Security Analysis and Portfolio Management
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Lecture 43
Mean Variance Portfolio Optimization - III

Welcome back. So, let us continue from where we left off but before that as usual A quick recap of the important points that we discussed in the last lecture. We started discussing the mean variance portfolio optimization theory and we defined certain terms and arrived at certain expressions for the portfolio possibilities curve.

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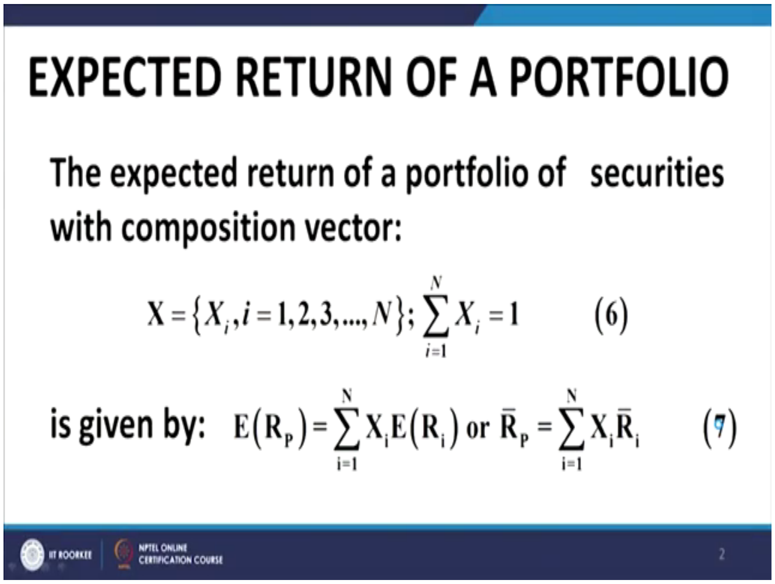

We started with the expected return of A portfolio with the expression that is given in equation 7 here on the slide with for the composition vector which is given in equation 6, composition vector means the weights attached to the various securities that form the portfolio in terms of money values. A fraction of money values of the total that is represented by the various securities that form the portfolio. The variance of the portfolio is defined by equation 8 here on the slide, there are three different ways in which we can express the variance of A portfolio.

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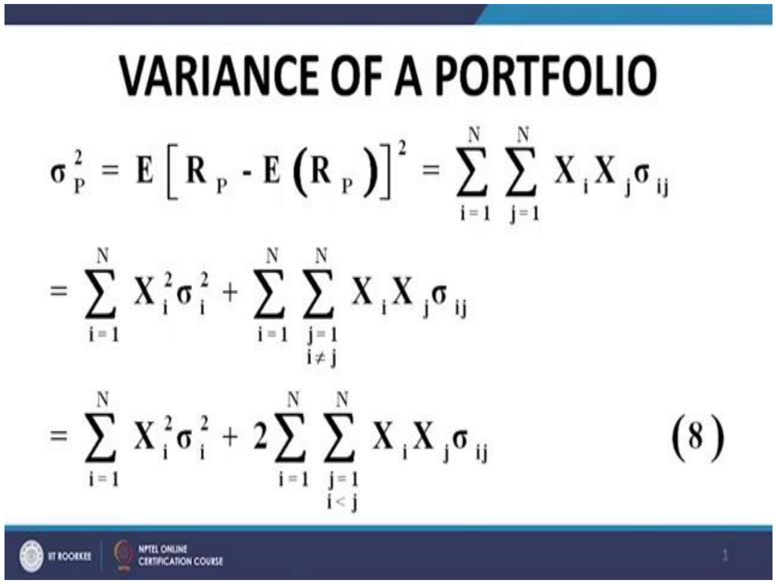

The first one is in the first equation, the second equation and the third equation. So, these three are different ways of expressing the same quantity, the variance of A portfolio of securities.

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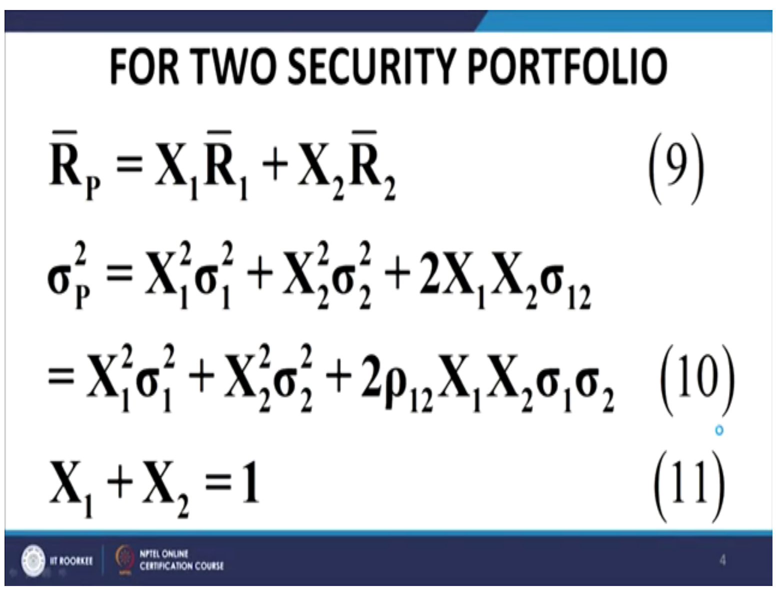

Now, then we moved over to the two-security problem in the two security problem the expressions that we saw just now A few seconds back translate to rather simple expressions in terms of the composition vector of x 1 and x 2 as the equations 9, 10 with the aggregate of the composition vectors being equal to one the sum of the composition vectors being sorry sum of the compositions rather being equal to 1. So, 9 represents the expected return of A two-security portfolio, 10 represents the variance of A 2 security portfolio and 11 gives you the sum of the compositions of the constituents of the composition vector.

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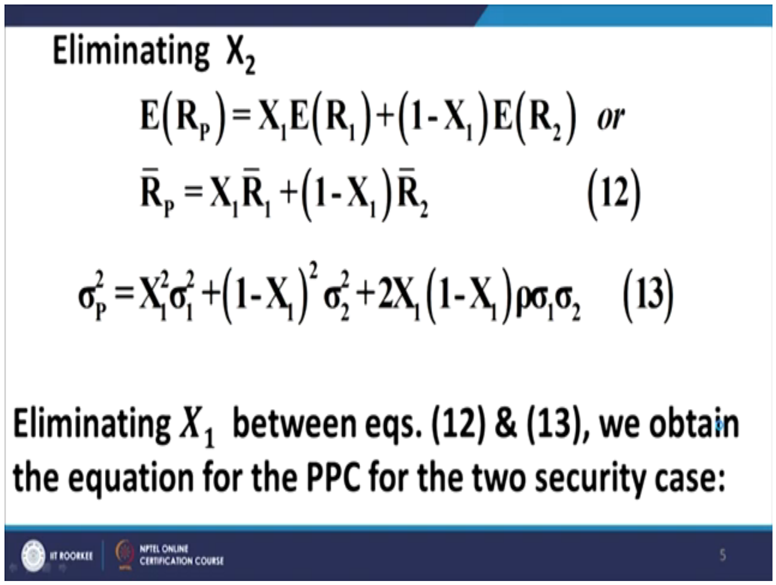

We if we eliminate X2 among these equations among this set of equations what we get is equation number 12 and the next equation gives you the variance of the two security portfolio in terms of X1 and using these two equations again we can eliminate X1 as well so we can eliminate both X1 and X2 and we get the equation of A curve in terms of as A functional relationship between the x the variance or standard deviation of the portfolio and the expected return of the portfolio. With the equation of that curve which represents the functional relationship between the standard deviation of the portfolio which is captured by x here abbreviated by x represented by x and the expected return which is represented by y is given in equation number 14 here on this slide.

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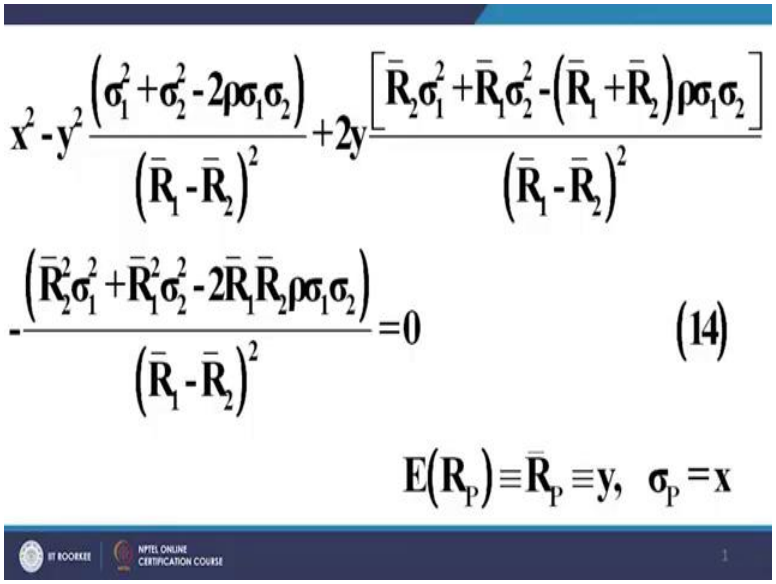

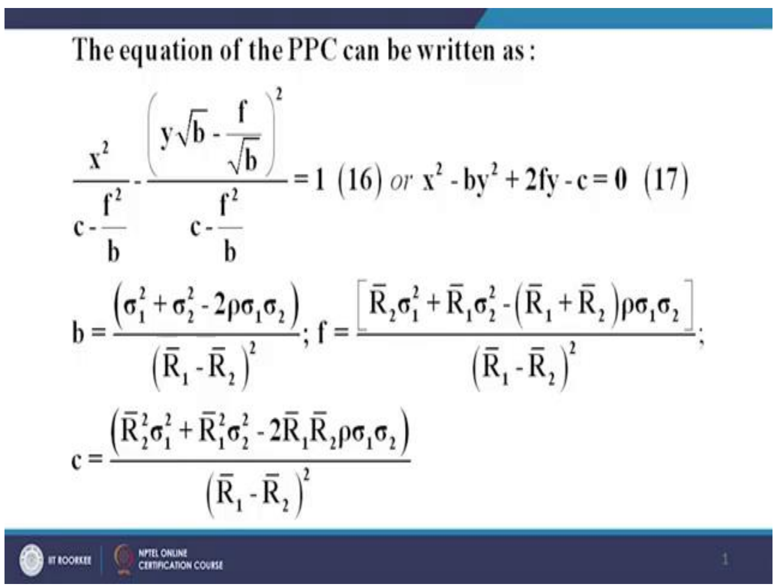

And we saw that this equation can also be written in the standard form of A hyperbola that is x square upon A square minus y square upon B square is equal to 1 which is equation 16 or in the form of A second degree equation as x square minus B y square plus 2 fc plus 2 F y minus C is equal to 0. I repeat x square minus B y square plus 2 F y minus e is equal to 0 which is equation number 17 where B F and C have the values assigned to them as shown in this particular slide.

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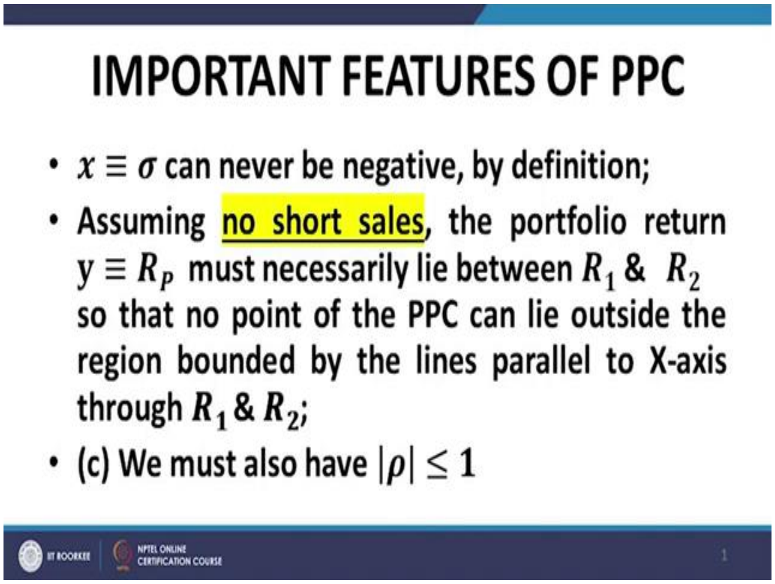

The important features of this portfolio possibilities curve which as I mentioned is A hyperbola represented by the equation that we saw just now equation number 14, first thing is that the portfolio possibilities curve must necessarily be confined to the right half plane because sigma cannot take values, negative values and hence there will be no point on the portfolio possibilities curve which lies to the left or lies in the left half plane. All the, well all the points on the portfolio possibilities curve must necessarily lie on the right half plane that is to the right of the origin with respect to the y axis.

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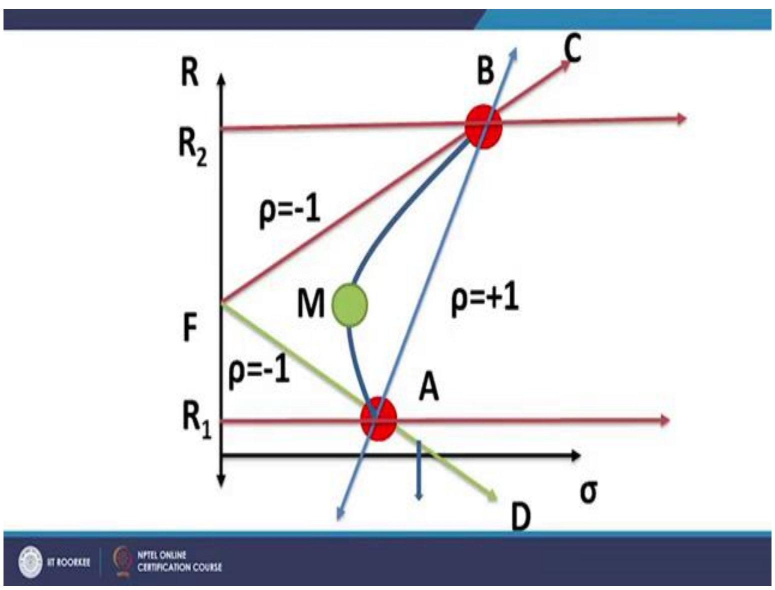

And if you assume that no short sales are allowed if you assume that no short sales are allowed then the portfolio return will be necessarily between the returns or the expected returns of the two securities constituting the portfolio that is E of R1 and E of R2 which i have abbreviated as R1 and R2 and therefore there would be no point on the portfolios possibilities curve that would lie outside the region or which is constrained by the straight lines through the points 0 comma R1 and 0 comma R2 parallel to the x axis.

Now, then we moved over to the limiting cases of rho ah the limiting cases of rho arise from the values of rho equal to plus 1 and rho equal to minus 1, we examine these two cases; rho equal to plus 1 gives us the perfectly correlated securities perfectly correlated assets and rho equal to minus 1 gives us the perfectly anti correlated assets.

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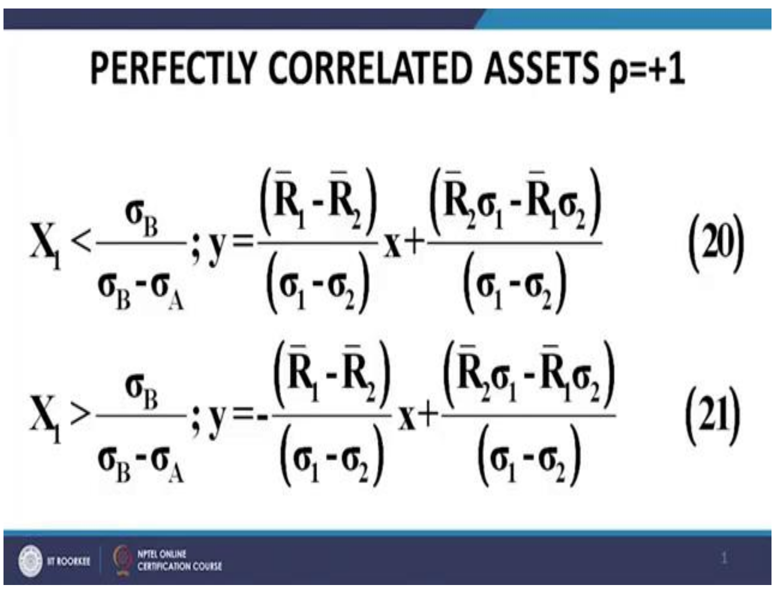

In the case of the perfectly correlated assets what we have is we have the equation of the portfolio possibilities curve represented by A pair of straight lines 20 and 21, the equation 20 would operate in the region where X1 is less than sigma B upon sigma B minus sigma A and equation 21 would operate in the region where X1 is greater than sigma B divided by sigma B minus sigma A. The pictorial representation of the portfolio possibilities curve in the case of rho equal to plus 1 is given in the slide.

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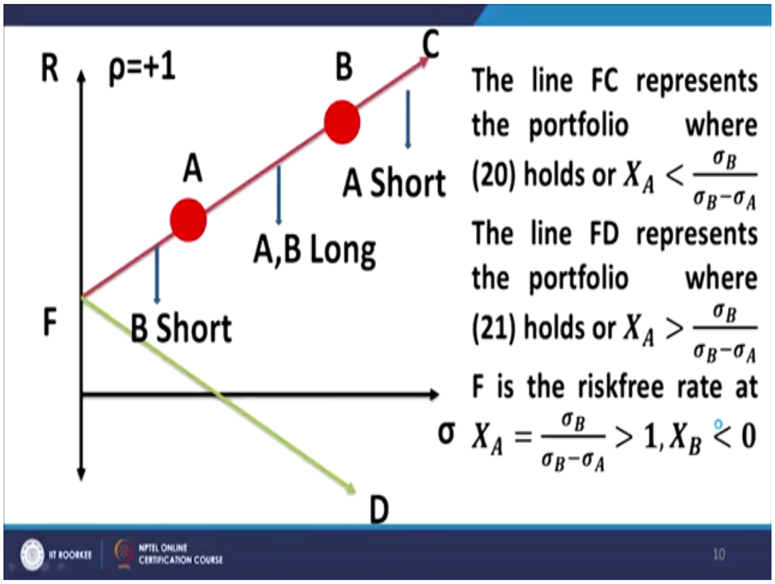

You can see that both the securities A and B lie on the upper straight line that is the straight line represented by equation 20 and the region in the region between A and B both A and A and B are long, both the securities A and B are long beyond B towards C B continues to be long A becomes short and beyond A towards F and along F d in the entire region A F d B is short and A is long.

So, in this case the portfolio possibilities curve translates to A pair of straight lines and the intersecting the y axis at the risk free rate which is given by which is given by R2 sigma 1 minus R1 sigma 1 divided by sigma 1 minus sigma 2 that is represented by the point F in this diagram. In the conclusion as far as this uh situation is concerned.

When we have got two assets, two risky securities perfectly correlated with each other those prices or whose returns are perfectly correlated then this situation or the result is that the portfolio of two perfectly correlated securities will be A pair of straight lines will be represented on A pair of straight lines and risk return space through A common y intercept which represents the risk free rate.

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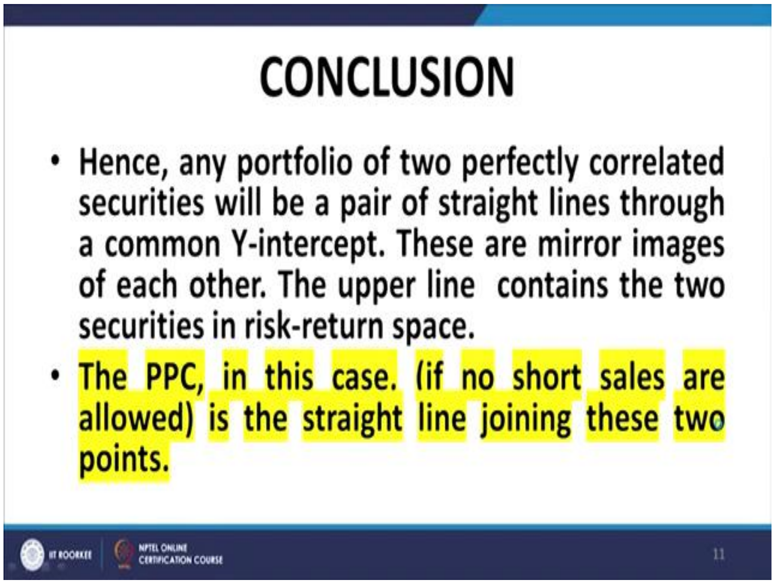

The two straight lines are mirror images of each other, the upper line contains the two securities and risk return space and the portfolio possibilities curve in this region. If no short sales are allowed then what happens? Then the portfolio possibilities curve is constrained or is restricted to the line segment A B joining the two straight line segment line A B joining the points representing the two securities A and B in risk returns space.

Then we move over to the perfectly anti correlated assets for which rho is equal to minus 1 here again we have A situation where we have A pair of straight lines in the region that X1 is less than sigma B divided by sigma A plus sigma B the equation 26 will hold and in the region where x 1 is greater than sigma B divided by sigma B plus sigma A equation 27 would hold.

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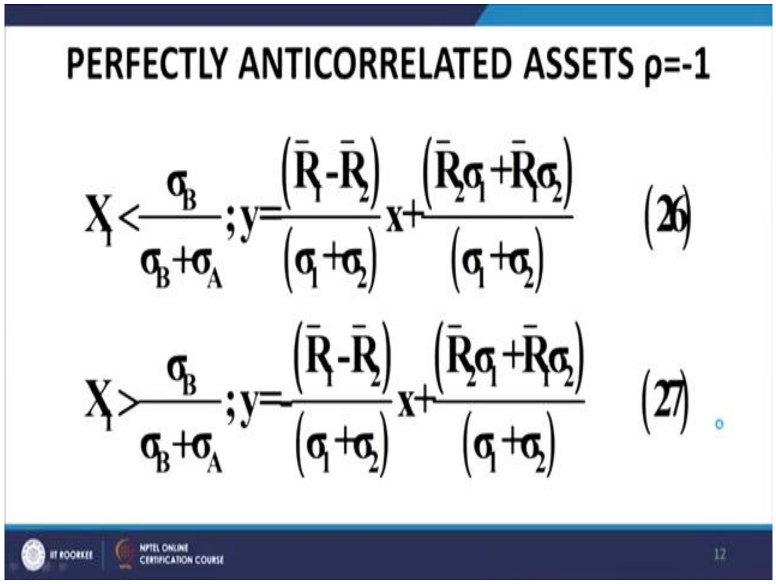

Again you see that these are pair of straight lines which are mirror images of each other about A line which is parallel to x axis through the point of intersection of the two lines with the y axis indeed the y intercept of both these lines is the same point, so the two lines and let us call them B F and A F intersect the y axis at the same point which is given by the point F which represents the risk free asset.

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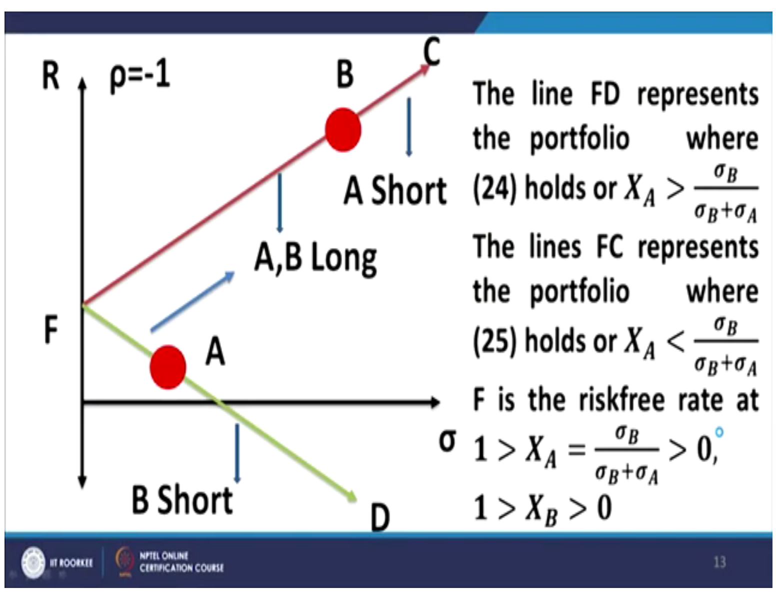

Now, so this is the pictorial diagram of the case when rho is equal to minus 1 when the two securities are perfectly anti-correlated the portfolio possibilities curve represent is represented by the pair of straight lines cf and ft where the region bc represents A situation where A is short and B is long, the region B F A or the portion of the two lines B F and af or fa represents the region where both A and B are long and then there is an A d represents where A is long and B is short.

The risk free rate is given by R1 sigma 2 plus R2 sigma 1 divided by sigma 1 and sigma 2 and this lies between R1 and R2 this is the interesting part, this is the important distinction between the perfectly correlated and the perfectly anti-correlated cases in the end perfect correlated case the risk-free rate was outside the interval R1 and R2 here the respirator lies between R1 and R2.

So, this is the diagram pictorial representation of the case where rho is equal to minus 1. So, A lies on0ft that is the lower, lower, assuming that sigma the security A has A lower standard deviation than security B lower return then security B then A lies on F d and B lies on fc and for the risk free rate as I mentioned there is free rate is given by R1 sigma 2 plus R2 sigma 1 divided by sigma 1 plus sigma 2 and this lies between the returns R1 and R2. This risk free rate is obviously unique.

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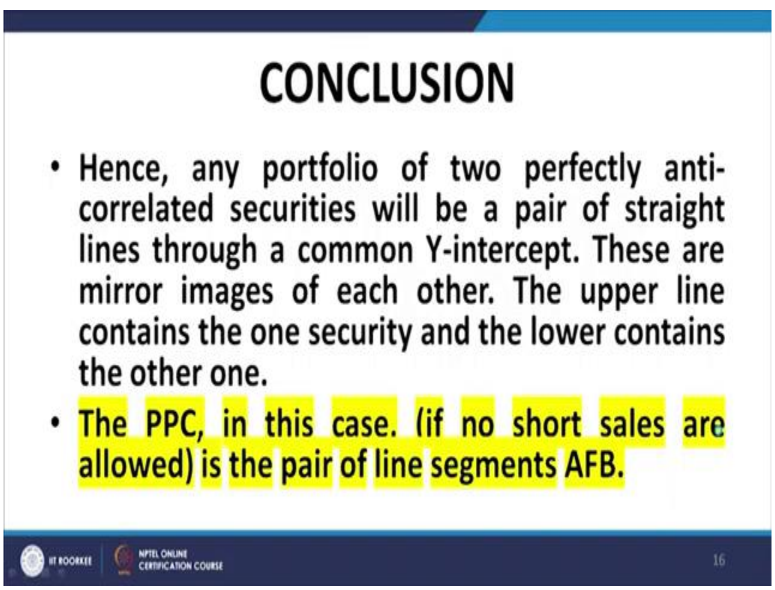

So, to conclude as far as the perfectly anti-correlated securities are concerned any portfolio of two perfectly anti-correlated securities will be A pair of straight lines through the common y intercept these are mirror images of each other and the upper one upper line contains one security and the lower line contains the other security. The portfolio possibilities curve if short sales are not allowed is the pair of line segments AF and FB.

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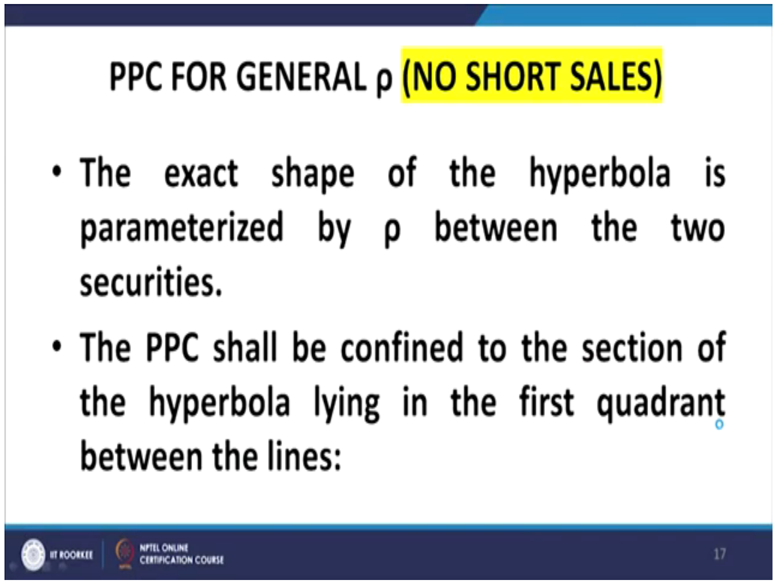

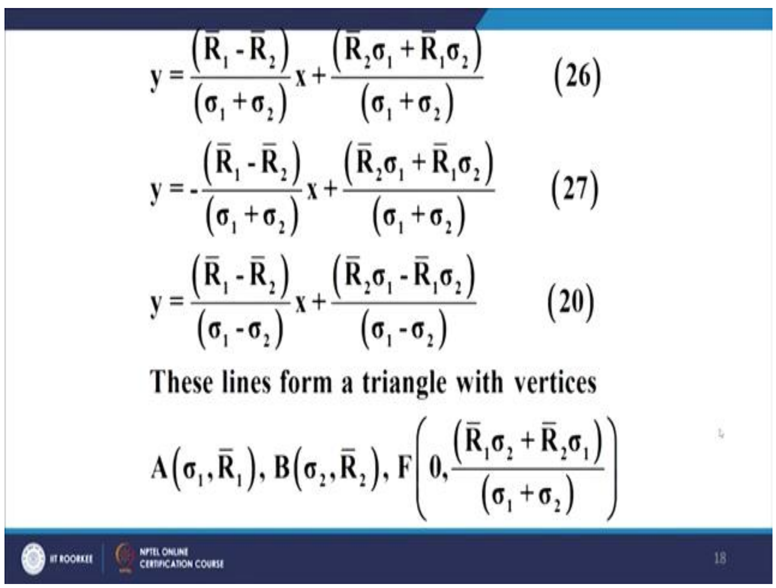

For general A row when no short cells are allowed what happens the exact shape of the hyperbola it is A hyperbola as we have discussed earlier it the exact shape of the hyperbola is determined by the value of rho, the closer the value of rho is 2 plus 1 the lesser would be the bulge and the closer the value of rho two minus 1 the greater would be the bulge.

The portfolio possibilities curve shall be confined to the section of the hyperbola that lies in the first quadrant between the lines that is given by the equation number 26, 27 and 20 and these three equations or the lines represented by these three equations intersect at three points which are given by A which represents one security sigma 1 R1, the B which represents the other security sigma 2 B2 and the risk-free rate which represent which is represented by r1 sigma 2 plus r2 sigma 1 divided by sigma 1 plus sigma 2.

Of course the x coordinate of F is 0 because it is the risk-free point or point representing the risk-free combination and by definition we assume that the standard deviation is A measure of risk and therefore risk free, risk freeness implies A zero value for the standard deviation.

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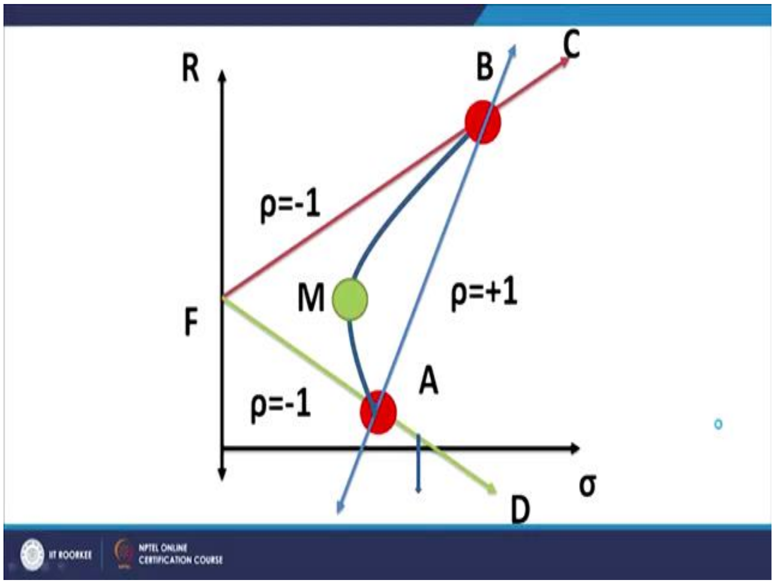

So, this is the diagram of the general portfolio possibilities curve of two securities you can see that m is A point in between these two will come back to the characteristics of m but the basic thing is that the portfolio possibilities curve in the event that short sales are not allowed is restricted to the arc of the hyperbola AMB.

And if short sales are allowed then of course the portfolio possibilities curve would extend in either direction along this arc of the hyperbola beyond B as well as beyond A. So, if we extrapolate the arc of the hyperbola beyond B that would be the region where B is long as short and if you extrapolate the region, extrapolate the arc of the hyperbola beyond A and then it would be the region where A is long and B is short. Then we talked about the portfolio possibilities curve with short sales for allowed which I have just discussed in the case of two risky assets

Now, we move to the agenda for today, we start with A situation where we have two securities we have so far discussed the situation where we have two securities and both are risky securities so both of them have non-zero standard deviations. Now, we talk about the situation where one of the two securities let us say the security B is A risk-free security and because it is A risk-free security by definition or by the assumption that standard deviation is A measure of risk and because B is assumed to be risk free therefore sigma B must necessarily be 0. Furthermore because the risk-free asset has no correlation with any risky security it must also be that rho would be 0 for any security A with the risk free security F, let us call the risk free security F, instead of B.

So, let the asset be renamed F where is free asset so that sigma 2 equal to rho is equal to 0 sigma B is equal to sigma 2 is equal to rho is equal to 0 and R2 is equal to RF, RF may not be 0 please note this point the risk free rate of return need not necessarily be 0 at all.

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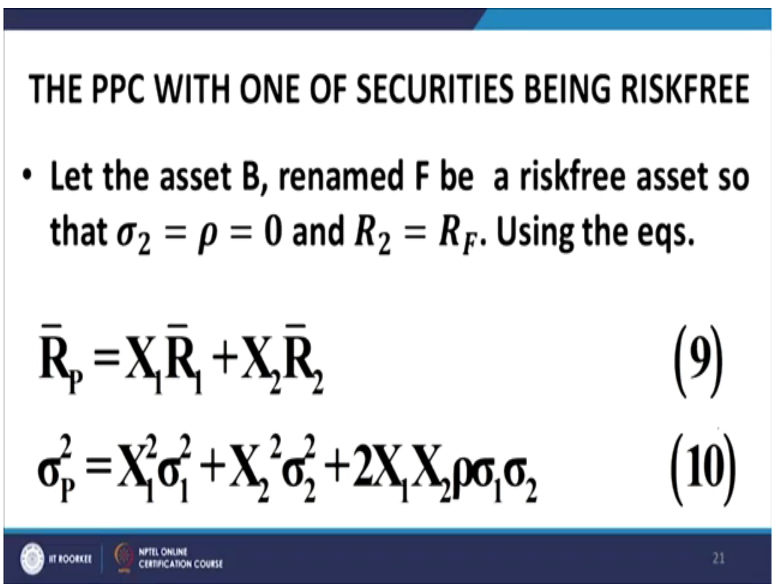

$$\bar{R}_P = X_A \bar{R}_A + (1 - X_1)\bar{R}_F ; \ \sigma_P = \pm X_A \sigma_A$$

$$\bar{R}_P = \frac{\bar{R}_A - \bar{R}_F}{\sigma_A} \sigma_P + R_F \ \text{ when } \ X_A = \frac{\sigma_P}{\sigma_A} > 0 \quad (32)$$

$$\bar{R}_P = -\frac{\bar{R}_A - \bar{R}_F}{\sigma_A} \sigma_P + R_F \ \text{ when } \ X_A = -\frac{\sigma_P}{\sigma_A} < 0 \quad (33)$$

So, in this case the standard portfolio equations become equation number 9 using equation number and 9 and 10 we have the situation where these two equations on eliminating on the composition fraction XA we get equations number 32 and 33. Equation number 32 would hold in the region where XA is positive and equation number 33 would hold in the region where XA is negative.

Furthermore, we can see that the security A, that is the risky security that we have lies on this the line represented by equation number 32 not on the line represented by equation number 33. Now, depending on the sign of RA minus RF it would determine whether which of these securities is upward sloping and which of these securities is downward sloping. If RA is greater than RF then what happens is that the equation 32 or the line represented by equation 32 would be upward sloping and the line represented by equation 33 would be downward sloping and the converse would be the case if RA is less than RF.

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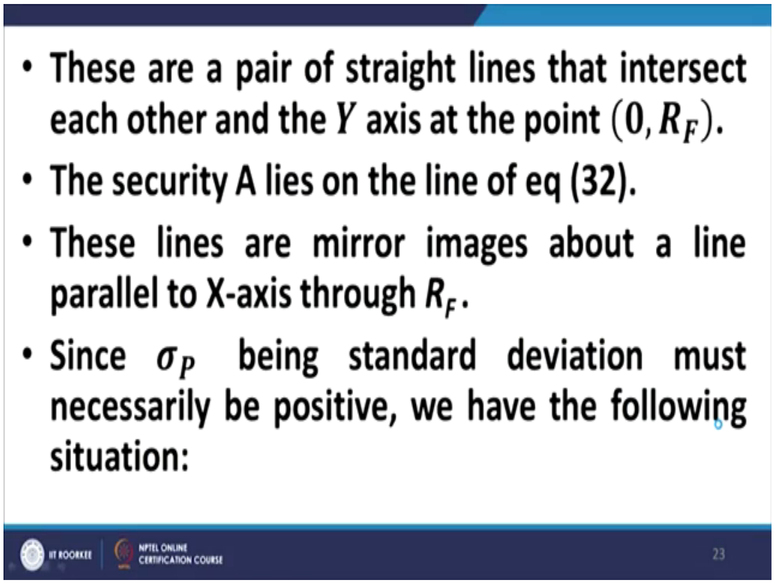

So, this is A pair of straight lines which intersect each other at the and the y axis at the point 0 comma RF, so this is A I repeat these are straight lines it is quite obvious that these are straight lines and the y intercept of both the straight lines is RF so the both these lines intersect the y-axis and among themselves at the point 0 comma RF.

The security A lies on equation 32 I have already mentioned that the lines are mirror images of each other about A line parallel to the x-axis through RF because this slopes are inverse of each other since sigma P being standard deviation must necessarily be positive we have the following situation.

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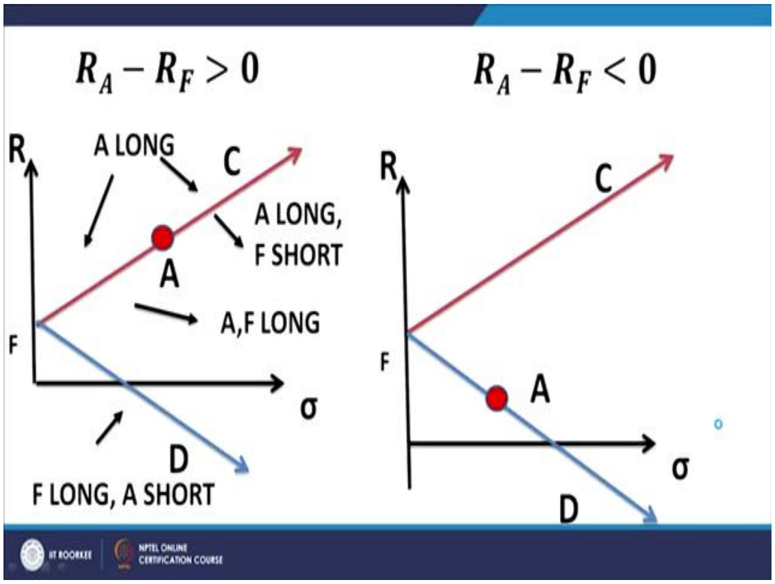

If RA minus RF is positive and then what happens that the equation number 32 would be upward sloping and equation number 33 would be downward sloping and as I mentioned security A lies on the line represented by equation 32 and therefore we have the left hand side figure in out of this pair of figures and as far as identifying the and the portfolio composition along the various points of various segments represented in this figure AF region holds or the portfolio would lie in the region AF if both A and F are long.

In other words we have risk-free lending coupled with investment in the security A along AC the security A remains long whereas we have risk free borrowing and along FD we have risk free lending at the cost of being short in securities. So, you are short selling security A and you are investing the proceeds in the risk-free asset that is represented by the section FT extended of course.

And if RA minus RF we have the opposite situation as depicted in the second figure or the right hand side figure on this slide. So, in the region A F both A and F are long let me repeat both A and F and longer in the region af along ac extended indefinitely A is long F is short; F is short means we have risk free borrowing and along F d extended we have A is short and F is long in other words we have, we are short selling A and doing risk free landing.

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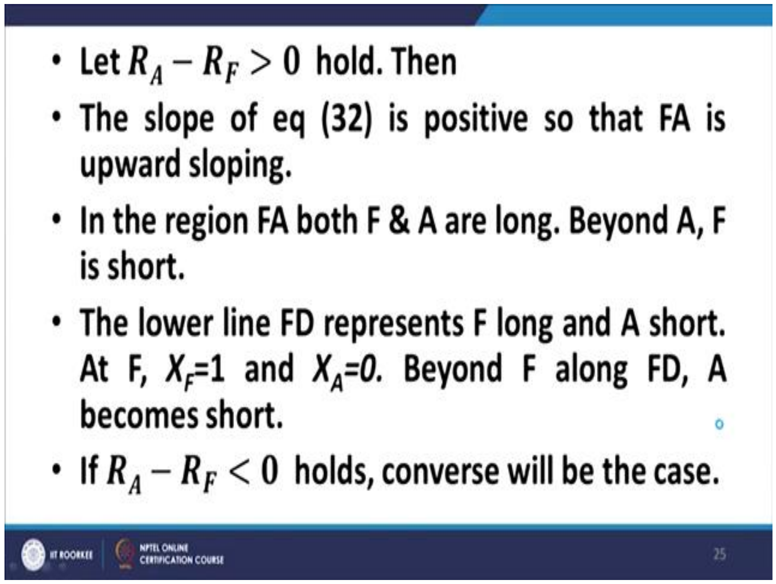

So, this is the description of that figure, the slope of equation 32 is positive so that fa is upward sloping as you saw in the diagram. In the region F F A both F and A are long beyond A F is short and the lower line represents F long and A short as I mentioned at F x F is x F is equal to 1; x A is equal to 0 beyond F and along fd A becomes short. If RA minus RF is less than 0 the converse will be the case as represented in the right hand side figure on the previous slide.

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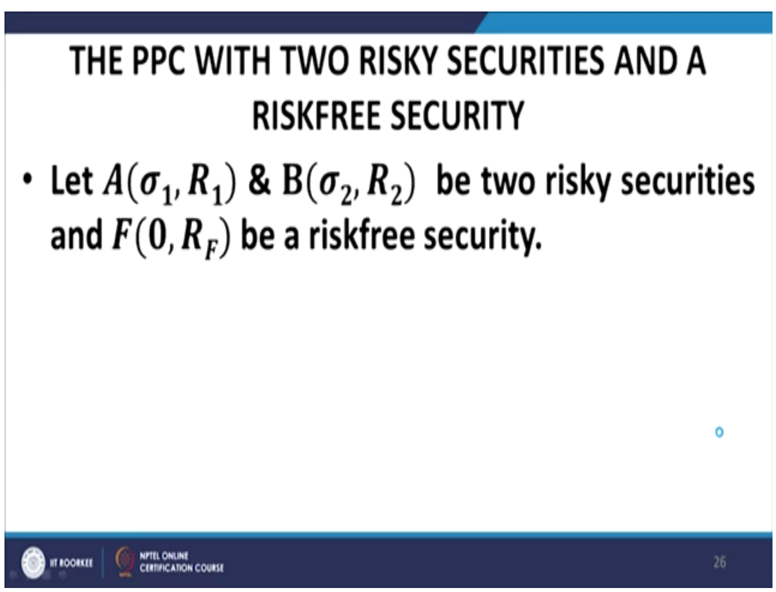

Now, we come to the nuances of portfolio so far we have been addressing the foundations of portfolio optimization or rather the shape and structure of the portfolio possibilities curve in rather simple simplistic situations we now get into the nuances of this theory. We now consider the portfolio possibilities curve or the portfolio possibilities region as you will soon find it to be with two risky securities and A risk-free security.

So, let A sigma 1 R1 be and B sigma 2 R2 be two risky securities and F 0 comma RF be the risk-free securities. Please note we are now considering the case where we have two risky securities and A risk-free security; two risky securities and A registry security and A sigma 1 R1 B sigma 2 R2 are the two risky securities and F 0 comma RF is the risk free security, RF is obviously the risk free rate of return. And this is the diagram which emanates when we you can see the complexity relative to the diagrams that we have observed earlier but I will explain each and every segment of this, this will take some time but let us start going through it.

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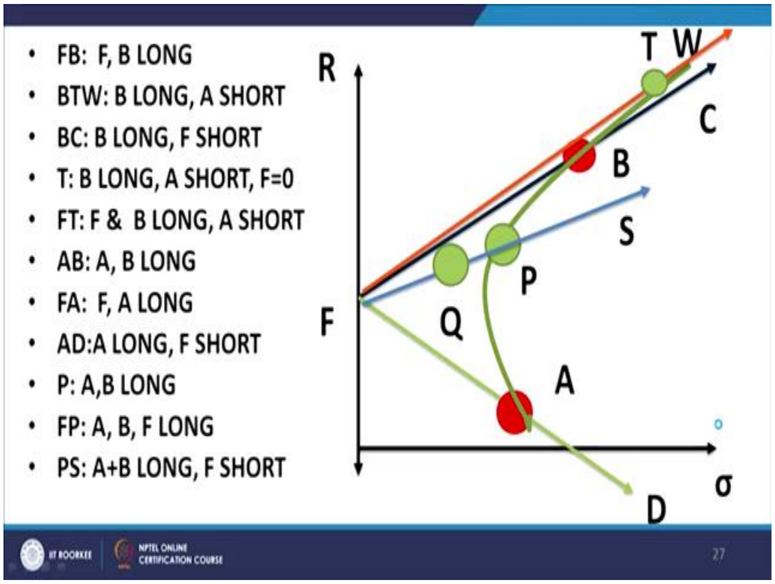

A and B obviously represent the two securities and F represents that is free asset, risk free securities please note we are having A situation where rho can be take and can take any values within minus 1 and plus 1 with excluding minus 1 and plus 1 because we have already discussed the cases of the two securities being perfectly correlated and perfectly anti-correlated.

So, in the region F B if you look at F B, now let me first explain this B C is the extension of F B, F B is the straight line joining F B and this is being extended along C. Now B T W; B T W is the extension of the hyperbolic arc that joins A and B. So, A P B T W is the arc of the hyperbola as far as d A is concerned it is the extension of straight line F A, so F A d is A straight line, P is an arbitrary point on A B.

S is an arbitrary point on fb extended and Q is an arbitrary point which lies between P and F. So, this is the description of the various symbols that we have on the slide. Now let us see what each of these lines and the symbols represent. So, as far as F B is concerned it is quite obvious that it has to be A linear combination of securities F and B with both securities being long, x F and x B will both be between 0 and 1 in the region F B.

Now, what about B T W? Along B T W, B T W is the extension of A B along the arc of the hyperbola A P B this continues onwards to T W. Therefore, because T and W lie on this arc of the hyperbola they will be combinations of A and B only. In fact beyond B if you go along B T W then B becomes longer B continues to be long but A starts becoming short at the point B; x B is equal to 1; x A is equal to 0 and there is of course no x F.

But as you move along B T W; along B T W along the hyperbolic arc I reiterate then it x A starts becoming short, x B continues to be long with being greater than 1 x B being greater than 1 but x F continues to be 0 because there is no involvement of F in so far as this hyperbolic arc is concerned. In other words I will come back to it except for the point T, there would be no point on this arc A B and even at the point T of course there be no point here where x F would be having A positive or negative value other than 0 along the arc of the hyperbola A B T W.

So, B T W represents B long and A short; B C now B C is what? B C is the extension of the straight line joining F P, so at the point F we have x F equal to 1, x B equal to 0; at the point B we have x B equal to 1, x F equal to 0 and if you move along the same straight line to along B C then x B becomes greater than 1 and x F becomes less than 0 which means that we are having risk-free borrowing.

So, what you are doing is along bc you are borrowing risk-free and you are investing in this security B. Then at the point T, what is the point T? First let me explain the point T is the point at which with the tangent is drawn from the point F tangent is drawn from the point F to the hyperbolic arc A B extended if required. I repeat T is the point at which we draw A tangent from the point F to the hyperbolic arc A B extended.

So, at the point T, F T is tangent to the arc A B T W. So, at the point T we have A long because it lies on the hyperbola A B so we have A long we have B I am sorry we have A short, we have A short we have B long and we have no involvement of risk-free landing or borrowing but beyond T if you go along the straight line F T beyond T then short risk free borrowing starts and then you have risk free borrowing together with the long position in B and the short position in A.

So, you have short positions in F and short position in A and A long position in B if you move along F T straight line extended. Where F T is the tangent I repeat F T is the tangent to the hyperbolic arc A B extended. So, F T, F and B long along the line F T of course what will you have; you will have you see now please note this interesting point along the line F T what will you have? You will have linear combination of F and T, at the point F, x F is 1; x T is equal to 0, at the point T; x F is 0, x T is equal to 1.

But what is x T? X T is x T lies on the arc of the hyperbola AB therefore it must be A combination of A and B with because it is extended beyond B that means x B and x A would be in such A proportion that x B is long and x A is short. So, along F T up to T F and B are long and A is short and what about A B? I have already discussed A B is the arc of the hyperbola and at any point along the hyperbolic arc we shall have A and B together either A long or B long or one of them long and one of them short.

But within A and B, along the r k B within A and B there would be no short sales and both A and B would be long. If you extend beyond B along the arc then B becomes A long A becomes short if you go beyond A along the arc then A becomes long B becomes short. So, what about F A; F A is the straight line where it is A linear combination of A and F so obviously in the segment F A, A is long F is also long in other words you have A combination of risk-free lending and investment in security.

But if you move beyond A along A d then what happens then A becomes continues to be long but F becomes short in other words you have risk-free borrowing and the investment of the borrowed proceeds in security A. AD I have discussed the what about the point P, now this is interesting, point B is an arbitrary point between A and B along the arc A and B I repeat point P is an arbitrary point along A B within A and B that is along the arc AB and it lies somewhere within A and B.

So, it must be A combination of securities A and B with both securities being long. Now, what about F P? F P is A straight line which is bounded by F and P so it has to be A linear combination of the portfolios represented by F and P that means at the point F obviously we have x F equal to 1, x P equal to 0 at the point P we have x P equal to 1 x F equal to 0 and if you take any arbitrary point Q in between F and P then it would be A linear combination of F and P.

But P itself is A combination of securities A and B both long therefore at any point Q which lies between F and P both or all the three securities F A and B would be long and please note this is any arbitrary point within the region B F A bounded by the lines line segments BF bounded by the line segment AF and bounded by the arc A B.

So, if you take any point bounded by these three points A, F, F B and the arc A B then you have A combination where all the three securities would be long. So, this region of space lying between F A, F B and the arc A B will represent combinations where all the three securities are long. Now, what about the point is S? S is the point outside the arc towards the right of the arc A B.

Now, obviously at point P we have x P equal to 1, x F equal to 0 but if you move along F P beyond P then what happens is that x F becomes negative x P becomes greater than 1 so the point S represents the risk-free borrowing and the investment of the proceeds in the combination represented by the point P. So, at the point P what do we have? I am sorry at the point S what do we have?

We have short in F and long and the securities A and B. So, I have discussed almost all these regions and all these positions of various points on this diagram it is very interesting and it will help us in understanding what is to follow from here on.

Now we can have different scenarios as I explained in the diagram previously we have different situations where one security can be long, the other can be short or two can be long and the third can be shorter all three can be long so on that basis I have classified the portfolio composition into different scenarios we have six scenarios here.

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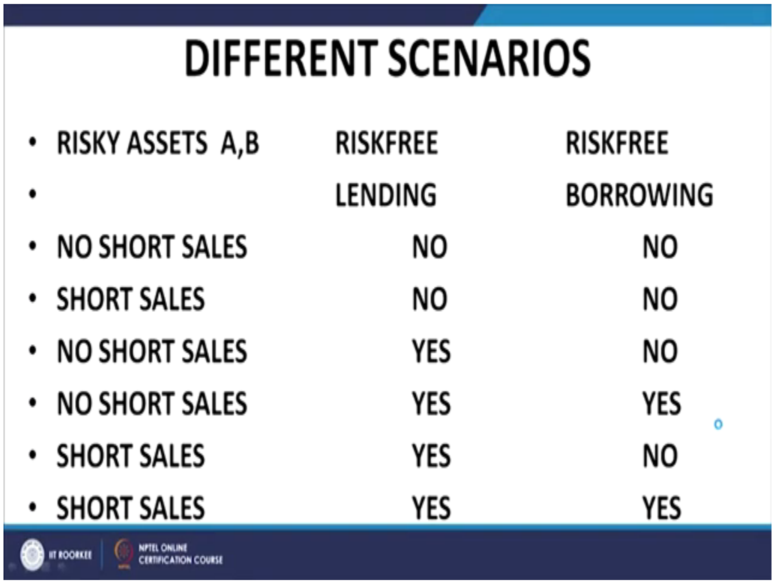

First of all, we have A situation where no short sales are allowed in A and B and there is no risk-free lending or borrowing then we have short sales in A and B but no risk-free lending and borrowing. Then we have no short sales in A and B but we do have risk free lending; then we have no short sales in A and B but we do have this free lending and borrowing then short sales with risk free lending but no borrowing and then finally the most general situation where we have short sales allowed as well as risk free lending and borrowing allowed.

So, let us take the first case now. In the first case we have no short sales allowed no risk-free asset as I mentioned earlier in this case the portfolio possibilities curve we have discussed in it in A lot of detail it will be the arc of the hyperbola confined between the points A and B. So, I repeat in this situation where there are no short sales in A and B and no risk-free asset is there that means there is no risk-free landing, there is no risk-free borrowing in that situation the point representing the portfolio will lie on the arc A and B.

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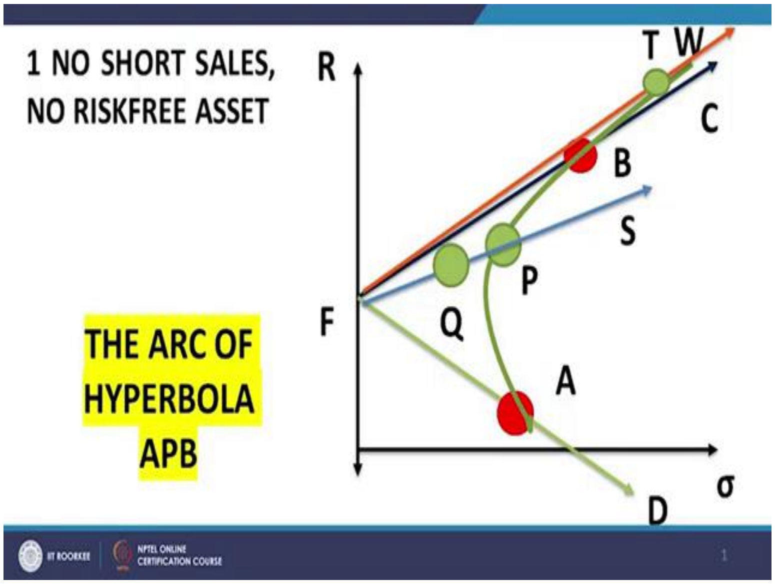

For example, the point P P is A typical point where we have combination of A and B both long with no involvement of the risk-free asset F. So, in the case of no short sales with no risk-free asset lending or borrowing either case we the portfolio possibilities curve is the arc of the hyperbola lying between the two securities A and B.

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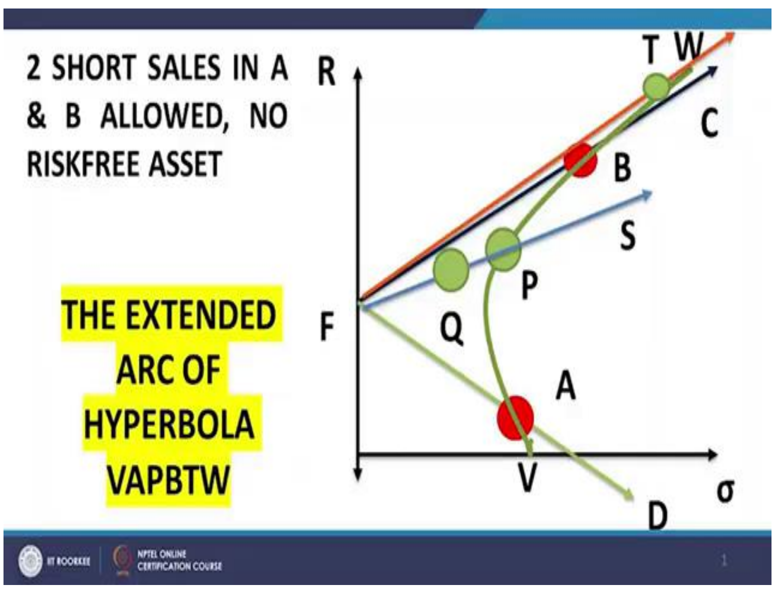

Then we move to case 2; in case 2 what do we have? Short sales in A and B are allowed but again we do not have A risk free lending that means no risk-free lending no risk-free borrowing but we do have uh short sales in A and B allowed. In this case what happens is the portfolio possibilities curve gets extended along the arc of the hyperbola beyond A and beyond B as well.

So, in other words the entire arc of the hyperbola A B extended in both sides extended beyond A along the arc of the hyperbola and extended beyond A as well along the arc of the hyperbola will form the portfolio possibilities curve. If you take A point beyond B on the arc of the hyperbola A B it will represent B long and A short and if you take any point beyond A along the arc of the hyperbola you will find it to be A long and B short. And of course, along A B, along the arc in the region A B both A and B would be long. Now, we take the next situation, where no short sales in A and B are allowed but risk-free lending is allowed. In this case what happens is I have explained briefly this situation the portfolio possibilities region you know this is not A curve it is A region of space.

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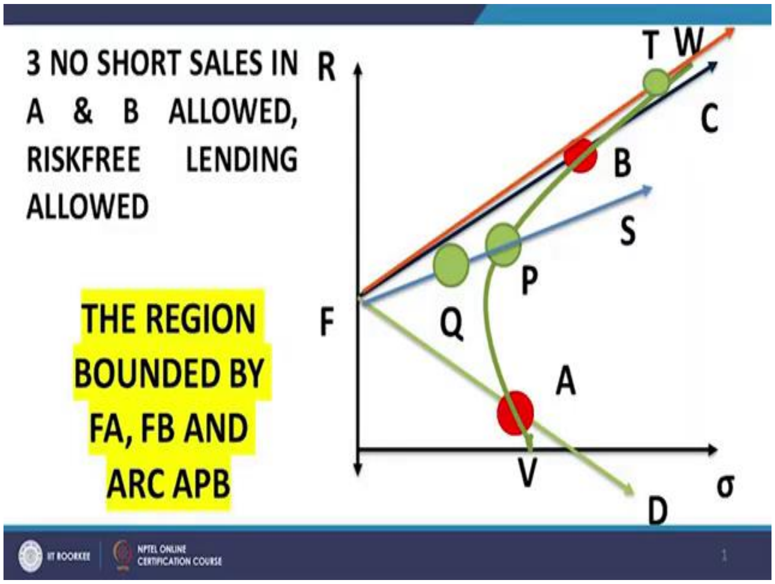

If the region of space is the space which is bounded by the straight lines or line segments BF or F P F A and the arc of the hyperbola A B, so this region which is bounded by these three to this pair of straight lines and the arc A B constitutes the feasible region constitutes the portfolio possibilities region in A situation where A neither any short sales are allowed nor risk-free borrowing is allowed but this will lending is allowed.

For example if you take any arbitrary point Q here, Q is A linear combination of F and P and P itself is A combination of A and B both being long please note because P lies between A and B therefore P is A combination of both A and B, both being long. Because Q lies in between F and P both F and P are long in Q and P is log in A and B so the net result is that for Q being any arbitrary point between B F A the three securities all the three securities would be long. We shall continue from here after the break. Thank you.