Security Analysis and Portfolio Management

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Lecture 49

Single Index Model 1

Welcome back. So, let us continue from where we left off. In the last lecture, I concluded the discussion on the Mean Variance Portfolio Theory which is the cornerstone of contemporary portfolio management. However, if you look at the Mean Variance Portfolio Theory as in fact the learners would have appreciated by now, the calculations are quite cumbersome.

And we now move on to the attempts that have been made in the literature towards simplifying the optimization process and arriving at some kind of a situation where we can implement it practically with lesser problems, lesser difficulties. So, the first model that I am going to discuss is the single index model, and then I will move on to the capital asset pricing model. Finally, I will take up the arbitrage pricing theory which is the contemporary benchmark of modern portfolio theory.

So, let us start with the single index model. Now, to appreciate the backdrop the appreciate to appreciate the background in which this single index model was propounded Harold Markowitz was one of the founders of this model. In fact, Harold Markowitz was awarded the Nobel Prize in 1994 for its mean variance portfolio optimization theory, the complete theory that we discussed in the last few lectures and this is an extension of that in the sense that it is a simplification it is an attempt to use that theory in a more practical environment.

So, you see, the problem arises between why we need to look at simplification processes is because, if you are tracking N assets, N risky assets in your portfolio, then let us understand what are the inputs that we require for formulation of our construction of the optimal portfolio. First of all we need N means or N expected returns of all those N assets and risky assets.

Then we need N variances for each and every one of the risky assets and then we need N into N minus 1 by 2 covariances between the various pairs of securities that are contained in those N risky assets or risky securities. So, we need to track a total of 2n plus N into N minus 1 upon 2 parameters to facilitate the optimization process. For example, if N is equal to 100 in other words, if we are tracking 100 securities, the number of parameters that we need to estimate as inputs to the portfolio optimization using the mean variance optimization framework that we have discussed is the totals to 4750.

And if N is equal to 1000, we will need to track 501 comma 500 parameters which is a gigantic amount and furthermore, the tracking of covariances involve time series analysis of two securities at a time which is even more cumbersome. So, we the amateurs and the practitioners in fact, who were involved in the portfolio optimization practice or trade, attempted to simplify or attempted to find ways of simplifying the situation.

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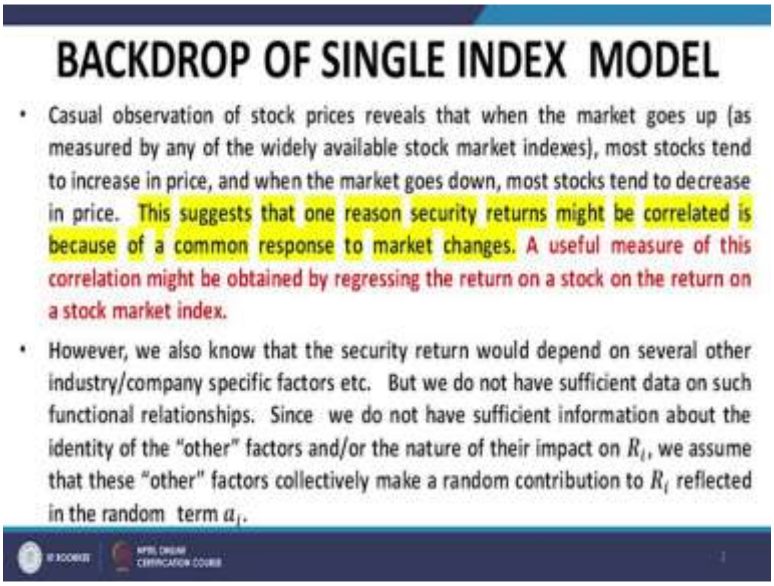

The first model that was advocated was the single index model, what is the backdrop of the single index model? Casual observations of stock prices reveals that when the market goes up market means market maybe epitomized by a broad-based index like we in India we have the nifty or we have the BSE SNP sent Sensex or some similar index which is representative of the market.

So, it is observed in most cases that stock prices when the market goes up, tend to go up together with the market indices. And similarly, when the market goes down as represented represented by the market indices, most of these stocks tend to go down in prices. So, this is the backdrop of the single index model that there is some intimacy some relationship between the market indices and the stock prices of individual securities.

That forms the backdrop of the single index model. So, the inference that was drawn from this empirical behaviour observed as far as prices of stocks are concerned, or prices of securities are concerned that the one reason that security returns might be correlated is because of a common response to market changes. So, the interrelationship, intercorrelations between securities, maybe due to their direct correlation or their direct association with a particular market index.

So, when the market changes most of the security change in tandem, although of course, with different amplitudes but the movement tangent, tandem with the origin in the direction of movement of the market indices, and therefore, they seem to be correlated among each other. That is the perception that went into the development of the single index model. And then the the, the advocates of single index, a propounded that a useful measure of this correlation might be obtained by regressing the return on a stock on the return on a stock market index.

This is the backbone of the single index model this particular statement, let me read it again. A useful measure of this correlation might be obtained by regressing the return on a stock on the return on a market index. So, that market index forms the independent variable and the return on the stock forms the dependent variable and if you run a regression between the two with the return on the market index along the x axis and the return on individual securities along the y axis, that the outcome would epitomise the single index model.

So, no, there is there is another issue that I need to highlight in this context, you see the return on a particular security agreed it may relate to the market index, I accept the fact that may relate to the market index. But it would not be totally explained by the market index, there would be some component of the return on the security on an individual security which depends on several other industry or firm's specific factors, which may not be captured by the market index or the relationship between the security and the market index.

But the problem would be that we may not have sufficient data either about the functional relationship between the security returns and these other factors which you may identify as industry factors or firm specific factors. So, what do we do? We do the next best thing that because we do not have sufficient information either about the functional relationship or the data in the context of the exploration of this relationship between the industry or firm specific factors and these individual securities return.

We model all these constituents into one term and we call this a random error term or a random term. So the net result is what of what I am saying trying to say the net takeaway of what I am trying to say is that we can model these individual securities return into two components, we can separate it into two components one is what emanates or what relates to the market variations or the variations in the market index.

Market variations as captured by the market index you may say and the other part which is independent of the market index in which we model as a random term, which encapsulates all of the singular features are in relation to the security like the industry specific or firm specific factors about either about which we do not have much information about the functional relationship between those factors and the and the security return.

Or we do not have enough data to explore there any kind of functional relationship or we are not interested in in, trying to follow up or trying to explore that relationship. In any case, whatever the situation may be, we model these composite factors as a single term as a random term, which contributes to the return on the individual security.

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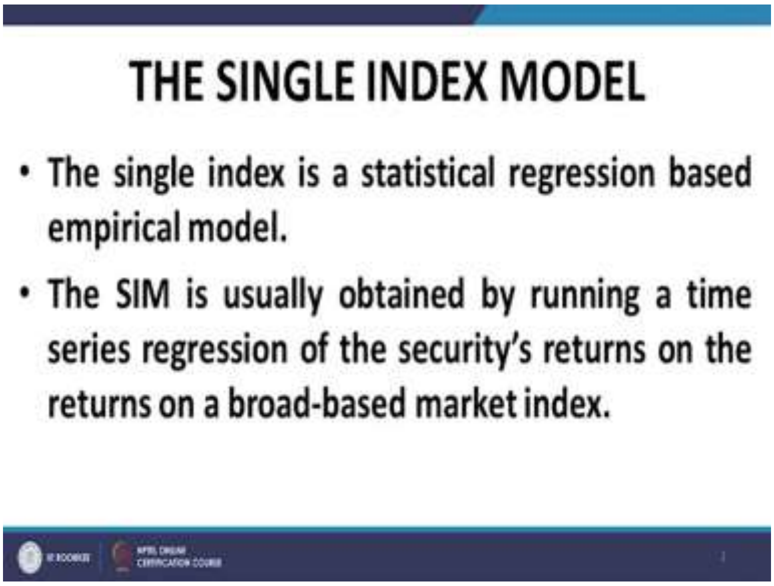

So, that is, in essence, the philosophy of the single index model. And let us now move on to the development of this model. As I mentioned at the outset, the single index model is a statistical regression based empirical model. I repeat, there is no backdrop theory or finance theory behind the single index model. In fact, this is one aspect which differentiates the single index model from the cap m model.

That we shall discuss after taking up this single index model after completing the single index model. So, the single index model is usually obtained by running a time series regression of the securities return on the returns of a broad-based market index, this is the philosophy of the single index model. Now, if you let me explain certain terms, which would be a which would be taken up again and again during the course of this discussion. So, if you run ordinary least square regression between the security return and the market returns.

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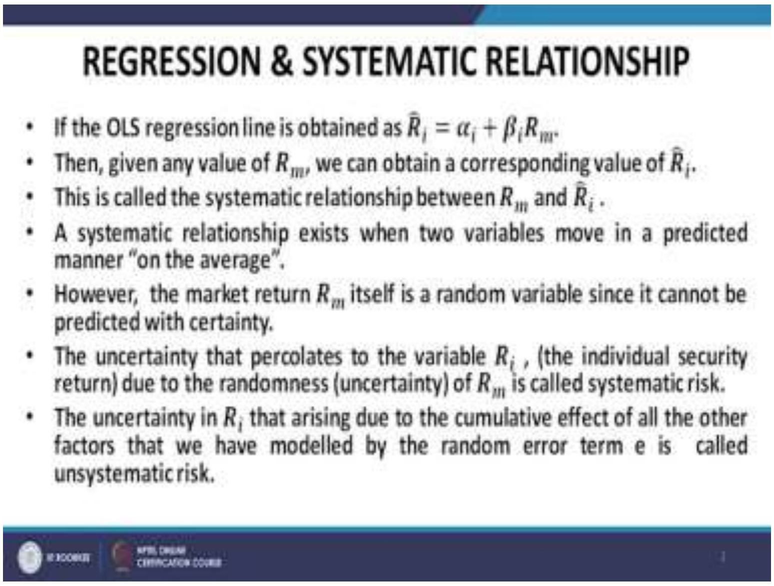

Let us assume that you get a straight line get represented by our R hat i is equal to alpha i plus beta i Rm. Now, as you can see here, please note R hat is different for R hat i is different from Ri, I will explain the difference in a minute. But get this by virtue of the regression relationship that we identify by running ordinary least squares regression is that given a value of Rm, we can obtain a corresponding value of Ri hat.

Now, this is called this relationship which arises out of regression is called a systematic relationship between Rm and R hat i or Ri. So, the OLS regression equation depicts a systematic relationship between the independent variable and the dependent variable. In other words, you can explain the systematic relationship as a relationship that exists between two variables when the two variables move in a predicted manner, predicted maybe by a regression equation.

So, they move in a predicted manner but they move in a predicted manner on the average; this on the average is again to be highlighted and emphasized. Why? Because it is not true that every value of Ri that we get in actual fact on actual observation would be contained or would be captured by this regression equation. Regression equation is the line of best fit as the learners would know.

The regression equation is the line of best fit there would, but the parts that represent actual data need not necessarily at all lie on the regression line totally. Some points maybe above the regression line, some points would be below the regression line, some points will lie on the regression line depending on the nature of data that we have available with us. But the point is, that regression line is the line of best fit and that line of best fit because it is a straight line is captured by this regression, regression equation Ri cap.

Now Ri cap is that point which corresponds to Rm which lies on the regression line. Ri is the point which is the actual value of the data point corresponding to a value of Rm. So, there is a difference between them. Let me repeat Ri cap or Ri hat as the case may be, it represents a point on the regression line which corresponds to Rm. However, Ri itself may be either above Ri cap or Ri, it may be below Ri cap.

But that Ri itself is a dead end actual data point which represents the value that the variable is taking corresponding to the value of Rm. Please note Ri is a random variable because it may take values which are not predetermined which are not predictable and absoluteness. So, there would be difference between Ri cap and Ri that is important. So, this systematic relationship is a relationship between two variables which are captured by a regression line and therefore which move in a predict as predicted by the regression line.

But that that relationship holds on the average that means, if you have a number of observations between the two points, and then the regression relationship would the systematic relationship would tend to predict reasonably good results. But any particular any single value of the actual data may not lie on the regression line may not satisfy this relationship in its absoluteness.

Please note this point this is very fundamental system and to repeat systematic relationship between two variables exist when there is some predictable relationship on the average not for each value. So, this is the this this particular term shall be a backbone in so far as this single index model and the cap m model is concerned. Now extending on this systematic relationship. Rm itself is also random variable, the market returns are also random variable.

You cannot predict with absolute certainty, what the market return is going to be say two days from now or one day from now or even one hour from now. So, the important thing is that the market returns themselves are random variables. Now, that means, they obviously have a probability distribution and an expected return and a variance. But that is not important at the moment the point is because of this randomness to this random nature of this independent variable Rm certain randomness will percolate down to the value of Ri.

In other words, a certain amount of randomness in Rm will percolate to the dependent variable which is Ri that component of risk or that component of randomness and the corresponding risk that arises from the randomness or the risk of Rm in Ri is termed as systematic risk. Let me repeat the statement it is very important, the uncertainty that percolates to the variable Ri that is the individual securities return due to the randomness or the uncertainty of Rm is called systematic risk.

This again is very important in so far as these models are concerned, the uncertainty in Ri that arises due to the cumulative effect of all other factors, which you have modelled by the random term is called unsystematic risk. So, again we are here we are bifurcating risk into two parts one part of the risk which arises or emanates from the randomness.

The uncertainty the risk embedded in the market returns and being observed in Ri that is called and that is called systematic risk of Ri and the part which arises from other factors which we have modelled by a single random term is called the unsystematic risk and in the security returns Ri.

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So, let us now get down to the model itself. The expected return under the single index model, the basic single index model equation simply breaks the return on a stock into two components as I mentioned just now. The part that is due to the market that can be identified as being associated with the market and the part that is independent of the market.

Therefore, we can write Ri as ai plus beta i Rm beta Rm is the part of the return which relates to the market and ai is the return that is generated by or that is independent from the market and that is generated due to other factors or other terms that are not captured by Rm. So, Rm is the rate of return on the market index and that is a random variable, beta is a constant that measures the expected change in expected.

Please notice the word expected I highlight emphasize this particular term, we are not talking about actual return we are talking about expected returns. So, beta is a is a constant that measures the expected change in Ri given a change in Rm and ai is a component of securities returns that is independent of the market performance which is again a random variable. So, both Rm and ai are random variables, both will have a certain probability distribution with means and variances. So, this is the this is the fundamental equation of the single index model.

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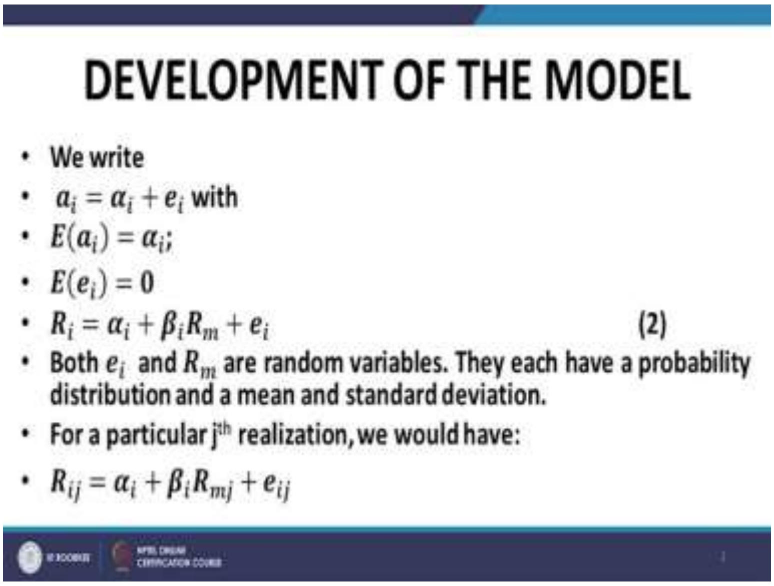

Now, let us talk about the development of this model we write ai is equal to alpha i plus ei, in other words, we split this unsystematic return or this return arising from the random factors industry specific or from specific practice we split it up into the mean of this particular term and the distribution around that mean or the dispersion around it mean and we write it as ai is equal to alpha i plus ei where the expected value of ai is given by alpha i.

And obviously, therefore, the expected value of ei would be 0 and we can write Ri in this with alpha i and ei embedded incorporated therein in the form of equation 2 as Ri is equal to alpha i plus beta i Rm plus ei. Now both now, clearly, the randomness that was there in ai is not being captured or is not being absorbed by the term ei and therefore, ei and Rm continue to be random variables.

And each of them of course, as I mentioned, has a probability distribution and a standard deviation. The j th realization of this particular equation, we will take the form that is given here Rij is equal to alpha i plus beta i Rmj plus eij.

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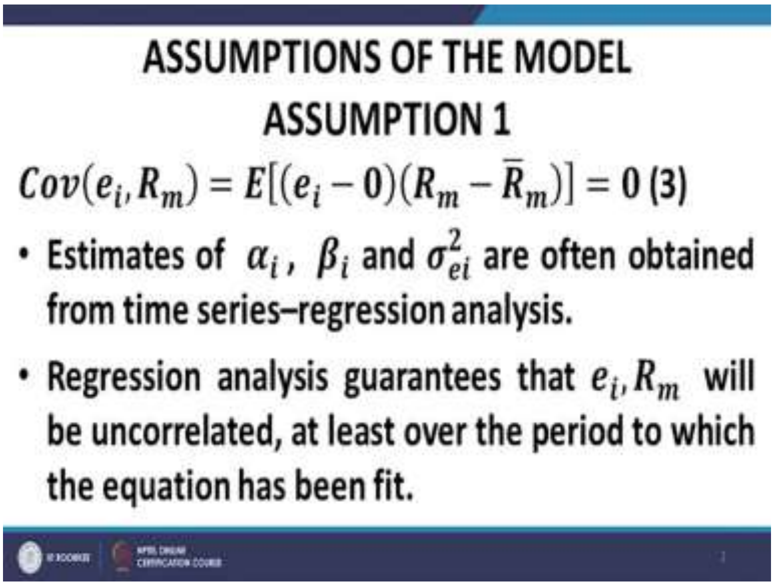

Now, assumptions about this model. The simplistic version of what the assumptions are, I have already explained let us get into the mathematical representation of those assumptions. The first assumption is that there is no correlation there is no covariance between the term ei and the Rm that is the two random terms which capture the market risk and the non-market risk are independent of each other.

They do not correlate with each other. This is the fundamental assumption but because we usually develop this model through a regression process, this assumption need not be explicitly made the very fact that we develop this model through regression the process of regression itself captures this assumption or makes this assumption formal into the into the regression process itself.

To repeat the covariance between the unsystematic relationship and the and the systematic relationship that is the two variable ei and the and Rm are 0 that is they are independent of each other. So, that is equation number 3 and estimates of alpha and beta I have just mentioned this before in fact, estimates of alpha beta and sigma e square are often obtained from time series regression analysis. Regression analysis can guarantees that ei and Rm will be uncorrelated at least over the period to which the equation has been fit.

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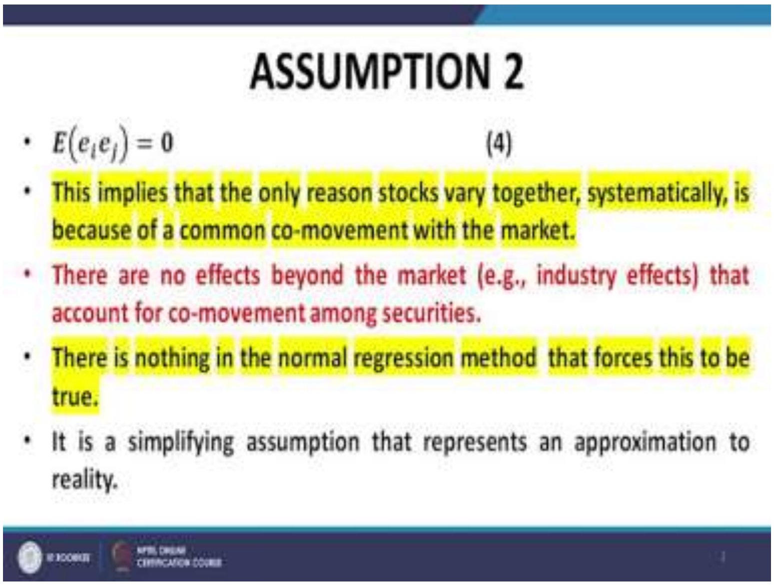

The second assumption is explicit we make this assumption because we need to simplify the remaining part of the model. In other words, to make this model more tractable to make this model fulfil the objective for which it is it is developed in the first place, that is to simplify the optimization process, we make this additional assumption. What is this additional assumption? The additional assumption is that the expected value of ei and ej is equal to 0 that means what?

That means that the only reason that the stocks vary together systematically is because of a common core movement with the market that the any two stocks in the market do not have any internal or have any direct functional relationship or as a result of which they move in tandem, they move in tandem only because of the relationship between stock i and the market and stock j and the market.

Because both these stocks are related systematically to the market, it follows that they would be moving together in some sense and that means, that means what? That means that, if we can afford to on the basis of this assumption rather on the basis of this assumption that the stocks move together only because they are moving along with Rm we can make this assumption that e of expected value of ei, ej will be 0.

So, in other words, what we are simply trying to say is there are no effects beyond the market that is industry effects that account for the co-movement among securities. Stocks move together, because they have a relationship with the market or the market index as represented by the market index. They do not move together because of any direct a relationships among themselves that is the underpinning of this particular assumption.

There is nothing in the, this is another issue. There is nothing in the normal regression method that forces this to be true in contrast to the former assumption assumption number 1, which mandated that the covariance between Rm and ei should be 0. That is there should be uncorrelated which is captured which is a part of the regression assumptions in fact. So, if you are doing this process if you are modelling this as a regression model.

Then that part becomes automatically satisfied or fulfilled. This particular assumption does not get automatically fulfilled, because there is nothing in the normal regression process that forces this to be true. It is a simplifying assumption that represents an approximation to reality. Now, we work out the various parameters relating to the security and the portfolio.

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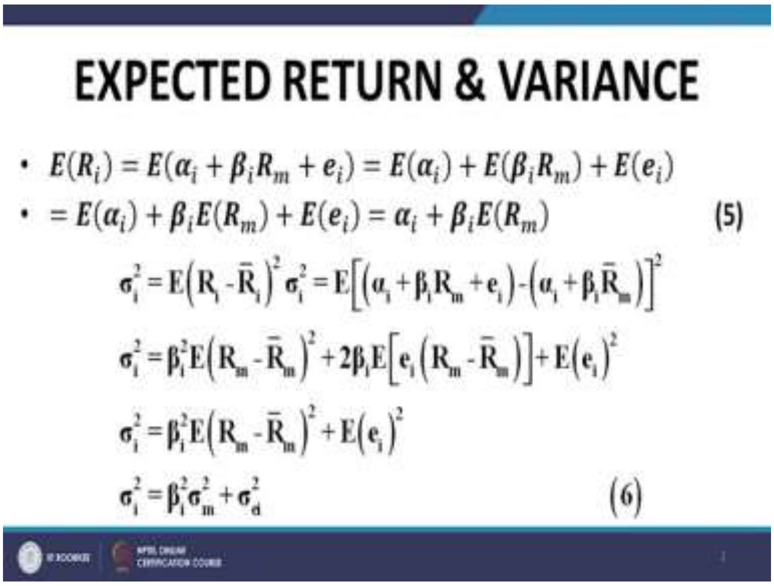

On the basis of the single index model, as far as the expected return is concerned, it is quite straightforward, the expected return on the individual security can be written as alpha i plus beta i into expected return on the market. I repeat, because the expected value of ei is 0, the expected value of the random error term is 0. Therefore, we directly obtain the expected return on the individual security is equal to alpha i plus beta i expected return on the market this is equation number 5.

As far as the variance of a single security is concerned under the single index model, the derivation is given on the slide it is again a straightforward derivation the sigma square is given by expected value of Ri minus Ri bar will square when you substitute the value of Ri and Ri bar, Ri bar we have just calculated that is equal to alpha i plus beta i Rm bar and Ri is equal to alpha i plus beta i Rm plus ei.

So, when you substitute these values and you do a bit of simplification, the outcome that we arrive at is sigma i square is equal to beta square sigma m square plus sigma ei square where sigma m square is the market variance and sigma ei square is the is the variance of the random error term. Let me repeat now, you can clearly see that the total risk of the security has been split up into two orthogonal components.

The first component that is related to the market which is captured by the first term beta i square sigma m square and the second term which is independent of the market will average arises from the other singular factors which we have not considered or which are not an engraved, ingrained in the market and which we represent as the variance of the random error term. So, just like the expected return, we have the variance split up into two parts the expected return error unique part alpha i and a market related part beta i Rm.

Similarly, the variance has the part the unique risk sigma ei square and the market related risk beta i square sigma m square. So, again both the risk and returns are being split up into two components a component which is related to the market which is associated with the market and a component which is independent of the market which is representing the random singular factors to which the return on the security relates.

Now, the covariance of securities as you can see in this in this slide, again is the derivation is quite simple and you have to take keep track of one more particular point that I would like to highlight in this derivation.

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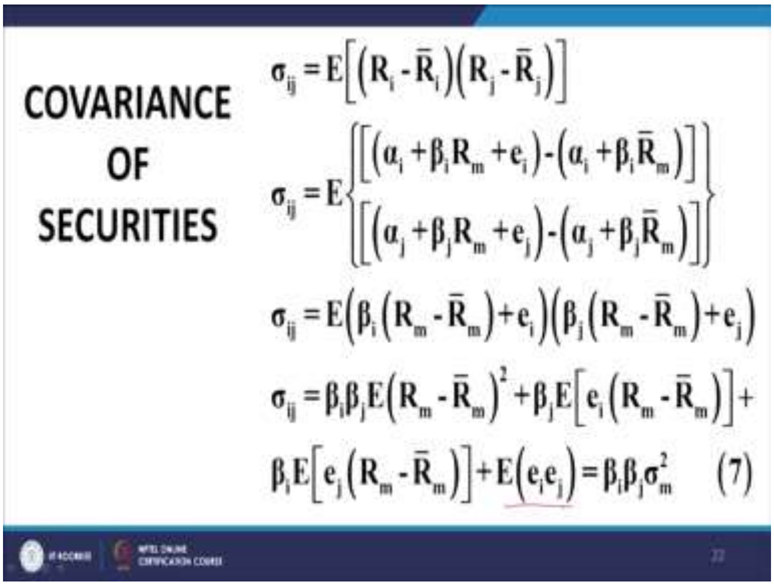

And that is that the expected value of ei ej is 0. Please note that this is assumption number 2 of the model that the any 2-security issue or co-movement only because of their common relationship with the market then do not have any direct relationship between them and therefore, the expected value of ei ej is equal to 0 and if you use this value, what we end up with is that the covariance between security i and security j is given by beta i, beta j, sigma m square.

I repeat the covariance between security i and security j is given by beta i, beta j, sigma m square. So, the covariance depends only on the market risk in this model as in this this equation or this expression for the covariance clearly justifies our argument that the securities move because of their relationship with the market index and they do not have any direct relationships between themselves.

Because the covariance the covariance term does not contain anything about the individual securities, it contains only the market variance. Thus, the single index model implies that the only reason securities move together is a common response to market movements.

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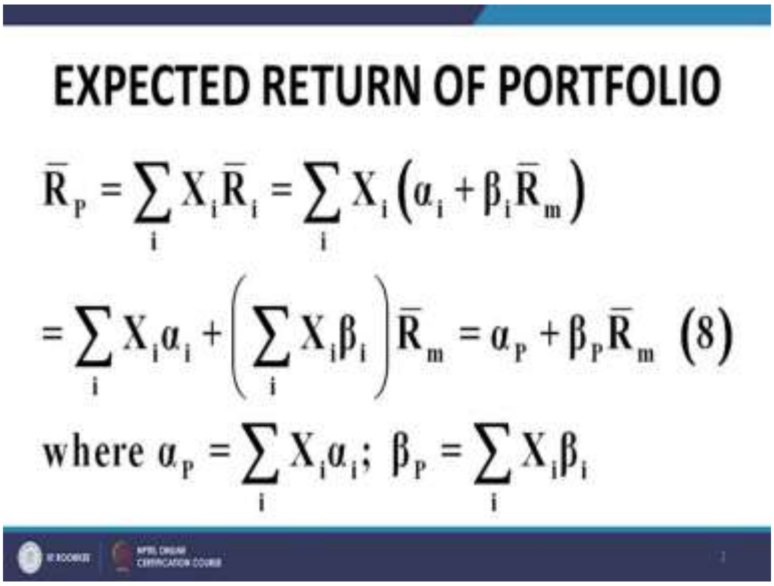

The expected return of portfolio, this is quite straightforward, I will not devote time to this, this is a replication of the process or the position as far as individual security is concerned. The expected return on a portfolio Rp or Rp bar is equal to alpha p plus beta p Rm bar where alpha p is the weighted average of the alphas of the constituent securities of the portfolio. And beta p is the weighted average betas of the constituent securities of the portfolio.

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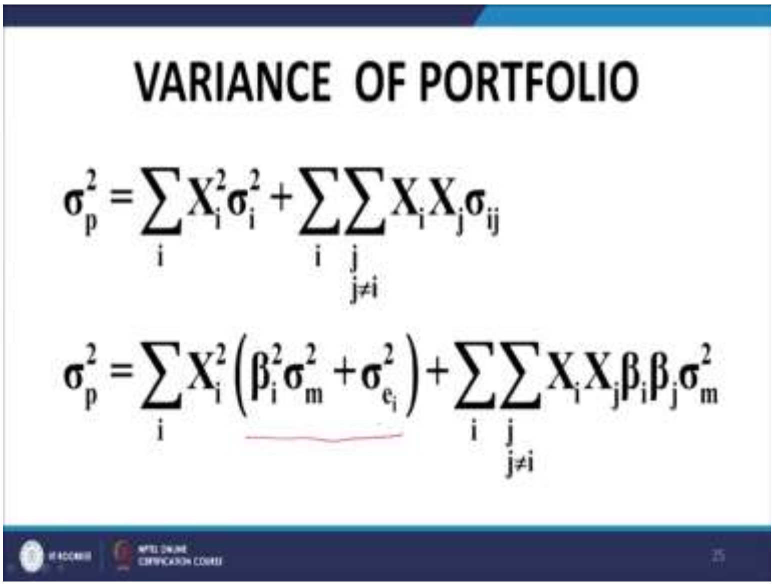

The variance of a portfolio well again we have expression which is absolutely akin to the expression that we have for our individual security. Although the derivation is a little bit more involved, we start with the standard expression for the for the variance of a portfolio of

N securities using the mean and the using these variants and the covariance matrix and then on simplification on introducing the expression for the total risk other variance of an individual security given by this expression which I have underlined.

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$$=\sigma_m^2\sum_i X_i^2\beta_i^2+\sigma_m^2\sum_i\sum_{\substack{j\\ j\neq i}} X_iX_j\beta_i\beta_j+\sum_i X_i^2\sigma_{e_i}^2$$

$$=\sigma_m^2\sum_i\sum_j X_iX_j\beta_i\beta_j+\sum_i X_i^2\sigma_{e_i}^2$$

$$=\sigma_m^2\sum_i\left(X_i\beta_i\right)\sum_j\left(X_j\beta_j\right)+\sum_i X_i^2\sigma_{e_i}^2=\beta_p^2\sigma_m^2+\sum_i X_i^2\sigma_{e_i}^2 \quad (9)$$

And further simplification we end up with the expression that is given in equation number 9. Beta p square sigma m square this is the systematic risk and summation sigma Xi square sigma ei square This is the unsystematic risk. Let me repeat beta p square sigma m square represents the systematic risk of the portfolio and summation of Xi square sigma ei square represents the unsystematic risk of the portfolio.

Thus, the total variance of the portfolio is again orthogonally divided into two parts. One part the systematic part are arising from the market totally from the market which is captured by the first term and the second part arising from the singular random variable or the singular term which captures the other factors.

So, I repeat the total variance is split up into two orthogonal parts beta p square sigma m square, which represents the randomness arising out of the market randomness and the other term arising from intrinsic randomness of the security.

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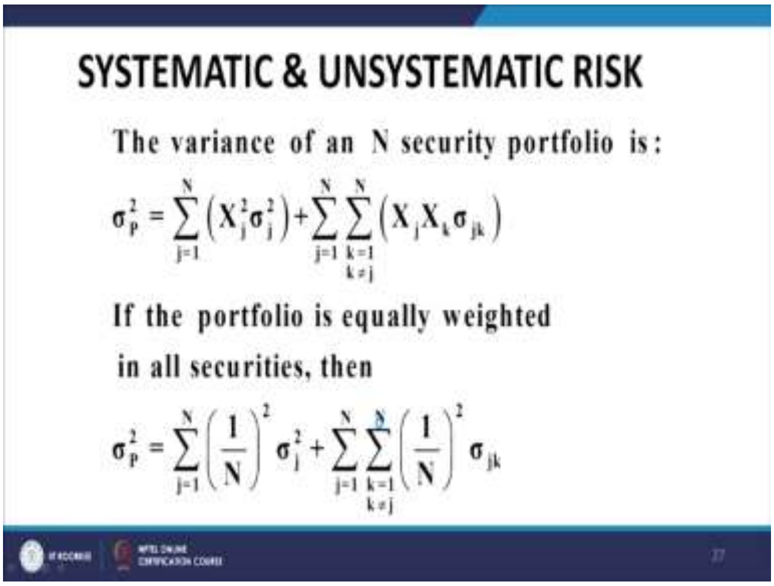

So, I just mentioned that beta p square sigma M square represents the system systematic risk and the other term sigma Xi square sigma ei square represents the unsystematic risk. Let us explore that this a little bit more, let us assume we know that the variance of an N security portfolio is given by this expression here on this slide. Let us assume that we have a portfolio which is equally weighted in all security and money amounts.

I repeat, we will start with the formula for the variance of an N security portfolio which is given by the first equation on the slide and then we assume that this portfolio that we are considering consists of equal weights of each of the constituent securities. So, for all the securities are equally constituted equally included in terms of money values, in terms of money values in the given portfolio.

So, the weight of each security will begin by 1 upon N because there are N securities and let us assume that the weight and the total value of investment is 1 unit. So, the amount of investment in each security will be equal to 1 upon N. So, then we if we substitute X1 equal to X2 equal to X3 for all X's, we substitute 1 upon N we arrive at this second equation on this slide.

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$$\sigma_P^2 = \left(\frac{1}{N}\right)\left(\frac{\sum_{j=1}^{N}\sigma_j^2}{N}\right) + \left(\frac{N-1}{N}\right)\left[\frac{\sum_{j=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N}\sigma_{jk}}{N(N-1)}\right]$$

$$\sigma_P^2 = \frac{1}{N}\bar{\sigma}_j^2 + \left(\frac{N-1}{N}\right)\bar{\sigma}_{jk} \Rightarrow \text{Lim}_{N\to\infty}\,\sigma_P^2 = \bar{\sigma}_{jk}$$

Now, the second equation that was there on the previous slide can be written in this form that is here as the first equation. In the first term, we have taken 1 upon N outside and we have written the first term that represents the intrinsic variance of the securities divided by N and the second term relates to the covariances. Now, if you look carefully, let us analyse the first term first.

The first term in the numerator contains sigma of summation of sigma square in other words, it is the sum of the variances of all the securities. Sum of the variances of all the securities and it has been divided by the number of security. So, in some sense, you can take it as the average variance of the of the portfolio or average variance or the various per security or the average variance for security of the portfolio.

That is the first term. Now, if you look at the second term, the second term in the numerator contains the summation of all covariances. Please note given N securities the number of covariances are N into N minus 1 although the number of unique covariances will be N into N minus 1 upon 2 because sigma ij is equal to sigma ji for any pair of securities, but the total number of covariances which include non-distinct covariances will be equal to N into N minus 1.

So, the the numerator in this term represent the sum of all the covariances and the denominator represents the number of covariances. So, we are dividing the sum of all covariances by the number of covariances. And what we get therefore is in some sense the average covariance. So, the first term already gives us the average variance per security and the second term gives us the average covariance.

Now, let us look at how the situation develops when we increase the number of securities in the portfolio. If we take the limit of this expression sigma p square in the form that we have written it in the first equation, as N tends to infinity, what we get is, we get limit and tending to infinity sigma p square is equal to sigma bar JK that is the first term will disappear will tend to 0 because of the free factor of 1 upon N as N tends to infinity, the prefactor will tend to 0 and therefore, the first term will vanish.

What about the second term if you look carefully, N minus 1 upon N the in the limit that N tends to infinity is equal to 1 and therefore, the second term does not vanish, the second term remains intact when even in the situation where N tends to infinity. So, the net outcome is that even after infinite diversification, we have a situation where one of the components of risk the component of risk that arises from the variances does disappear.

The component of the risk that arises from mutual covariances does not disappear and the part that disappears is called diversifiable risk or unsystematic risk and the part that does not disappear, which arises from mutual co correlations or mutual interactions is the part that does not disappear and it is called undiversifiable risk or systematic risk. So, the outcome of this particular exposition is is summarized in this slide.

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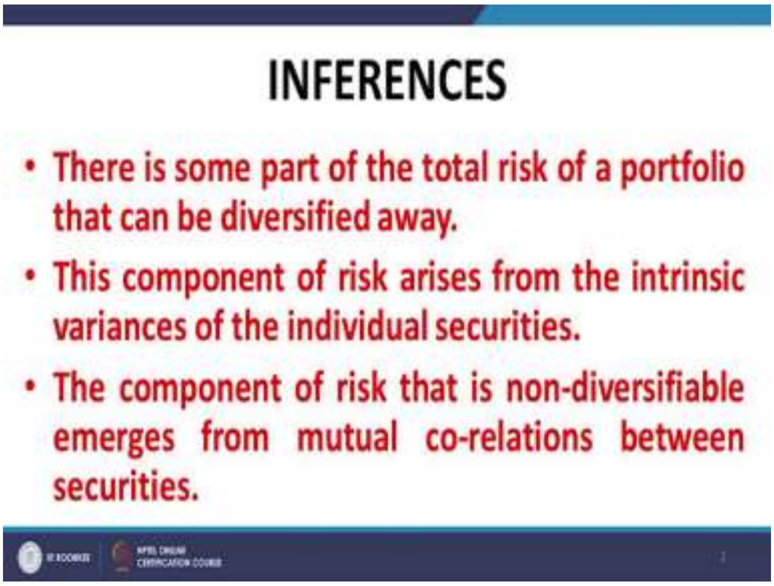

Let us read through it, there is some part of the total risk of a portfolio that can be diversified away the first component that was part of the equation the first term of the equation, the component of the is this component of the risk arises from the intrinsic variances of individual securities. This is the unsystematic risk it arises from the individual variances of the securities and the component of risk that is non diversifiable that is systematic risk arises from mutual correlations between the securities. So, we shall continue from here after the break. Thank you.