Security Analysis and Portfolio Management
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Lecture 45
Mean Variance Portfolio Optimization - V

Welcome back. So, let us continue from where we left off but as usual A quick recap. We were talking in the last lecture about the two risky securities problem, two risky securities with and without or without and with A risk-free asset that is risk-free lending or borrowing.

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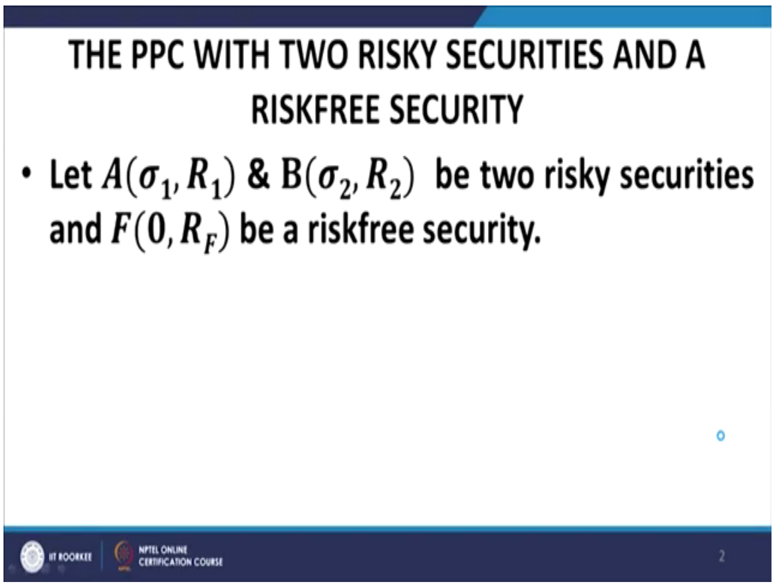

We assumed A and B to be the two risky assets and F to be A risk-free security and we examined the feasible region in the context of various combinations of these three securities. This is the primary diagram, this is the real essence of the two security problems so let us recap this once again.

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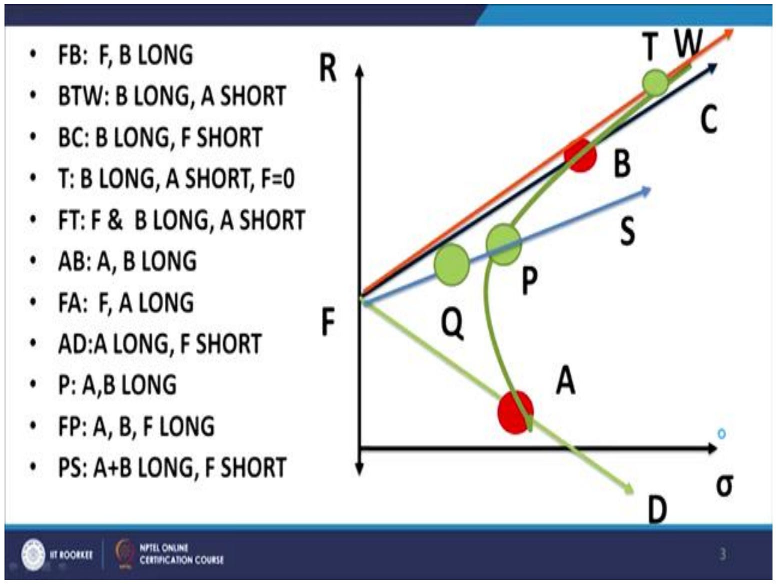

If you look at FB it is A linear combination of F and B so in this case both are long as far as FB is concerned if you extrapolate this B to C then in the region BC we have risk-free borrowing and the investment of the risk free borrowing in the asset B then you have the line B T W where T is the tangent to the arc A P B of the hyperbola extended if required and F T is A combination of the security, the risky security combination of A and B.

Obviously B would be long A would be short represented by the point T together with the risk-free lending or borrowing as the case may be. Within the region F T or with within the straight line F T; F would be long that is you would have risk free lending together ways, the long position in security B and A short position in security A which is represented by the point T on the hyperbolic curve extended beyond V.

And beyond T you will have risk-free borrowing and the investment of the risk-free borings in the combination represented by the point T of A and B obviously A short and B long. Thus so at the point T; B is long A is short and F is 0 then F T I have already discussed A B is the arc of the hyperbola which represents the portfolio possibilities curve of the two security problem, two risky security problem let me be more precise and therefore within the region A B if you pick up any arbitrary point P it represents A long combination of A and B.

Now, suppose you join F P, then what does F P represent? F would represent A linear combination of F and P that means it would comprise of risk free lending together with A and B both long securities. For example, point Q here in this diagram represents A position where all the three securities A B and F are in long positions.

However if you extrapolate this line FP beyond P towards s then you enter the realm of risk free borrowing and the investment of the risk free borrowing amount in the combination represented by P which is A long position in A and B. So, as far as Fa is concerned it is quite straightforward it is A linear combination of F and A with B being absent and if you extend that A straight line A beyond D then it represents A position where we are having risk-free borrowing and the investment of the risk proceeds in the security A.

So, that is more or less about this diagram, the details are given in the slide for the convenience of the learners. Now, we take up the different scenarios one by one, obviously we have six scenarios in this case.

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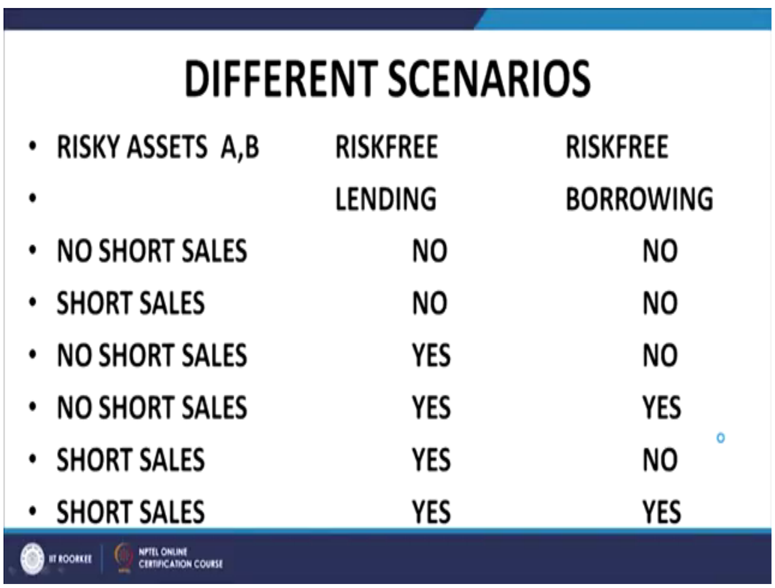

First of all when there is no risk free lending or borrowing that is the risk-free asset is absent and we have no short sales in A and B and then the rest of the scenarios. Let us go through them one by one to refresh our memory.

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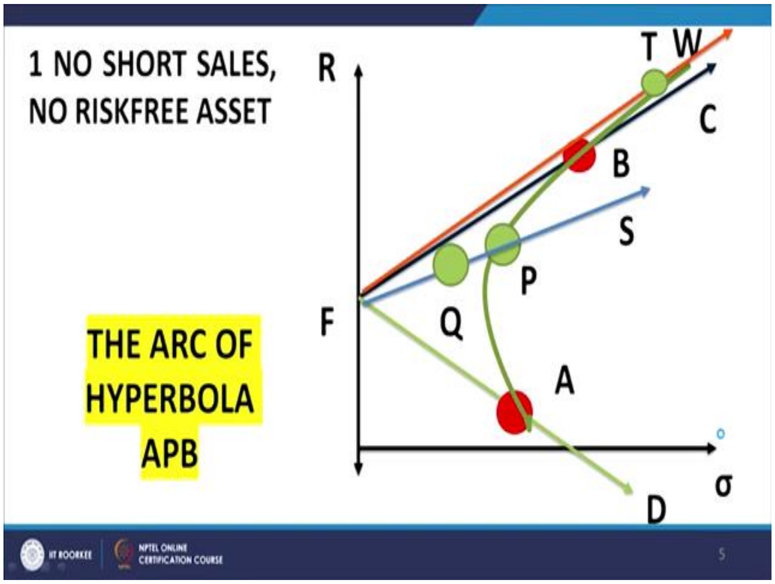

So, the first case is when there are no short sales allowed and there is no risk-free asset then obviously the way it resolves to a two security risky problem which we know is represented by the arc of the hyperbola A B. Please note there would be no extensions either way because we are not having short positions in A or in B being allowed.

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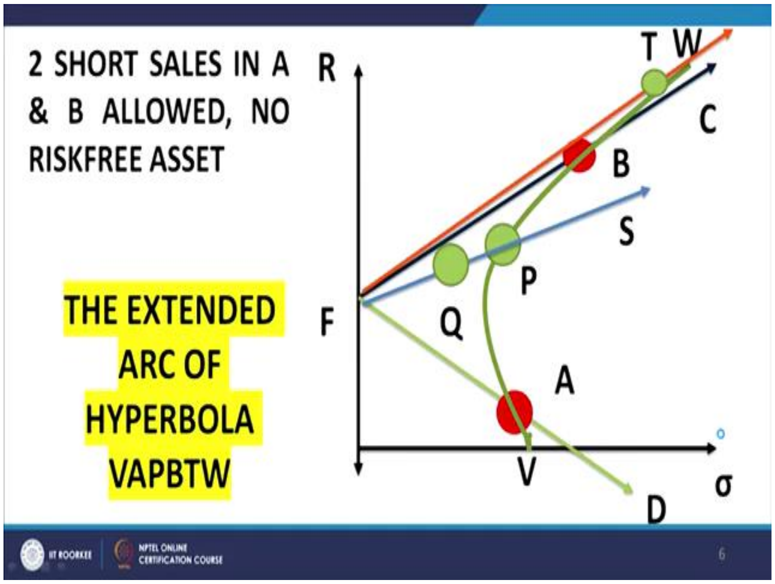

Now, if short sales in A and B are allowed then the arc gets extended, the arc of the hyperbola A B gets extended in both directions along Bw on the one side and along A V and on the other side so now the situation is that the portfolio possibilities curve becomes the arc of the hyperbola A B extended in both directions beyond A and B on B indefinitely.

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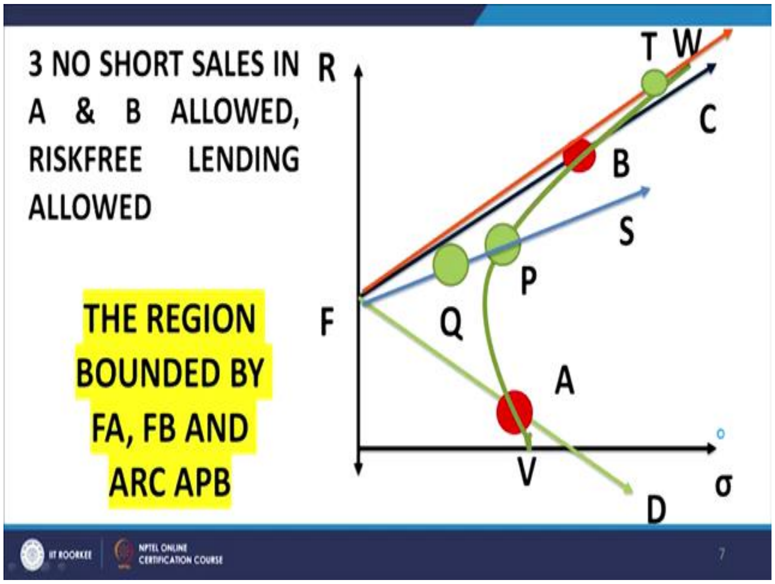

Then the third situation was when no short sales in A and B are allowed but we have risk free lending allowed. Now, this is now the feasibility region not takes the form of A surface it is the surface which is enclosed within the straight lines FA and the FB and the arc of the hyperbola A B so the region which is confined within the surface A F B enclosed within the straight lines AF BF and the arc A B; Q is A typical point in this region and q represents obviously A and B in the proportion represented by the point P and A long position that is together with risk-free lending.

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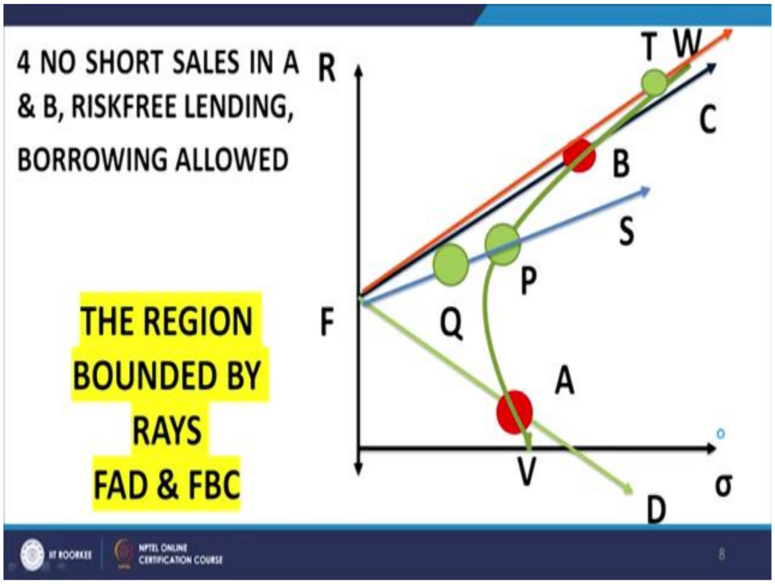

Now, the fourth scenario is when no short sales in A and B are allowed but risk-free lending and borrowing are allowed. This is typified by this the points q and the point S as mentioned in the previous case q represents the situation where we have the combination of A and B both long and together with risk free lending.

Whereas the point s represents the combination of F short that is risky borrowing and the investment of the risk free proceeds in the combination represented by the point P which is A long and B long. So, the entire region that is bounded by the rays F A and F B would represent the feasible region in this case.

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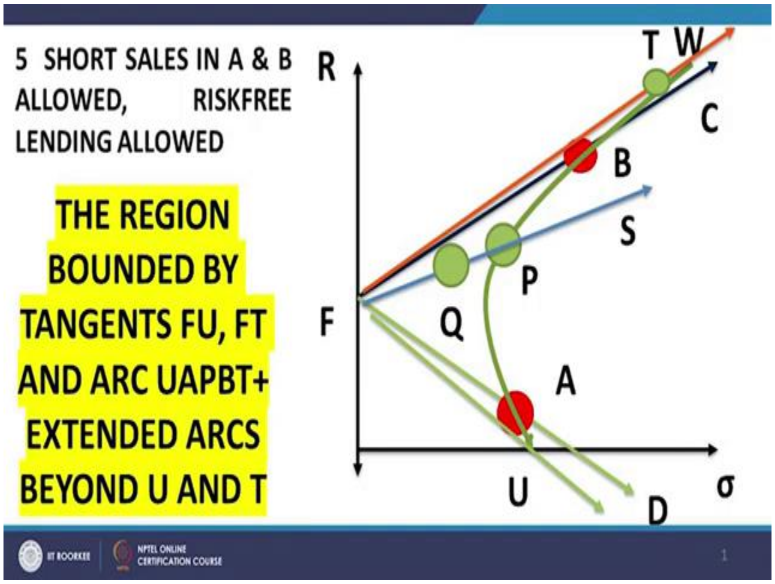

Now, this situation 5 is A bit tricky situation where we have short sales in A and B allowed but only risk free lending is allowed. So, because risk free lending is allowed only we have to confine ourselves up to the point F T where T is the tangent to the curve A and B but please note here that as far as the securities A and B are concerned short sales in A and B are allowed.

So, we are able to go beyond the point B along the arc and draw the tangent and the feasible region would include the region enclosed by the tangent F T and F U where U is the tangent into the arc of the hyperbola B A extended beyond A on the downside.

So, in this case what happens is the region or the feasible region is the region bounded by the tangents F U and F T and the arc of the hyperbola T B P A U, TBPAU and beyond this because short cells in A and B are allowed we go along the arc of the hyperbola T W onwards and also beyond U also we go along the arc of the hyperbola A U.

So, let me repeat this, this is the probably the most tricky case in this case the feasible region would include the region bounded by include the region bounded by F T where T is the tangent to the arc of the hyperbola A P B extended beyond B and the line F U where U is also F U is also the tangent to the arc of the hyperbola A P B extended beyond A and the area enclosed by the arc of the hyperbola T B P A U.

Together with in addition to this region because A and B are short positions and A and B are allowed if you extend the arc of the hyperbola beyond T then any point on that arc of the hyperbola would also represent a feasible combination where B would be long and A would be short. Similarly, if you extend the arc of the hyperbola beyond U along A U then any point on this arc extended beyond A U would also represent A feasible combination where A is long and B is short and we do not have any F.

The restriction that F is or risk free lending is allowed but not risk free borrowing is which is not allowed in means that we will not have any point means that we will not have any point along F T beyond T along the straight line ft extended beyond T because beyond T any point would represent risk free borrowing and the investment of that amount in the combination of A and B represented by the point T.

So, you will not extend the feasible region beyond F T, beyond F T you have to go back to the arc of the hyperbola extend the arc of the hyperbola beyond T and that extension of the arc of the hyperbola beyond T would give you feasible points. Similar is the case on the downside with the extension beyond U. You will not have any points along the straight line F U beyond U because here again risk-free borrowing comes into play but you will definitely have feasible points along the arc of the hyperbola beyond U because that represents situations where A is long and B is short but no F.

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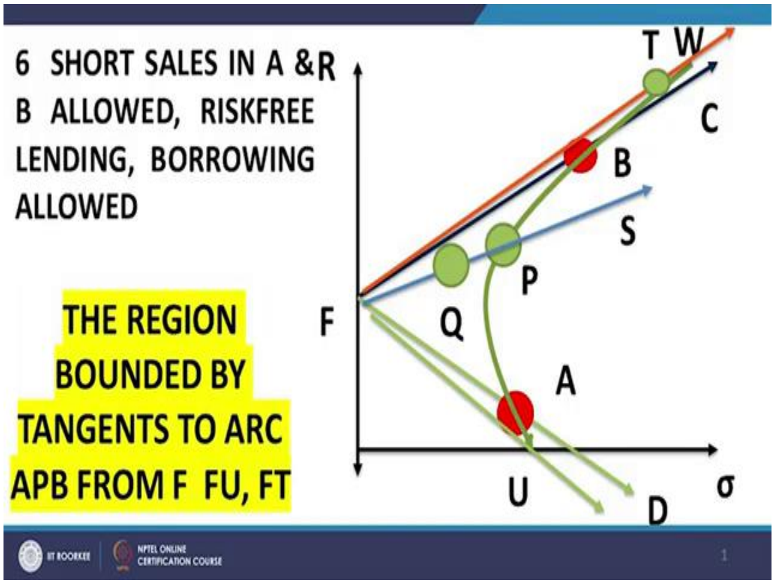

Then we come to the last case which is A relatively simple case where short sales in A and B are allowed together with risk free lending as well as risk-free borrowing. In this situation the feasible region is very simple it is the region to the right of the V which is enclosed by the rays from F, F T and F U which are tangents to the arc of the hyperbola T B P A U extended on both sides.

So, in this case which is the simplest case possibly and which is the most general case you may say where short sales in both the securities are allowed and the risk free landing as well as borrowing is allowed the feasible region is the V shaped region beyond A v shaped region enclosed within the rays F T and F U where T and U are respectively the tangents from F to the arc of the hyperbola A B extended on both sides beyond A and beyond B. Now, as far as the coordinates of T calculation were concerned we know let us see the next diagram.

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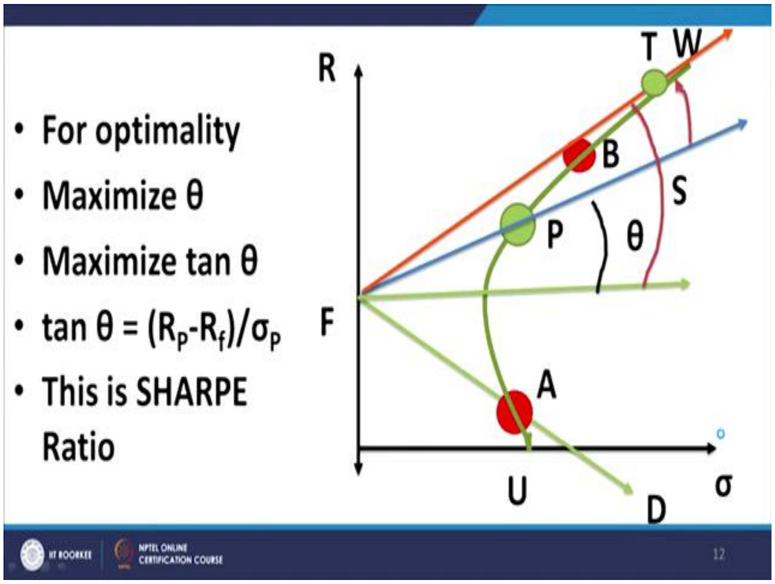

We can see here that how do we calculate T; how do we arrive at the point T; if you let us say we have the straight line FP now this if this straight line FP which makes an angle of theta or which has A slope of theta then if the straight line is to become A tangent to the hyperbola then what should happen to theta; what should be the impact on theta?

Obviously if we move along the anticlockwise direction and increase theta then we will gradually move from F P or that straight line F P will then move towards the straight line F T and at A certain point when theta is attains its maximum value we will reach the, we will reach the tangent F T. In other words what I am simply trying to say is that if I want to make the straight line F P tangent to the given arc of the hyperbola what I have to do is simply to maximize theta.

If i maximize theta or if when theta is maximized this line F P will then coincide with the line F T. Now, maximizing theta means obviously maximizing tan theta because we are in the first quadrant and maximizing tan theta means maximizing R P minus R F divided by sigma P. This ratio is A very significant ratio and will play A significant role in the rest of our studies particularly when we talk about the cap M model.

So, and this ratio is given A specific name it is called the sharp ratio. Now, in other words to arrive at this equation of the tangent or the point T coordinates of the point T what we need to do is we need to maximize theta, maximizing theta means maximizing tan theta because we are in the first quadrant where tan theta is A monotonic function of theta and therefore maximizing tan theta means what? Means maximizing rp minus r F divided by sigma P which is called the sharp ratio. So, then we do a little bit of algebra as I discussed in the last lecture let us not devote time to this and we end up with the equations which are represented by equation number 37.

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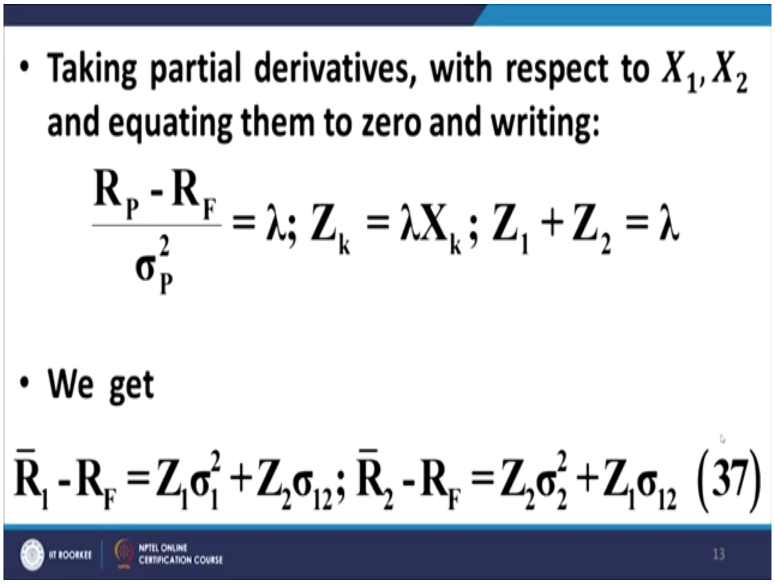

If we solve this two; these are two equations and two unknowns z1 and z2, if we solve these equations we can arrive at the value of z1 and z2 and therefrom we can work out the value of x1 and x2.

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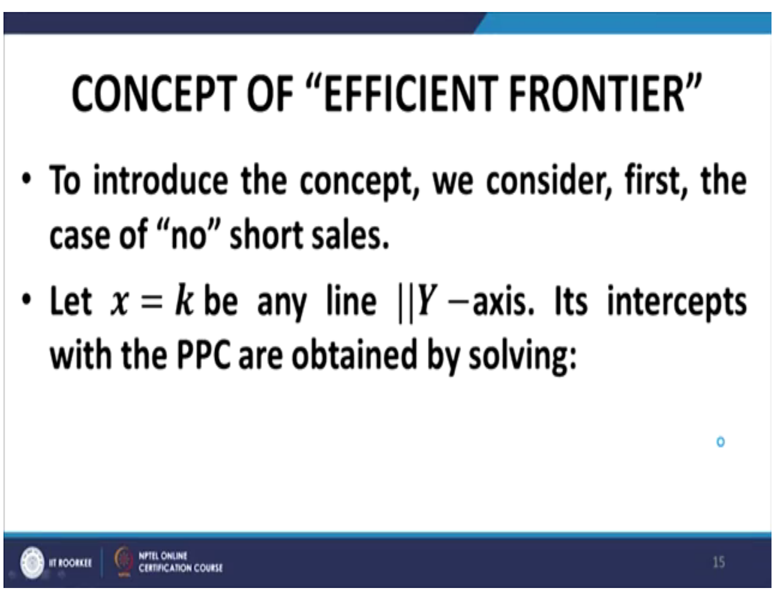

So, knowing the value of x1 and x2 we can work out the coordinates of T and then from the coordinates of T we can work out the equation of the straight line F T and similarly for F U and we can arrive at the feasible region in this case of the most general case where short selling is allowed as well as risk-free lending and borrowing is allowed.

So, F T in fact why I have emphasized F T is because F T is going to play A vital role in the development of the theory as we move on to the capital asset pricing model that is the reason I have focused on the derivation of the equation of FT in the expression for the coordinates of the point T at which the line straight line from F that is free point is tangent to the given arc of the hyperbola. Now, let us look at A new concept which also I briefly touched upon in the last lecture which is the concept of efficient frontier.

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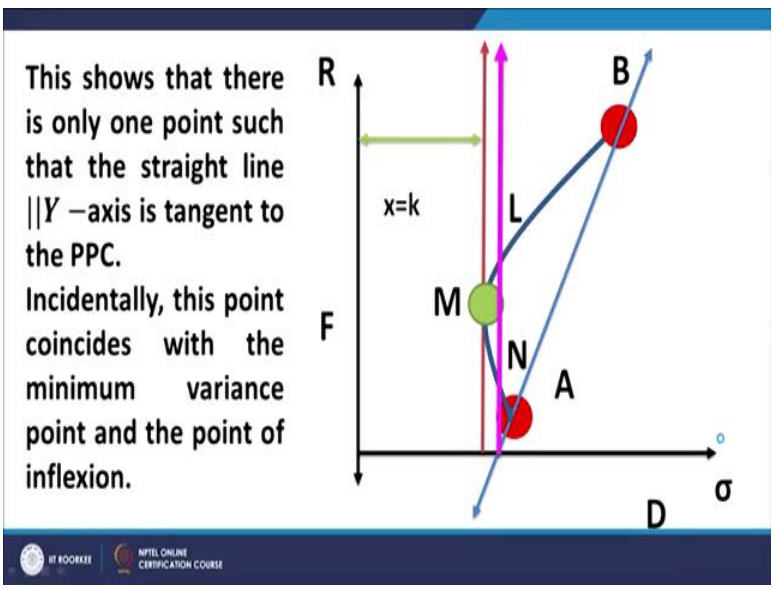

To motivate this concept let us try to let us look at this diagram this diagram what I am trying to do here is that I have taken A line let us say which is parallel to the y axis let it be represented by x is equal to k okay. So, we have taken A line straight line parallel to the y axis represented by x equal to T, x equal to k I am sorry.

Now, what I have tried to do is I have tried to obtain the points of intersection of this line x equal to k with the r with the hyperbola or the arc of the hyperbola that represents our portfolio possibilities curve for the case of two securities. What we find is very interesting; let us go to another slide.

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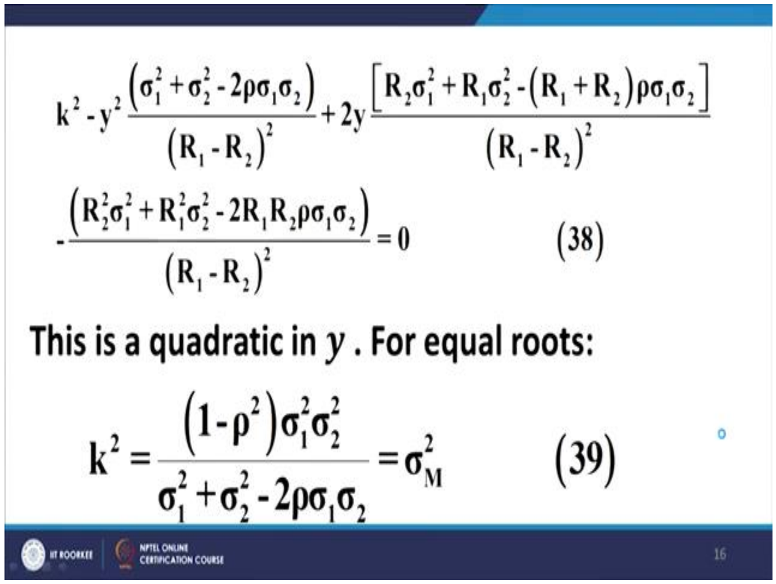

Now, this represents this is this was equation 14 basically but in which I have substituted x is equal to k because we are trying to evaluate the points of intersection of the straight line x equal to k with this particular hyperbola which is represented by this equation which was equation number 14 and which is now equation number 38; after substituting x equal to k.

Now, this is an equation which is quadratic in y if I want to have equal roots in other words if I want this line x equal to k to be tangent to this arc of the hyperbola then I need to have the discriminant of this equation which is A quadratic in y to be 0 and when we solve this what we get is the expression for k square is given in equation number 39 which you will immediately recognize that this k square equation is clearly or is immediately seen recognizable as the point of minimum variance as the combination of the two risky securities which form A minimum variance portfolio.

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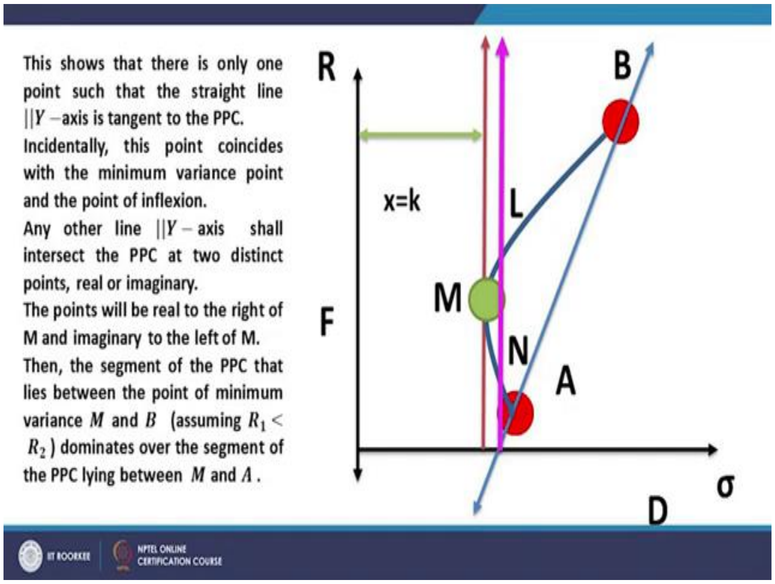

So, the bottom line of what I have said is; what explained just now is that if I try to work out the condition for the tangency of that line x equal to k that is parallel to the y axis to be tangent to our given arc of the hyperbola then the tangency occurs at the point of minimum variance. The tangency occurs at the point of minimum variance which incidentally we have also shown to be the point of inflection of the arc.

So, the point of minimum variance is that single point, solitary point at which this line x equal to k is happens to be tangent to the given arc of the hyperbola that means what that means if I move this if I change the value of k either to the right if I increase the value of k or I decrease the value of k what I will get is I will get the intersection of this arc with two points which may be real which may be imaginary. If I increase the value of k that is if I move to the right-hand side then the intersection is at two real points as you can see on this diagram these are the points L and N; L and N.

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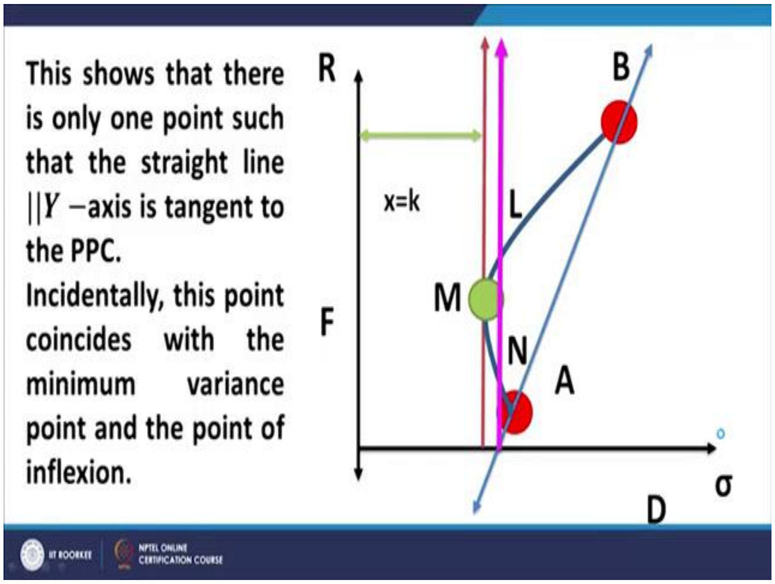

If I increase the value of k from the original value at which the x equal to k was tangent to the given arc then I have the line intersects the arc of the hyperbola at two distinct real points which are given by l and n. And of course, if I decrease the value of k I will get the intersection at two imaginary points which is irrelevant for our discussion. So, the point is the point is accept at m now the basic thing is except at the point m if I move, if I intersect the straight line parallel to the y axis with the arc of the hyperbola the intersection occurs at two points; two distinct points L and N.

Now, if you look carefully at the points L and N they have A very beautiful or A very interesting property. The points L and N obviously coincide to the; obviously are identical in terms of risk because the risk is plotted along the x axis as you can see and L and N have the same x intercept that means they have the same level of risk. but what about the expected return?

When you look at the expected return the point L is definitely giving you A higher expected return compared to the point N. What does it, what will it result in? It would result in the fact that the portfolio represented by the point L would dominate over the portfolio represented by the point N. No investor or no rational investor would in would be interested in investing at A port in A portfolio which is represented by the point N.

All investors having A level of risk represented by this straight-line x equal to k intersecting at any point on the x axis which represents the risk tolerance of the investor would it be invested in the portfolio represented by L. Now, this is the situation not only for any specific L for but for all points along the arc of the hyperbola mb compared to the arc m A.

In other words, all the points represented on the arc m A accept m except the point m which is singular which is specific which represents the point of minimum variance all the parts along arc m A will be dominated by A corresponding point on the arc mb just like the point n is dominated by the point L.

Similarly, if you explain let us say you move this line more to the right you will get another point let us say L dash which will dominate the corresponding intersect of the straight line on the arc M A extended if required and which you would let us say which is n dash. So, corresponding to every point on the arc M A or extended if required which is which is epitomized by the point n there would be A point L there will be A corresponding point L on the upper arc m B which would dominate the portfolio represented by the point n.

So, this arc MB in this present case it is an arc the arc mb is called the efficient frontier because it is efficient relative to the other portfolios that are possible. In other words, the portfolios that are represented on the arc m B cannot be improved upon they are the best possible portfolios that can be generated given the risky assets A and B.

You can generate other portfolios also which would lie along the arc ma but they would not be good portfolios because by improving or by redistributing the composition between A and B you can end up with A portfolio having A superior expected return compared to that other portfolio and with both the portfolios having the same level of risk.

So, this is the important thing M B represents the efficient frontier which is the set of the best portfolio best possible portfolios that you can get for at A given level of risk and comprising of the securities A and B. The portfolios that are represented in the arc M A extended if required would not find favor with the investors. So, as far as the portfolio optimization problem now is concerned we are more concerned with the nature or the structure of the sufficient frontier then the portfolio possibilities curve.

We have now narrowed down our selection from the entire feasible region to A region or to A region or A shape which is called the efficient frontier and which comprises of portfolios which are the best portfolios comprise relevant to or related to A given level of risk let us now explore more about the sufficient frontier. So, now we discuss the efficient frontier for each of the six cases that we discussed earlier.

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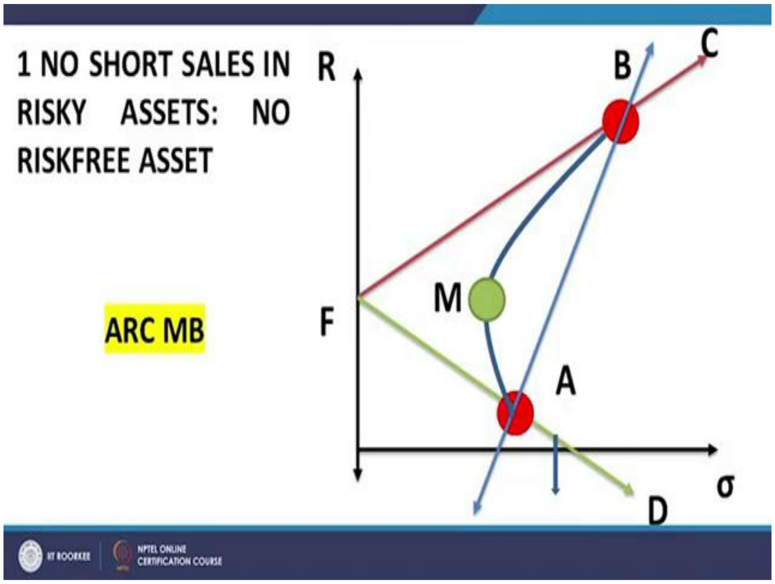

Now, the first case when we have no short sales in the risky assets no risk-free asset that is A and B are the only two securities you have no risk-free lending you have no risk-free borrowing this is the most elementary case. In this case as i have explained just now I reiterate that although the portfolio possibilities curve is the arc of the hyperbola from A to B, the efficient frontier is only the arc from M to B. M to A is cut out is removed because it is subordinate in terms of performance to the portfolios or the corresponding portfolios lying on the arc M to B. So, in this situation the efficient frontier is aimed to be the arc of the hyperbola along M to B.

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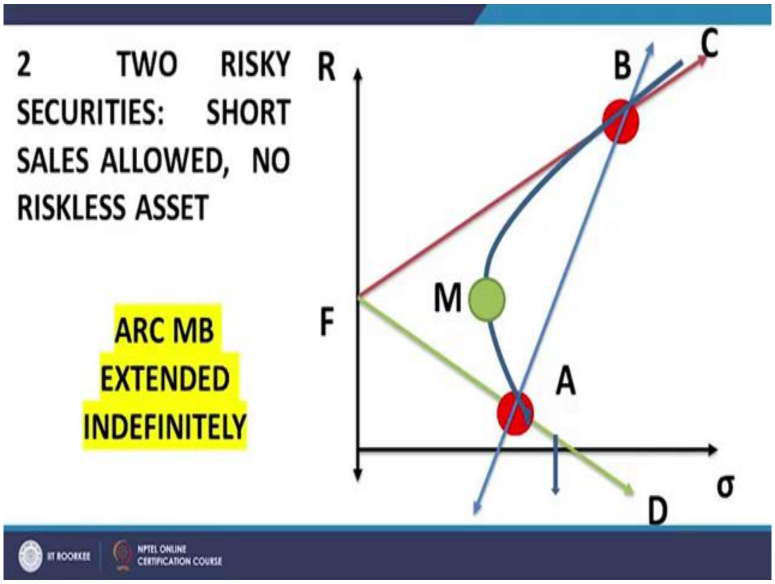

Now, we look at the second case what is the second case the second case is the case of two risky securities short sales are allowed and no risk-free asset. Short sales are allowed means the arc of the hyperbola mb will be extended indefinitely that is the only change that will occur in the efficient frontier of case one.

In the case of efficient frontier of case one because short sales were not allowed we could not go beyond B along the arc but here because you are being allowed short sales you can go beyond B and your efficient frontier will comprise of the indefinite arc of the hyperbola starting from m and continuing indefinitely beyond B.

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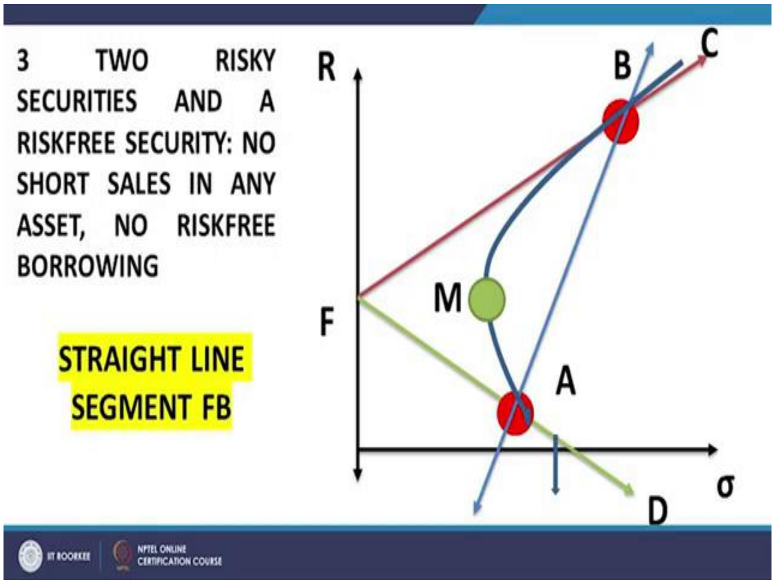

Then we take the third case; two risky securities and A risk-free asset no short sales in any asset. No risk-free borrowing so in this situation what happens what was the feasible region let us recapitulate; the feasible region was the region that was encompassed between that was enclosed between the straight lines FB, FA and the arc of the hyperbola AMB.

Now, if you look at this feasible region you immediately find that the efficient frontier will be nothing but the straight-line FB. At any point that lies within this feasible region would be dominate, dominated by A corresponding point on the straight line FB which is now A part of the feasible region because risk-free lending is allowed if risk-free lending is allowed FB becomes A viable proposition and therefore if you take any point within this enclosure FB, FA and AB then you are dominated or you have A point on FP which dominates that combination.

So, in this case the efficient frontier becomes A straight line segment FP why? Please note is the straight-line segment it is not the ray that means you have to terminate at FB; why you have to terminate at FB? Because number 1; you are not having short sales in A and B and number 2; you are not having risk free lending risk free borrowing as well. So, beyond B either you will have risk free borrowing or you will have short sales in security A. So, we have to stop at B it is the line segment between F and B.

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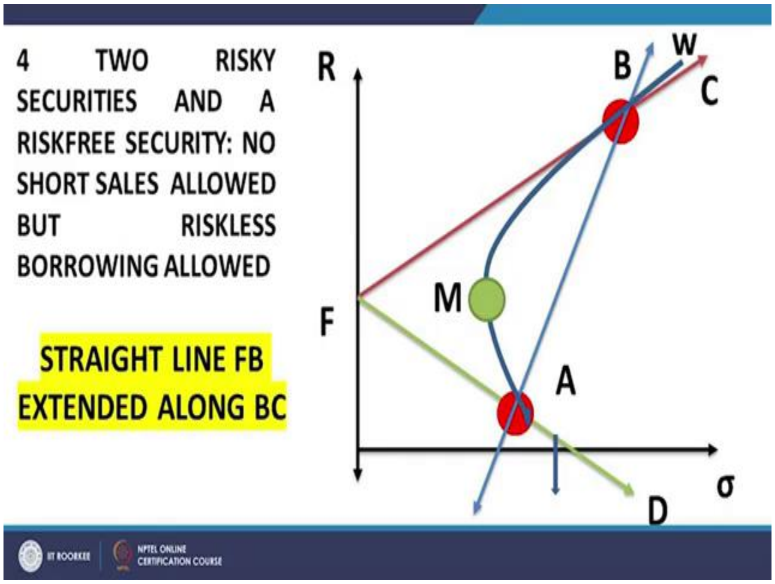

Now, we look at case number 4; two risky securities and arrest free security no short sales allowed but risk-free borrowing is allowed. Now, in this case what was the feasible region? The feasible region was the entire region please note this entire region which is bounded by the rays FB extended along x C and indefinitely beyond C and F A extended along d and beyond d as well.

So, the entire v that is enclosed within the rays FB and FA constitutes the feasible region. Now, what is the efficient frontier? Again, on the same rational that I discussed for the previous case; case number 3 we end up with the feasible region being the straight-line FB extended indefinitely along C and beyond C.

Between F and B obviously you have linear combinations of F and B that is risk-free landing and B and beyond B you have risk free borrowing and the investment in the security B. So, in a situation where you have two risky assets and risk-free security but you have no short sales in these in either of the securities risky securities but you are allowed risk free borrowing the portfolio possibilities I am sorry; the efficient frontier changes from FB the line segment FP to the ray if we extended beyond B indefinitely.

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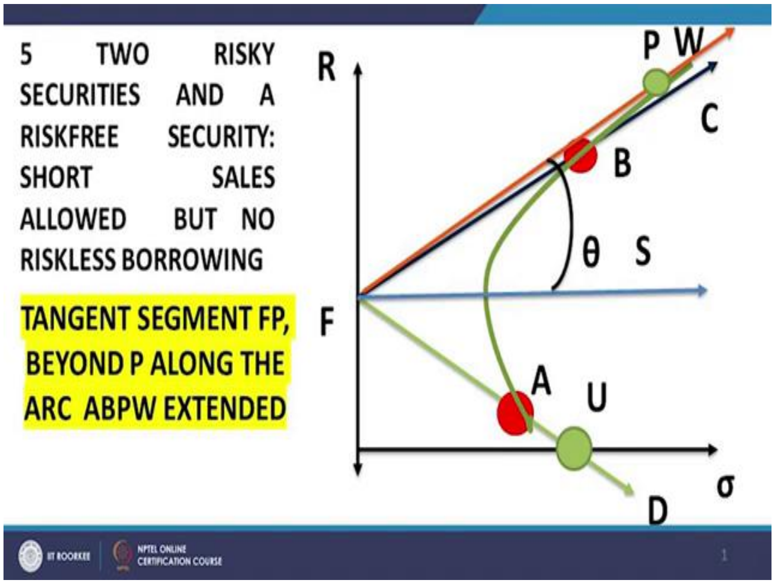

Now, we move to case number 5; what is case number 5? We have two risky securities and risk-free security risk-free asset short sales are allowed but no risk-free borrowing. If you recall this was the most tricky case, in this case what was the feasible region? The feasible reason was the region enclosed now, listen to me carefully; the region enclosed by the tangents F P and the tangents F U extended on the arcs A B beyond B and beyond A this was the feasible region up to P and U.

And then beyond P and U we had the arc of the hyperbola extended ah along the arc itself in other words the feasible region was the region confined or enclosed within the straight lines F P up to P not extended F q up to U not extended where P and U are tangents from F to the arc A B and the arc P B; A U this was the feasible region and then beyond this there is an additional portion which also is feasible which is the arc of the hyperbola A B.

So, this is A two-part feasible region, what about the efficient frontier? The efficient frontier is clearly the straight line FP the line segment FP, the line segment FP where I repeat P is the tangent to the arc A B, so the r this line segment FP and then beyond that we have the arc of the hyperbola which extends beyond B P and W. So, let me repeat the efficient frontier in this case is the line segment, the straight line segment F P plus the arc of the hyperbola beyond P extended along B P W and thereafter indefinitely.

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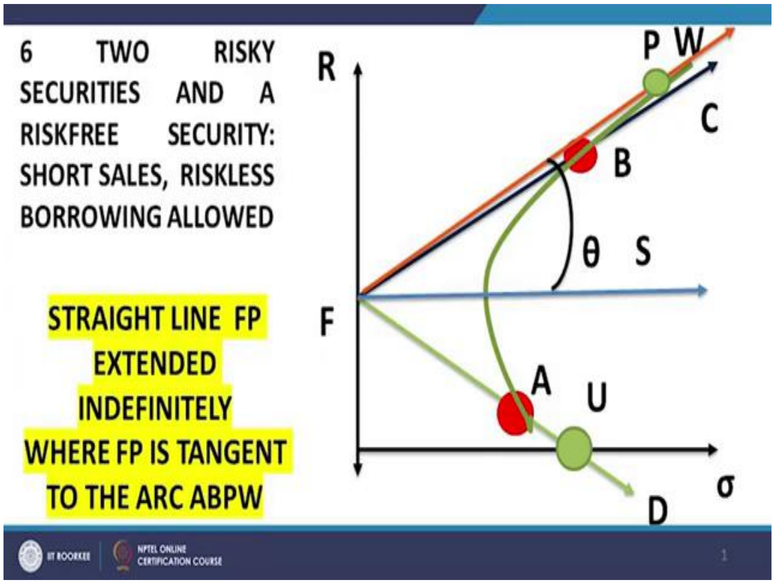

Then the last and the possibly the simplest case that is the case where both risky assets can be short sold and the risk-free borrowing can also be resorted to. In this case as I mentioned the feasible region is the entire V that is enclosed by the by the rays F P and F U where these are tangents to A B and in this case the efficient frontier is obviously the tangent F P extended indefinitely.

Beyond P you will have risk free borrowing and the investment represented by at the point P by A combination of A and B with A being short and B being long. So, these are the shapes of the efficient frontier corresponding to the various scenarios that exist for combination of two risky assets under risk-free asset. Now, we take up the case of 3-risky securities which I will start after the break. Thank you.