Security Analysis and Portfolio Management
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Lecture 44
Mean Variance Portfolio Optimization - IV

Welcome back. So, let us continue from where we left off. In the fourth scenario that we were discussing in the third scenario we have already discussed now we to talk about the fourth scenario.

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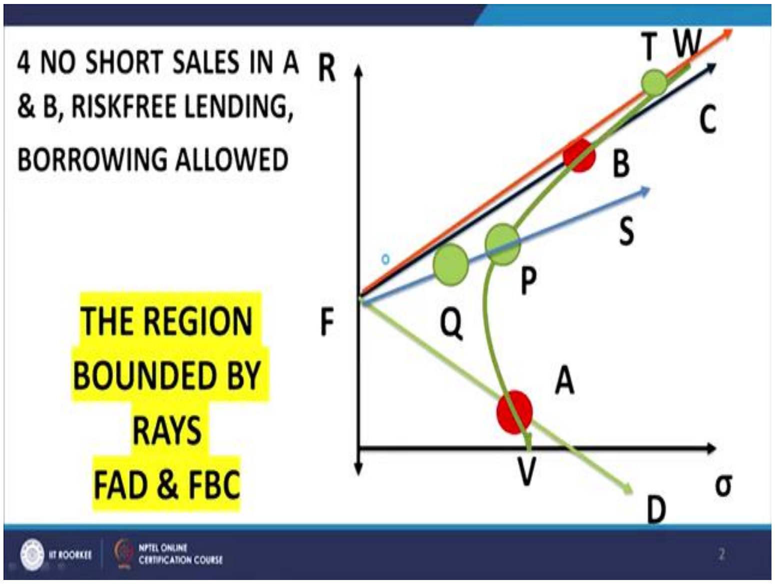

We have no short sales in A and B but we have riskless lending and borrowing both allowed. I repeat we have no short sales in A and B but we have riskless lending and borrowing both allowed. In this case what would be the portfolio possibilities curve or the portfolio possibilities region rather?

Obviously, it would include the lines fb and fa and the RKB that is very much A part because all three securities are long in this region so it will include FP F A and A B but because you have risk-free borrowing also the lines BC would also be admissible portfolios and similarly the line A D will also be admissible portfolios. Because along B C what is happening is you are having B long and F short.

In other words, you are having risk free borrowing but you are not having risk free you are not having A short A, A is at 0 at the point B and beyond B along BC A is 0 X B is greater than 1 and X F is less than 0. In other words we have risk-free borrowing and with the investment of those borrowed proceeds in security B. So, you have the line fc extended and similarly you have the line F D extended which the region A D would consist of A long B0 and F as borrowings.

So, in the region A D what will happen is you will borrow at the risk-free rate and you would invest in the security A that is F D and of course any point in between these two rays starting from F that is F C and F T both of them extended indefinitely would be an admissible portfolio. For example, if you look at the point S here, S is any arbitrary point within the region C F D extended indefinitely then F consists S consists of what?

S consists of P long and together with risk-free borrowing but P itself is A combination of A and B both long therefore along P S what we have is A long both long and F short that is S B borrowing and of course along P F we have all the three securities long. So, in this case when we have no short sales in A and B but we do have risk free lending and borrowing the portfolio possibilities region is the entire region bounded by the race FC and FT extended indefinitely.

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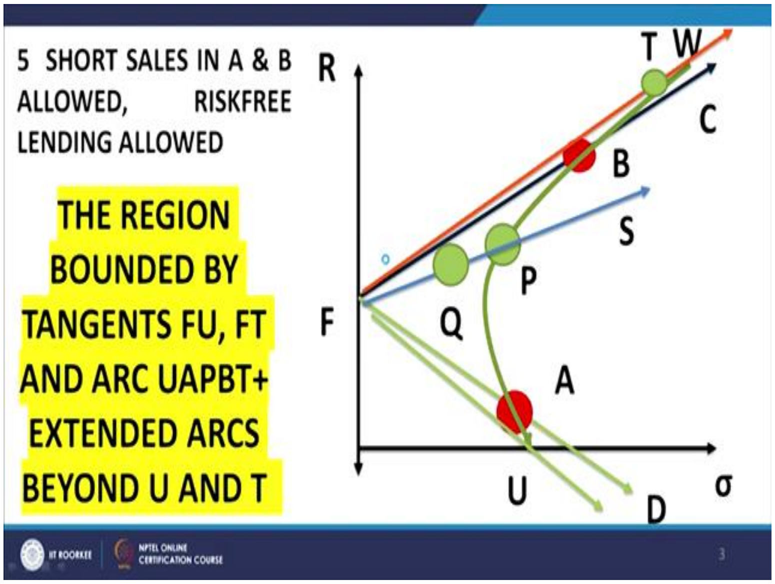

You know the fifth situation where we have short sales in A and B allowed but risk-free borrowing is not allowed, risk free lending is allowed. Now, in this case what happens let us try to understand because short sales in A and B are allowed the entire arc of the hyperbola A B extended beyond B and beyond A would definitely be A part of the portfolio possibilities curve.

So, if you extend the arc of the hyperbola along the hyperbolic arc beyond B that is along T and then W that this would be parts of the portfolio possibilities region. And similarly, in the case of A anything beyond A along the hyperbolic arc would be also forming part of admissible regions. Now, what about the of course what would be the composition here in the case of points lying between B and W?

It would consist of B long and A short and similarly in the case of extensions of the arc beyond A it would comprise of A long and B short but because risk free lending is also allowed we would also have points along FT the entire region bounded by FT and FU where T and U are tangents. So, what it means it that the entire region which is bounded by the lines FT and FU where T and u are the tangents drawn from F to the hyperbolic arc AB extended both sides would form part of the portfolio possibilities region.

And then beyond T beyond T what happens is the extended arc of the hyperbola T W and similarly beyond u the extended arc of the hyperbola would form admissible portfolio so up to T and U we have the entire region and then beyond T and U we have the arcs of the hyperbola. So, that is what the region would seem the region bounded by the tangents FU and FT and arc be extended beyond U and T.

So, this would be in this case the portfolio possibilities region in this case it becomes slightly complex but that is what it is. So, let me repeat it is the region which is bounded by FT FU and the arc of the hyperbola T B P A U and then beyond T the arc extended only the arc extended at TW and similarly the arc extended beyond U along A U.

So, that is the situation in the case where both short sales in A and B are allowed but risk-free lend borrowing is not allowed please note points like point S would not be allowed because this represents risk-free borrowings. The points which would be allowed in this region would be confined to the arc up to the arc T B P A U.

So, now let us move to the sixth scenario when short sales are also allowed and risk-free lending and borrowing is allowed. In this case the portfolio possibilities region is very simple it is the entire region which is bounded by the rays FT and FU. So, on the entire region of space which is bounded by these two rays FT and FU forms admissible portfolios every point here would be admissible either because it would comprise of either long or short positions in the two securities with risk free landing or with risk borrowing.

I have already discussed that in an earlier section so we will not go back to it and this is what the situation is. So, these are the six scenarios and in and I have demarcated or explained the portfolio possibilities curve or the region as the case may be in each of these six possible situations. So, let us now, let us recapitulate what we have done so far before we move to the next section.

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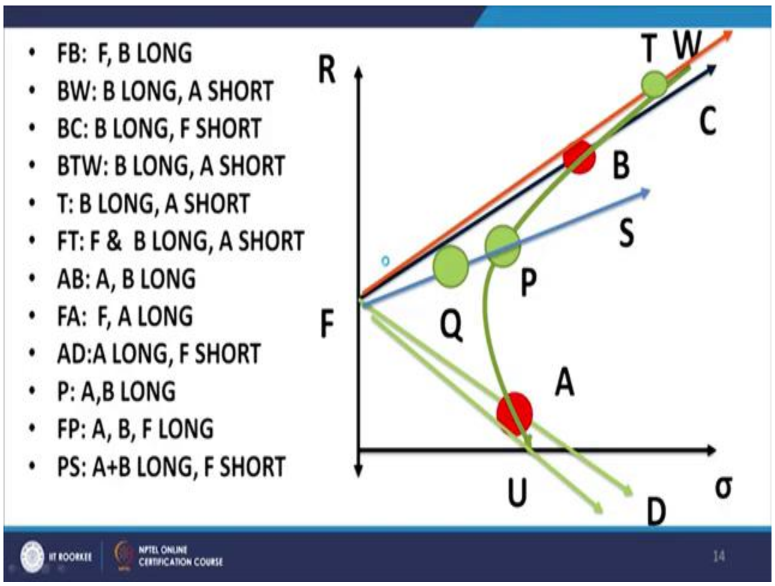

As far as FB is concerned what is FB? FB is F long and B long as far as BW is concerned please note BW is along the arc of the hyperbola AB and therefore BW means B long and a short, BC; BC is along the straight line, FB produced so FB means B long and F short B T W as I have explained earlier is B long and A short T at the point T we have B long and A short as mentioned just now.

Along FT we have what? We have F long and T long and what is T? T is B long and A short so FT represents F long B long and A short AB; AB is the arc of the hyperbola and along this arc of the hyperbola both A and B are long bounded by the points A and B then we have FA; FA is the straight line joining F and A so it is A linear combination of F and A; AD along AD, AD is an extension of FA, so beyond A we have A long and F short that is borrowings.

The point P, point P is an arbitrary point on the arc AB so it represents A and B both long, the point F the line F P where P is any arbitrary point between A and B represents what represents A linear combination of F and P, P is long in A and B so any point between FB, FA and AB represents all the three securities long.

So, as far as F as any arbitrary point S is concerned S means what? S means risk-free borrowings plus long in A and B which is represented by the point P. So, coordinates of T not, T is A very important point here what is T? T is the tangent to the hyperbolic arc AB extended from the point F, F represents the risk-free rate so FT is the tangent to the hyperbolic arc AB from the point F.

We need to locate exactly the point T because it is as you will see it plays A very important role in much of the development of mean variance portfolio theory and indeed it is it encapsulates the region as FT and FU encapsulate the entire portfolio possibilities written in the most general situation when A and B are allowed together with risk-free lending and borrowing. So, we need to determine the coordinates of the point T for further development of the theory this is very important.

So, there are two approaches to this; the first is the coordinate geometry method. We assume that Y is equal to mx plus rf because the tangent has to start from F so the Y intercept would be rf and therefore we can write the equation of the tangent in the form Y is equal to mx plus rf and they we assume that the equation of the hyperbola can be put hyperbola representing the portfolio possibilities curve can be put in the form X square minus P Y square plus 2 F Y minus C is equal to 0, we have already discussed this equation 17.

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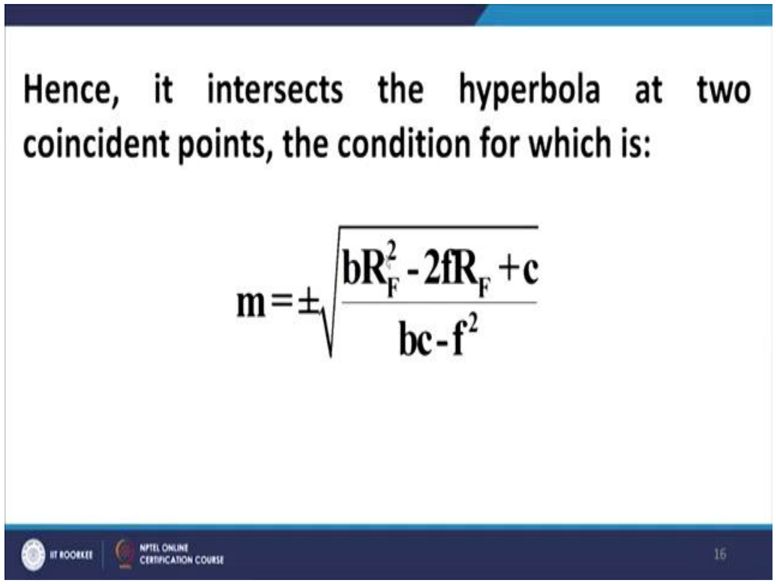

And then we find the points of intersection and what we find and then we require that the points of intersection be coincidental. In other words the tangent is determined by determining the points of intersection and then requiring the points to be coincidental in which case with the condition for which comes out to be m is equal to plus minus bRF square minus 2 fRF plus c divided by bc minus f square. There are two points obviously because we will have two tangents from F to the hyperbola one in intersection at T and the other intersecting at u as you have seen in the earlier figure and what will be the equation of the tangent?

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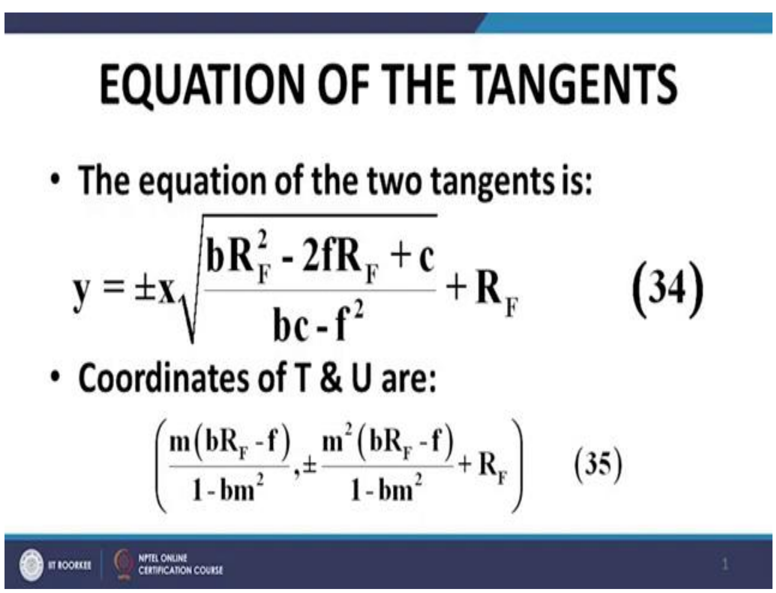

The equation of the tangent would be given by Y is equal to plus minus m X plus F r F where we have already determined m in the previous slide as being the expression under root B A r F square minus two F r F plus C divided by B C minus F square and what are the coordinates of this point of intersection or the point of contact of the tangent F T with the hyperbolic arc AB extended they are given by equation number 35.

Please note we have plus minus x of the plus would correspond to the point T and the minus would correspond to the point U. The other method is I would also like to highlight another method for calculating the composition and the coordinates of T because also it is immediate ability to be generalized to the form, to the case of n security. So, let us quickly look at that.

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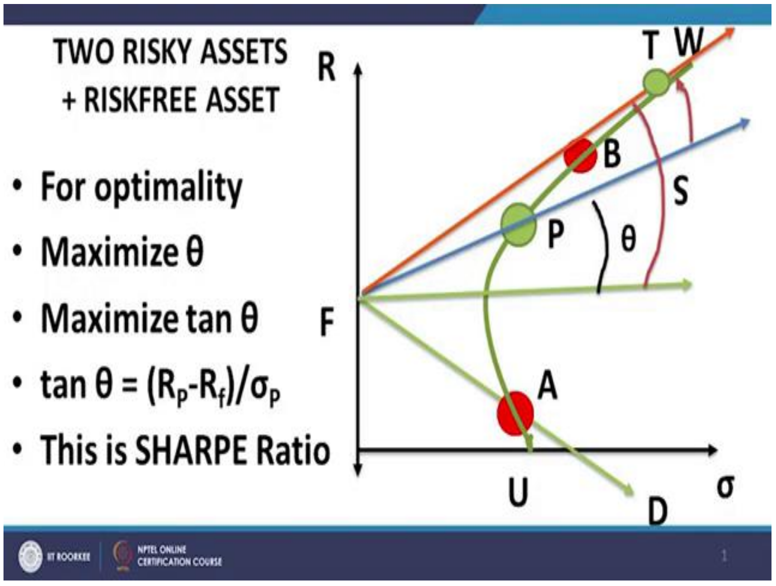

Now, what we see from the diagrams here is that if you look at FT, the property of FT is that it maximizes the angle between the any line, it any line intersecting the hyperbola from the point F and the X axis and the line parallel to the X axis rather. If you look at this particular diagram here and you look at the property of FT what you find is that FT maximizes the angle that the A straight line through F intersects the hyperbola there must be an intersection point with the hyperbola because if you take any point beyond T along the, above T, about the line FT then that would not form A feasible portfolio at all so that is redundant from our consideration. So, we must have A point of intersection of F with the hyperbola.

So, let if there is any arbitrary line starting from F intersecting the hyperbola and then the angle that this line that arbitrary line makes with the straight line parallel to the X-axis through F or you may say the slope of that arbitrary line has to be maximized and that means theta of any arbitrary line from F intersecting the hyperbola needs to be maximized.

If you can maximize the slope then what will get is the line FT you can see this very well from the diagram here and what if you see tan theta is A monotonic function of theta in the first quadrant so maximizing theta (())(15:39) maximizing tan theta and what is tan theta in this case; tan theta would be nothing but RP minus RF divided by sigma P.

So, if you are trying to maximize theta you all you can do with maximization of tan theta because the tan theta is the monotonic function of theta in the first quadrant and if theta increases tan theta increases; theta decreases tan theta decreases in the first quadrant. So, at the end of the day what I want to convey is that we can come to the expression for T or we can work out the slope of FT where T is a tangent by maximizing the objective function Rp minus Rf divided by sigma P.

So, that is the theory behind the second method if you maximize the slope of any line which starts from F and intersects the in the hyperbola you maximize the slope of that line then you end up with that line being tangent to the hyperbola. So, if you take an arbitrary line, calculate it slope which is Rp minus Rf divided by sigma P you maximize this then you end up with the line FT, so that is what is done in the second method.

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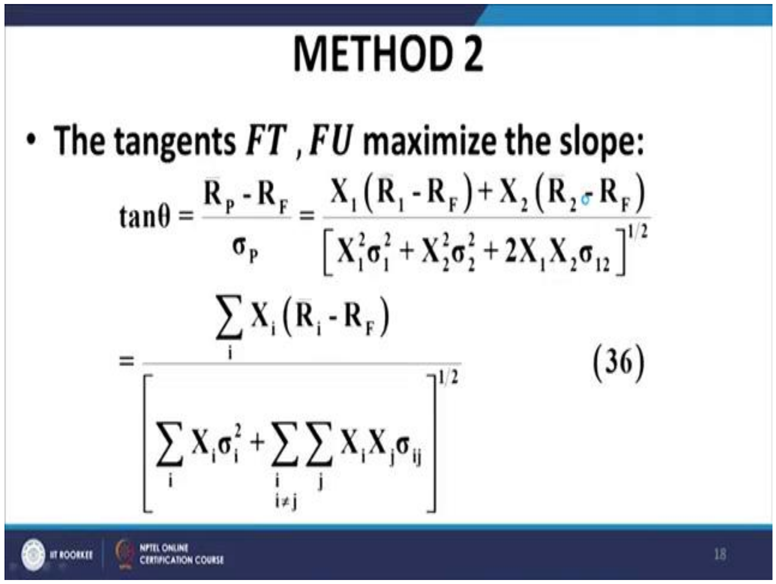

So, we work out RP minus RF upon sigma P and we use the expressions that we have worked out earlier for the return on A portfolio X1 Rn R1 plus X2 R2 and because X1 plus X2 is equal to 1 we can write RP minus RF in the form that is given here in this equation, equation number 36 where we are simply substituted the expressions for the two security portfolio.

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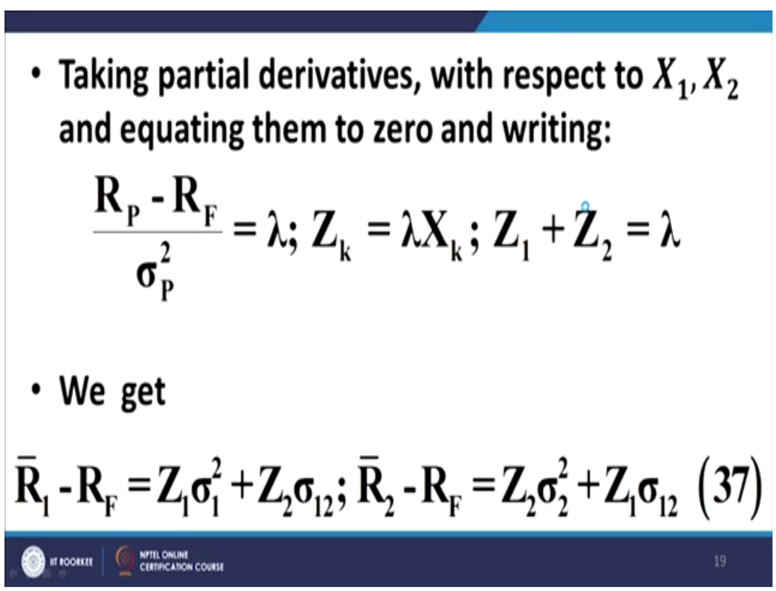

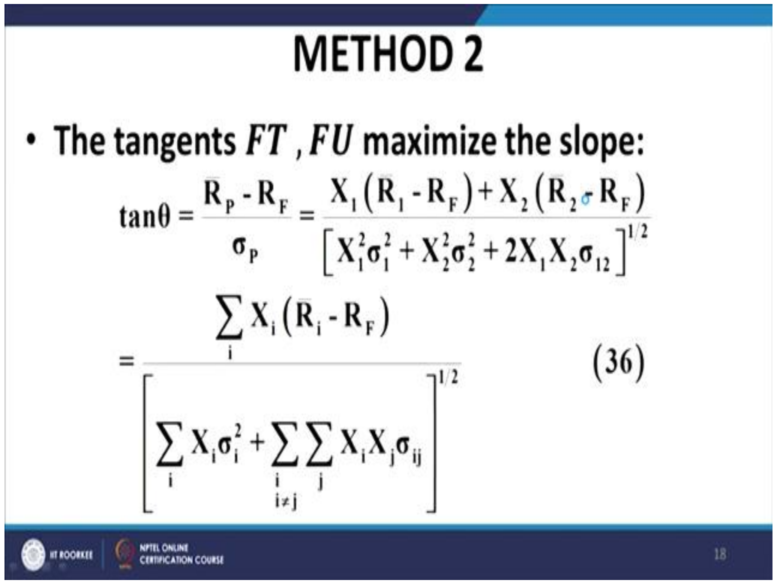

Now, if you take partial derivatives of this expression partial derivatives of equation 36 with respect to X 1 and X 2 and put in some abbreviations RP minus RF upon sigma square is equal to lambda Z k is equal to lambda xk Z1 plus Z2 is equal to lambda if you make all these substitutions what we end up with at the end of the calculations which are again quite straightforward differentiation calculations and then making this algebraic substitutions. We end up with the equations R1 minus RF is equal to Z1 sigma square plus Z2 sigma 1 and sigma 1 2 I am sorry R1 minus RF is equal to Z1 sigma 1 square plus Z2 sigma 1 2 and R2 minus RF is equal to Z2 sigma 2 square plus Z1 sigma 1 2 this is set this set of equations is equation number 37.

So, I repeat we partially differentiate equation number 36 which is here on the slide with respect to X1 and X2 and then we equate them to 0 and then we make the substitutions rp minus rf upon sigma P square equal to lambda Zk is equal to lambda Xk and Z1 plus Z2 equal to lambda. We make these substitutions and after doing all the simplifications, doing all this algebra and we end up with equation number 37.

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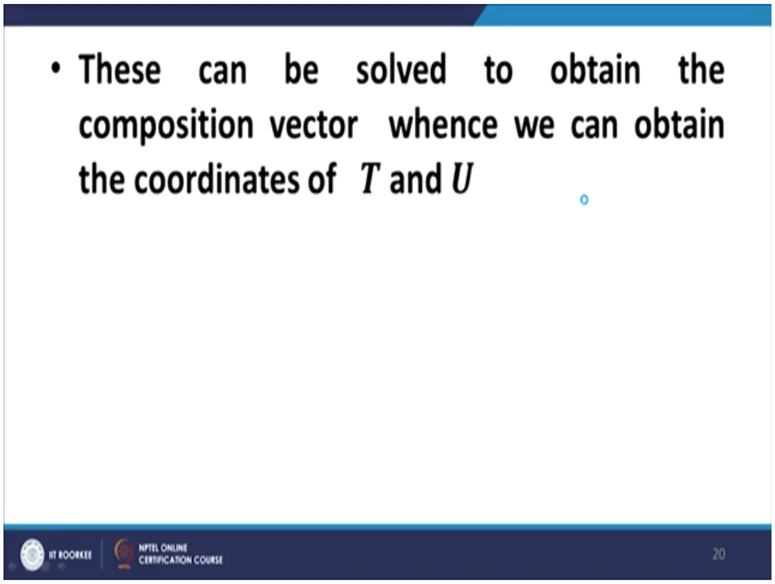

Now, equation number 37 can easily be solved it is a pair of equations and two unknowns Z 1 and Z 2 we can solve this to obtain the expression for Z 1 and Z 2 and knowing the expressions for Z 1 and Z 2 you can calculate the expressions for X 1 and X 2 straight away. So, at the end of the day the this set of equations 37 enables us to work out the composition vectors of the composition that is represented by the points T and u in the diagram that we discussed earlier.

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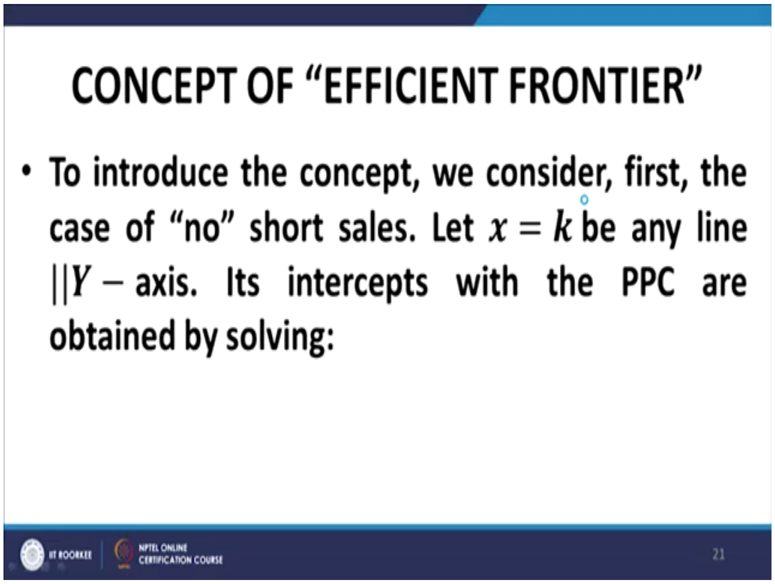

So, once we have the combination, once we have the composition of T and U it is elementary to work out the coordinates of T and U in the diagram and which enables us to determine the equation of the straight lines FT and FU, so that is the second method we maximize tan theta tan theta is equal to RP minus r F upon sigma P we substitute the expressions for rp in terms of what we have done earlier for A two security risky portfolio and after doing that we the result that we arrive at we differentiate partially with respect to X1 and X2 and equate the differential derivatives to 0 and then make some elementary substitutions, algebraic substitutions to make the results compact and that enables us to get the equations which is represented by the set 37.

Which and there are two equations and two unknowns Z1 and Z2 we can solve them for obtaining the value of Z1 and Z2 which gives us the value of X1 and X2 knowing the composition of A and B that is X1 and X2 we can determine the coordinates of the point P, point T I am sorry and on the basis of knowing the coordinates of the point T we can determine the equation FT.

Now, we come to a very important concept which is called the concept of efficient frontier. So, we have discussed in a lot of detail the portfolio possibilities curve, portfolio possibilities region corresponding to different scenarios now we take up the case of efficient frontier. Now, efficient frontier is a part of the portfolio possibilities curve it is a subset of the portfolio possibilities curve. What is the specialty of efficient frontier let us look into that.

Let us we know that the equation of the portfolio possibilities curve in the simplest case let me illustrate the efficient frontier with reference to the simplest case that is the case of two risky securities. For that purpose, let us try to find out the points of intersection of a straight line which is parallel to the Y-axis with the portfolio possibilities curve which for a pair of securities pair of risky securities with no risk-free lending or borrowing which is represented by the equation 14.

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$$x^2 - y^2\frac{\left(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2\right)}{\left(\bar{R}_1-\bar{R}_2\right)^2}+2y\frac{\left[\bar{R}_2\sigma_1^2+\bar{R}_1\sigma_2^2-\left(\bar{R}_1+\bar{R}_2\right)\rho\sigma_1\sigma_2\right]}{\left(\bar{R}_1-\bar{R}_2\right)^2}-\frac{\left(\bar{R}_2^2\sigma_1^2+\bar{R}_1^2\sigma_2^2-2\bar{R}_1\bar{R}_2\rho\sigma_1\sigma_2\right)}{\left(\bar{R}_1-\bar{R}_2\right)^2}=0 \qquad (14)$$

$$E\left(R_P\right)\equiv\bar{R}_P\equiv y, \quad \sigma_P = x$$

$$k^2 - y^2\frac{\left(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2\right)}{\left(R_1-R_2\right)^2}+2y\frac{\left[R_2\sigma_1^2+R_1\sigma_2^2-\left(R_1+R_2\right)\rho\sigma_1\sigma_2\right]}{\left(R_1-R_2\right)^2}-\frac{\left(R_2^2\sigma_1^2+R_1^2\sigma_2^2-2R_1R_2\rho\sigma_1\sigma_2\right)}{\left(R_1-R_2\right)^2}=0 \qquad (38)$$

This is a quadratic in y. For equal roots:

$$k^2=\frac{\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}=\sigma_M^2 \qquad (39)$$

So, we try to find out the point of intersection of X is equal to k with the portfolio possibilities curve. So, let us substitute X equal to k in equation number 14 what we get is equation number 38 and this is obviously a quadratic in Y. Now, for equal roots that is for a coincident point of intersection what we get is k square must be equal to the expression given by 39 which we immediately recognize as the minimum variance point.

So, what does it mean? It means that the straight line parallel to the Y axis, the straight line parallel to the Y axis intersects the hyperbola that the arc of the hyperbola that represents the portfolio possibilities curve at a single point which is the minimum variance point. In all other cases, in all the other cases the straight line parallel to the Y axis intersects the arc of the hyperbola either at two real points or two imaginary points.

So, that means what? That means except for the point of minimum variance which is a unique point which is in a sense which is tangent to the; which is tangent or a straight line parallel to the Y axis is tangent to that hyperbola at that particular point of minimum variance which is also the point of inflection at any other point the, at any other any other line rather which is parallel to the Y axis would intersect the hyperbola at two points.

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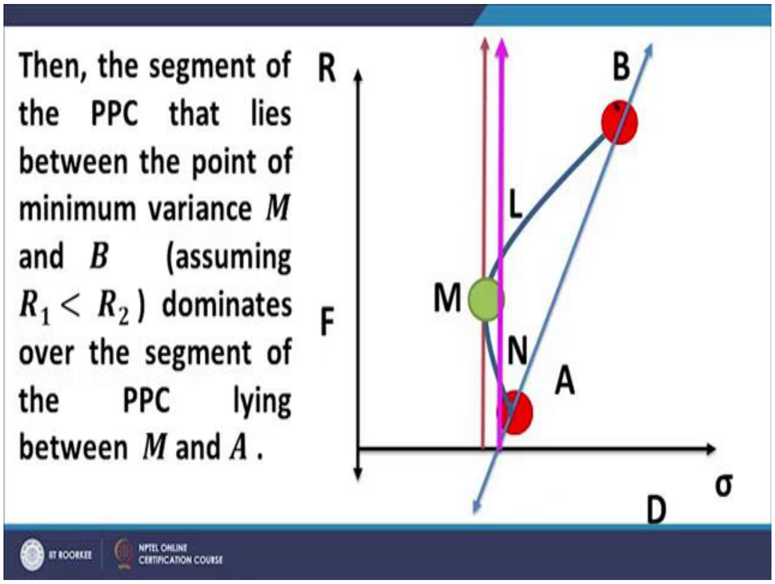

So, let us that means what; that means if you take any point here, any straight line let us look at this diagram. Now, in this diagram if you have any straight line parallel to the Y axis there is only one straight line parallel to the Y axis that is represented or that is tangent to the arc of the hyperbola AB and that is tangent at the point m which is the minimum variance point.

If you take any other line parallel to the Y axis then the points of intersection of that line with the arc of the hyperbola AB extended of course if required in both sides then the points of intersection would either be imaginary or would be two real points. So, for example if you can see it has, this, so the off root is or the take away is that if you have except for that special case when we have the minimum variance point when we have a straight line parallel to the Y-axis through the minimum variance point in which case that line is tangent to the arc of the hyperbola any other line to the right of m, to the right of m would intersect the arc of the hyperbola A B at two distinct points.

What does it mean? Let us say the points of are given by say L and N. Let us say the points are given by L and N. Now, what happens if you look at the point L and N there is a very interesting inference that we have. The inference that we have is that L and N have the same level of risk in terms of standard deviation they have the same level of risk because this line L N is parallel to the Y axis.

Because it is parallel to the Y axis the two points L and N are equidistant from the Y axis that means they have the same level of risk. But if you compare the expected returns of L and N the expected return on L is definitely more than the expected return on N. And that means what; that means that the point L dominates over the point N.

In other words if an investor was to choose between L and N he would invariably choose the point the L because the point L gives you a higher expected return for the same level of risk for the same level of risk as the point N. So, the net result is that all the points that lie on the arc AB extended, that lie on the arc AB extended from the point M onwards along B that is M B and onwards along the arc extended dominate over the points that lie along the arc M A extended.

So, therefore, nobody would be willing to although the portfolio possibilities curve admits points along M A and extended along a extended beyond A along the arc MA although the portfolio possibilities actually allows this but the rationality of the investor would intervene and mandate that the investor invest along the curve M B rather than the curve M A.

In other words the portion M A be of the portfolio possibilities curve with extension becomes totally redundant because corresponding to any point along this curve M A and extensions we have A point along M B an extension which for the same level of risk gives you A much superior expected return. So, this region or this curve M B extended if required is called the efficient frontier.

And all efficient portfolios that is all portfolios which are dominant which cannot be improved upon in A sense under the mean variance framework would lie along this curve M B but if you select A portfolio along M A you can definitely improve the situation by taking A corresponding point on the line parallel to the Y axis which intersects the curve M B at the point of intersection you can take that point of intersection will give you A superior portfolio compared to the point along M A that you have chosen. So, this is the concept of efficient frontier. So, let us quickly read through the theory if there is something which I have missed out.

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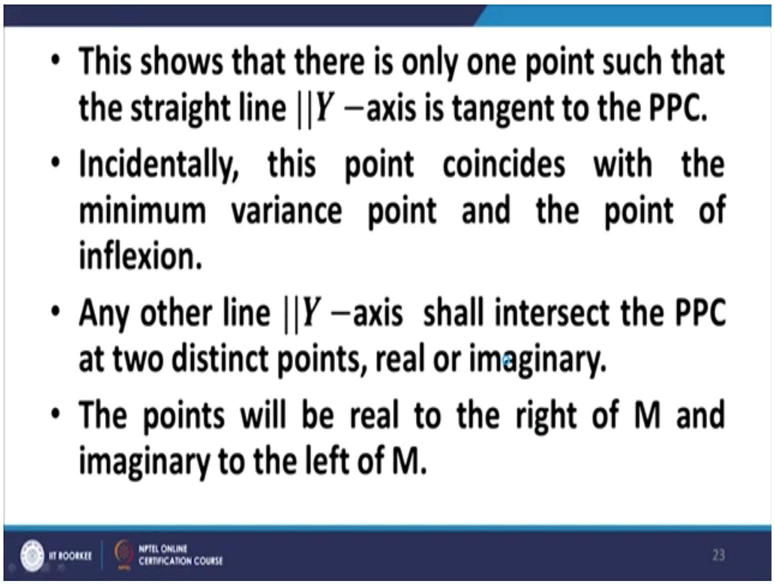

So, this shows that there is only one point which is the minimum variance point such that the straight line parallel to the Y axis is tangent to the PPC. Incidentally this point coincides with the minimum variance point and the point of inflection. Any other line parallel to the Y axis shall intersect the PPC at two distinct points real and imaginary. If you move to the right of M the points will be real as far as the arc AB is concerned, if you move to the left of M the points would be imaginary.

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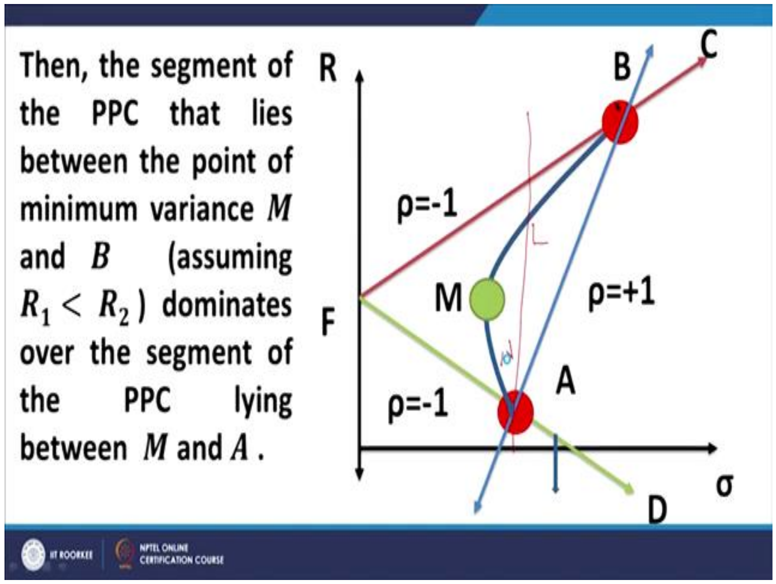

Now, this is what I have shown then the line segment of the PPC that lies between the point of minimum variance m and B and B assuming R1 is less than R2 which is as per the diagram dominates over the segment of the PPC lying between M and A.

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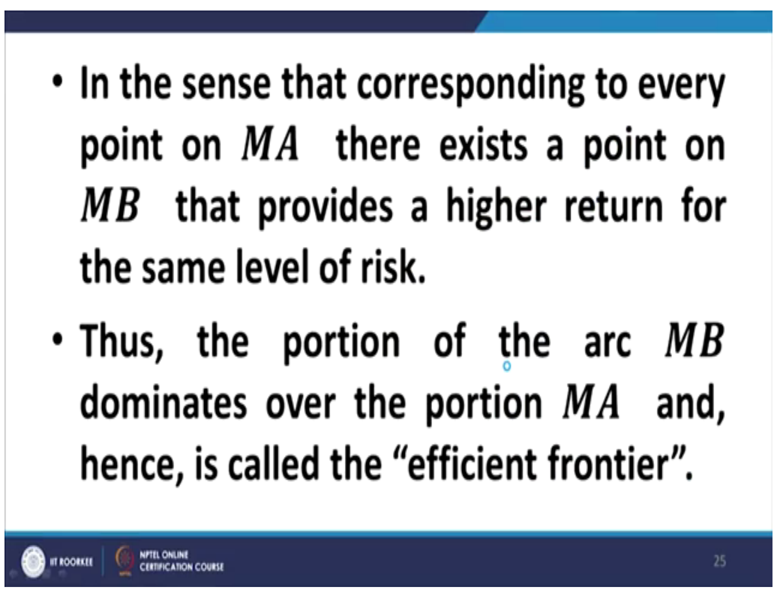

In the sense that corresponding to every point on M A there exists A point on M B that provides A higher expected return for the same level of risk. Thus, the portion of the arc M B dominates over the portion M B and hence is called the efficient frontier. Now, in the next lecture I will take up the efficient frontier in the context of each of the six scenarios that I have discussed in today's class in so far as the portfolio possibilities curve is concerned. Thank you very much.