Security Analysis & Portfolio Management
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Lecture 55
Arbitrage Pricing Model II

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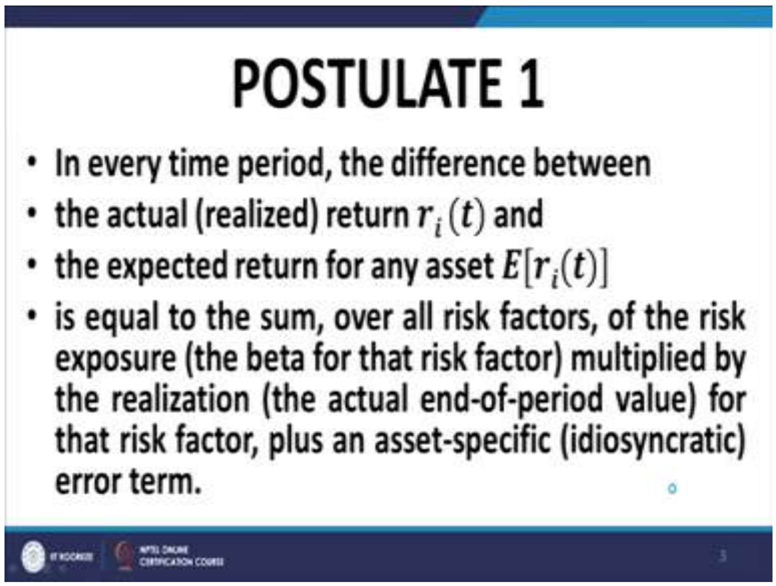

Welcome back, so let us continue from where we left off. Let us quickly recap the arbitrage pricing theory. Postulate 1 of the arbitrage pricing theory, says that in every time period, the difference between the actual, or the realized return, that is ri t. And the expected return, or the expected value of that return as calculated at the beginning of that time period, that is t equal to 0 is equal to the sum over all risk factors of the risk exposure, captured by the beta for that risk factor, multiplied by the realization, that is the actual end of period value, for that risk factor plus an asset specific idiosyncratic error term.

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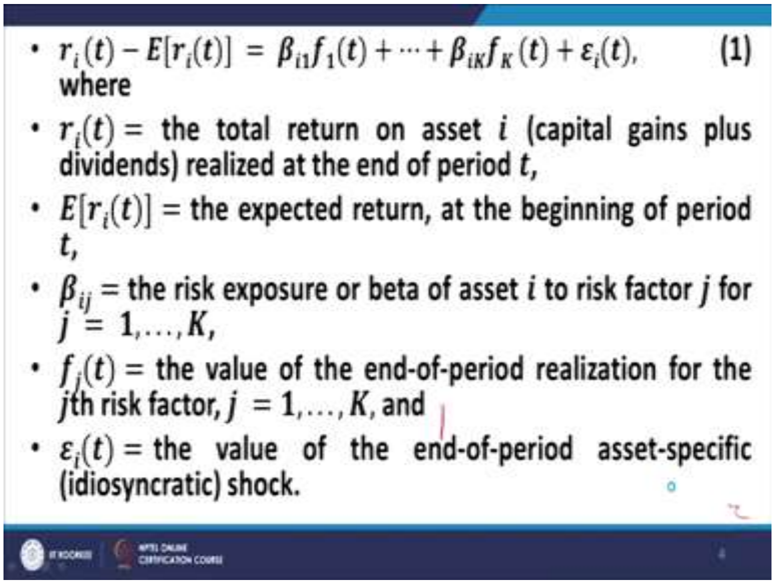

So, that this statement is captured by the equation, that is given at the top of your slide, that is equation number 1. Where the terms of the meaning that I explained in the last class, ri t, is the actual return, the realized return, the total return on asset i, which includes capital gains, as well as dividends, realized at the end of the period t.

E of ri t, is the expected value of this return as worked out as t equal to 0. Beta ij is the exposure, or beta of asset i to risk factor j, fj t is the value of the end of period realization for the jth risk factor and epsilon i t is the value of the end of period asset specific idiosyncratic risk, or shock.

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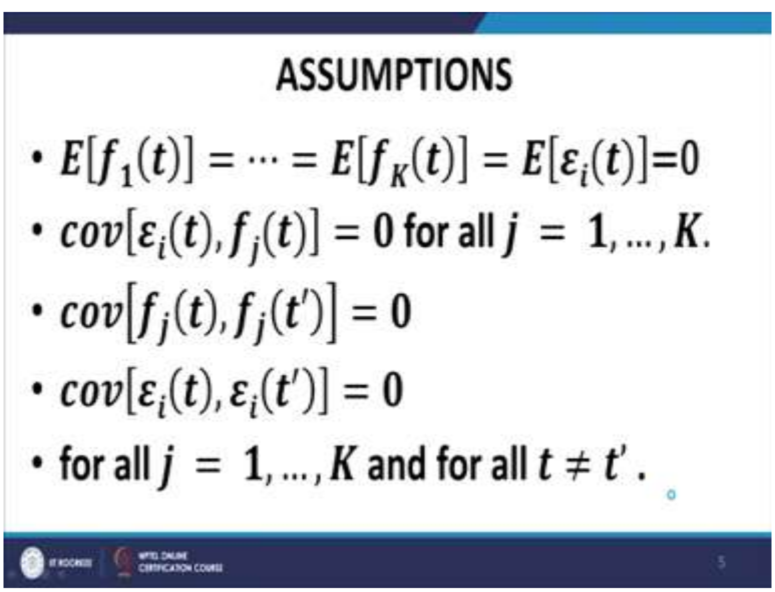

The assumptions that go into this postulate, that the expected value of each of these risk factors, as worked out at t equal to 0, the workout to 0. And the expected value of the idiosyncratic shock also, at the point at t equal to 0 is equal to 0. All these values worked out at t equal to 0 for the period 0 to t work out to 0.

The covariance between epsilon t and fj t is 0, that means none of the factors risk factors is correlated with the idiosyncratic risk, that is an assumption that we also make in the CAMP model. And then we have that fj t and fj t dash is equal to 0, that means the value of the various risk factors, at different points in time are uncorrelated with each other.

Similarly, the value of the idiosyncratic term, or the epsilon term worked out at different time periods t and t dash, bear no correlation amongst themselves. So, these are the assumptions that go into postulate number 1. And these conditions as summarized in the previous slide imply, that the asset returns are generated by a linear factor model LFM.

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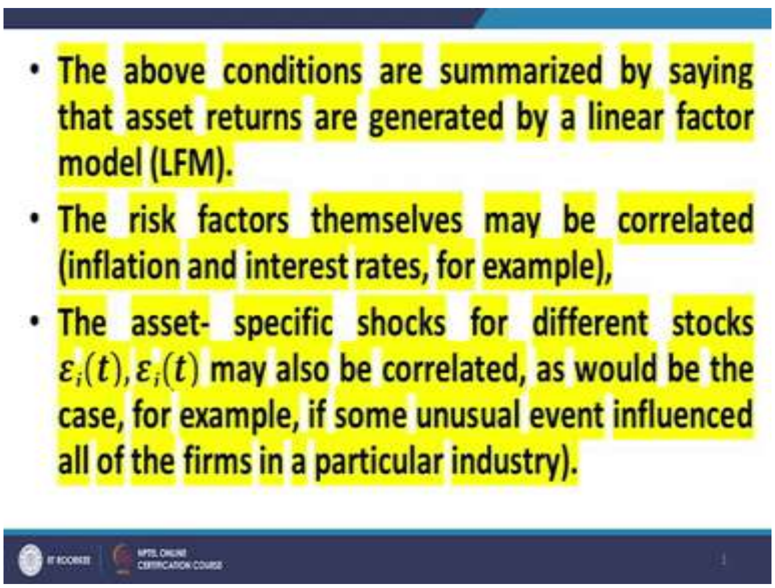

The risk factors themselves may be correlated, please note this important point, we have nowhere assumed that, the covariance between fi t and fj t is equal to 0, we have not assumed that. Therefore, the risk factors themselves may be correlated like inflation and interest rates. The asset specific shocks for different stocks Ei t, or epsilon i t and epsilon j t, may also be correlated.

Now this assumption is at variance with the assumption, that we make in the CAPM model and the single index model. In the CAPM model, and in the single index model, we assume that, the residual risk of any two securities i and j is equal to 0, but that assumption is not made in this APT model. And to that extent, the APT model is a generalization of those earlier models.

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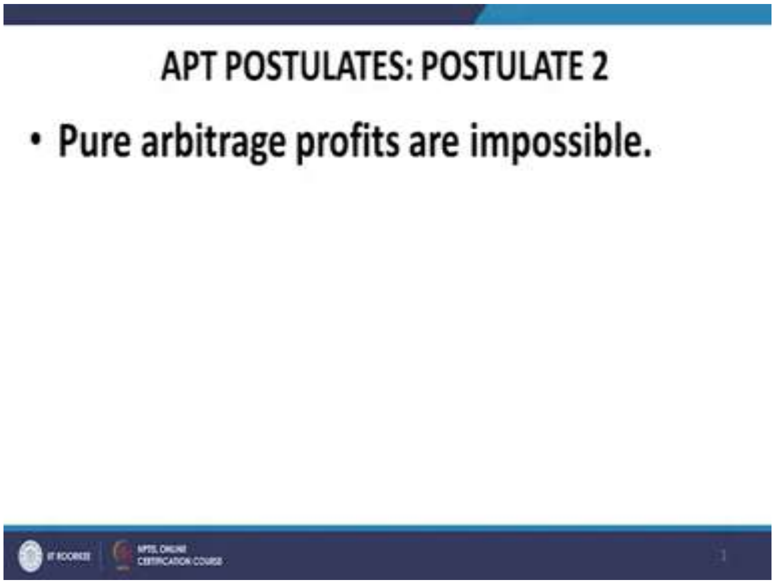

Postulate number 2 is very straightforward. The markets are efficient enough for us to assume, that pure arbitrage profits are impossible. That means, we are working on a platform, where the efficiency of the markets is good enough for us to assume, that pure arbitrage profits are impossible.

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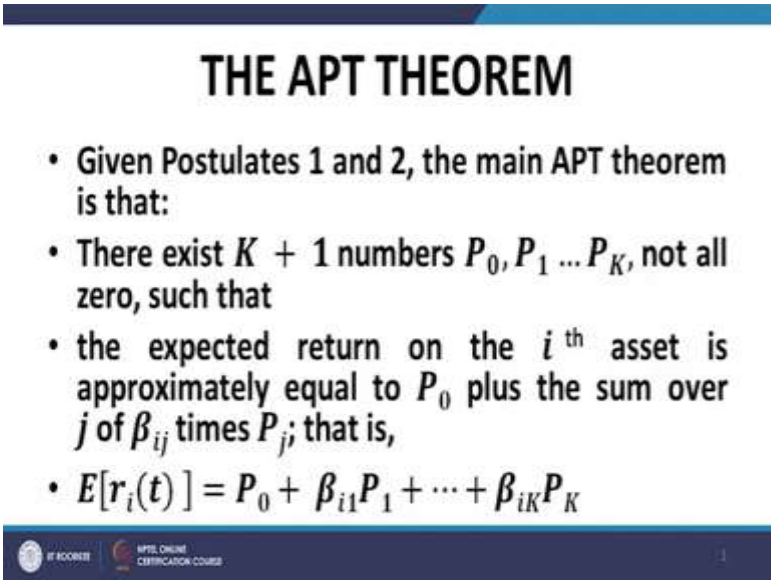

Now we come to the APT theorem. Given postulates 1 and 2, the main APT theorem says, that there exist K plus 1 numbers P0, P1, P2, up to Pk, not all 0, such that. The expected return on the ith asset is approximately equal to P0 plus the sum over j of beta ij times P j.

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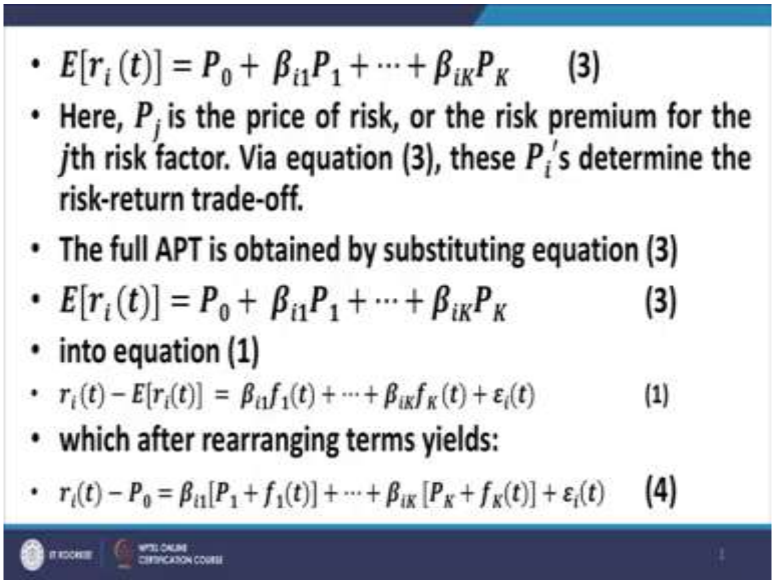

Or equation number 3, that is at the top of this slide. The expected value of ri t, is equal to P 0, plus beta i1 P1 plus beta i2, P2 and up to beta ik PK, please note the first suffix, or the first subscript a beta is the security identity and the second subscript of beta is the factor identity. So, and Pj or P1, P2 and so on are call the price of risk, or the risk premium for the jth risk factor, please not P0 represents the risk free rate of return, as I explained in the last lecture. So, P0 is the risk free rate of return and P1, P2 and up to PK are the risk factor, risk premia associated with the appropriate risk factor.

So, why question number 3, that is the equation at the top of the slide, these Pi's determine the risk return trade off of the security. The full APT is obtained by substituting equation number 3, that is the first equation on your slide, into equation number 1, which was the original equation representing the postulate number 1. So, if I substitute from equation number 3, in equation number 1, if I substitute the value of E of ri t from equation number 3 and equation number 1. And simplify a little bit what I end up is equation number 4.

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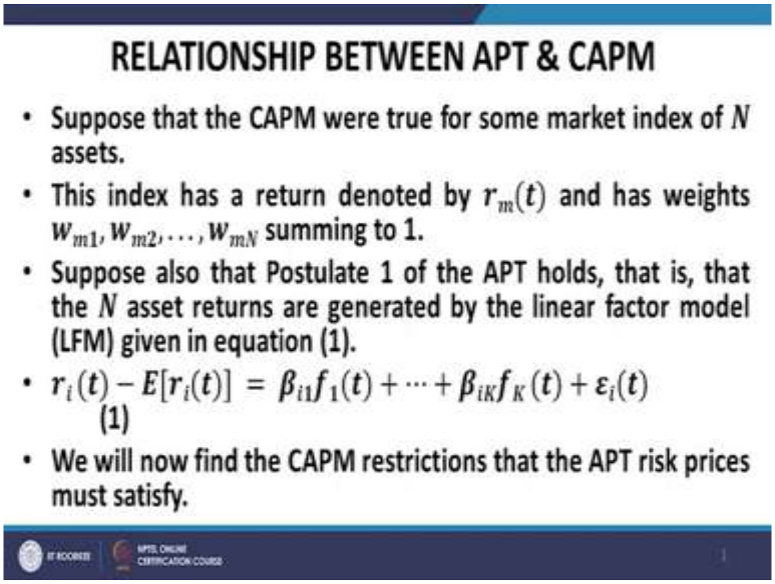

Now, we talk about the relationship between the APT and the CAPM model. The APT betas and the CAPM beta. For that purpose, we assume, that the in that, the CAPM is true for some market index comprising of N assets. And this index has a return, which is denoted by r mt and has weights W m1, W m2, up to WN with some up to 1.

Now, suppose that also that postulate number 1 of the APT holds, that is the N asset returns r1 t, r2 t, and so on, generated are generated by a linear factor model and it is given by equation number 1. So, using equation number 1, we can write the return on the ith security in the form, which is given here and equation number 1 on this slide. We now find the CAPM restrictions that the APT risk prices must satisfy. In order that CAPM hold.

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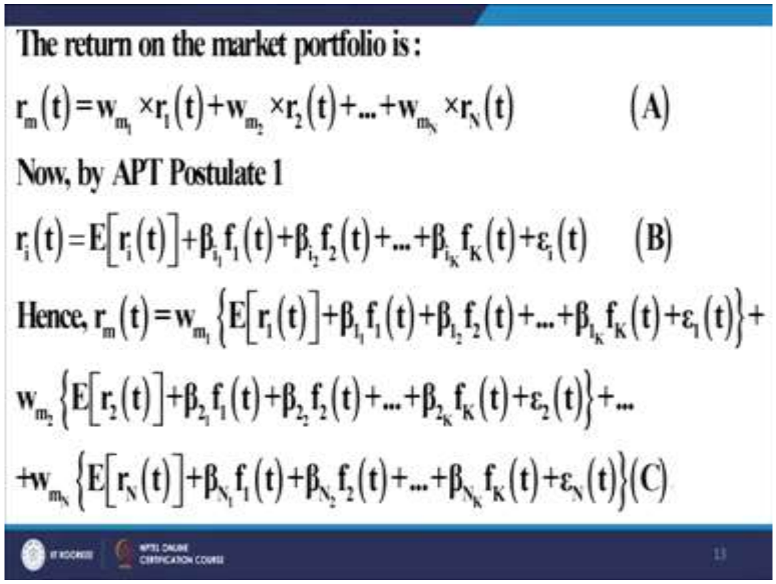

Now, the return on the market portfolio is given by rm t is the weighted average return of the co constituent securities. So, that is what is represented in equation number A, on this slide. Now, by APT postulate number 1, which you saw just now, ri t is equal to if I take the expected value of ri t to the right hand side from the left hand side, I can write ri t, as the expected value of ri t plus beta i1, f1 t plus 1. So, up to beta ik fk t plus the idiosyncratic term, which is equation number B.

So, r mt is equal to now I substitute from equation B for r1 t, r2 t, r3 t, and up to rn t from equation number B, I substitute in equation number A. What I get is equation right at the bottom of your slid, it is an extensive equation. But it is simply obtained by substituting equation B, as it becomes in the context of security 1, 2, 3, up to N in equation number A.

In other words is substitute the expressions for r1 t, r2 t, r3 t, up to rN t, as obtained from equation number B, in equation number A and we arrive at equation number C at the bottom of your slide.

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$$r_m(t)=\left[w_{m_1}E\left[r_1(t)\right]+\ldots+w_{m_N}E\left[r_N(t)\right]\right]+$$
$$\left[w_{m_1}\varepsilon_1(t)+\ldots+w_{m_N}\varepsilon_N(t)\right]+$$
$$\left(w_{m_1}\beta_{1_1}+w_{m_2}\beta_{2_1}+\ldots+w_{m_N}\beta_{N_1}\right)f_1(t)+$$
$$\left(w_{m_1}\beta_{1_2}+w_{m_2}\beta_{2_2}+\ldots+w_{m_N}\beta_{N_2}\right)f_2(t)+\ldots+$$
$$\left(w_{m_1}\beta_{1_K}+w_{m_2}\beta_{2_K}+\ldots+w_{m_N}\beta_{N_K}\right)f_K(t)\quad(D)$$

A little bit of rearrangement of equation number C, gives me equation number D, please note there is nothing else except algebraic rearrangement, no principles involved except algebraic rearrangement as I mentioned and we get equation number D.

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Setting $\beta_{m_j}=w_{m_1}\beta_{1_j}+w_{m_2}\beta_{2_j}+\ldots+w_{m_N}\beta_{N_j}$

for $j=1,2,\ldots,K$

$$r_m(t)=\left\{E\left[w_{m_1}r_1(t)+\ldots+w_{m_N}r_N(t)\right]\right\}$$
$$+\varepsilon_m(t)+\beta_{m_1}f_1(t)+\ldots+\beta_{m_K}f_K(t)$$
$$r_m(t)=\left\{E\left[r_m(t)\right]\right\}+\varepsilon_m(t)+\beta_{m_1}f_1(t)+\ldots+\beta_{m_K}f_K(t)\quad(E)$$

Now, if we said beta m j equal to W m1 beta 1j plus W m2 beta 2j, please note as I mentioned, the first subscript of beta identifies the security, the second subscript of beta that is j identifies the risk factor. So, beta 1j, identify security 1, and the risk factors j. So, we make a substitution for, we make this substitution, or an abbreviation into the equation number D, that we had in the previous slide and we what we get is equation number E.

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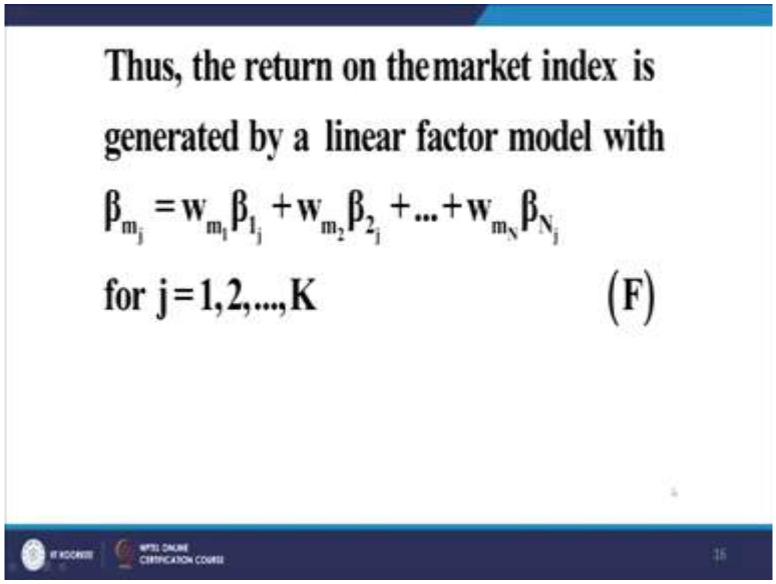

Now, if we observe equation number E carefully, what we find is that the market returns are also generated by a linear factor model with the betas, which are given in equation number F. Let me repeat, the market returns, or the return on the market index is also generated by a linear factor model with the beta, that is given by equation number F. So, that is one conclusion that we arrive at.

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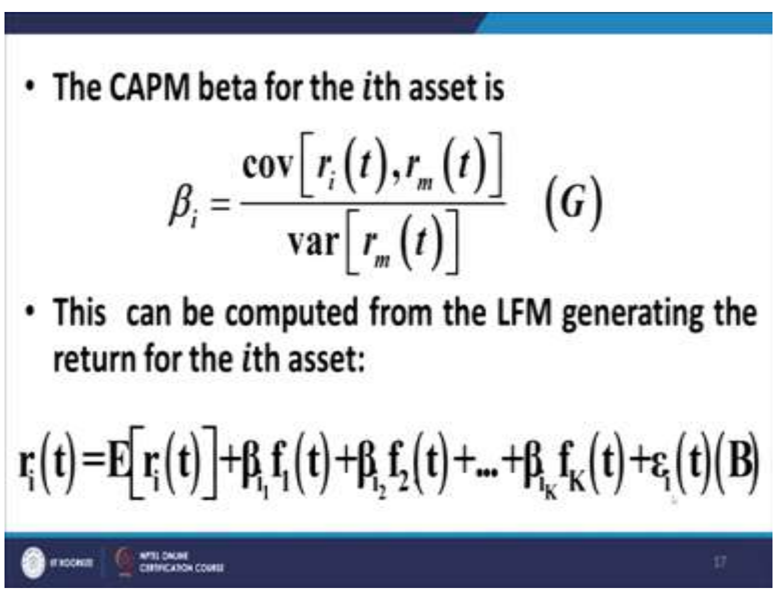

$$\beta_i = \frac{\beta_{i_1} \times \operatorname{cov}\left[f_1(t), r_m(t)\right]}{\operatorname{var}\left[r_m(t)\right]} + \cdots$$

$$+ \frac{\beta_{i_K} \times \operatorname{cov}\left[f_K(t), r_m(t)\right]}{\operatorname{var}\left[r_m(t)\right]}$$

$$+ \frac{\operatorname{cov}\left[\varepsilon_i(t), r_m(t)\right]}{\operatorname{var}\left[r_m(t)\right]} \quad (H)$$

Now, the CAPM beta for the ith asset is given by equation number G. By definition beta is a regression coefficient, between the market returns, or the returns on the market index and the returns of the security, the security returns are regressed on the return on the market index. And therefore, beta i is given by the expression, that is here on equation number G.

Now, if we substitute the value of ri t, in terms of the expression, that is given in equation number B. What we get is? It the equation number H, that is here, we are simply substituted the expression for ri t from equation number B, into the equation number G, to arrive at equation number H. Now, we will look at the last term of equation number H as we investigate the last term of equation number H.

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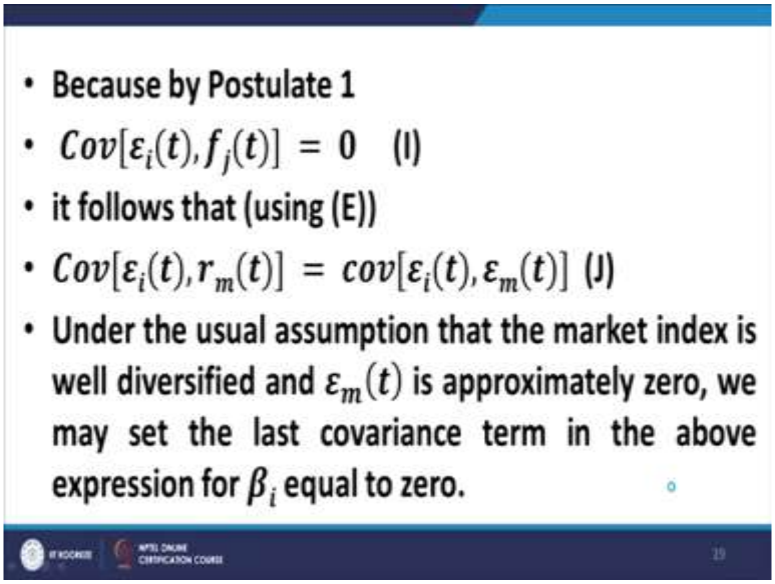

Now, we know that rm t is a linear factor model of comprising of the generators f1 t, f2 t, f3 t and so on. But we also know, that epsilon i t with the covariance rather of epsilon i t with any of the factors fj t is equal to 0. Let me repeat there are two things, which are relevant number one, we know we have just proved that rm t, the market return is a linear factor model generated by the various risk factors f1 t, f2 t, f3 t and so on.

So, if I substitute the expression for rm t in terms of these generators, then what I find is that there would be terms comprising of the covariances between epsilon it f1 t, epsilon it f2 t and so on up to epsilon it fk t. By the assumption of this model, we have that the covariance between epsilon i t and epsilon jt, fj t is equal to 0.

Let me repeat, the covariance between epsilon i t and fj t t is equal to 0. So, all the terms would vanish except for the last term, which is epsilon i t and epsilon m t. In other words, all the terms in the covariant this covariant, let me underline it, this covariant when I substitute rm t, in terms of its linear factors, which are f1 t, f2 t, f3 t, and so on generators f1 t, f2 t, f3 t, and so on.

Then we what we get is epsilon it f1 t, covariance, covariance epsilon it f2 t and so on up to fk t. So, each of these terms as 0, but we have not assumed, that the covariance between epsilon it and epsilon mt is 0, we have not assumed that. So, that term will survive, otherwise all the other terms will go to, because of the restriction, that is contained in equation number I. So, that being the case, we end up with the equation number J.

Now, under the usual assumption, that the market index is well diversified, what we can, what we can assume is a very good approximation is that epsilon m t is approximately 0. And therefore, this term epsilon it rm t ends up as being equal to 0, on the premise of equation number 1, equation number I, I am sorry, and in the mandate, that the idiosyncratic risk of the market portfolio is 0.

If we have made these two assumptions, then what we end up with is that the covariance between the epsilon i t and rm t is equal to 0. And therefore, what we end up with beta i is equal to the expression, that is given on the right hand side of equation number K.

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$$\text{Thus, } \beta_i = \frac{\beta_{i_1} \times \operatorname{cov}\left[f_1(t), r_m(t)\right]}{\operatorname{var}\left[r_m(t)\right]} + \cdots + \frac{\beta_{i_K} \times \operatorname{cov}\left[f_K(t), r_m(t)\right]}{\operatorname{var}\left[r_m(t)\right]} \quad (K)$$

$$\text{By CAPM: } E\left[r_i(t) - TB(t)\right] = \beta_i E\left[r_m(t) - TB(t)\right] \quad (L)$$

$$= \left\{ \frac{\beta_{i_1} \times \operatorname{cov}\left[f_1(t), r_m(t)\right]}{\operatorname{var}\left[r_m(t)\right]} + \cdots + \frac{\beta_{i_K} \times \operatorname{cov}\left[f_K(t), r_m(t)\right]}{\operatorname{var}\left[r_m(t)\right]} \right\} E\left[r_m(t) - TB(t)\right] (M)$$

$$\text{By APT: } E\left[r_i(t) - TB(t)\right] = \beta_{i_1} \times P_1 + \ldots + \beta_{i_K} \times P_K \quad (N) \text{ From eq } (3)$$

$$\text{Hence, } P_j = \frac{\operatorname{cov}\left[f_j(t), r_m(t)\right]}{\operatorname{var}\left[r_m(t)\right]} E\left[r_m(t) - TB(t)\right] \quad (O)$$

Now, we also known by the CAPM equation, that equation L holds right in the context of the CAPM model, or the expected excess return on a security is equal to beta times, the expected return on the market, which is what is given an equation number L. Substituting the value of beta, we can write it in the form of equation number M.

In other words, from equation L to M, we have simply arrived at by substituting the value of beta i as given by equation number K. If we make a substitution from equation number K, in equation number L, we arrive at equation number M. But by the arbitrary pricing theorem, that is equation number 3, we get this expression for the, for the excess return on a given security in terms of the APT risk prices. I repeat, the equation number N, that is shown on your slide is the APT equation, where the excess return on a security is represented in terms of the risk prices, or risk premia.

So, using that fact and using the treasury bill rate TBT, as the proxy for the risk free rate P0 TBT is the proxy for P0, we can arrive, or we can write. And the expected value of ri t minus P0, that is equal to expected value of ri t minus TBT in the form of equation number 9. Equation number N I am sorry.

So, if we compare equation number N, and equation number M, the relationship that we arrive at between the CAPM beta and the risk prices of the APT model is captured by equation number O. So, this is what is the relationship between this CAPM model and the APT model.

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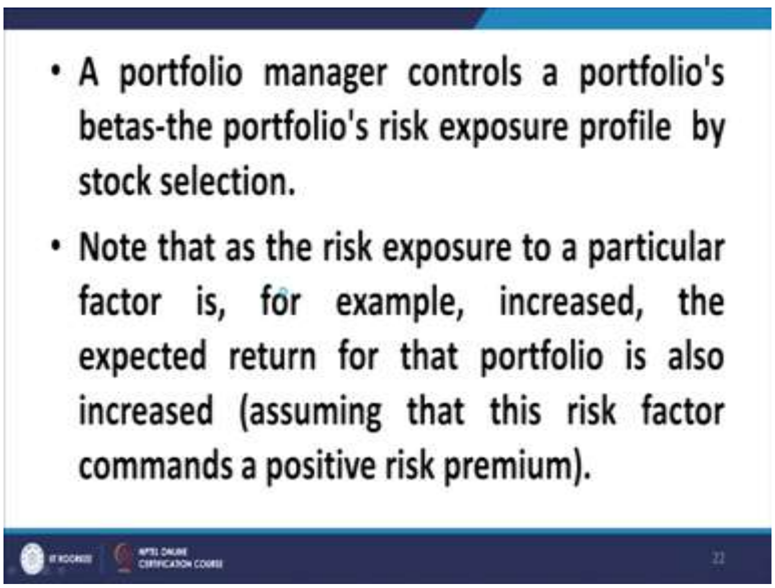

And therefore, it follows that the CAPM leads to the APT model, if the risk prices of the APT model satisfy the equation, that is given on this slide. Conversely, if the APT is true and the above K, K restrictions, or the risk prices of the APT model satisfies, the equation that is given on the slide. Then the CAPM will also be true. So, this is the relationship between the APT model and the CAPM model.

Now, a portfolio manager controls a portfolios beatas, by choosing the choosing the assets of choosing the securities, that constitute the portfolio, the portfolio manager is able to control the portfolio's betas. The portfolio's risk exposure profile, which is represented by the portfolio's betas as we discussed earlier.

Now, as the risk exposure of a particular factor is for example, increased the expected return for that portfolios also increase assuming, that this risk factor commands a positive risk premium. So, if the risk exposure with reference to a particular risk factors increase, a corresponding increase in expected return can be expected under the APT model provided of course, that risk factor commands a positive risk premium.

Now, the risk exposures and as the implied expected return for a portfolio are determined by a manager stock selection. Now, when we talk about the approaches to estimate in the APT model, what you see the APT model is much too general, it is extremely general. So, we need to find out approaches, which can be used for identifying the risk factors, that is what we are coming to.

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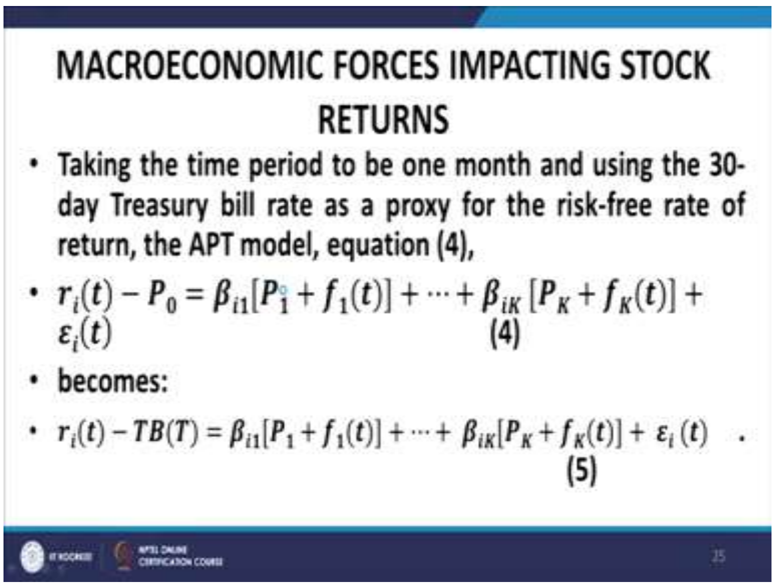

We take the time period to be 1 month and use the 30 day treasury bill rate as a proxy for the risk free rate of return. Then we can write the APT model as In fact, we discussed it a few minutes back as well, that ri t minus P0 will be equal to the expression, that is given in equation number 4, and when we substitute period P0, which represents the risk free rate in terms of the returns on the trial 30 day Treasury, because we are consisting our time horizon of 30 days.

So, if we consider the return on the 30 days Treasury bill as the proxy for the risk free rate P 0, then we can write equation number 4, as equation number 5.

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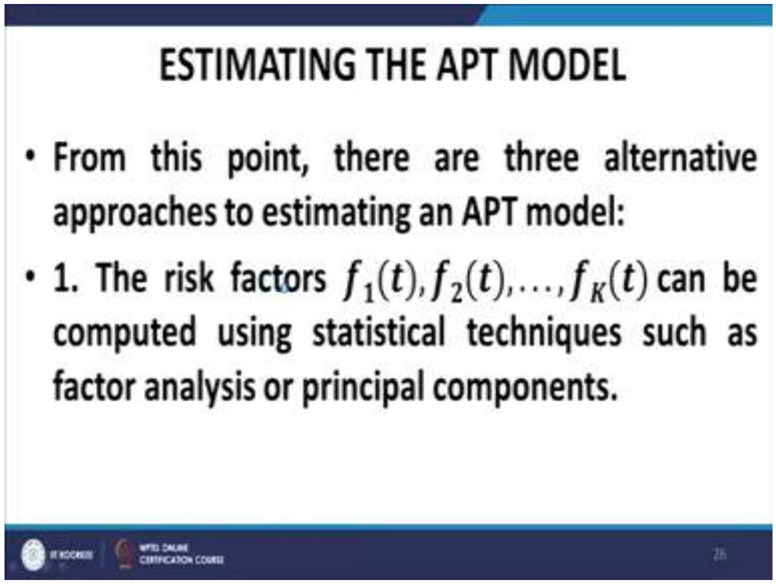

Now, we talk about the estimation part. Now, from this point there are three alternative approaches to estimating an APT model, that is we have to estimate, or we have to find some way of identifying the resources, or the risk factors. The risk factors f1, f2, and fk t can be computed using statistical techniques, like principal component analysis, or factor analysis.

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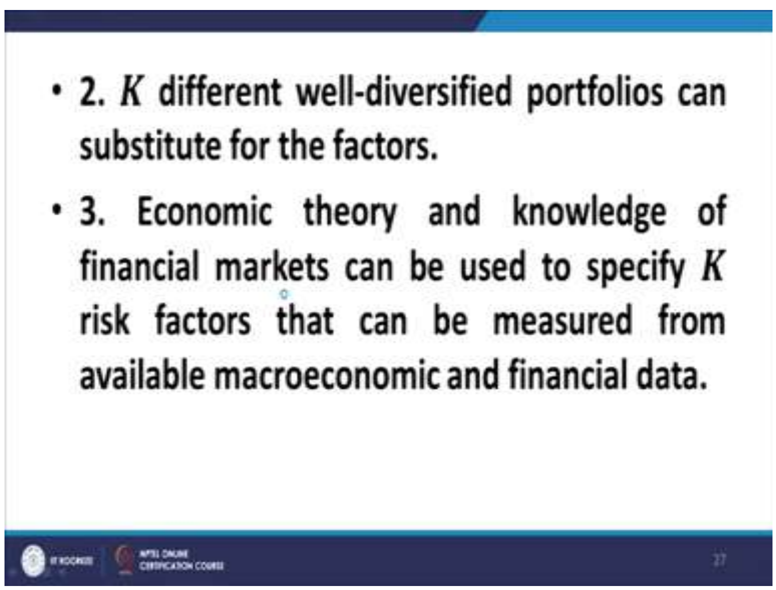

Number 2, the second approach can be by using K different well diversified portfolios, which we can substitute as the risk factors, or the third is a rather subjective approach, where we use economy theory and knowledge of financial markets, to specify the K risk factors, that can measured from available macroeconomic and financial data. So, then we will discussed the issues, or the shortcomings of the factor analysis, or principal components.

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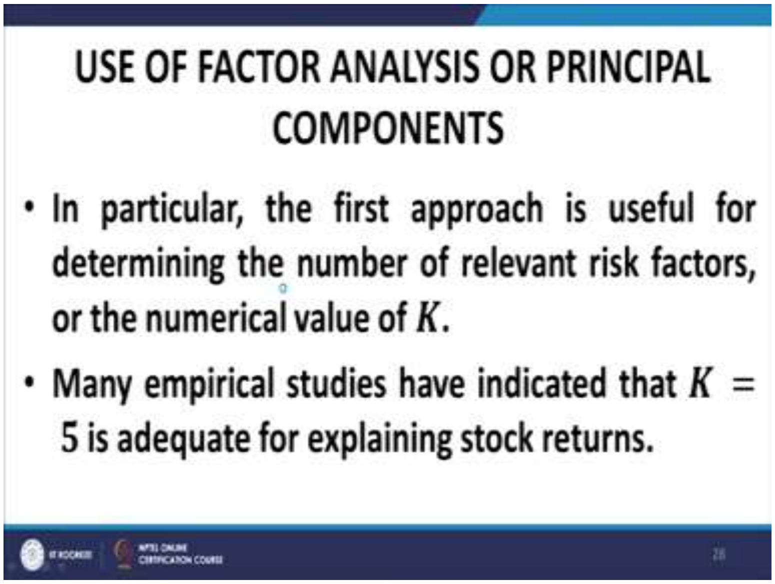

In particular this first approach is useful for determining the number of relevant risk factors, the numerical value of K that is, how many risk factors are good enough to give you a suitable description of the risk return trade off in relation to the APT model, that can be determined by using factor analysis, or principal component analysis.

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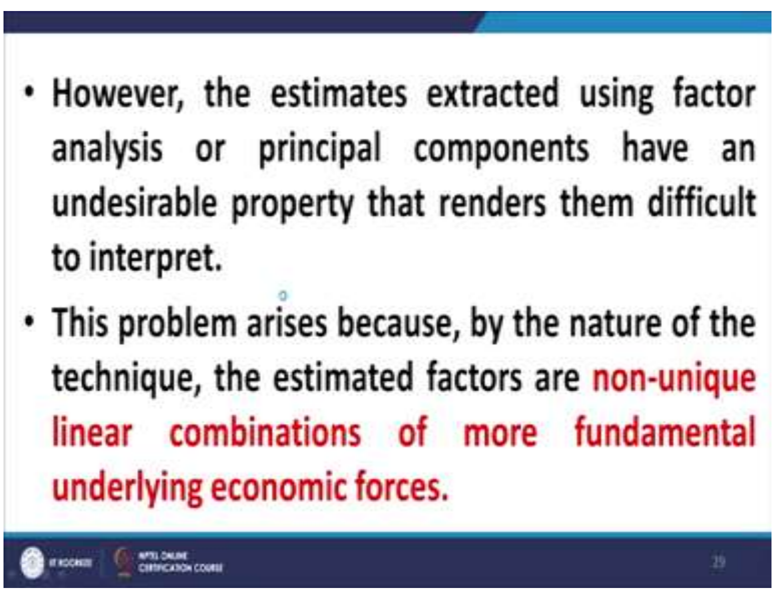

However, there is a major drawback of the use of the factor model, or the principal components approach to the determination of the risk factors. The problem is that, the estimates extracted using factor and this is a principal components have an undesirable property, that renders them difficult to interpret.

What is that undesirable property? The undesirable property is that because by the nature of this technique, the estimated factors are non-unique linear combinations of more fundamental underlying economic forces.

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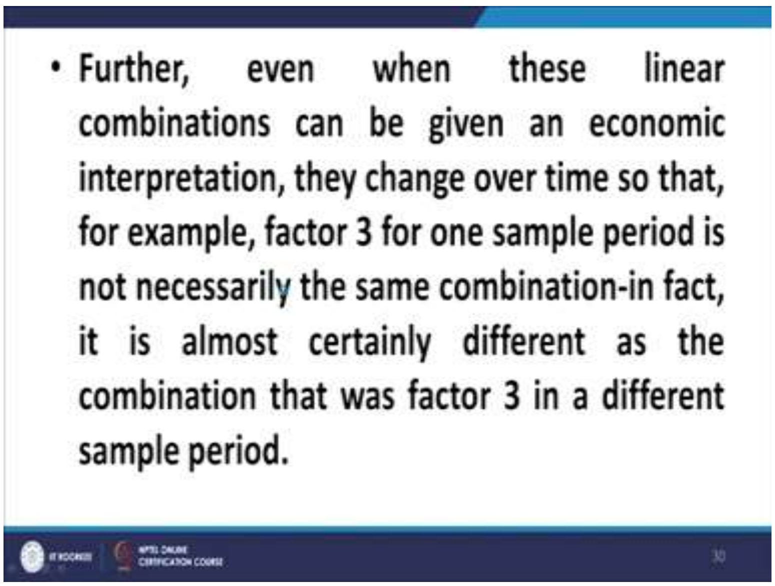

Furthermore, in addition to the shortcoming, even when these linear combinations can be given an economic meaning, can be given an economic interpretation, that change over time. So, that for example, factor number 3, for one sample period is not necessarily the same combination. In fact, it is most probably, most likely a different combination, as the combination that was factor number 3, in a different sample period. So, that makes the comparison problem written.

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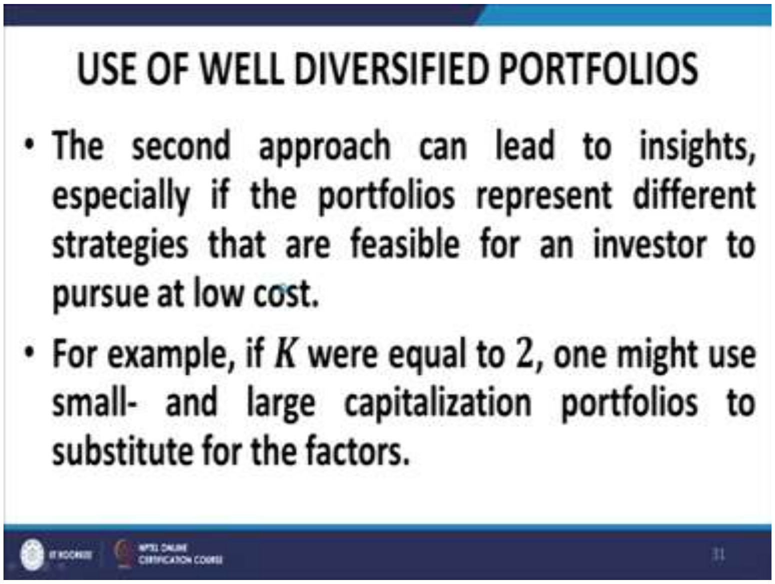

Now, take the second approach, the use of well diversified portfolios. The second approach can lead to insights, specially if the portfolios represent different strategies, that are feasible for an investor to pursue at low cost. For example, if K were equal to 2, if we had a two factor model, one could use small and large capitalization portfolios to substitute for the factors.

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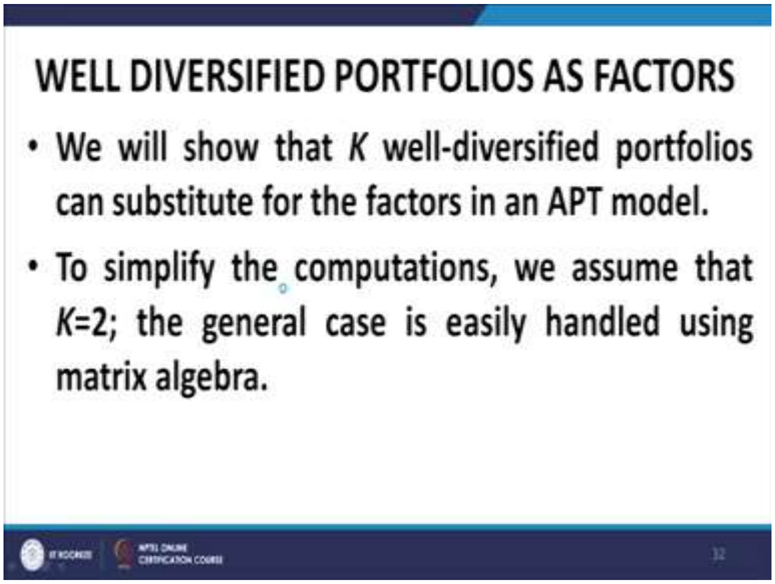

Now, we look at a derivation to justify the use of the well diversified portfolios as factors. We algebraically arrive at the justification a rationale of using the well diversified portfolios as factors in risk factors, in the APT model. For this purpose, we assume that K equal K is equal to

2, for simplicity to keep the exposition simple and tractable. Of course, for larger K, we can always take requests from matrix algebra and prove the results. So, we will show here that K well-diversified portfolios can substitute for the factors in an APT model, the proof is rather simple.

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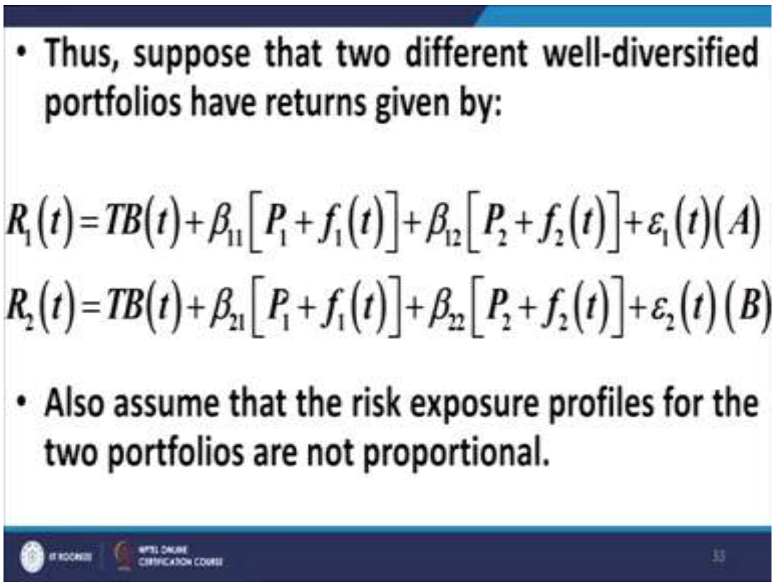

Suppose, we consider the K equal to 2 as I mentioned just now, and the two different well diversified portfolios of returns given by the expression, that I given in equation number A and equation number B. Please note, this is these are representations as obtained from the APT model, which we had discussed a few minutes back and these are simple representations obtained from the APT model r1 t is given by equation A and r2 t is given by equation B.

These are obtained as the expressions under the APT model of two well-diversified portfolios, let us call them 1 and 2. Now, we also assume, that the risk exposure profiles of the two portfolios are not proportional. Why is that? That is relevant as you shall see in a few minutes.

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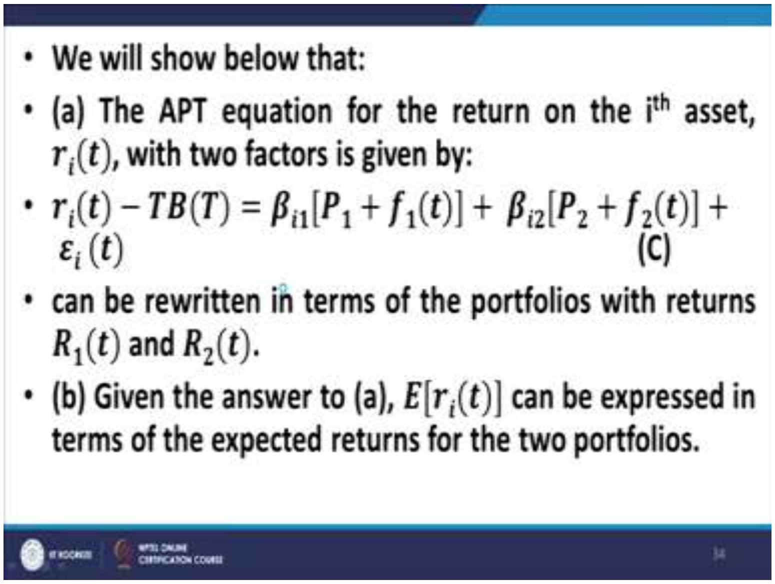

APT equation for the return on the ith asset, that is obtained from the APT model can be rewritten in terms of the portfolio's will returns capital R1 t and capital R2 t, these are the returns on the given portfolios. In other words, the return on any arbitrary security, or an arbitrary security i, which can be written in terms of the APT, as equation number C, can also be written now in terms of the well-diversified portfolios number 1 and 2, whose returns are given by capital R1 t and capital R2 t.

And the second part that we are going to prove that is that, given the answer to a, that is the proof of all, than the expected value of ri t can be expressed in terms of the expected returns on the two portfolios. That is E of capital R1 t and E of capital R2 t.

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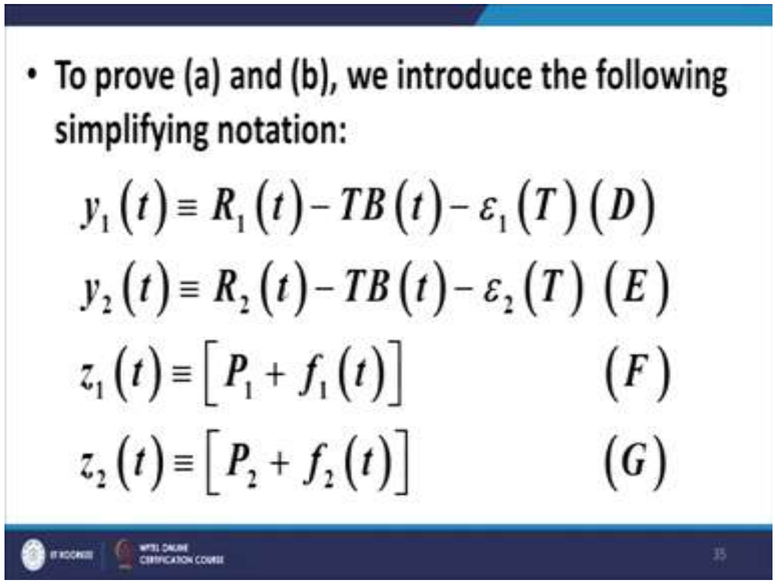

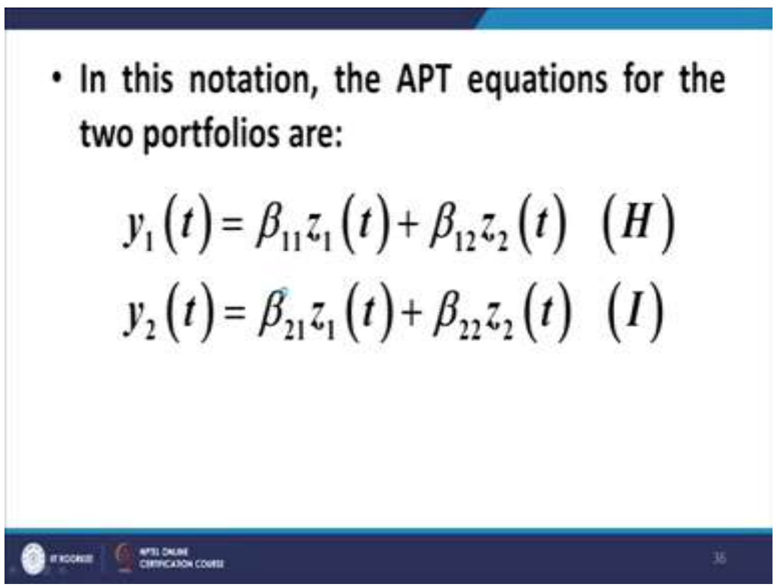

To prove this A and B, we use the following simplification notation, this is given an equation D, E, F and G. This is simply abbreviation, so that the calculations do not get cumbersome the representations of various equations do not get cumbersome, we abbreviate various expressions by single numbers.

Then using these abbreviations, or using these expressions, that are contained in D, E, F, and G, we can write the APT equation that is, that is the APT equation, which is a and b equations A and B, in terms of these abbreviations, which are captured in D, E, F, and G. In the form given an

H and I, given an equation H and I. So, in terms of H and I, what we have is equations A and B represented in a different set of variables for simplicity of notation.

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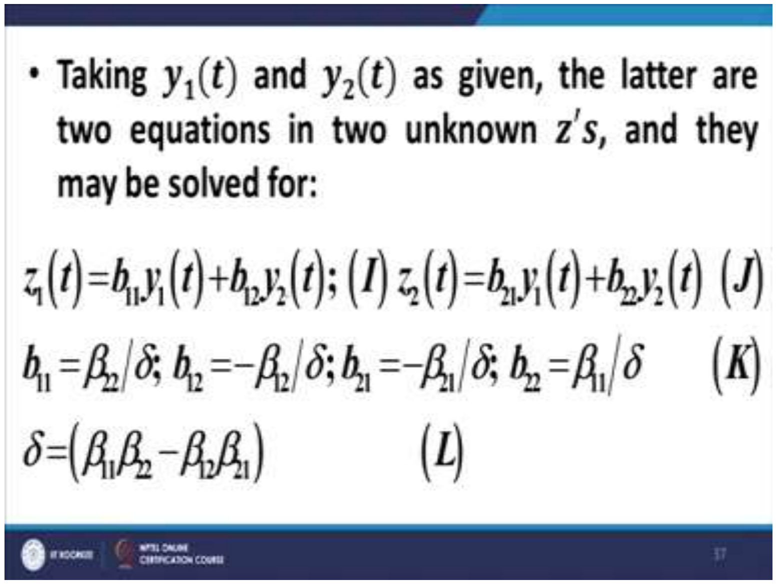

Now taking y1 t and y2 t as given, we can solve for the z's and when we solve for the z's, what we arrive at is equation number I and equation number J, where the various B's that is B 11, B 12, B 21 and B 22 are given by equation number K. And delta is given by beta 11, beta 22, minus beta 12, beta 21.

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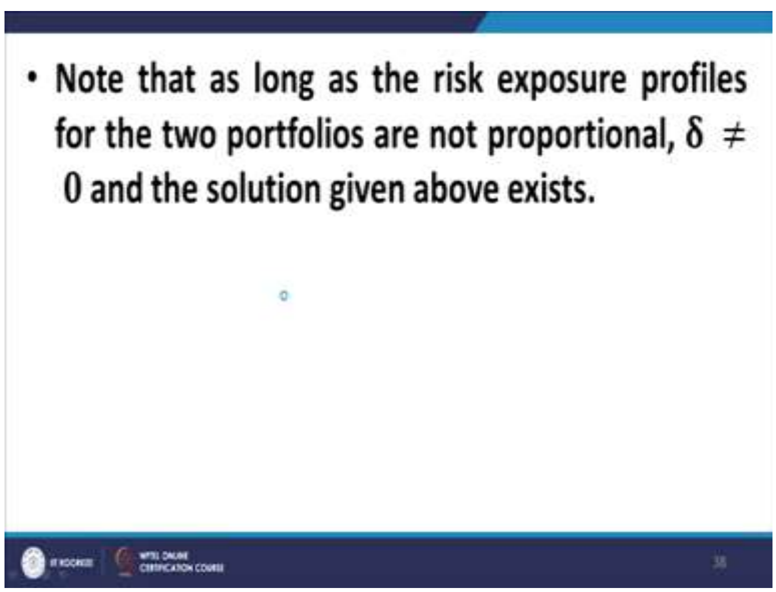

Now, the important thing here is that, I just mentioned that the risk exposure profiles for the two portfolios should not be proportional. Why they should not be proportional is because, we want delta to be unequal to 0, otherwise we would not get a singular solutions.

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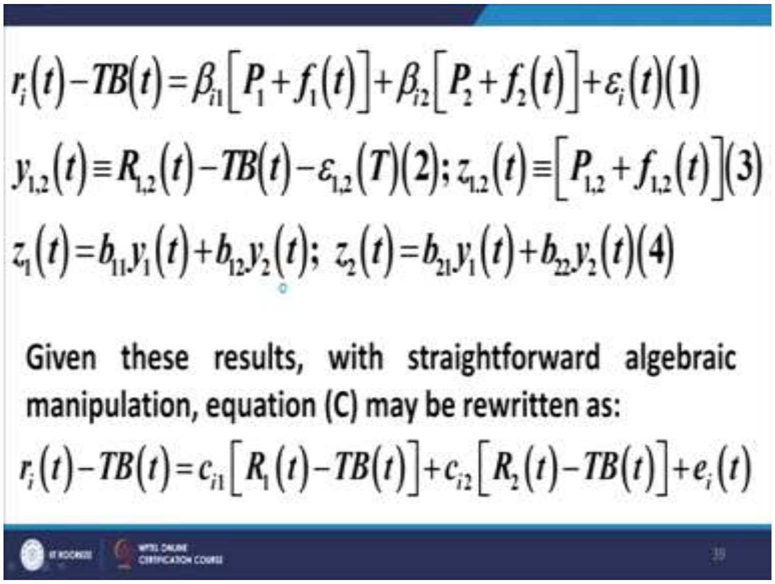

Let us now see the complete picture as given on this slide, we have the expression for the return on an arbitrary security i in terms of the APT model, that is equation number 1 here. Then, we have the expression for y1 and y1 t and y2 t, that is equation number 2 here. We have expression for z1 t and z2 t, that is equation number 3 here. And solving for z1 t and z2 t in terms of y1 t and y2 t, we have the equation, which is equation number 4 here.

Now, it can it is obvious that the first thing is that in equation number 1, I can make this substitution from equation number 3, and what I will get is, a relationship between ri t, that is the return on security i and the various z, z1 t and z2 t. But z1 t and z2 t are both expressible in terms of y1 t and y2 t, as shown in equation number 4.

And what is y1 t and y2 t? y1 t and y2 t contain the representation of the returns on the portfolio 1, and portfolio 2, which we need to establish as the factor components, or the factor risks, risk factors. So, given the results, given the sequence of inputs that I explained just now, I elucidated just now, we can easily obtain the expression for ri t in terms of the returns on the security, or the portfolio 1, and the portfolio 2, and of course, an error term, which is given here as ei t in the fall which is given in the bottom equation here on this slide.

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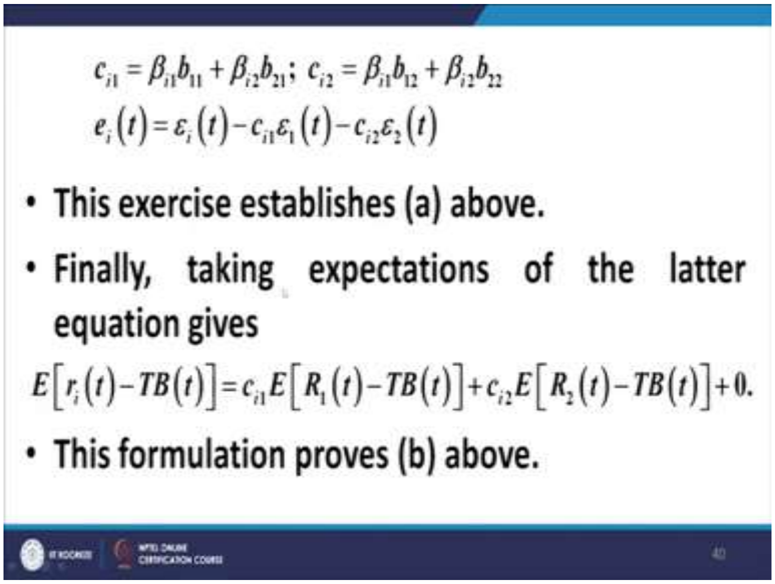

Where ci 1 is equal to the expressions that are given in the top equation on this slide and ei t which is the error term is given by this second equation here. So, this x by this x says whatever you establish that the return on any arbitrary security i can be represented as a linear factor model, where we are substituting capital R1 t and capital R2 t that is the returns on the well diversified portfolios 1 and 2, as the factor components.

Now, then by taking the expectation of the in this equation, the last equation that is there on the previous slide, this particular equation, the last equation here on this slide, we can establish the second part of our contention. So, we are able to establish the fact that if we have two diversified portfolios, or if we have K diversified portfolios, we can use those K diversified portfolios for the inputs, in terms of the risk factors into the APT model.

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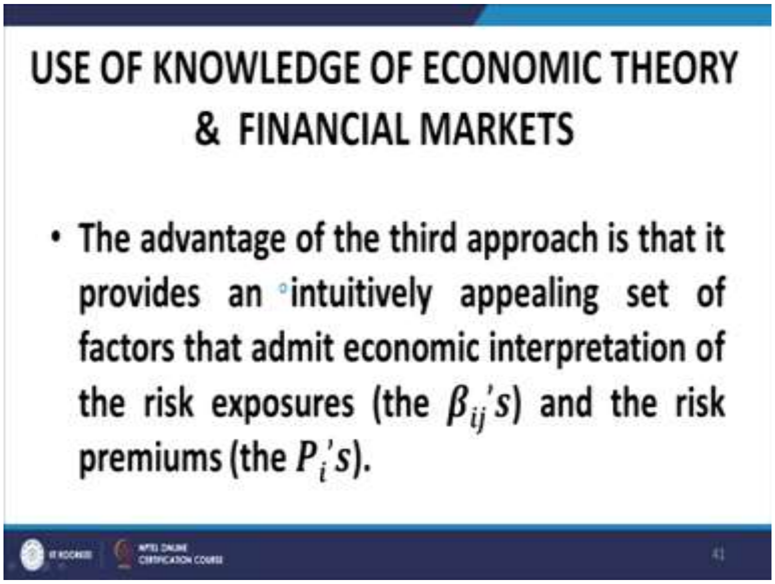

Then we come to the third approach, that is the use of knowledge, or economy theory and financial market. The advantage of the third approach is that, it provides an intuitively appealing set of factors, that admit a current economic interpretation of the risk exposures, that is the beta ij's and the risk premium, that is the Pi's.

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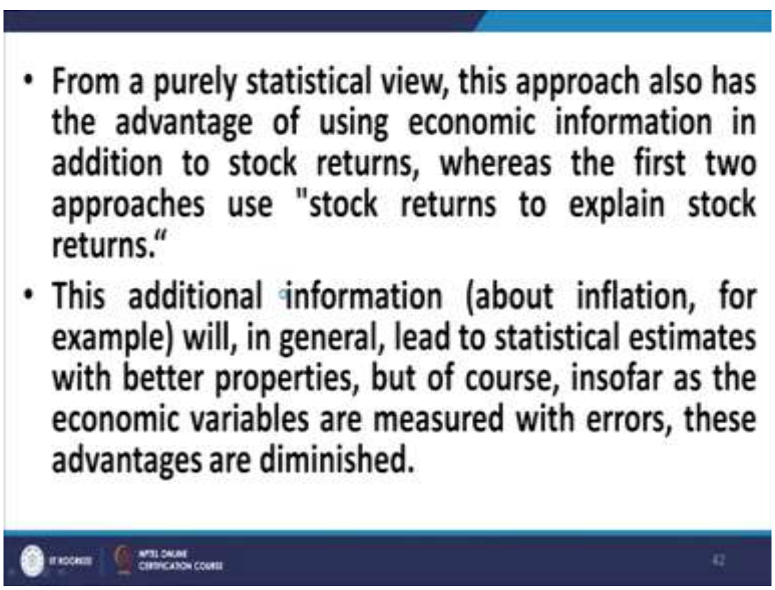

From a purely statistical view, this approach also has the advantage of using economic information in addition to stock returns. Whereas, the first two approaches were trying to establish, or trying to explain stock returns by using the stock returns. The additional information about inflation etcetera, will in general lead to statistical estimates with better properties. But of course, in so far as the economy variables are measured with errors, these advantages are diminished.

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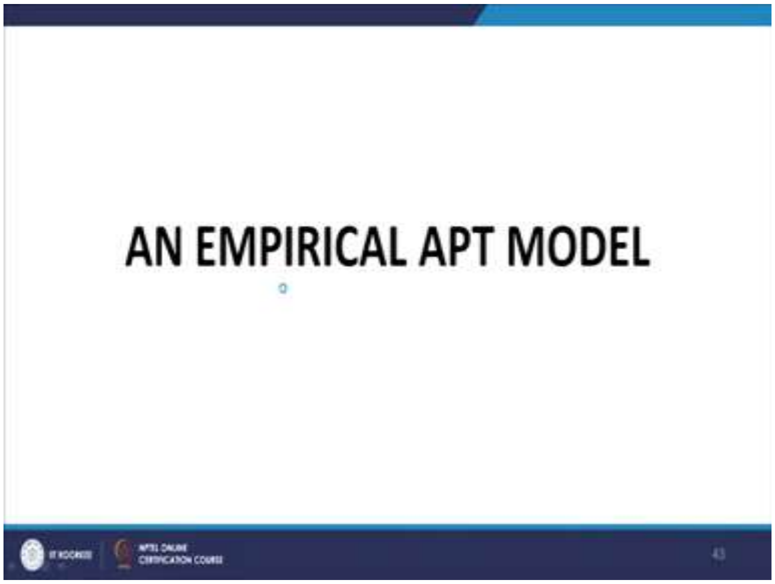

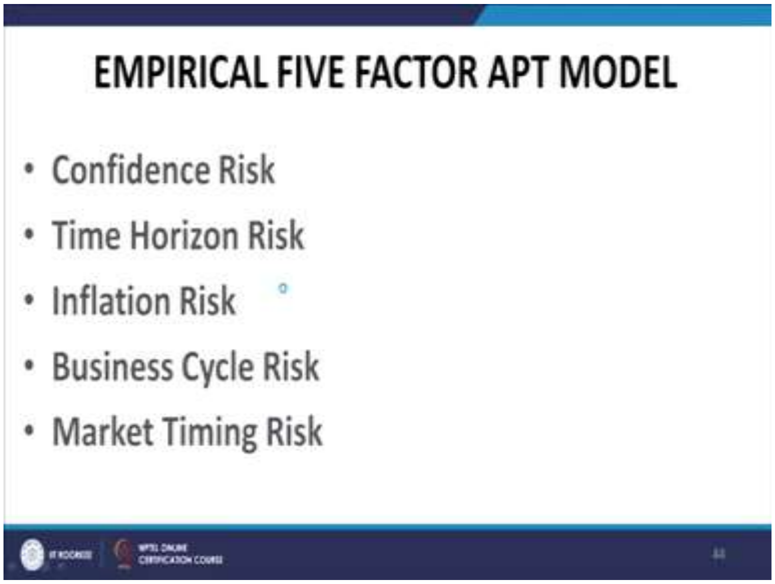

So, we now look at an empirical APT model, in which has been estimated, or which has been empirically tested and found to be reasonably good. Now, the risk factors, that constitute this model are number 1, confidence risk, number 2, time horizon risk, number 3 inflation risk, number 4 business cycle risk, and number 5 market timing risk.

So, this is a five-factor model, which has been empirically tested to be quite good, quite satisfactory and it comprises of the following risk factors, the confidence risk, the time horizon risk, the inflation risk, business cycle risk, and market timing risk. I shall take up this model in detail after a break. Thank you.