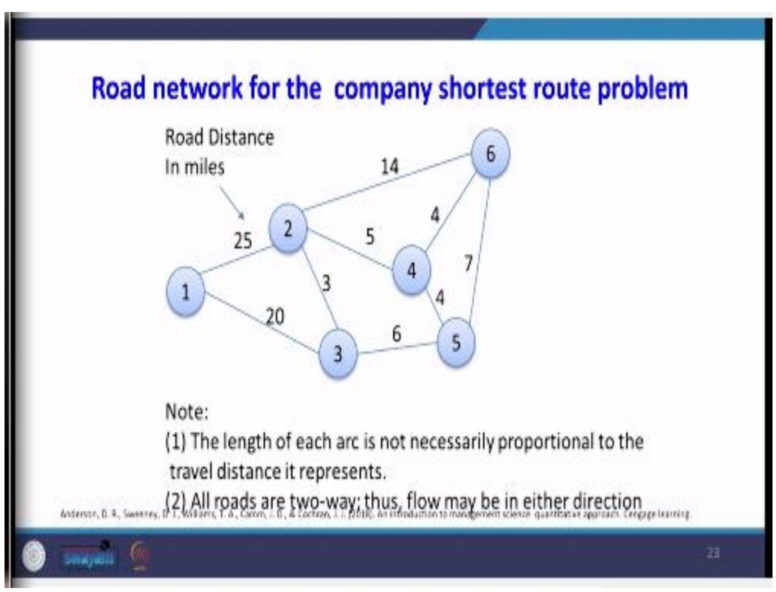


What is the shortest path problem? So, I have taken a sample problem from this book by Anderson et al.. The problem is that a company has several construction sites located throughout, say, the area. With multiple daily trips carrying personnel, equipment, and supplies from the company's office, this location is office from 1 to construction sites. The costs associated with the transportation activities are substantial.

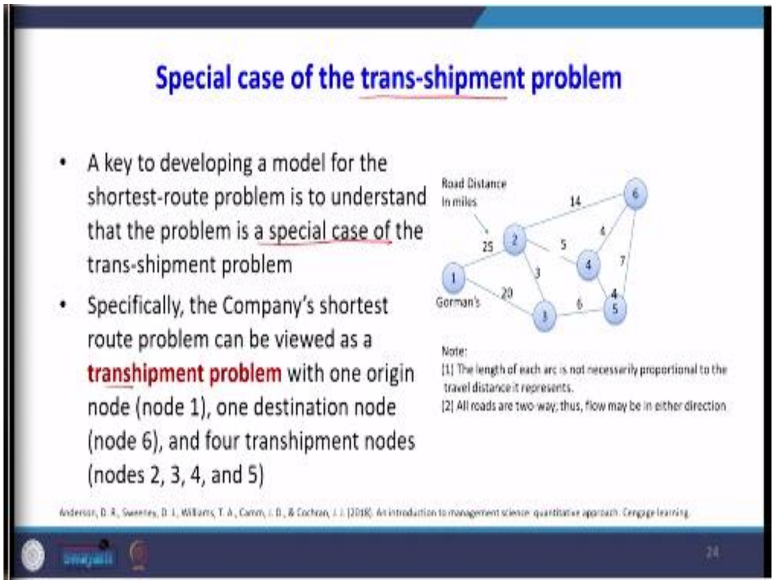
The travel alternatives between the company’s office and each construction site can be described by the road network shown in the figure. Suppose this is he can go via 2 to 6; for example, he has to start from 1 he has to reach 6; there are different possibilities 1 to 6, 1, 3, 5, 6, or 1 to 46 like this. Remember, the length of the arc is not necessarily proportional to the travel distance it represents; we have not drawn this picture to scale. So, all roads are 2-way, and the flow may be in either direction; this is the assumption.



What is the objective of a shortest-path problem? The road distances in miles between the nodes are shown above the corresponding arc. For example, this 25 represents the road distance between nodes 1 and 2 is 25 miles. In this application, the company would like to determine the route that will minimize the total travel distance between the company's office, which is node 1, and the construction site located at a 6. So, between 1 and 6, we should find the shortest path, which is an objective of the shortest path problem.

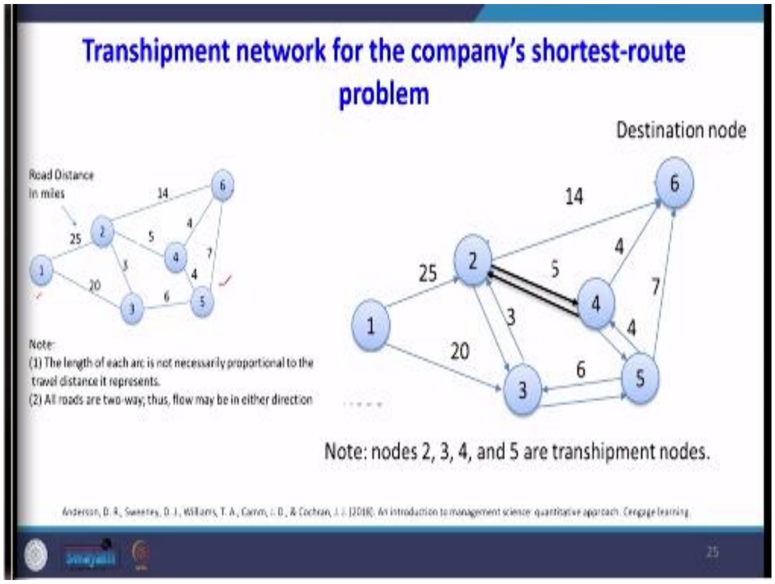


This is a road network for the company's shortest route problem.

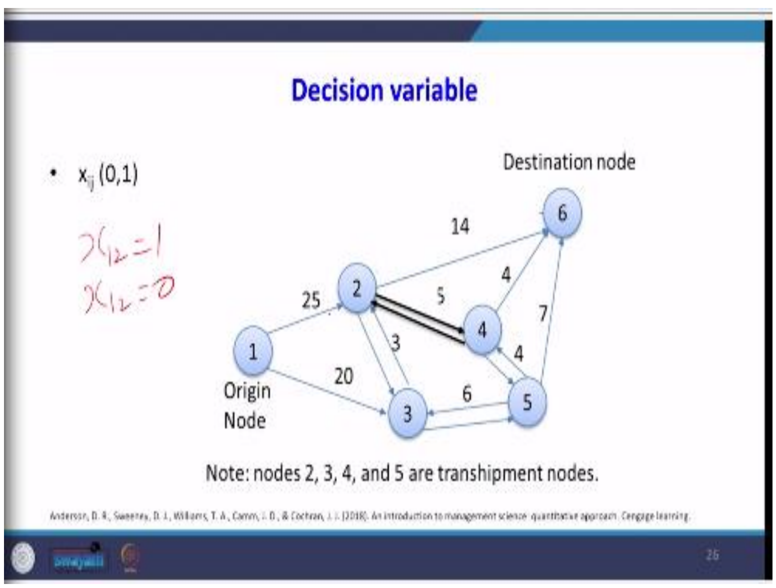


Now we are going to introduce a concept here. So, the shortest path problem is this special case of the trans-shipment problem. We have seen that we have already solved what is a trans-shipment problem; how do we write the constraint for a trans-shipment problem? So, here the important point is that the shortest path problem is the special case of our trans-shipment problem. What is the meaning?

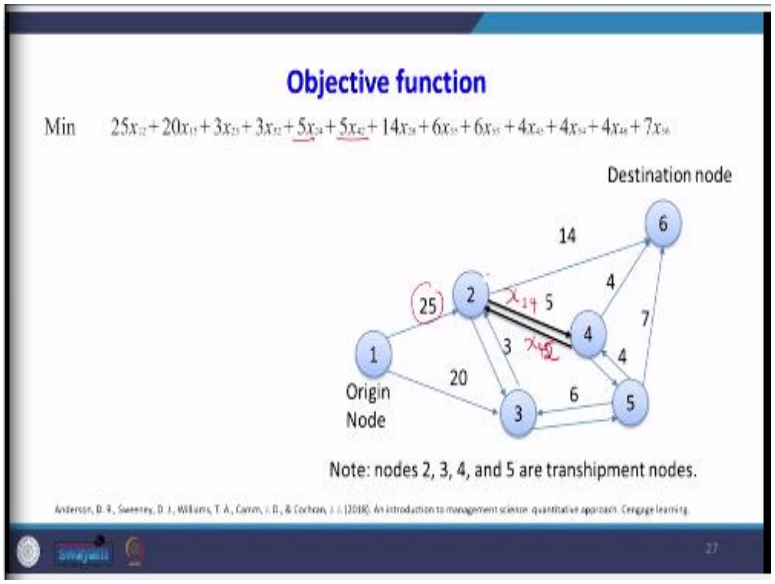
All intermediate nodes, for example, 2, 3, 4, and 5, we are going to consider as a trans-shipment node; now let us see. The key to developing a model for the shortest route problem is to understand the problem is a special case of the transshipment problem. Specifically, the company's shortest route problem can be viewed as a transshipment problem with 1 origin, which is node 1, and 1 destination node 6, and 4 transshipment nodes 2, 3, 4, and 5. So, by having this idea in your mind we are going to solve this problem.



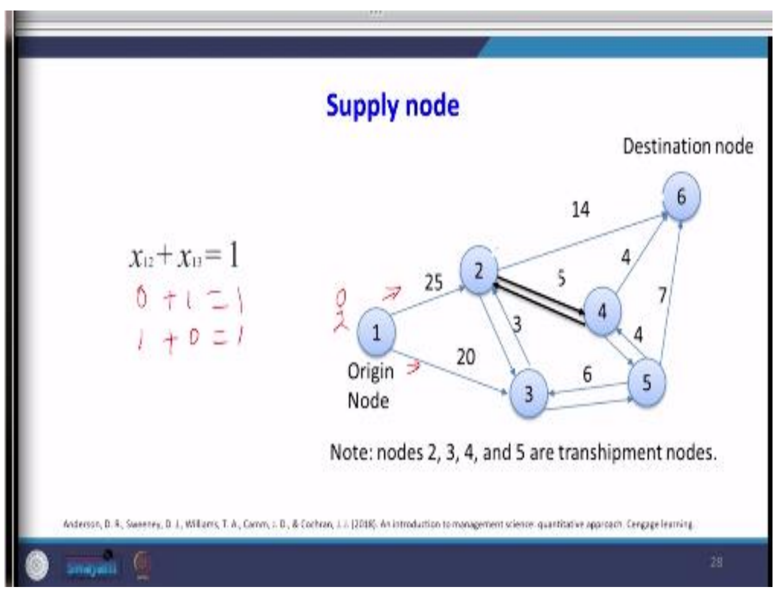
Now, I have brought up our transportation problem, which I am going to convert into a transshipment problem; what is the conversion taking place? You see that there is a starting node there is a destination node, so 1 and 6 are there. And in between, there are certain intermediate nodes, for example, 2, 4, 5, and 3. So, I have added a double arrow for these transshipment nodes; what are they? 2 to 3, I have put a double arrow; 2 to 4, I have put an arrow in both directions; 4 to 5, there is an arrow in both directions. Because our assumption says that all roads are 2 ways thus flow may be in either direction.



The decision variable for a transshipment problem is xij, it is 0 and 1. The decision variable is binary; if the value of, say, for example, x12 = 1, that means that we are traveling on the root from 1 to 2. If it is x12 = 0, I am not traveling; that is the meaning of our decision variables.



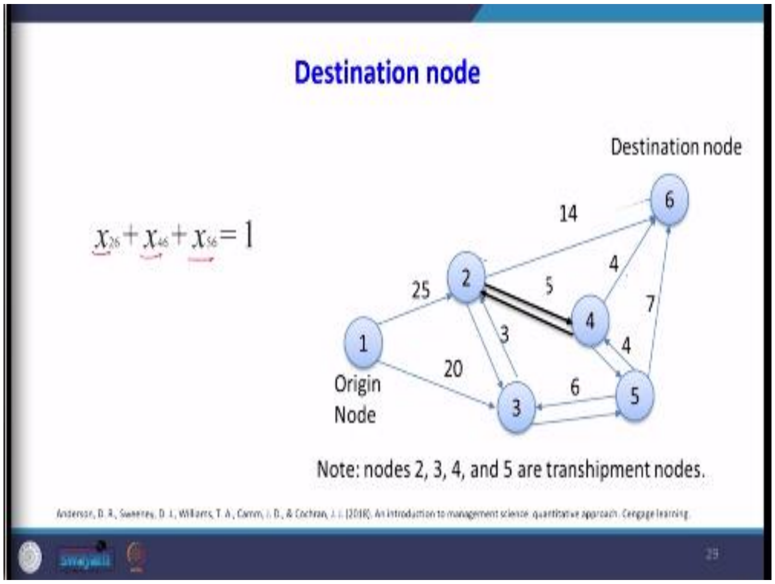
So, now we have to write the objective functions. So, we have to multiply each decision variable by corresponding to its distance. For example, here it is a 25, so x12 is 25. Here, when you look at this, this is x24, and the bottom one is x45, so 5x24 and 5x42 like I have written x42, and I have written the decision variables for all the nodes, then I have multiplied by corresponding the distance. So, then I am getting the complete objective function.



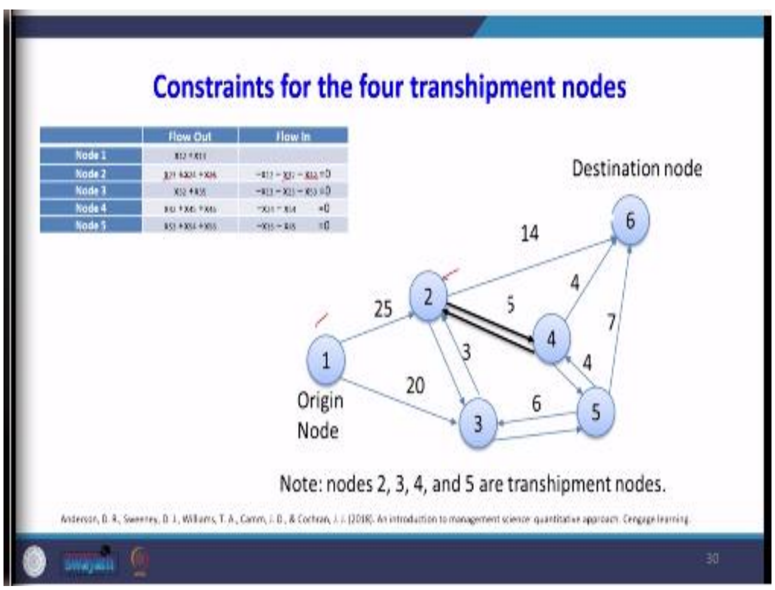
Now let us write supply nodes. Here, 1 is supply nodes; somebody standing here, for example, standing here can take route 1 to 2 or 1 to 3, so how to write this decision in the form of an equation?

So, x12 + x13 = 1.

Since it is binary, a person can go x13 or x12; then only we will get equal to 1; that is why we have written supply nodes.

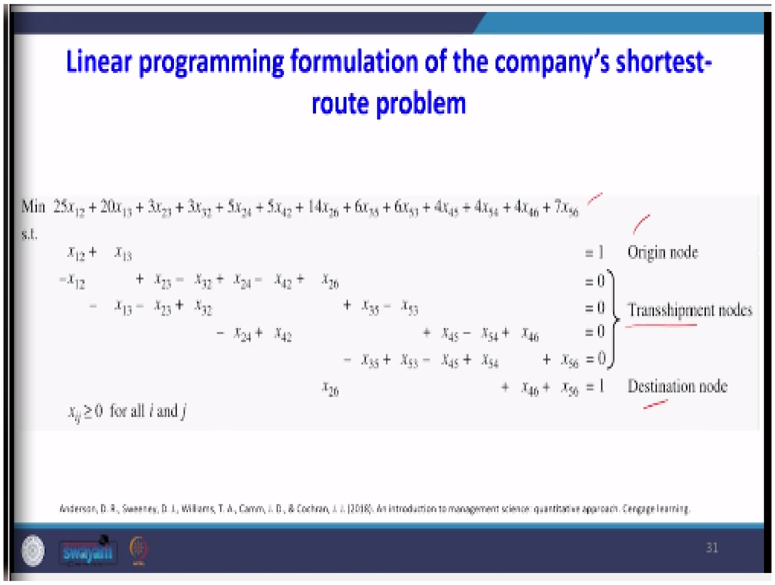


The same idea is for the destination node also. There are 3 ways to reach 6. What have they X26, X46, and X56 is equal to 1. So, that means if we can reach node 6 by any one root, that is why it is equal to 1.



Now, I will write about the transshipment node constraints. For example, node 1 we have written already, see node 2 is the transshipment node. As I have discussed already, the outgoing errors are plus, and the incoming arrows are negative, equal to 0. So, what are the arrows which are going outside? X24 is there, then X26 is there, then X23 is there. So, for node 2, what are the incoming arrows? X42 is there, so a -X42, so then another incoming arrow is X32 is there, and X

12 also is there, that also incoming arrows. So, a number of incoming arrows or the sum of incoming arrows is equal to the sum of outgoing arrows. When you bring in the left-hand side, it will be negative. Similarly, I have written for all other transshipment nodes, that is for 3, 4, and 5.

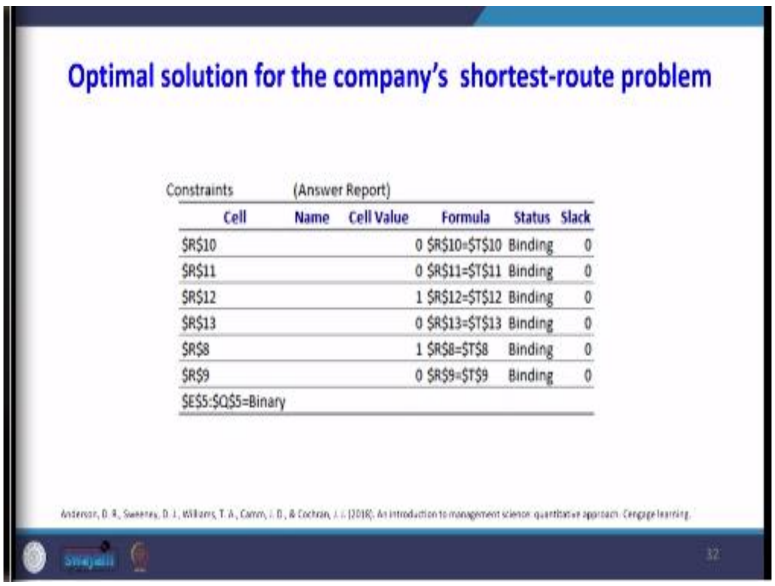


Now this is my complete LP formulation for our transshipment node. So, the objective function is minimization type, so this is the origin node and destination node corresponding constraint. There are 4 transshipment nodes, there are 4 transshipment constraints. Now, I am going to solve this one with the help of a solver. Now, I have built an Excel model to solve this shortest path problem; the idea is the concept of a transshipment node.

As usual, I have written decision variables, then the coefficient of the objective function, and then the value of the objective function is written in S4. So, if we go to data, then go to the solver; look at this. It is a minimization problem. Here, I did not include the binary constraint. Even if you include the binary constraint, it will not affect the result. But at present, I will solve the same problem without including binary constraints.

Then, including binary constraint, what will happen? Now we will solve this problem without including binary constraints, so solve it. Now, the value of the objective function is 32, so we got some values of 1; for example, x13 is 1, then x32 is 1, then x24 is 1, then x46 is 1; this is a path that has to be followed. Now I am going to add a binary constraint here, and then I am going to solve it again; let us see if there is any difference.

So, go to data solver, so go to add constraint, you select F5 to Q5, select that, then you select is a binary, bin binary, ok binary constraint, now we solve it. Now you see, even though I am not adding binary constraints, all the values are integers. So, it will not affect; it is better to have a binary constraint, but even if you do not have a binary constraint, this will not affect the result. But you have to make sure that you are getting 11 there.



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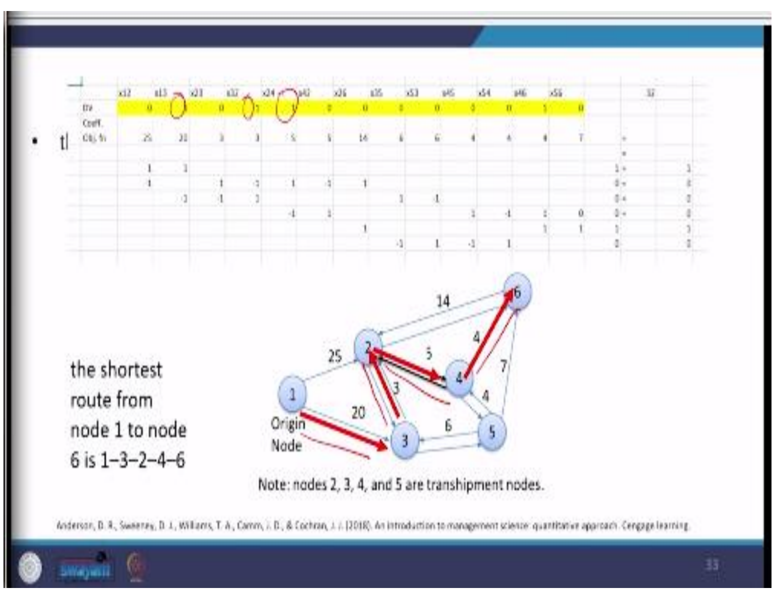
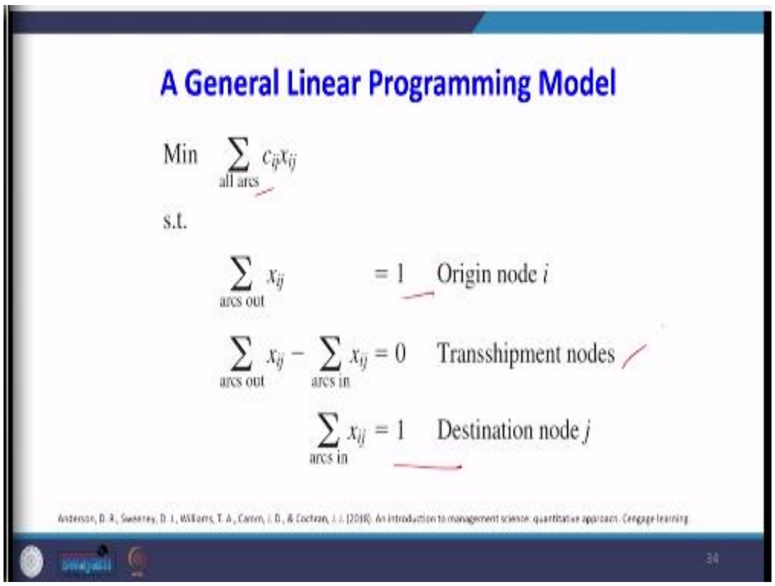
Now we will go to our presentation. So, the constraint output says all the constraints are binding constraints.



Now look at the answer. So, what is the shortest route? I must start from 1 to 3, so I have started 1 to 3, then 3 to 2, then I have to go to 3 to 2 this one, then I have to go to 2 to 4, then 2 to 4, then 4 to 6.

So, the total distance is 32 = 20 + 3, 23 + 5= 28 + 4= 32.

It is a 32, so this is the shortest route; what is the shortest route? 1, 3, 2, 4, 6, so this is a way to solve a shortest path problem. What idea do you have to remember? That the shortest path problem is a special case of your transshipment problem is an important point that has to be noted.



So, the general linear programming formulation for the shortest path problem is it is a minimization function. The sum of the origin nodes should be 1, and similarly, the sum of the destination nodes should be 1. Then, you have to include transshipment nodes. In this lecture, we have taken 2 problems: one is the assignment problem, and the other is the shortest path problem. In the assignment problem I have formulated in the form of network.

From the network, I formulated a linear programming model, and then I solved it with the help of Excel. In the same way, I have solved the shortest path problem, what assumption have I made? The shortest path problem is the special case of your transshipment problem. So, I have included some more constraints for the concept of transshipment; then I have solved them with the help of a solver; thank you very much.