Security Analysis and Portfolio Management
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Lecture 47
Mean Variance Portfolio Optimization VII

Welcome back, so let us continue from where we left off. In the last lecture towards the end I was discussing the case of three securities, three risky assets. Let us recap the salient features of that development and then we will continue from there on. So in the case of three security assets the problem becomes more intriguing.

You see the point is when we are talking about two risky assets, then we have got four unknowns basically the standard deviation of the portfolio, the expected return of the portfolio and X1 and X2 which is the composition vector of the components of the composition vector comprising the portfolio. We have 3 feasibility equations as you can see in the right hand panel of your slide.

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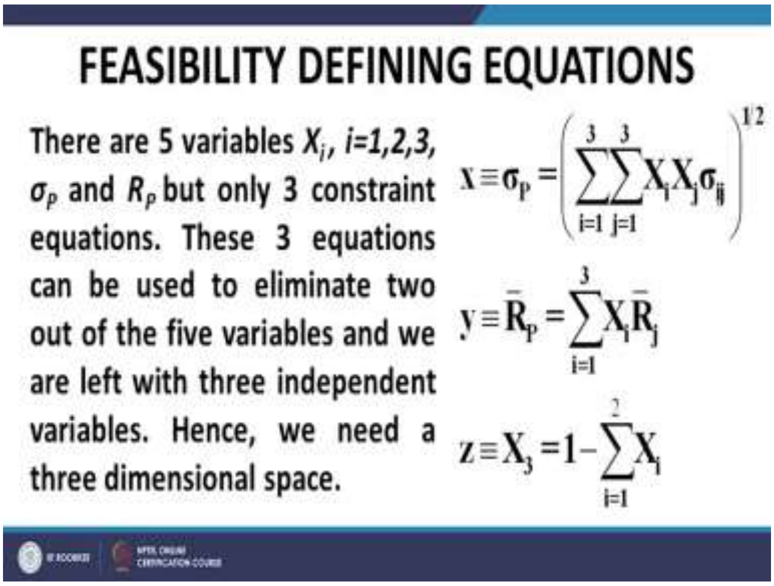

Now using the three feasibility equations we could eliminate both X1 and X2 and we arrived at a functional relationship between this expected return and the standard deviation or vice versa, this could be plotted in a two dimensional space then we got a hyperbola. Every point on this arc of the hyperbola that I am talking about represents a feasible portfolio and if you have a particular value of sigma P you can work out or you can obtain the corresponding value of Rp and vice versa and you can define the portfolio completely.

In other words every point on this hyperbola corresponds to a feasible portfolio and we when you look at the arc of the hyperbola upwards beyond the point of minimum variance m onwards then you arrive at the efficient frontier. So the basic thing that I want to convey here is that in that case the entire problem could be handled on a two dimensional framework, because the two dimensional representation provided a complete description of the problem given any point we had a unique solution to the problem.

However, when we move to more than two securities the simplest case being of three securities what we have is five unknowns, we have sigma P and Rp which is the portfolio standard deviation in the portfolio expected return and then we also have the components of the composition vector X1 X2 and X3 which represent the constitution of the portfolio in terms of securities A, B and C.

Now we have three equations or the three equations representing the feasibility conditions as in the earlier case therefore, at best we can eliminate two unknowns. Let us say we eliminate two unknowns X1 and X2, in that case what happens is we have still three unknowns or three degrees of freedom this sigma P expected return Rp and the the composition or one component of the composition vector let us take let us call it X3. Let us assume that we eliminate x one and x 2 and we retain x 3 as a part of the free variables or 3 degrees of freedom.

Therefore, we now need a 3 dimensional space for a complete representation of the problem, so that is the first step towards understanding the situation. Now, when we talk about a three dimensional problem or a three dimensional space as you can see here if I take this particular pack of cards it can be, it is a three dimensional object and the three dimensional space that I am talking about which represents this pack of cards can be considered as a series of planes parallel for example to the x y plane.

Let us assume that the origin is on the left hand side corner of this pack, the x axis is the horizontal axis emanating from the origin, the y axis is the vertical axis and the z axis is oriented towards me in the positive direction and away from me in the negative direction. Now you can see that this three dimensional object or this three dimensional space you may assume it can be represented by a series of planes which are parallel to the xy plane.

If I pick up any arbitrary plane what happens is the plane would be represented by its perpendicular distance from the xy plane which in the case of orthogonal coordinates in the case when x y and z are orthogonal would be nothing but the z intercept of the plane as you can see, let us say this is the point at which I make a cut off this is an arbitrary plane and then this distance from the origin along the z axis, because z axis is oriented along me or towards me the cutoff point of this plane with the z axis is sufficient to completely identify this plane.

The important thing is all the points on this plane which let us say the cut off point is z is equal to k then all the points that lie on this plane z is equal to k have the z coordinate of k, obviously x and y coordinate should vary with the position of the point on this plane, but so long as the point lies on this plane z is equal to k the z coordinate will remain at k. Now in our problem the z axis represents the constitution of the security C in our portfolio.

Therefore, if we have a point on this particular plane z equal to k that means we have fixed the composition of security C in our portfolio at k that means if I pick up any point on this particular plane that would comprise of comprise of security C to the extent of k and of course it would be a mix of securities A and B as well.

As far as this plane is concerned the composition of security C is fixed that means in effect the problem has been reduced to a problem with two degrees of freedom sigma and Rp and therefore when we if we plot sigma and Rp by varying the composition of A and B keeping the composition of C fixed at k what we will get is a two security problem.

We have already dealt with the two security problem the two security problem results in a hyperbola or the arc of the hyperbola that we have talked about in a lot of detail and when we talk about the efficient frontier in this situation, the efficient frontier is the tangent from the line emanating from the point F which represents a risk free asset. In this case the coordinates of F would be what, because it lies on the on the plane z equal to k, so the z coordinate has to be k because it is a risk-free asset.

The sigma P or the standard deviation would be 0 therefore x would be 0 and the y coordinate would represent the risk free return which would be Rf. So the coordinates of the point f which lies on this plane z equal to k would be 0 comma Rf comma k. So you take you draw a tangent from this point F to the arc of the hyperbola which represents the two security problem in this plane because we have fixed, I reiterate, because we have fixed the composition of the third security in this case because we are lying on this plane and it would be the tangent on this particular hyperbola drawn from the point F.

Let us say it intersects, the point intersects the hyperbola at the point of contact of the tangent and the hyperbola is the point P k, so that being the case then FP k represents the efficient frontier. That means what? That means given a value of k or given the value of the third security we can solve this problem as a two dimensional problem and the efficient frontier would be the line that would be the tangent to the hyperbola lying in the particular plane which is indicated by or which is recognized by the composition of the third security.

But our problem is different, our problem is an extended version of this extended in the sense that we also allow the composition of the security C to vary. That means what that means the value of z equal to k is not fixed in our full problem. The value of z that is the composition of security C can vary from minus infinity to plus infinity assuming that short sales are allowed.

In that case the value of z can vary from minus infinity to plus infinity, thus k can, thus k can vary from minus infinity to plus infinity. That means what? That means we need to work out the efficient frontier for each value of k lying between minus infinity to plus infinity and this would give us an infinite set of hyperbolas and corresponding to every value of k we will do the same exercise.

Let us say we take another value of k equal to l that is we identify a plane given by z is equal to l and then we do the same exercise again, we will obviously end up with a different hyperbola and that lies on the plane z equal to l and then we can draw a tangent from that plane from that point F which now would be what the coordinates of F would be new point F would be 0 comma RF comma l, because now it lies on the on the plane z equal to l.

We draw a tangent from this point F new point F let us call it Fl to the hyperbola which is the new hyperbola lying on the lying on the plane z equal to l let us call it Hl and then we identify the point of contact let us call it Pl, so we get another efficient frontier.

This sufficient frontier corresponds to what it corresponds to the combination of securities A and B in various proportions with the composition of security C fixed at l. So this is the process that we that we would follow for each and every security composition in the portfolio that means each and every value of z from minus infinity to plus infinity.

The next step the second step in this exercise is to find out that that plane that value of z for which this tangent that we now discover let us call this value of z equal to alpha, now a plane out of all these planes that are that we have from minus infinity to plus infinity let us identify a particular plane by what characteristics, by the characteristic that the slope of the tangent that is the slope of the line F alpha where F alpha is the point 0 comma RF comma alpha, z is equal to alpha it lies on the plane z is equal to alpha.

From this point to the point P alpha which is the point of contact of the hyperbola H alpha with the tangent drawn from the point F alpha. Now the property of this particular hyperbola or this particular plane rather must be that the slope that I am talking about the slope that is tan theta l alpha must be the maximum of all the slopes that we have worked out.

For example, tan theta l tan theta k and so many other values of k that we have taken of values of z that we are taken and we have for each value of z we have identified a hyperbola we worked out the tangents and we worked out the slope of the tangent let us call them tan theta k tan theta l and so on. We find that value of theta or that plane such that the l and the slope happen to be the maximum tan theta alpha is the maximum of all the tan theta.

That constitutes the optimal hyperbola and the efficient frontier is what? The efficient frontier is the line joining F alpha with P alpha where the line F alpha P alpha is tangent to the hyperbola H alpha which lies on the plane z is equal to alpha, so this is the geometrical description of the problem. Let us quickly recap what I have tried to state and then we will move on to further development of the problem.

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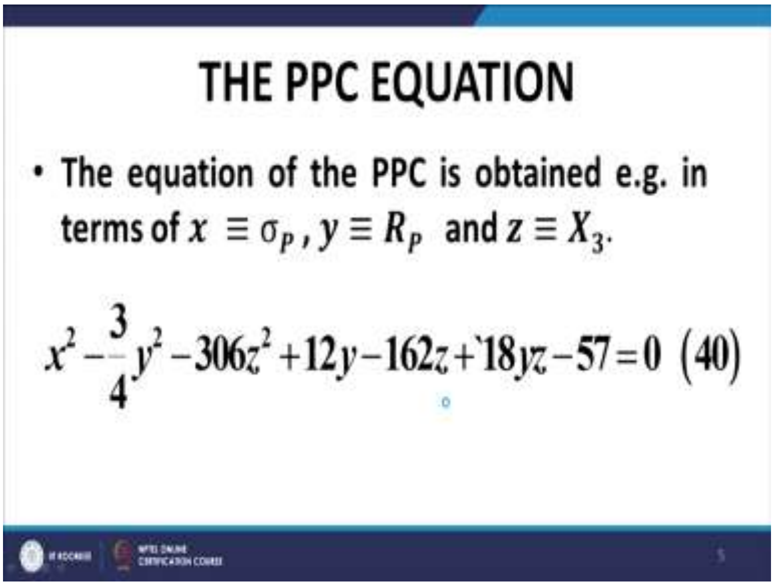

The portfolio possibilities equation that I explained in the last lecture is given by this expression this is obtained by setting sigma P equal to x, expected return on the portfolio equal to y and X3 equal to z and eliminating X1 and X2, the result is equation number 40. The shape as I mentioned, I have explained in a lot of detail just now, the shape would be hyperbola on each plane parallel to the x y plane identified by a particular value of the intercept of the plane with the z axis.

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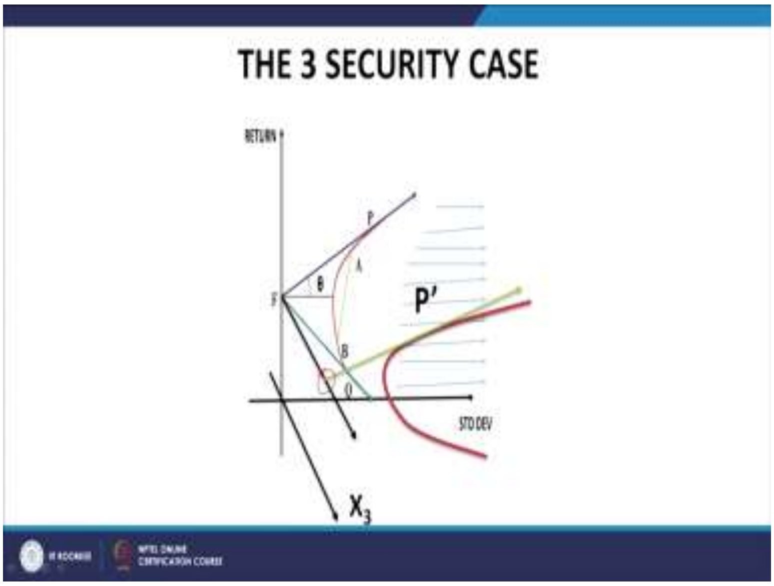

This is the geometry you have a plane which is parallel to the x y plane, the x y plane is the base then you take a plane in the positive x direction towards me and you have a hyperbola, you have this tangent drawn from this point…this intersection, this point to the point P dash and that represents the efficient frontier. The coordinates of this point here which I have marked with the pen are given by the value of z at this particular point.

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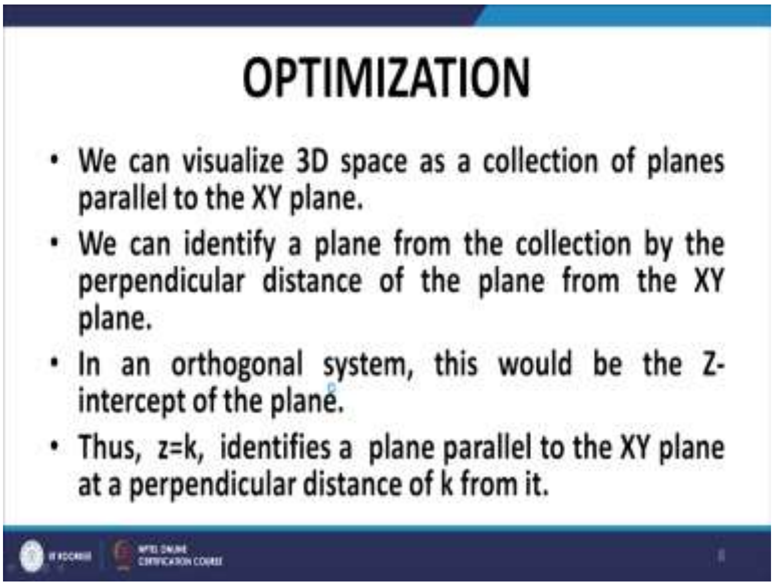

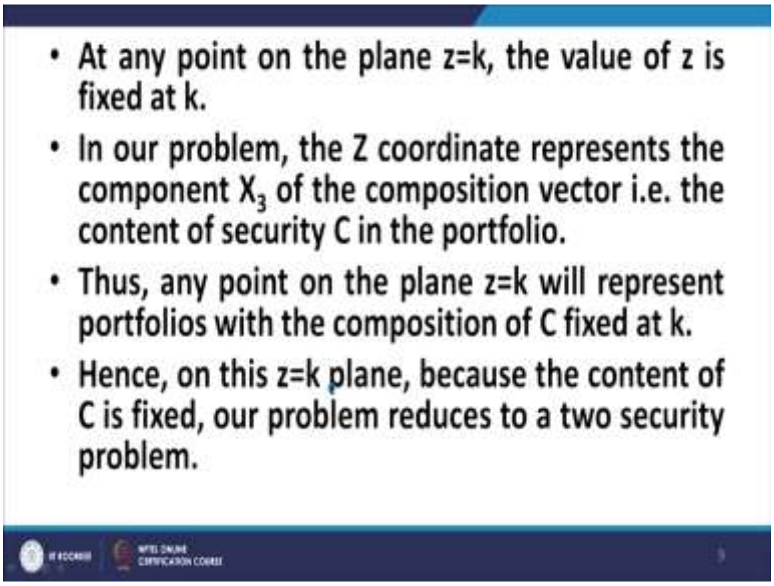

Now the optimization process let me explain what I have explained just now, at the beginning of this class. We can visualize the three dimensional space as a collection of planes parallel to the x y plane, that is the description of the three dimensional space. We can identify a plane from this collection by the perpendicular distance of the plane from the x y plane.

In an orthogonal system this would be the z intercept of the plane, the intersection of the plane with the z axis. Thus z equal to k identifies a plane parallel to the x y plane which is at a perpendicular distance of k from the x y plane. At any point on the plane z equal to k the value of z is fixed at k. In our problem the z coordinate represents the component X3 of the composition vector that is the content of security C in the portfolio.

Thus any point on the plane z equal to k will represent portfolios with the composition of C fixed at k. Hence on the this z equal to k plane because the content of C is fixed our problem reduces to a two security problem, because one the composition of one security is fixed so long as we lie on a particular plane. It is given by the intercept of the plane which is parallel to the x y plane with the z coordinate or z axis.

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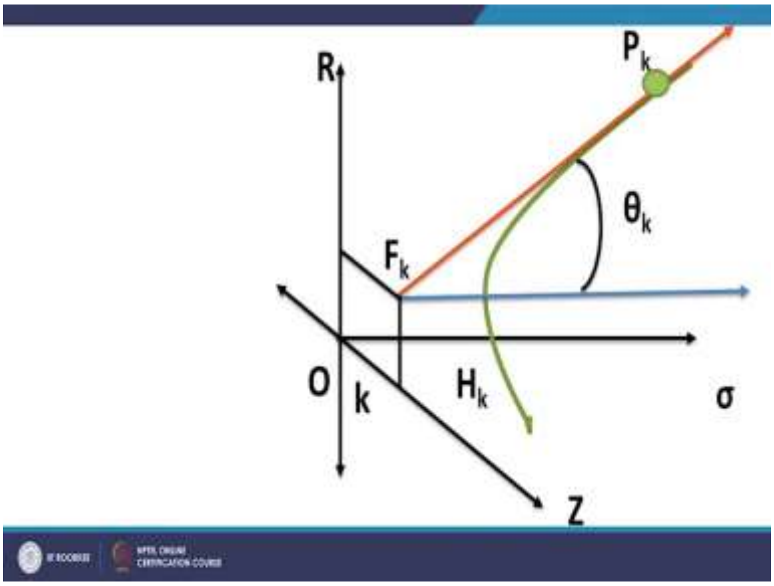

The plot of x and y on the plane z is equal to k would be a hyperbola Hk as in the two security problem. For this value of z equal to x 3 equal to k the efficient frontier will be the tangent line from the risk free point, risk free point lying on this plane z equal to k, please note this point.

It is not the respiratory point lying on the on the plane z equal to zero or which refers to the origin, we are obtaining the tangent from the point which lies on the plane on which the hyperbola is lying. So the points F and the arc of the hyperbola and the tangent as well are all coplanar. For the point of z equal to x three equal to k the efficient frontier would be the tangent line from the risk free point 0 comma RF comma k to the hyperbola Hk, let the slope of this tangent be tan theta k.

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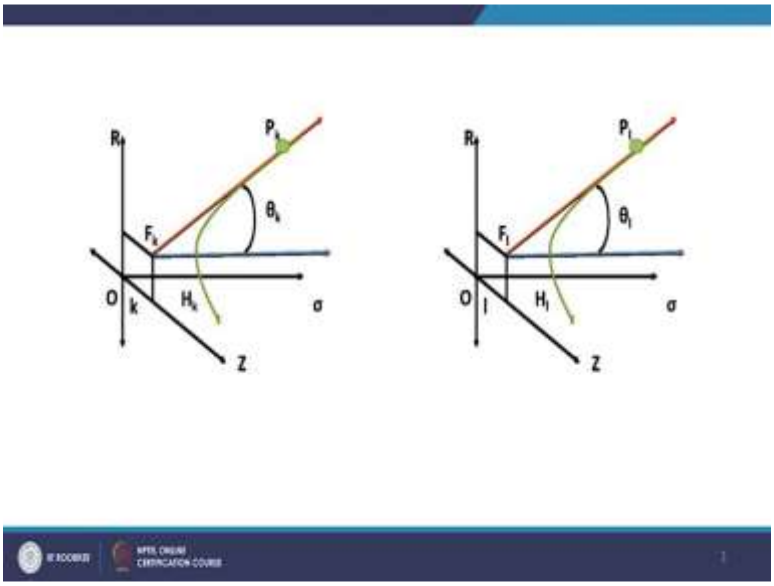

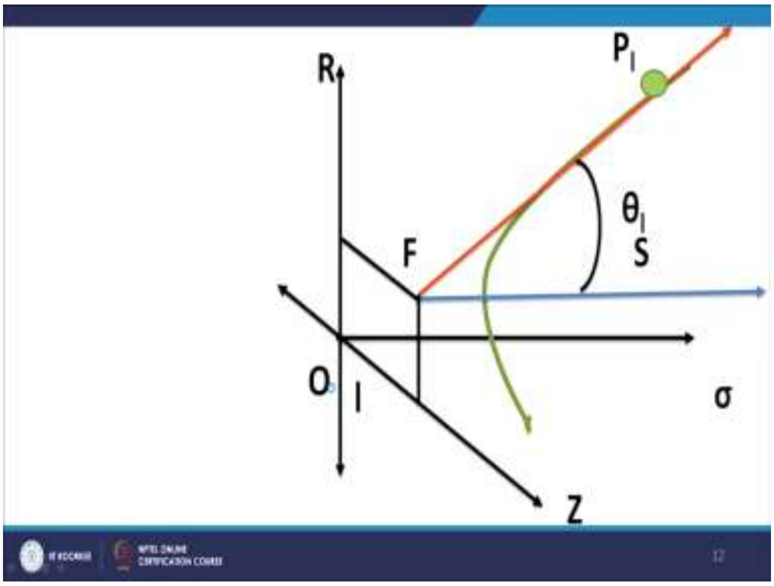

This is the diagram as you can see here and this is the point F, the coordinates of the point F are 0, RF and k and from here we draw a tangent to the arc of the hyperbola which is the portfolio possibilities curve. Given that the composition of the security is fixed at z is equal to k or X3 is equal to k and we obtain the tangent as the line FP and theta k is the angle that the tangent makes with the x axis and tan theta k is the slope of that line of the tangent.

Then we take another point as you can see in this diagram this is this is a new hyperbola that lies on the plane z is equal to l and on this plane we again do the same exercise, we find a tangent from the point 0 RF l, now please note it is different from the point 0 RF k. It is coplanar with the plane on which the hyperbola is lying.

So we find the tangent from the point F and that let the point of contact of the tangent from the point F to the hyperbola be given by Pl. In this case the point F let us call it Fl P l and the arc of the hyperbola lie on the same plane and tan theta l is the slope of this tangent.

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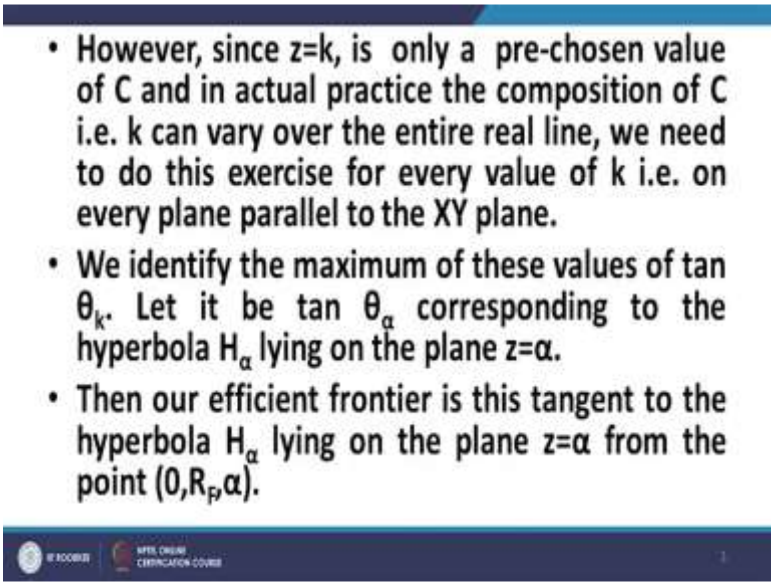

However since z is equal to k is only a pre chosen value of C, now we come to the important point we do this excess for everything y because z equal to k is only a pre chosen value of C and in actual practice the composition of C that is k can vary over the entire real line we need to do this exercise for every value of k that is on every plane parallel to the x y plane , because the composition of the security C in our portfolio is also unrestricted.

When we did the earlier exercise when we fixed z is equal to k, we have fixed the composition of C and now we are varying the composition of C over the entire real axis and therefore we are doing this exercise for every plane conceivable parallel to the x y plane. We identify the maximum of these values of tan theta k.

For every hyperbola and for every plane will get a value of tan theta k we select the maximum of this values of tan theta k let us call it tan theta alpha let us call the hyperbola H alpha and let us call the point F with the coordinates 0 comma RF comma alpha be the point of the risk free rate in that particular plane. Then our efficient frontier is the line F alpha comma P alpha that is tangent to the hyperbola H alpha and all these things lie on the plane z is equal to alpha for which the slope of the tangent is the maximum.

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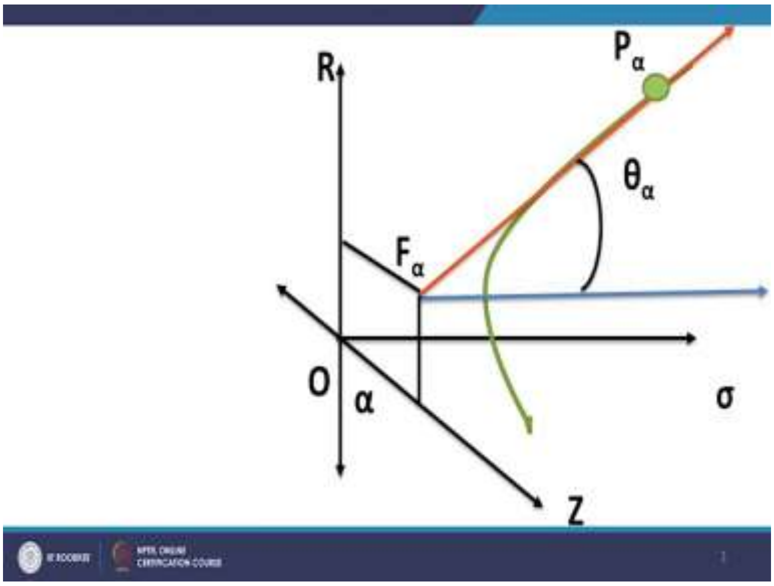

For optimality, Maximize $\tan\theta = \dfrac{\bar{R}_P - R_F}{\sigma_P}$

$$\tan\theta = \frac{\sum_{i=1}^{N} X_i(\bar{R}_i - R_F)}{\left[\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n} X_i X_j \sigma_{ij}\right]^{1/2}} \qquad (41)$$

We obtain the following equation for the composition vector :

$$\bar{R}_i - R_F = Z_i\sigma_i^2 + \sum_{j=1, j\neq i}^{N} Z_j\sigma_{ij}, i = 1,2,3,...,N \qquad (42)$$

This is the situation as far as the optimal plane is concerned, we identify this by z is equal to alpha and the theta that we have here is the maximum value of all the thetas that we have worked out in this process corresponding to z is equal to minus infinity to z is equal to plus infinity. Now obviously we cannot do it mechanically, we cannot do the exercise mechanically we need to have some math to help us.

The math is given in this following slides, the maximize tan theta is equal to RP minus RF upon sigma P that is straight forward you can see from here, theta alpha or any theta for corresponding to this is the tangent of that theta is given by RP minus RF divided by sigma P.

If you take the tan of theta alpha here it is nothing but RP where P is the expected return corresponding to point P minus RF divided by sigma P.

Here sigma P is the standard deviation corresponding to the point P, so when we do this exercise, when we do this maximization how do we do it? We differentiate with respect to the various degrees of freedom, in this case because we have three securities we can differentiate with respect to X1, X2, and X3.

When we differentiate them when we differentiate and obtain the partial derivatives and when we equate them to 0 what and we do some algebra some substitutions we end up with the set of equations that is represented by equation 42. The equation 42 gives us the set of equations which are called the fundamental optimization equations.

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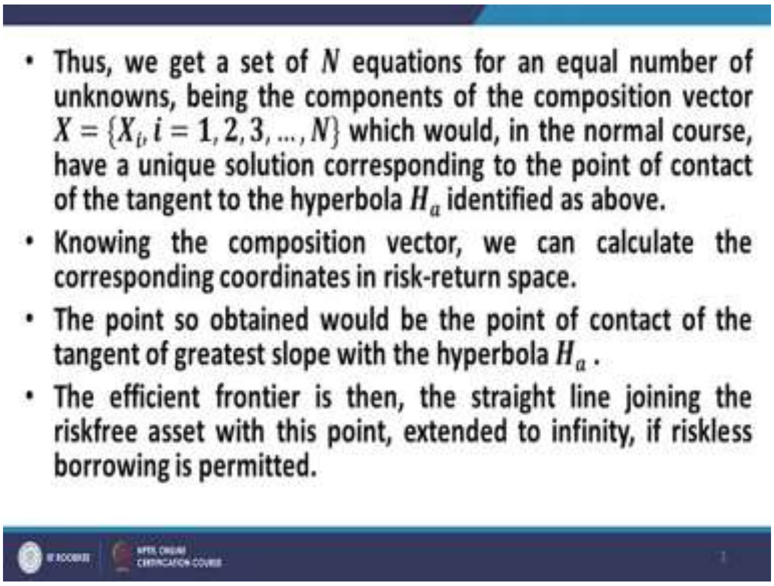

Thus we get a set of N equations for an equal number of unknowns for three unknowns we will have three equations being the components of the composition vector X equal to Xi I equal to 1, 2, 3 up to n which would in the normal course have a unique solution corresponding to the point of contact of the tangent that is the point P to the hyperbola H alpha described in the earlier part of this lecture.

Knowing the composition vector we can obviously calculate the corresponding coordinates and arrive at the coordinates of the point P. Knowing the point P and the point F we can work out the equation of the point of the straight line FP which represents the efficient frontier ah in our three security case or indeed the end security case. Let me reiterate at this point let me emphasize that the framework that we have not developed in the case of a three security case can be extended immediately to the n security case without any variations whatsoever.

You simply need to differentiate tan theta with respect to all the composition vectors X1, X2, X3 partially differentiate them and then equate them to zero and we get the same set of fundamental optimization equations Ri minus RF is equal to z I sigma square plus summation z j sigma I j I n equal to j or j n equal to I and then this gives us the set of n equations that we need. So let us now go back to our three security problem.

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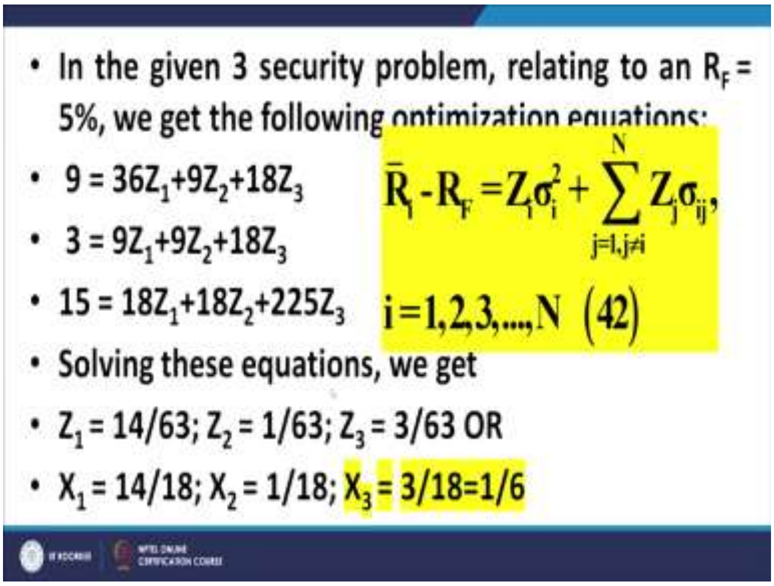

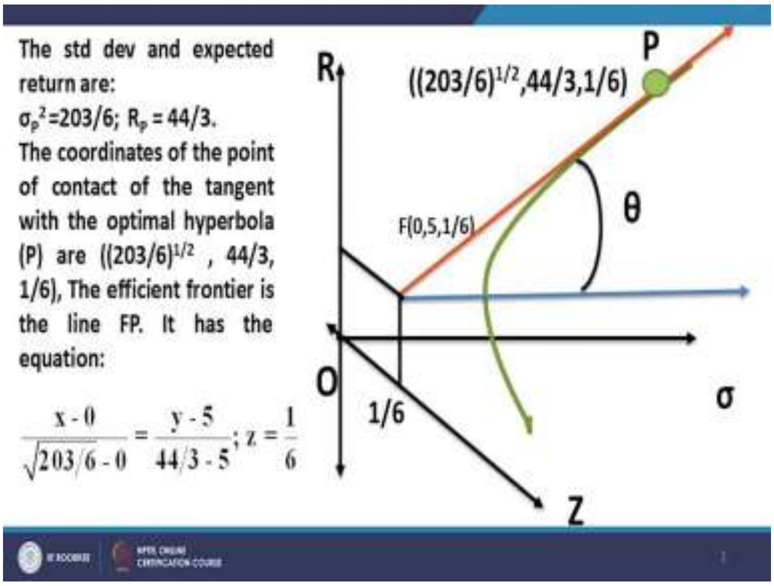

In our given three security problem relating to RF equal to 5 percent. If we choose RF equal to 5 percent the set of equations that we get is by substituting values in the set of equations 42 which is the highlighted panel we get is the set of equations that are given in the left hand side of this particular slide 9 is equal to 36 Z1 plus 9Z2 plus 18Z3 and so on.

Solving these equations we get the coordinates of P, I am sorry we get the composition vector corresponding to the point P not the coordinates of P we get the composition vector corresponding to the point P corresponding to the point of contact of the tangent from the risk free rate to the arc of the hyperbola.

And we get these as X1 is equal to 14 upon 18, X2 is equal to 1 upon 18, X3 is equal to 1 by 6. So this is what it is, the efficient frontier in this case the line FP which now you can see is lying on the plane z is equal to 1 by 6, X3 is equal to 1 by 6. So the composition of X3 in the optimal portfolio in the efficient portfolio is 1 by 6 and the composition of X1 and X2 are also obtained by solving the optimization equations and we get these values as 14 by 18 and 1 by 18.

Using all this information using the correlation coefficients and everything we are able to calculate the coordinates of the point P and they turn out to be 203 upon 6 to the power 1 by 2, 44 by 3 for the expected return and 1 by 6 because this point lies on the plane that is equal to 1 by 6 which represents the composition of security C in the efficient frontier.

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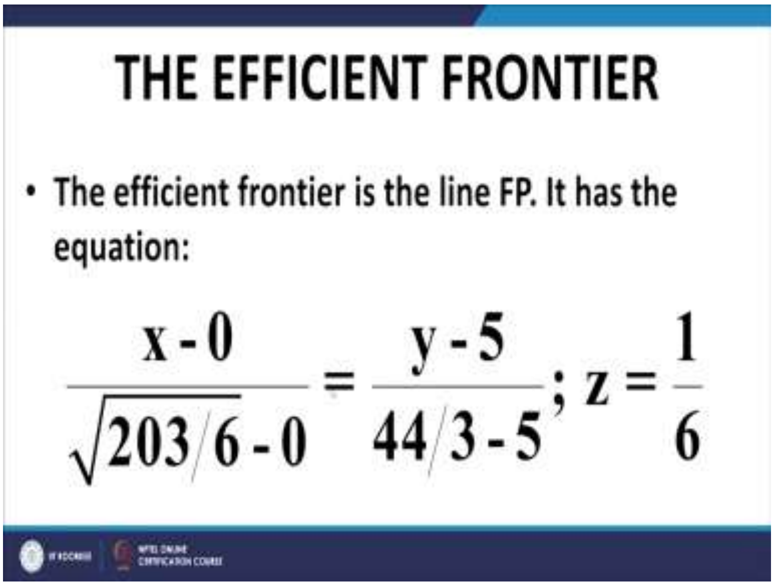

The efficient frontier, the equation of the efficient frontier is simply the equation of the line FP which is given by this expression here on this slide. Now we move on to the second section of this problem, the second section of the problem deals with the tracing of the entire efficient frontier that is if you are given three securities A B and C and let us say you are not having risk free landing on borrowing what would be the situation what would be the shape of the efficient frontier? Let us now tackle this problem, this I will explain step wise.

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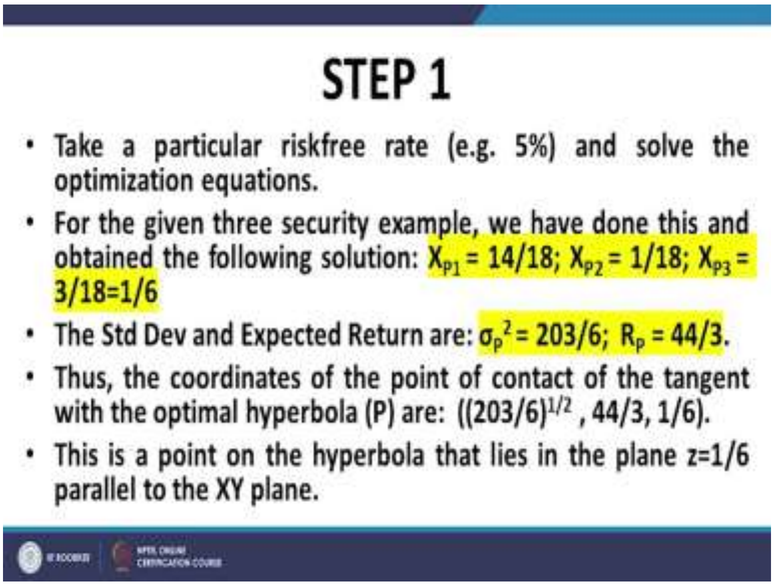

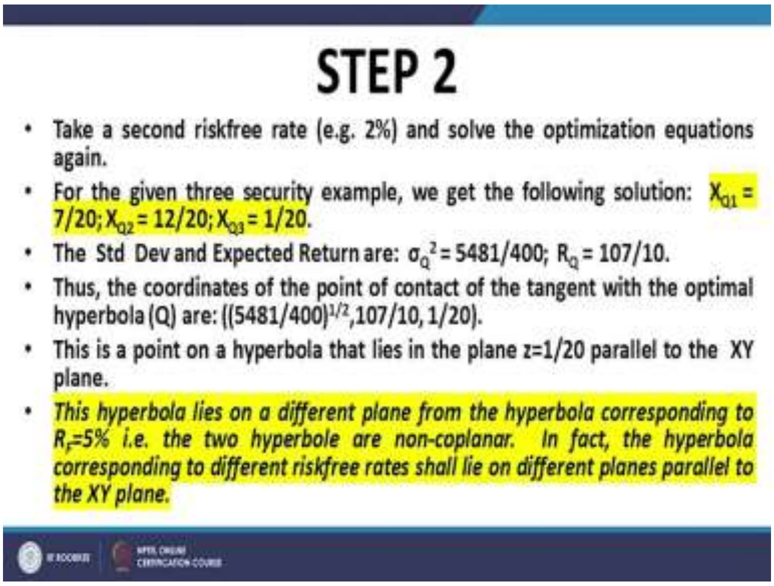

The first step is to solve these optimization equations fundamental optimization corresponding to a particular risk free rate we have already done it for the risk free rate of 5 percent and using the risk free rate of 5 percent we have arrived at the composition of the point of contact of the tangent to the hyperbola in the optimal plane that is z is equal to 1 by 6 or X3 is equal to 1 by 6 and we get the composition as XP1 that is the content of security A in the portfolio P as 14 by 18, the content of security B in the portfolio P is 1 by 18 and the content of security C in the portfolio P as 1 by 6.

We know fully the composition of the portfolio P and using that composition, using the information about the variance correlation, covariance matrix we can arrive at the coordinates of the point P including the expected return which is also given for the individual securities A B and C, so we can work out the expected return corresponding to the security P or the point P. Hence we can identify the point P completely in the space.

Now the next step is that we take a second risk free rate let us say we take a second risk free rate say 2 percent RF equal to 2 percent and again solve the same set of equations .When we solve the same set of equations taking the risk free rate as 2 percent the solution that we arrive at let us say the corresponding point is Q.

In other words let us say the point of contact of the tangent from the point F dash this is the new point which corresponds to a risk free rate of 2 percent with the hyperbola which has the maximum slope is the point which is represented by the composition vector XQ1 is equal to 7 by 20, XQ2 is equal to 12 by 20 and XQ3 is equal to 1 by 20.

Please note this point lies on a different hyperbola a different plane altogether and because it corresponds to a different risk free rate so the inferences that correspond to different risk free rates we get the optimal hyperbola, the optimal plane as being different planes parallel to the xy plane. So using the risk free rate of 2 percent we we re did the entire exercise that we have done so far and on completing the entire exercise what we end up with is XQ1 is equal to 7 by 20, XQ2 is equal to 12 by 20 and XQ3 is equal to 1 by 20.

The standard deviations and expected returns can be calculated because we know the variance covariance matrix and we also know the expected returns on the securities A B and C we know their respective compositions content in the portfolio Q and therefore we can work out the standard deviation and expected return of Q and we find the standard deviation of Q to be 5481 divided by 400 square root and the expected return to be 107 divided by 10.

So the coordinates of the point Q are identified as 5481 upon 400 to the power 1 by 2, 107 upon 10 and 1 by 20 and it lies on the plane that is equal to 1 by 20. Please note the earlier point and the point P lie on the plane z is equal to 1 by 6, so they lie on different planes parallel to the x y plane, please note this point.

This is what I have emphasized again let me read it out once more because this is important the hyperbola this hyperbola lies on a different plane from the hyperbola corresponding to RF equal to 5 percent as it that is the two hyperbola and non coplanar in fact the hyperbola corresponding to different risk free rates shall lie on different planes parallel to the x y plane.

I repeat the hyperbola corresponding to different risk free rates shall lie on different planes parallel to the x y plane. The next step, I also explained in the last lecture is that linear combinations of two efficient portfolios is efficient, linear combinations of two efficient portfolios is efficient. The proof is quite simple we take two efficient portfolios in other words two portfolios P and Q which satisfy the fundamental optimization equations because they are efficient portfolios, so they satisfy the fundamental optimization equations.

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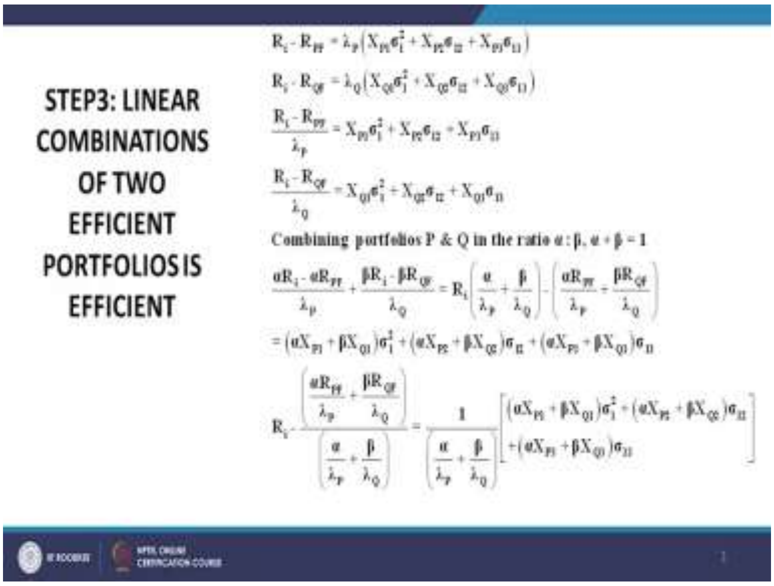

Then we form a new portfolio let us say a different portfolio which comprises of P and Q in the ratio alpha is to beta that is alpha P plus beta Q where alpha plus beta is normalized to 1 for convenience, it is not necessary but we do it for convenience. So alpha plus beta equal to 1 is taken to simplify calculations and using this fact we end up with the inference that the new portfolio that we have formed also satisfies the optimization equations albeit with a different risk free rate.

What is the inference? The inference is that a combination of two efficient portfolios is efficient is also efficient but it corresponds to a different value of the risk free rate, because the new portfolio the linear combination of P and Q also satisfies the fundamental optimization equations but with a different risk free rate. I shall continue from here after the break. Thank you.