Security Analysis and Portfolio Management
Professor J. P. Singh
Department of Management Studies
Indian Institute of Technology, Roorkee
Lecture 50
Single Index Model 2

Let us continue from where we left off. Before the break immediately before the break I arrived at the expression for the total risk and I showed that there is one component of total risk which arises from individual variances, intrinsic variances of the securities which tends to 0 as we diversify more and more. Gradually as we diversify this particular component of the risk tends to 0.

However, there is another component which it does not tend to 0, which is not completely eliminated by even infinite diversification. Even if you diversify totally, this component of risk does not vanish, it still makes the finite contribution to total risk. And therefore, the total is given in the event of infinite diversification tends to a finite value which is non-zero. This was one inference that I give just now.

And the second inference that was I do was that the component of rick that diversifies to 0 arises from the intrinsic variances and the component of risk that does not diversify to 0 arises out of mutual interactions. In other words, it is the mutual interactions between the securities that gives rise to this systematic risk.

Now, the learners may consider it as you may find this particular inference is a bit of a contradiction to what we discussed in the context of the single index model. So, I need to clarify this point, when I am talking about mutual interactions in the general situation, it is without reference to the single index model.

But however, when you move from the general framework into the single index model, what we find is that these mutual interactions are modelled completely by the relationships of the securities with the market index and we do not have any direct mutual interactions within the premise of the single index model.

So, therefore, the expression that represented the systematic risk that arose from mutual interactions is now captured by the term that represents the interrelationship of the securities with the market index that was beta p square sigma m square. So, let me repeat the covariance term that does not tend to 0 even in infinite diversification that term in turns gives you a finite value non-zero value for the risk is modelled in the single index model as being as arising due to interactions between individual securities with the market index separately.

And the single index model forbids any mutual interactions in the sense that the two securities interact directly with each other. This is a clarification I thought I need to give because, apparently the results seem to be conflicting. So, but please note in the last revision before the break, we never assumed the validity of the single index model. So, if we feed in the single index model into the derivation.

We will find that the systematic risk component arises from interactions of its security with the market index rather than direct mutual interactions between securities. And the reason, as I have emphasized again, and again, that securities move together or move in tandem with each other is due to their relationship with the market index, rather than any direct relationship with each other.

Because we have assumed in the single index model that expected value of ei, ej is equal to 0. This is the mathematical representation of the assumption that I have just explained. Now, we move on to portfolio optimization in the single index model. I mentioned at the outset at the time of introducing this model, that the fundamental objective of the single index model was to simplify the portfolio optimization process. So let us see how this has been achieved under this single index model.

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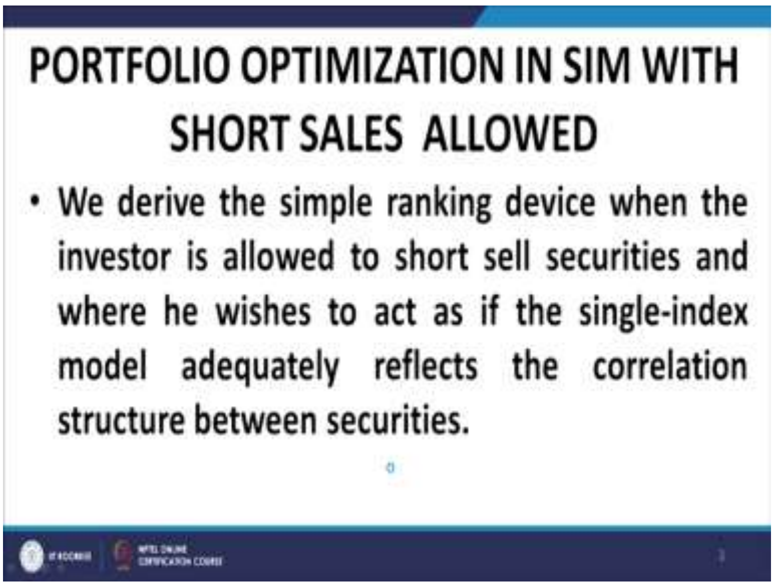

We derived a simple ranking device when the investor is allowed, we should explore two situations. Firstly, when short sales in the securities are allowed and the second situation when short sales are not allowed. We will illustrate them with examples. So first of all, let us take a situation where short sales are allowed, which is rather simple. We derive the simple ranking device when the investor is allowed to short sell securities, where he wishes to act as if the single index model adequately reflects the correlation structure between securities.

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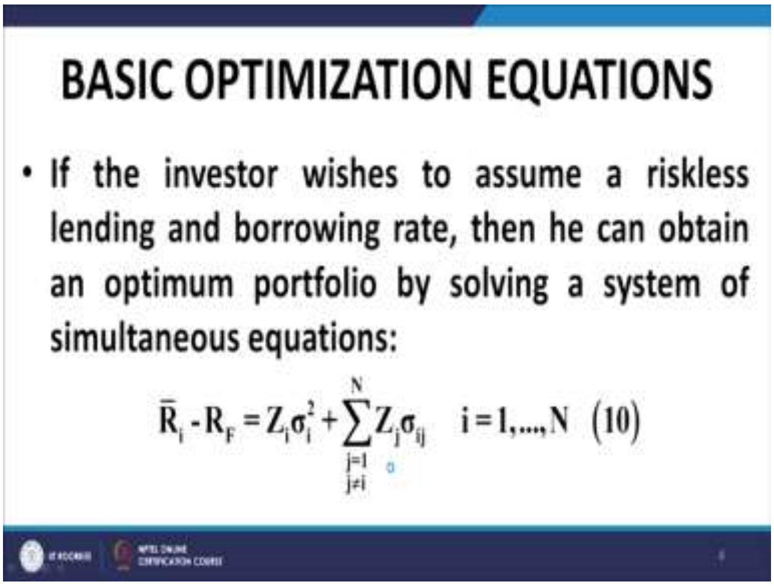

Now, we have this equation, which we have discussed in a lot of detail in the mean variance framework. This is the optimization equation the fundamental optimize equation optimizing equation when we talk about the mean variance framework. So, if the investor wishes to assume as riskless lending and borrowing rate that means, we are also allowing riskless lending and borrowing within this model.

So, we are allowing riskless lending and borrowing together with short selling of the risky securities. Then you can obtain an optimum portfolio by solving a system of simultaneous equation given by equation number 10. These are this is a set of 10 equations in N unknowns z1, z1 and z3 to all up to zN and hence we can find a unique solution. Now, let us quickly recall the single index model.

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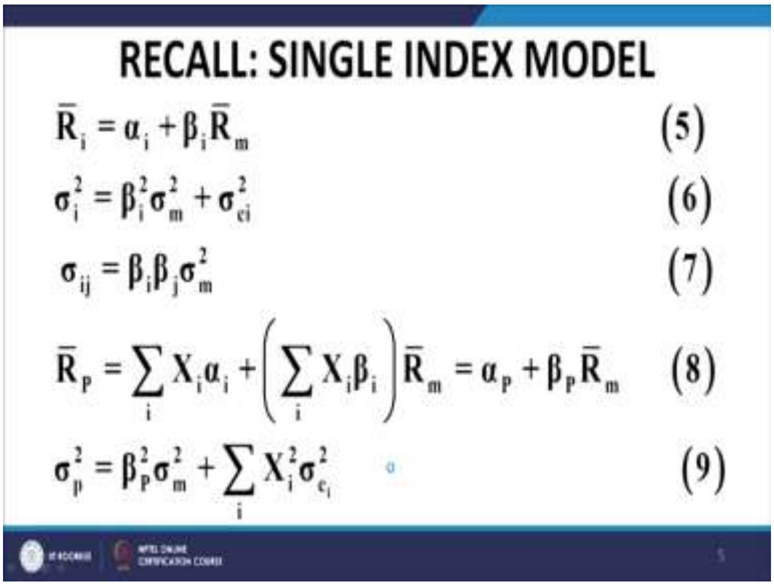

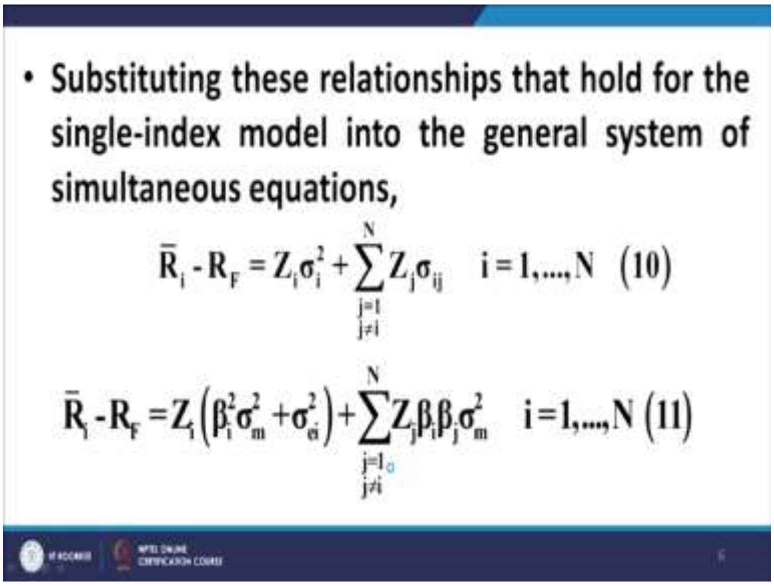

Equations are given on the slide the expression for the expected return on a security, the expression for the variance of a security, the expression for the covariance, the expression for the expected return on a portfolio and then the expected expression for the variance of a portfolio all these expressions we have discussed in detail and this is only a summary of the results that we have already obtained.

Substituting these relationships that hold for the single index model into the general equation a general system of simultaneous equations. Excuse me, what we get is equation number 11. When we substitute equation number, let me go back equation number 6 when we substitute equation number six in equation number 10 what we get is equation number 1.

Sigma i square is replaced by beta i square sigma m square plus sigma ei square and the covariance from sigma ij is replaced by beta i beta j sigma m square which is equation number 7 and this. Now, here we are injecting the single index model into our analysis.

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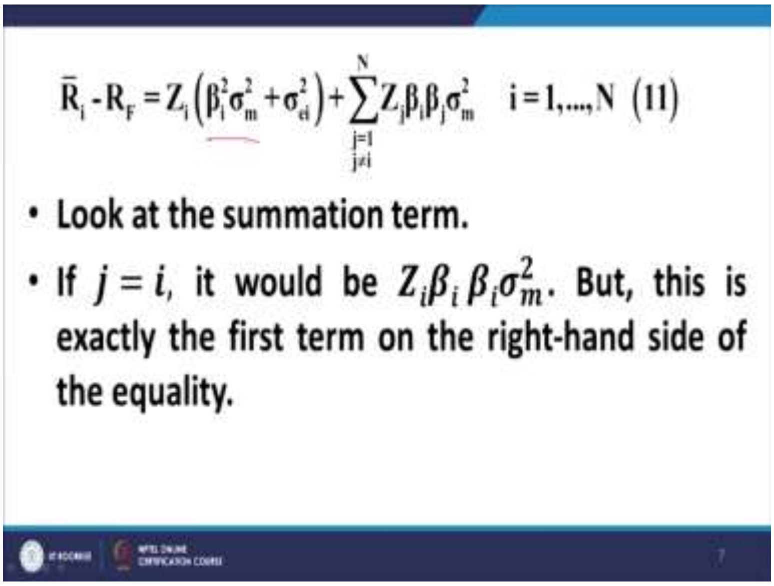

Let us see what we get from here. This is equation number 11 that I brought forward from the previous slide, let us look at this summation term. If you look at the summation term let us see it contains the caveat it contains the constraint that j should not be equal to i. In other words, the summation should be with all values of j except the value of i the summation does not extend or does not include the term for which j is equal to i.

It is explored what, what we get when we substitute j equal to i. When we substitute j equal to i what we get? zi beta i square sigma m square. If you substitute j equal to i in the term within the summation you get zi beta i square sigma m square. And please note, this is the first term on the right-hand side of the equation that is the term that I am underlined in equation number 11.

In other words, let me repeat what I have done if this summation extends over all values of j except the value of j equal to i that is what the summation term represents. Now, if in the summation term I was to remove this constraint, in other words, if I was to allow j is equal to i in this particular summation, then that means, the additional term that would come into the picture that would come into the summation would be the term for which j is equal to i and what is that term?

That term is zi beta i square sigma m square that is nothing but the term that I have underlined that is the first term on the right-hand side. So, in other words, what I can do is to simplify it, let us say it is the case of simplifying the notation, what I can simply do is I can incorporate this underlined term within the summation and remove the constraint j unequal to i because by incorporating or by removing the constraint j equal to i.

I will allow for j equal to i and the term that I get when j equal to I is simply the first term or the underlined term in equation number 1. So, the bottom line is if I remove the constraint j equal to i, then I need not write the first term the underlined term separately. it automatically gets embedded in the summation term.

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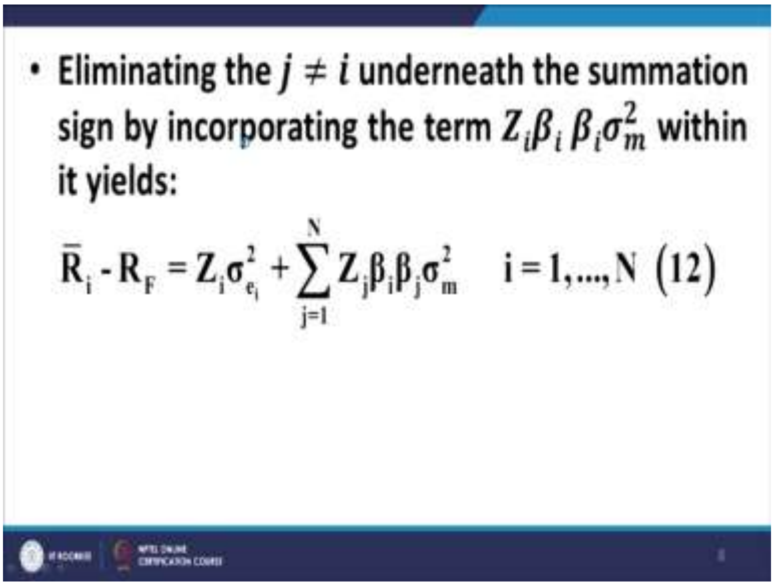

So, let us rewrite that equation by making these changes of notation. That gives me equation number 12. Now in equation number 12, what do we have?

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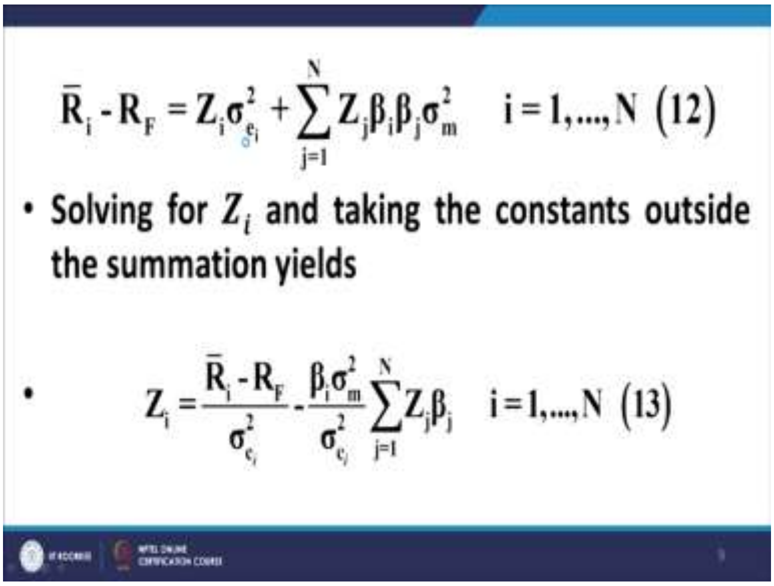

We have expressions for zi so, we can solve for zi if we solve for zi, what we get is the next equation that is right at the bottom of your slide. See, we simply solve equation number 12 for zi made making appropriate transpositions and we get the next equation at the bottom of your slide.

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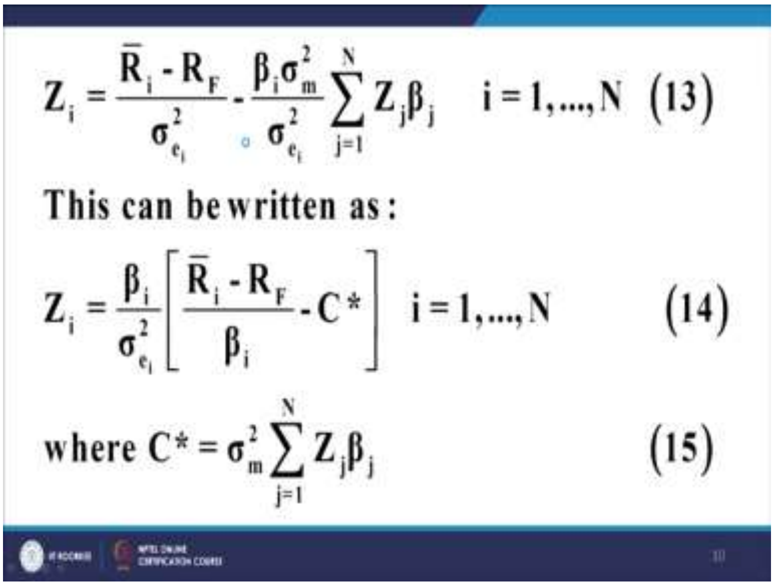

What we have is zi is equal to this Ri bar minus RF upon sigma square minus beta i sigma m square upon sigma ei square summation zj and beta j. Now you can write this in a slightly more helpful form slightly more convenient form by taking beta i upon sigma i square as a pre factored and within the square brackets, we have Ri minus RF upon beta i that represents the first term minus C star.

Where C star is nothing but C star is nothing but sigma m square summation zj beta j because the value or the appropriateness of writing this equation in this form in the form of equation number 14, will become apparent very soon.

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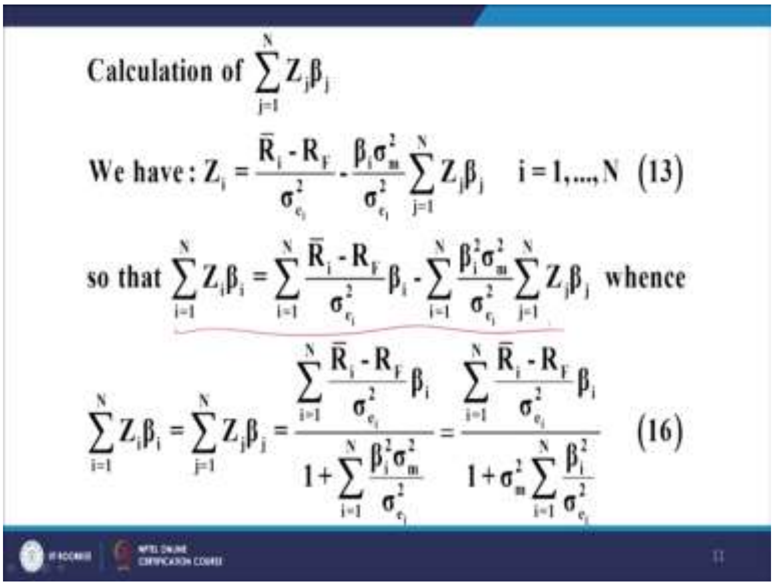

Now, we have to we do not know that term that we do not know in this equation number 14 is the summation term summations zj beta j. So, we have to somehow express the summation zj beta j, in terms of the inputs that are available with us, we do it in a very exquisite manner. What we do is we multiply this equation is zi is equal to Ri bar minus RF upon sigma ei square minus beta i sigma m square sigma ei square summations zj beta j.

We multiplied by beta i, and then we sum over all values of i. I repeat, we multiply this equation by beta i, and we sum over all values of i, what we get is summation zi beta i is equal to the expression that is on the right-hand side. I repeat, we have simply multiplied this equation by beta i. And we have summed over all values of i.

Now if you look at this equation, which I am again underlining, what we find is that there is summation zj beta j on the right-hand side, and then is the mission zi beta i. On the left-hand side, in both cases, the summation index runs from 1 equal to 1 to N I am sorry. So, in both cases, we have summation z beta summed over all values from 1 to N.

So, obviously, these two terms would be the same, the only thing is the naming of the index in one case it is i the other in other case it is j. The summing range is the same, and the term that is to be summed over is also the same in both cases. In other words, what I am saying is summation zi beta i is equal to summation zj beta j. In each case, the summing is over 1 to N.

So, I can take this or I can solve this equation the equation that I have underlined for summation zj beta j or zi beta i, you can take any index you can take k, a, b, j, whatever. And what we have is equation number 16. In other words, summation zj beta j or summations zi beta i is given by the right-hand side of equation number 16. This enables us to immediately work out the value of C star. So, C star is equal to summation.

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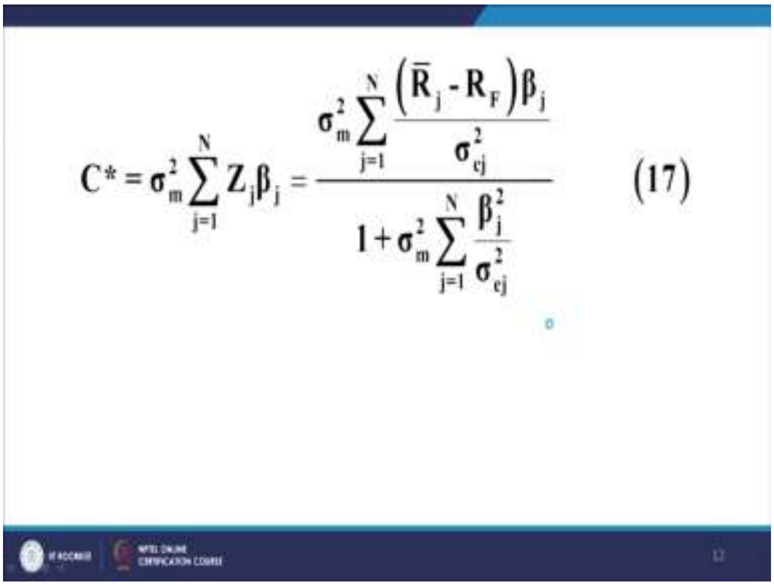

Sorry sigma m square summations zj beta j and which takes the form of equation number 17.

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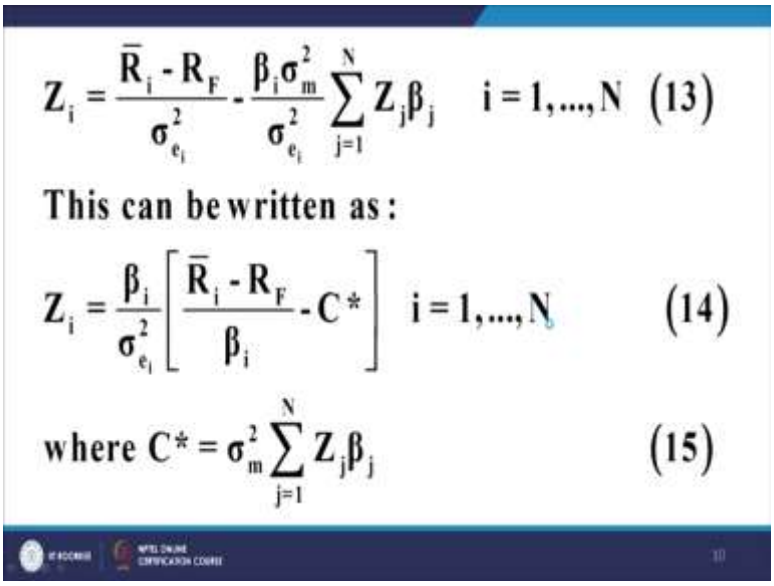

Having worked out the value of C star in terms of the known parameters, we are now in a position to work out each value of zi and then normalize this to obtain the values of Xi, X1 X2, X3 which form the composition vector of the optimal portfolio. So, that is the optimization process in the case where short sales are allowed together with risk free lending and borrowing in the single index model. So, you can see how simple this process has become by incorporating the assumptions of the single index model.

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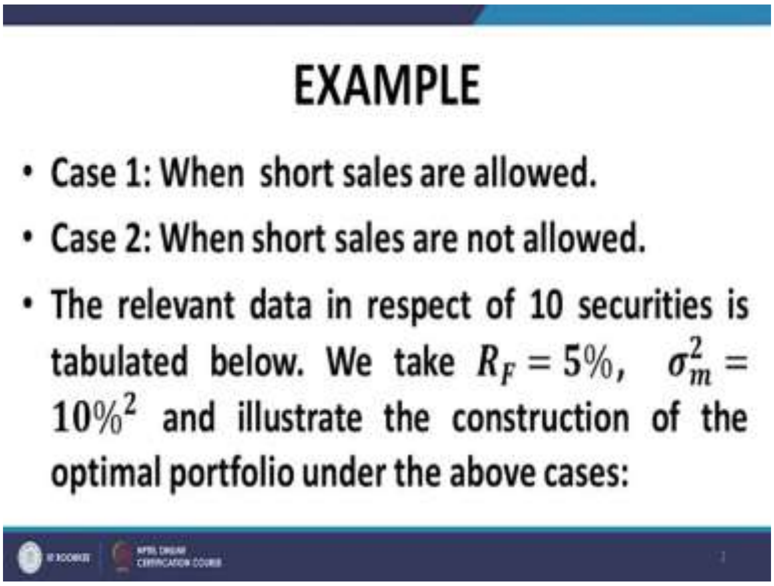

Let us look at an example. So, case 1 where short sales are allowed irrelevant data in respect of 10 securities is tabulated on the next slide, we take RF equal to 5 percent and we assume that sigma m square that is the market variance is equal to 10 percentage square and we illustrate the construction of the optimal portfolio when short sales are allowed and risk free lending and borrowing is also allowed.

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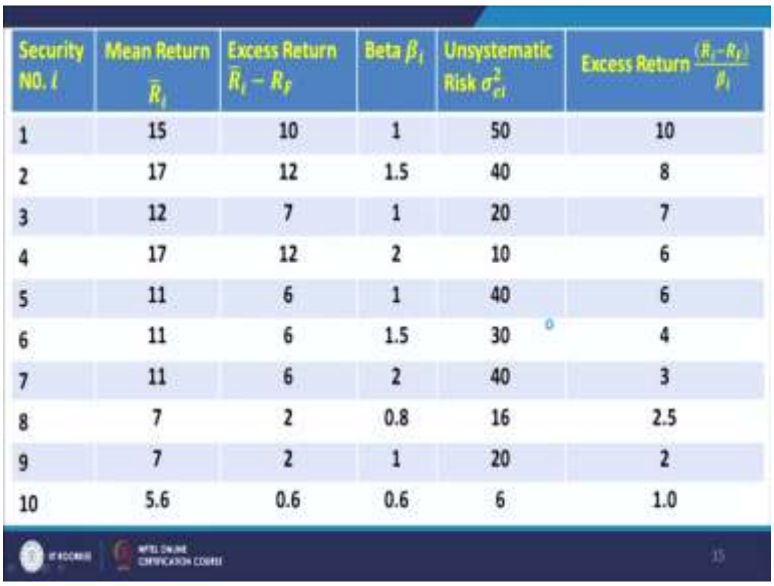

Security NO. i	Mean Return $\bar{R}_i$	Excess Return $\bar{R}_i - R_F$	Beta β_i	Unsystematic Risk σ^2_{ei}	Excess Return $\frac{(\bar{R}_i - R_F)}{\beta_i}$
1	15	10	1	50	10
2	17	12	1.5	40	8
3	12	7	1	20	7
4	17	12	2	10	6
5	11	6	1	40	6
6	11	6	1.5	30	4
7	11	6	2	40	3
8	7	2	0.8	16	2.5
9	7	2	1	20	2
10	5.6	0.6	0.6	6	1.0

So, this is the data that is given to us up to column number 1, 2, 3, 4, 5. And from the information that is contained in column number 1, 2, 3, 4, 5, we can work out the excess return that is column 6.

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Security NO. i	$\frac{(\bar{R}_i - R_f)}{\beta_i}$	$\frac{(\bar{R}_i - R_F)\beta_i}{\sigma_{ei}^2}$	$\frac{\beta_i^2}{\sigma_{ei}^2}$	$\sum_{j=1}^{i}\frac{(\bar{R}_j - R_F)\beta_j}{\sigma_{ej}^2}$	$\sum_{j=1}^{i}\frac{\beta_j^2}{\sigma_{ej}^2}$	C_i
1	10	2/10	1/100	2/10	2/100	1.67
2	8	4.5/10	5.625/100	6.5/10	7.625/100	3.69
3	7	3.5/10	5/100	10/10	12.625/100	4.42
4	6	24/10	40/100	34/10	52.625/100	5.43
5	6	1.5/10	2.5/100	35.5/10	55.125/100	5.45
6	4	3/10	7.5/100	38.5/10	62.625/100	5.30
7	3	3/10	10/100	41.5/10	72.625/100	5.02
8	2.5	1/10	4/100	42.5/10	76.625/100	4.91
9	2.0	1/10	5/100	43.5/10	81.625/100	4.75
10	1.0	0.6/10	6/100	44.1/10	87.625/100	4.52

And we can work out the other parameters as well. We can work out the excess return per unit of systematic risk and we can work out other expressions which are required for the calculation of C star.

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Case 1: When short sales are allowed.

$$C^* = \frac{\sigma_m^2 \sum_{j=1}^{N} \frac{(\bar{R}_j - R_F)\beta_j}{\sigma_{ej}^2}}{1 + \sigma_m^2 \sum_{j=1}^{N} \frac{\beta_j^2}{\sigma_{ej}^2}}; \; R_F = 5; \; \sigma_m^2 = 10; \sum_{j=1}^{N} \frac{(\bar{R}_j - R_F)\beta_j}{\sigma_{ej}^2} = 4.41;$$

$$\sum_{j=1}^{N} \frac{\beta_j^2}{\sigma_{ej}^2} = 0.87625; \; C^* = \frac{\sigma_m^2 \sum_{j=1}^{N} \frac{(\bar{R}_j - R_F)\beta_j}{\sigma_{ej}^2}}{1 + \sigma_m^2 \sum_{j=1}^{N} \frac{\beta_j^2}{\sigma_{ej}^2}} = \frac{10 \times 4.41}{1 + 10 \times 0.87625} = 4.52$$

What are the inputs that are required for C star? That are given in this slide. Sigma m square is required which in our case has known it is 10, RF is given as 5, summation of Ri minus RF beta i upon sigma ei square, this is all this is given in the table. You have to sum over all possible securities or all given securities in our case, and what we get here is, if you look at this, this is equal to 44.1 divided by 10 that is equal to 4.41.

What else is required, we need summation of beta square upon sigma ei square summation over all possible values of i, which can also be obtained from the index from the table I am sorry, and that is equal to 87.625 divided by 100 that is equal to 0.87625. And so, we can work out the value of C star. We have got all the inputs here, you find that C star is equal to 4.52 knowing the value of C star you can directly work out the values of z1 to z10.

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$$Z_i = \frac{\beta_i}{\sigma_{e_i}^2}\left[\frac{\bar{R}_i - R_F}{\beta_i} - C^*\right] \quad i = 1, \ldots, N; \quad X_i = \frac{Z_i}{\sum_j Z_j} \qquad (14)$$

Which is which is given in the next slide and knowing the value of z1 to z10 we can work out the corresponding values of X1, X2, X3 up to X10 by using equation number 14.

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$$Z_1 = \frac{1}{50}[10 - 4.52] = 0.110 \qquad Z_7 = \frac{2}{40}[3 - 4.52] = -0.076$$

$$Z_2 = \frac{1.5}{40}[8 - 4.52] = 0.131 \qquad Z_8 = \frac{0.8}{16}[2.5 - 4.52] = -0.101$$

$$Z_3 = \frac{1}{20}[7 - 4.52] = 0.124 \qquad Z_9 = \frac{1}{20}[2 - 4.52] = -0.126$$

$$Z_4 = \frac{2}{10}[6 - 4.52] = 0.296 \qquad Z_{10} = \frac{0.6}{6}[1.0 - 4.52] = -0.352$$

$$Z_5 = \frac{1}{40}[6 - 4.52] = 0.037$$

$$Z_6 = \frac{1.5}{30}[4 - 4.52] = -0.026 \qquad \sum_{i=1}^{10} Z_i = 0.017$$

So, the results are tabulated in this slide, we have z1 is equal to for example, that is given beta 1 upon sigma e1 square into R1 minus RF upon beta 1 minus C star. Where C star is equal to 4.52 when you substitute all these values for security number 1, what we get is that z1 is equal to 0.110 and similarly, you can work out all the other values of z.

So, this is how the case where short sales in the securities are allowed together with riskless lending and borrowing has to be handled within the domain of the single index model. Now, we look at the second case, where short sales are not allowed. The process is slightly more involved. So, let us go through it step by step.

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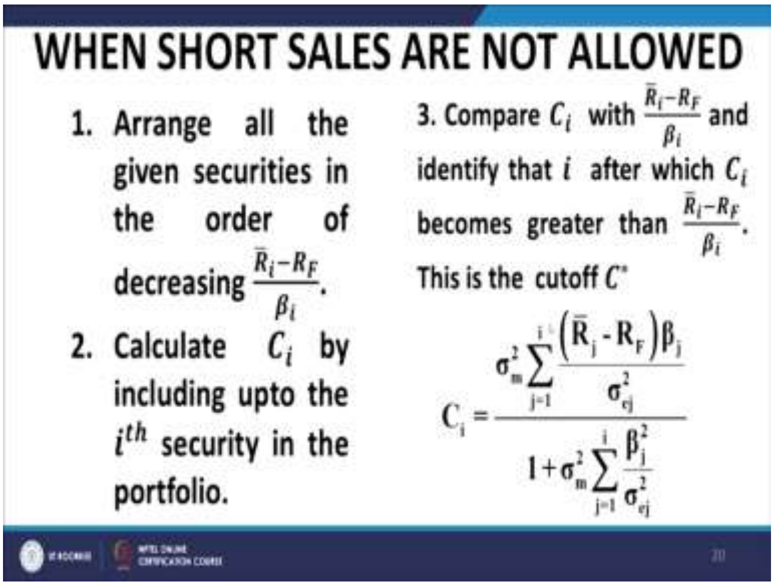

The first step is that we arrange all the given securities in the order of decreasing Ri bar minus RF upon which I repeat, we arrange all the given securities in the order of decreasing Ri bar minus RF upon beta. In the previous case, when short sales was allowed, this order need not be maintained, you had no need to arrange the securities in this descending order as per by this parameter.

Why, because we are, we were, we were summing over all these securities in the portfolio, some could be long, some could be short, we were allowed short sales and therefore, we could take negative values of X's also. And that meant that meant that there was no need to organize the securities, align these securities in the order of this decreasing value of the excess return to beta.

Then, after organizing them or after aligning them, in the order of decreasing Ri minus RF upon beta i. We calculate Ci for each security by including by including that security and all securities above them in the in the feasible set or in the optimal portfolio. Let us say, we assume that only the security with the highest Ri bar minus RF upon beta i is contained in the portfolio. Then we work our Ci on the premise that this one security alone is in a portfolio and we work out the value of Ci.

Similarly, if we assume that security 1 and security 2 as per the ranking are going to form the portfolio, we again work out the value of Ci, but this time on the premise that the portfolio consists of two securities. Similarly, we work out the value of Ci that is C3 on the present premise that security 1, 2 and 3 in the ranking is included in the portfolio. So, we work this out for all the 10 securities, all the N securities in our portfolio.

We work out C1, C2, C3, C4, where each Ci is based on the premise that that particular securities for example, if it is c4, then security number 4, and all securities lying above it in the ranking would be included in the portfolio. If we are working on security 6 that means what that means what? That means all securities including security 6 as per the ranking would form the portfolio.

If we are working on C10, we assume that the portfolio consists of all the securities and we work out C10 accordingly. We then compare the all the Ci that is C1, C2, C3 that we have worked out with the values the corresponding values of Ri minus RF upon beta i. So, the next step I repeat, we have got all the values of Ci, we compare these values of Ci with the corresponding values of Ri minus RF up on beta i and we locate that value of Ci such that below which below that value.

So, below that value, what happens is that the value of Ci becomes greater than the value of Ri minus RF upon beta i. That means, up to the values of Ci above and including that security, the value of Ri minus RF upon beta i will be critical, then the Ci that we have worked out. So, that is the cut-off Ci, we call it C star.

And based on the C star, the rest of the process resembles absolutely the steps that we followed in the previous example, when short sales are allowed, except for the fact that here securities which are giving a negative X will not be included in the portfolio, because you are not allowing short sale. So, only up to that security at the point of which we have C star would form part of the optimal portfolio. Securities below that C star level will not form part of the optimal portfolio. So now, let me illustrate this with an example. So, we have got 10 securities here.

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Security NO. i	Mean Return $\bar{R}_i$	Excess Return $\bar{R}_i - R_F$	Beta β_i	Unsystematic Risk σ^2_{ei}	Excess Return $\frac{(\bar{R}_i - R_F)}{\beta_i}$
1	15	10	1	50	10
2	17	12	1.5	40	8
3	12	7	1	20	7
4	17	12	2	10	6
5	11	6	1	40	6
6	11	6	1.5	30	4
7	11	6	2	40	3
8	7	2	0.8	16	2.5
9	7	2	1	20	2
10	5.6	0.6	0.6	6	1.0

And the first step is that we are ranked these securities in terms of Ri minus RF upon beta i. You can see here that Ri minus RF upon beta i is decreasing as we go along this particular column 10, 8, 7, 6, 6, 4, 3, 2.5, 2, 1. So it is decreasing as you go along this particular column downwards. In other words, we have ranked the securities in the order of decreasing Ri minus RF beta i.

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Security NO. i	$\frac{(\bar{R}_i - R_F)}{\beta_i}$	$\frac{(\bar{R}_i - R_F)\beta_i}{\sigma^2_{ei}}$	$\frac{\beta^2_i}{\sigma^2_{ei}}$	$\sum_{j=1}^{i}\frac{(\bar{R}_j - R_F)\beta_j}{\sigma^2_{ej}}$	$\sum_{j=1}^{i}\frac{\beta^2_j}{\sigma^2_{ej}}$	C_i
1	10	2/10	1/100	2/10	2/100	1.67
2	8	4.5/10	5.625/100	6.5/10	7.625/100	3.69
3	7	3.5/10	5/100	10/10	12.625/100	4.42
4	6	24/10	40/100	34/10	52.625/100	5.43
5	6	1.5/10	2.5/100	35.5/10	55.125/100	5.45
6	4	3/10	7.5/100	38.5/10	62.625/100	5.30
7	3	3/10	10/100	41.5/10	72.625/100	5.02
8	2.5	1/10	4/100	42.5/10	76.625/100	4.91
9	2.0	1/10	5/100	43.5/10	81.625/100	4.75
10	1.0	0.6/10	6/100	44.1/10	87.625/100	4.52

The next step is we calculate C1 on the premise that we have only one security security number 1 in our portfolio, we calculate C1. Then we calculate C2. C2 is worked out on both premise on the premise the security number 1 and security number 2 form our portfolio, the values of the various parameters are worked out accordingly and the value of C2 is worked out. We will take only these two values here and these two values here.

I am sorry, the summation is already done. So we shall take only this value. And this value is for the purposes of working out C2. And when we work on C3, what will we do? We will assume that our portfolio consists of only three security 1, 2 and 3. And therefore, we will use the values that are given here 10 by 10 and 12.625 by 100, when we work out the value of C3. Similarly, for C4, we will use the values 34 by 10 and 52.625 divided by 100.

When we work out the value of C4, similarly, for C5, 6, C7, C8 and C9 and C10. Now we compare the last column with the second column. What do we find? We find that it is at the point number 5, that is the fifth Security that beyond which that below which we find that the value of Ri minus RF upon beta i is less than the value of the corresponding Ci, you can see for security 6 Ri minus RF upon beta i is equal to 4.

But C6 is equal to 5.30. And let us look at for security 5, what we find is Ri minus RF upon beta i is equal to 6, but Ci is equal to 5.45. So, C5 is the cut-off point, because at that point, the it is the last value for which Ri minus RF upon beta i is greater than Ci. Below this for all values, Ri minus RF upon beta i is less than the corresponding Ci and therefore, we C5 forms our cut off, what is C5? C5 is 5.45. So, the cut-off value of all for a purpose. The value of C star for our purpose works out to 5.45. The rest is easy. We have the value of 5.45.

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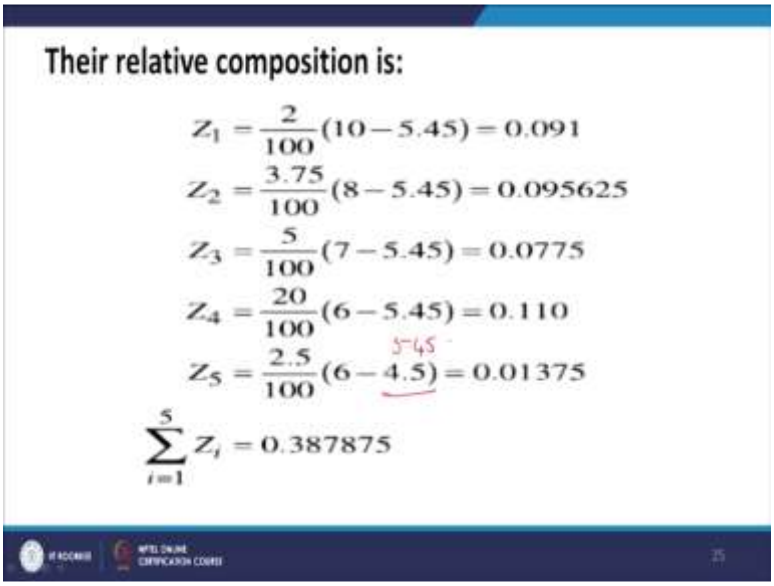

Workout the value of z1, z2, z3, z4, and z5 as we did in the earlier case and we will stop at z5 we will not go beyond z5 because beyond z5 if you if you look carefully, you will get a negative value. And therefore, you cannot consider that negative value you are not allowed to consider that negative value. There is a correction here I am sorry, this is 5.4,

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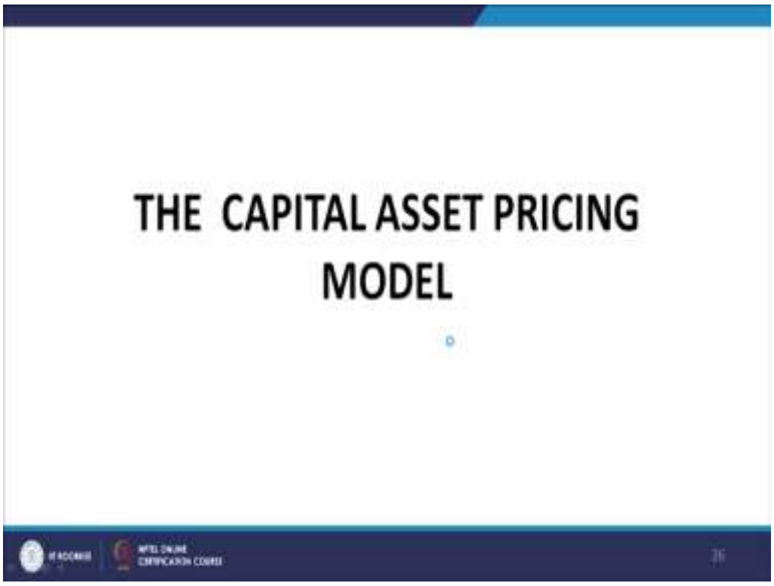

Now, we move on to the capital asset pricing model. Now, we have talked about the single index model. As I mentioned at the beginning, the single index model is an empirical model. It is an empirical regression based best fit model. It is a model where we simply do a statistical fitting of of a straight-line regression line in our data.

And on that premise, we were called certain simplifications in the portfolio optimization process and gather information that we can extract about the portfolio. And the second thing is that the model does make some radical assumptions which may not hold in practice. The third thing is that we do not have an underlying finance theory which justifies, which we indicates, which supports the assumptions that are fed into this single index model.

The assumptions are rather empirical, rather a priori and on those cases, we have arrived at certain results with nonetheless are useful. The capital asset pricing model is also a single parameter model, but it is or a single factor model rather. But it is much more rational in the sense that it has a finance theory behind it a finance basis behind it. It tells us a justifiable logic attached to it on the basis of which certain results have been obtained. So, let us let us at least introduce this in today's class.

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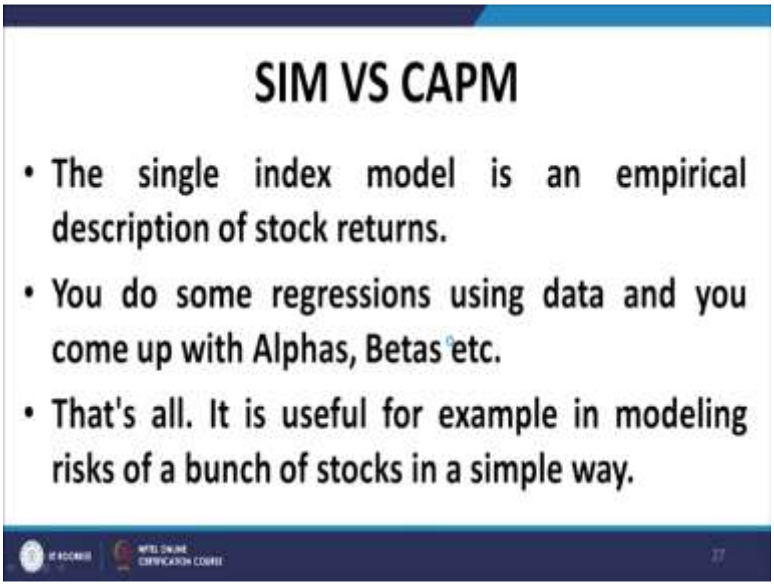

You see, as I mentioned just now, the single index model is an empirical description of stock returns. You do some regressions using data and you come up with alphas and betas for the securities and the portfolios of those securities. That is all. It is useful to for example, in modelling risk of a bunch of securities in a simple way.

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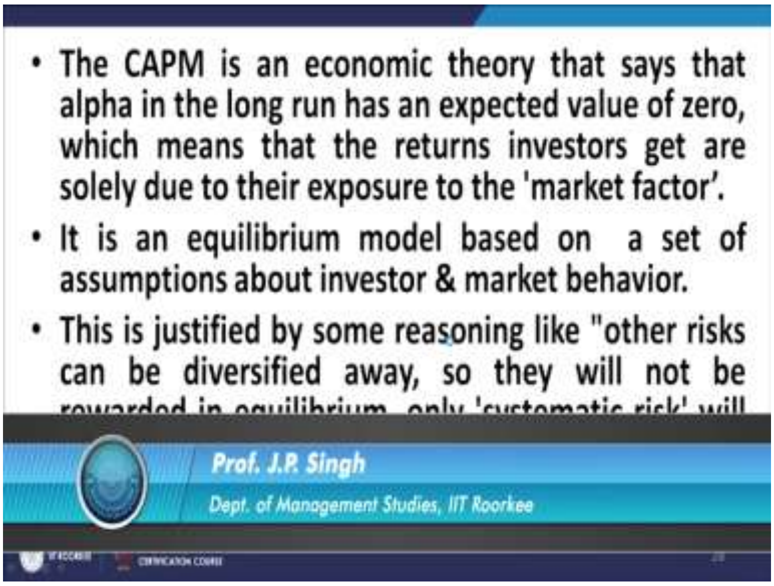

However, the capital asset pricing model as I explained has sound rational to it, it has certain strong certain assumptions on the basis on the behaviour of the investors and the market as a whole. So, this CAPM model is an economic theory that says alpha or that states or that proves, that establishes, that alpha in the long run as an expected value of 0. Which means that the returns investor get are solely due to their exposure to the market factor.

It is an equilibrium model. The CAPM model is an equilibrium model based on a set of assumptions about investor in market behaviour. This is justified by some reasoning like other risk can be diversified away so they will not be rewarded in equilibrium or unsystematic risk will be rewarded. This there is a rational finance sense attached to or embedded in the capital asset pricing model. We shall continue from here in the next lecture. Thank you.