Security Analysis and Portfolio Management
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Lecture 42
Mean Variance Portfolio Optimization - II

Welcome back. So, before the break we discussed the portfolio possibilities curve of two risky securities and we found the expression.

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$$x^2 - y^2\frac{\left(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2\right)}{\left(\bar{R}_1-\bar{R}_2\right)^2}+2y\frac{\left[\bar{R}_2\sigma_1^2+\bar{R}_1\sigma_2^2-\left(\bar{R}_1+\bar{R}_2\right)\rho\sigma_1\sigma_2\right]}{\left(\bar{R}_1-\bar{R}_2\right)^2}$$

$$-\frac{\left(\bar{R}_2^2\sigma_1^2+\bar{R}_1^2\sigma_2^2-2\bar{R}_1\bar{R}_2\rho\sigma_1\sigma_2\right)}{\left(\bar{R}_1-\bar{R}_2\right)^2}=0 \qquad (14)$$

$$E\left(R_P\right)\equiv\bar{R}_P\equiv y, \quad \sigma_P=x$$

The explicit expression for the portfolio possibilities curve to be equation 14 which is shown on your slide. Now, the important thing is that all feasible portfolios for all viable combinations of these two securities one and two will lie on this particular curve which is A hyperbola. Furthermore, we shall be confining ourselves to the section of the hyperbola that lies on the right half plane because sigma must necessarily be positive. The standard deviation of the portfolio must necessarily be positive by definition then we move on to discuss the or we moved on to discuss the various features of this particular hyperbola or the section of the hyperbola and now what we do is we take up the situation in the two extreme cases.

We study the two extreme cases where rho is equal to plus 1 and rho is equal to minus 1. What is the situation or what kind of changes emerge when the two securities are either perfectly correlated or perfectly anti-correlated. So, these are you see this this study of this particular cases of these specific cases of these extreme cases is very important why? Because they give you A feel of what to expect ah for the remaining values of the correlation coefficient for values which lie between this extreme (struct) extreme values of minus 1 and plus 1.

So, let us first tackle the case of rho equal to plus 1 that is the two securities are perfectly correlated. Now, when the two securities are perfectly correlated the expression for the variance becomes A perfect square and therefore we can take the square root of both sides and what we get is the standard deviation is given either by equation 18 or by equation 19 which of these two equations will hold will depend on the sign of the expression.

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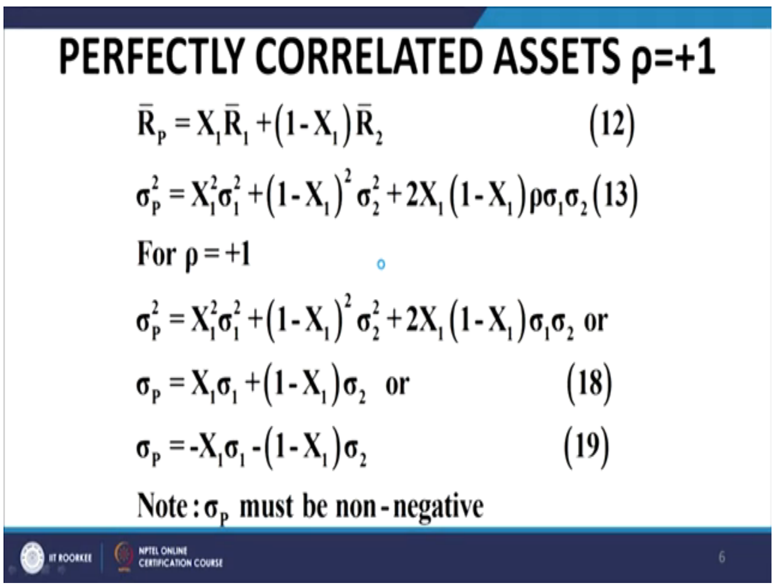

If the sign of the expression is positive if the sign of 18 is positive then 18 would hold and if the sign of 19 is positive then 19 will hold. So, either 18 would hold or 19 would hold depending on the sign of the equation X1 sigma 1 plus 1 minus X1 into sigma 2. So, if this is positive and then we or to the extent that this is positive equation 18 holds and when the 18 becomes negative or the right-hand side of 18 becomes negative then equation 19 takes over.

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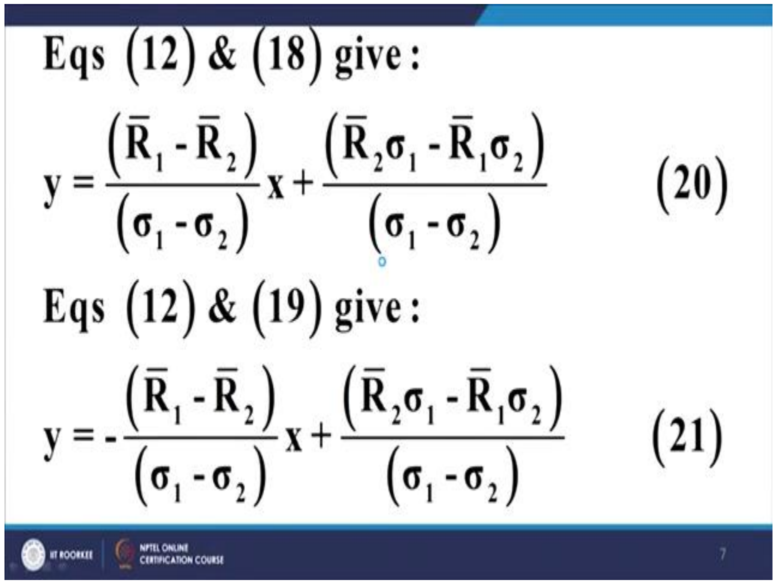

So, if the right-hand side of equation 18 is positive then eliminating X1 between 12 and 18 what we get is the expression that is in equation 20. And if equation 18 turns out to the right hand side of equation 18 is negative so that equation 19 holds together with equation 12; you eliminate X1 between equation 12 and equation 19 you get what is given in equation 21.

Now, there are certain interesting features about these two equations. The first is both of them are straight lines, you can easily see that their first-degree equation and therefore they are straight lines the and the second interesting feature is that the Y intercept in both cases is the same. They intersect the Y axis at the same point which is given by R2 sigma 1 minus R1 sigma 2 divided by sigma 1 minus sigma 2.

And the third observation is that the slopes are negative of each other that means they are mirror images of each other about A line which is parallel to the X axis and which passes through the point R2 sigma 1 minus R1 sigma 2 upon sigma 1 minus sigma 2. So, these are the three features of the portfolio possibilities curve now it is not A curve it is A pair of straight lines in fact A portfolio possibilities come as become A pair of straight lines in the event when the securities are perfectly correlated. The first equation holds in the region, in the region where the equation 18 holds and the second equation holds in the reservoir equation 19 holes.

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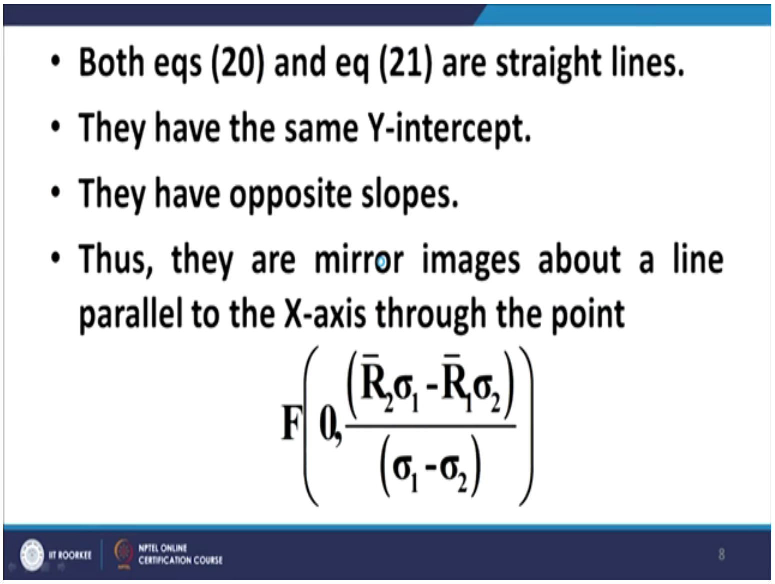

And so, these are the salient features of the this, the case where we have the, we have the perfectly correlated securities both 20 and 21 are straight lines, they have the same Y-intercept, they have opposite slopes and they are mirror images about A straight line parallel to the X axis through the point F where the ordinate of F is 2R sigma 1 minus 1 R1 sigma 2 divided by sigma 1 minus sigma 2.

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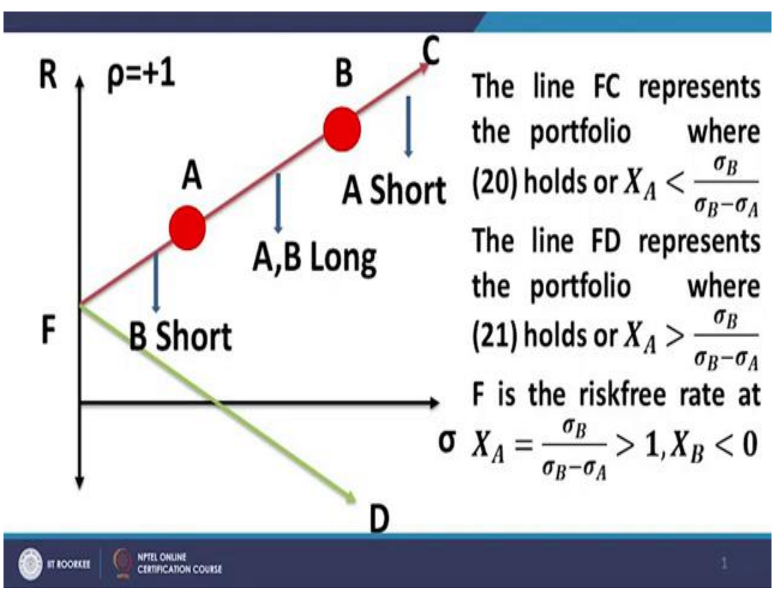

This is the figure that is there and please note both A and A both the securities A and B lie on the security the straight line represented by equation 20. If you put X equal to sigma 1 or you put X equal to sigma 2 you get R1 and R2 on the left-hand side as can be easily checked by explicit calculations. So, both the securities A and B lie on the on the straight line which is represented by equation 20 as is shown in this diagram.

So, the line fc represents the portfolio when 18 holds and 18 will hold in the region where X is less than sigma B divided by sigma B minus sigma A. In this region where so long as X is less than this figure the equation fc will hold and the line fd will hold F t will be the portfolio possibilities region when xa is greater than sigma B upon sigma B minus sigma A. F is the risk-free rate it is the intersection of the two lines with the Y axis which represents 0 standard deviation and therefore the risk-free situation.

Then this this intersection occurs when X A is equal to sigma B divided by sigma B minus sigma A which is obviously greater than 1 and therefore XA is long and XB is short at this particular point F when the two securities are ah perfectly correlated. Now, as far as the regions are concerned the first thing is therefore what we infer is that in the case of perfectly correlated securities the portfolio possibilities curve is A straight line joining the two securities and the mirror image of that line.

And between A and B both A and B are long beyond B A becomes short B becomes greater than 1 and beyond A that is towards F and along F D A becomes long greater than 1 and B becomes short so that is the distribution of the securities A and B in at various points on this portfolio possibilities pair of straight line.

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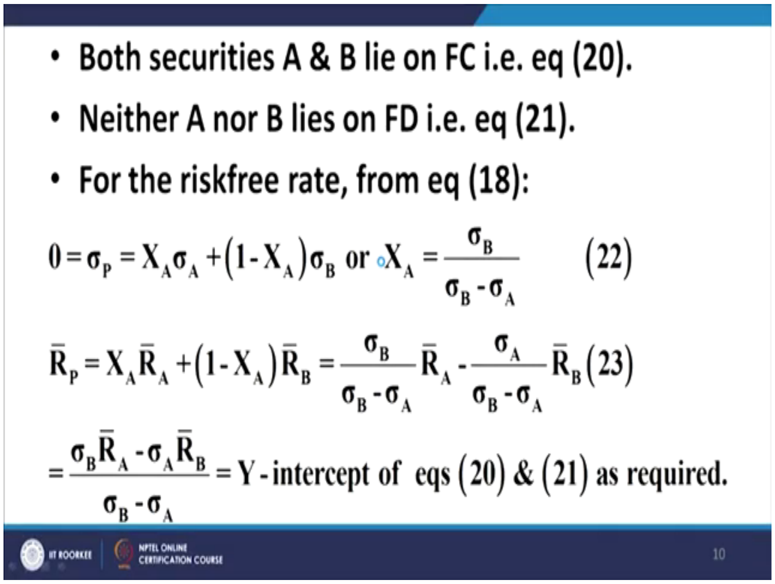

So, let us recap both securities A and B lie on fc that is equation 20 neither A nor B lies on F D that is 21 for the risky rate as I mentioned just now we can solve and simplify and what we find is that the risk-free rate is equal to sigma B RA minus sigma A Rb divided by sigma B minus sigma A which is the Y intercept of equations 20 and 21 as required. So, this is the derivation of the risk free rate which occurs when sigma A is equal to sigma B divided by sigma B minus sigma A which is greater than 1 and therefore sigma B is less than 1.

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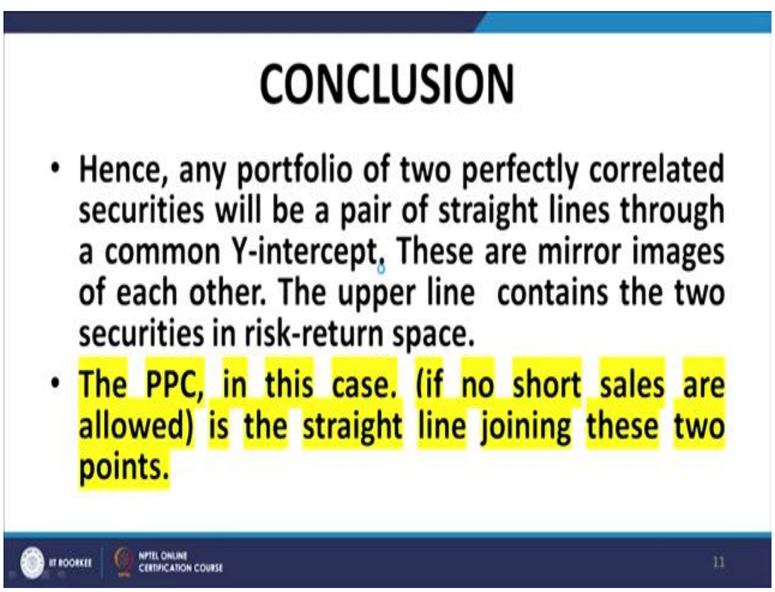

Hence to conclude to summarize any portfolio of two perfectly correlated securities will be a pair of straight lines through a common Y-intercept these are mirror images of each other, the upper line contains the two securities and risk return space. The portfolio possibilities curve in this case if no short sales are allowed; if no short cells are not allowed please note with a straight line joining these two points. This is easily seen from the diagram let us go back.

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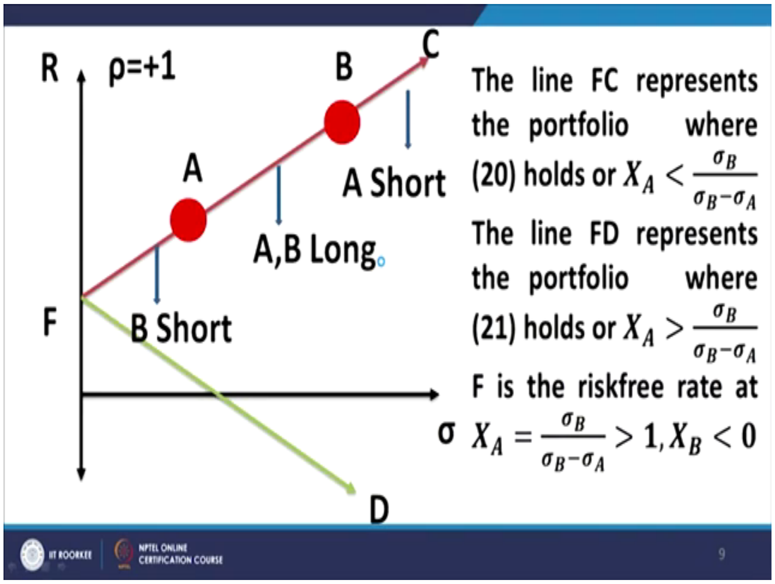

In this diagram as you can see between A and B there are not no short sales both A and B are long but as you move towards c or you move towards F and along F t the securities B and security A becomes short and security B becomes short respectively.

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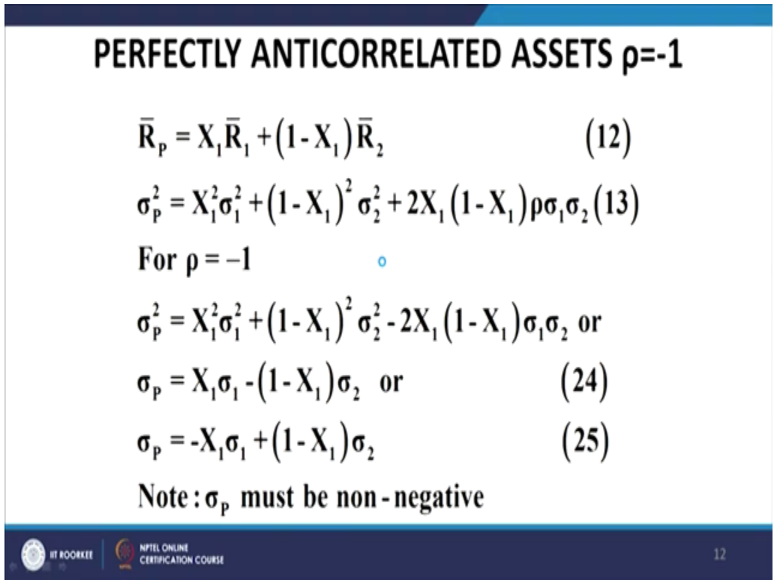

Now, we look at perfectly anti-correlated securities that is rho is equal to minus 1. Again. we start with the basic equations of expected return of a portfolio and the variance and what we find is that in this case again the variance of the expression for the variance becomes a perfect square and therefore we can take square roots of both the sides and what we end up with is equation number 24 and 25. So, in the region where equation number 24 holds we will have; in the region where the right-hand side of equation number 24 is positive equation 24 will hold and in the region where the right hand side of equation number 25 is positive; 25 will hold.

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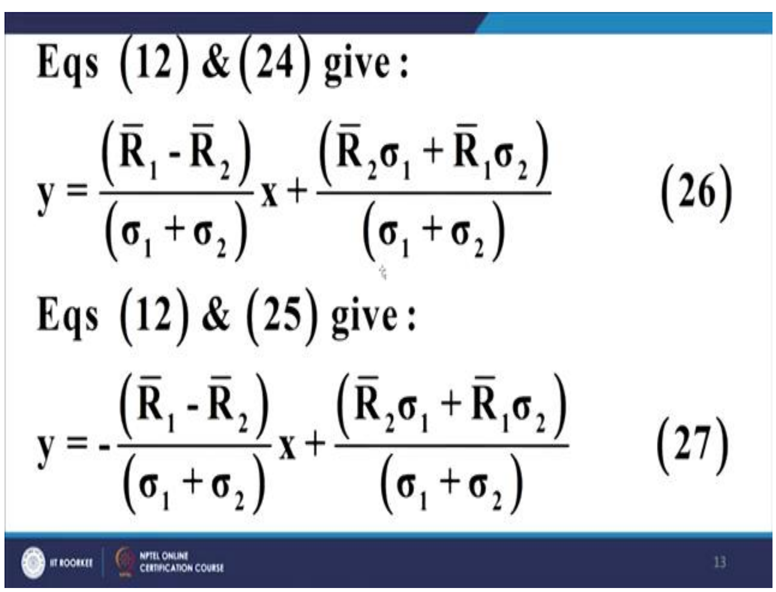

So, if we solve equation number 12 and equation number 24 what we get is equation number 26 which is again A straight line with the Y-intercept r2 sigma 1 plus r1 sigma two divided by sigma 1 plus sigma 2 and with the slope R one minus R two divided by sigma 1 plus sigma 2 and if we solve equation if we eliminate X one between twelve and twenty five what we get is the same intersection its again A straight line and the point of intersection with the Y axis is also the same as in the case of 26; 26 and 27 both are straight lines both have A common Y intercept.

Here again we have two straight lines intersecting at the Y axis and the slopes are again you know opposites of each other showing that the two lines are mirror images of each other mirror images of each other about A line A straight line parallel to the X axis through the point of intersection of the two lines with the Y axis that is r2 sigma 1 plus r1 sigma 2 divided by sigma 1 plus sigma 2.

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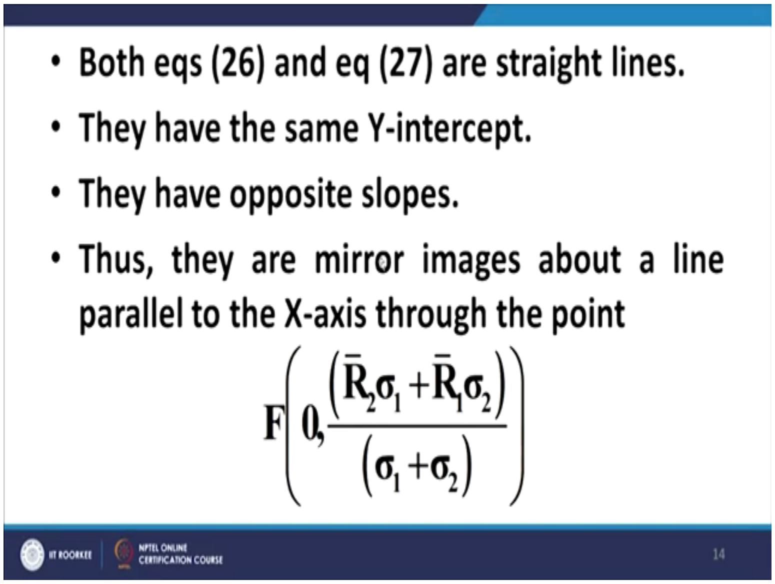

So, both these two lines are straight lines they have the same Y-intercept they have opposite slopes therefore the images about the line through the parallel to the X axis through the point of intersection with the Y axis.

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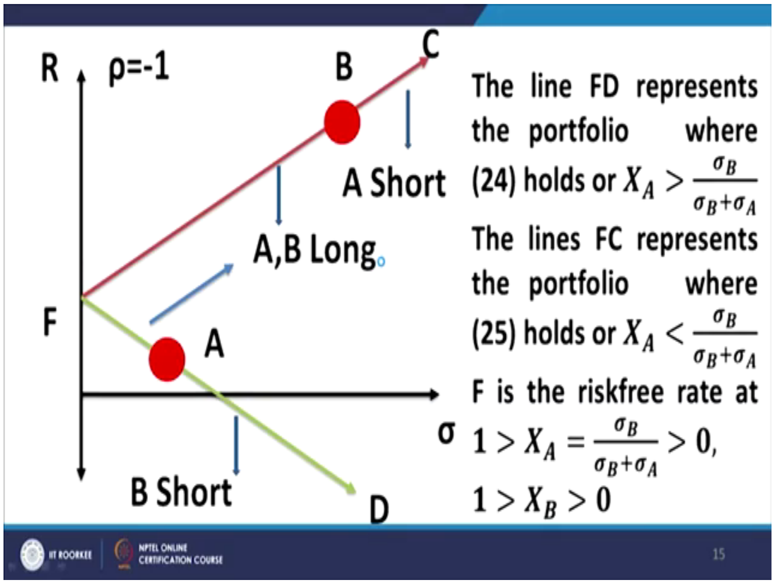

This is the pictorial representation of the portfolio possibilities curve which now again takes A pair of which now again becomes A pair of straight lines and in this situation what we find is there are some differences, there are certain differences between the case where we have rho equal to plus 1 in this case the line F t represents the portfolio where 24 over F t is the line where where XA is greater than sigma B upon sigma B plus sigma A which is obviously, which is obviously less than 1 and in XA is greater than this expression this is less than 1 but XA has to be greater than this so that means XA can be greater than 1 as well but XA has to be greater than sigma B upon sigma B plus sigma A.

And along F C the represents the region of the portfolio where X is less than sigma B divided by sigma B upon sigma 0, so in this region FC X is surely less than 1 and in the region F T X is initially less than 1 up to point A and then beyond point A Xa becomes greater than 1. So, moreover ah in the region B C A is short B is long in the region A, in the region B F A; B F A both are long and in the region A D A B is short A is long.

Now, the interesting feature is in contrast to the situation where we had when the two securities were perfectly correlated in this case the risk-free rate turns out to be between the expected rates of returns of the securities A and B you know you can see that the point F lies between RA and RB and therefore the risk free combination of A and B will have A return which is in between the returns of A and the return of B. This is the situation when the two securities are perfectly anti-correlated. However if the two securities were perfectly correlated then the risk free rate would be less than the rates of return on both the securities.

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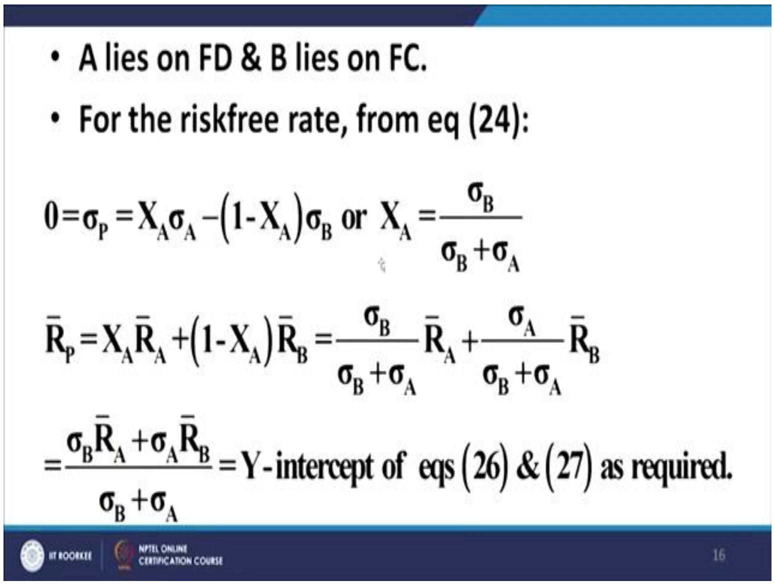

So, to recap A lies on F t B lies on fc as you can see in the figure and for the risk-free rate what we have is sigma X A I am sorry sigma p is equal to 0 sigma p equal to 0 gives me X A is equal to sigma B divided by sigma A plus sigma B and X B will be equal to what; sigma A divided by sigma A plus sigma B therefore both securities are long in the risk-free combination, please note this in the case of perfectly correlated assets one security was long, one security was short in the in the risk free combination. Here both securities are long in the risk-free combination

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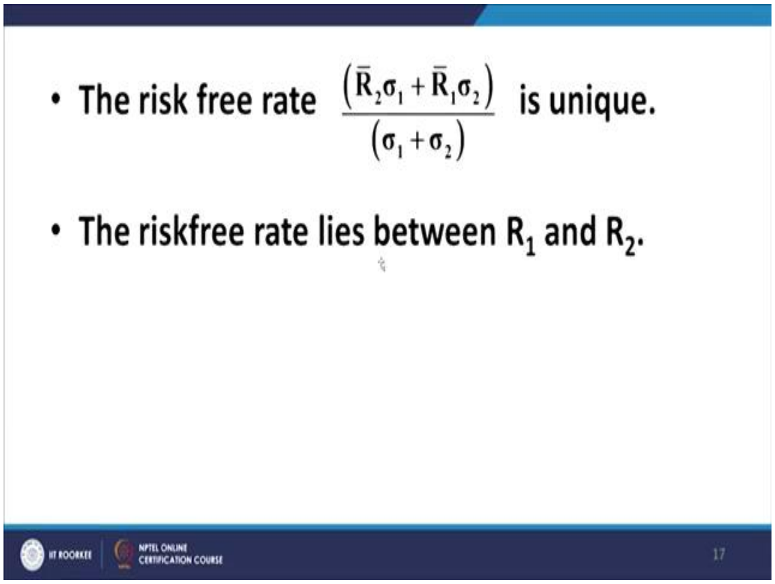

The risk-free rate is given by R 2 sigma 1 plus R 1 sigma 2 divided by sigma 1 plus sigma 2 it is unique it lies between R1 and R2.

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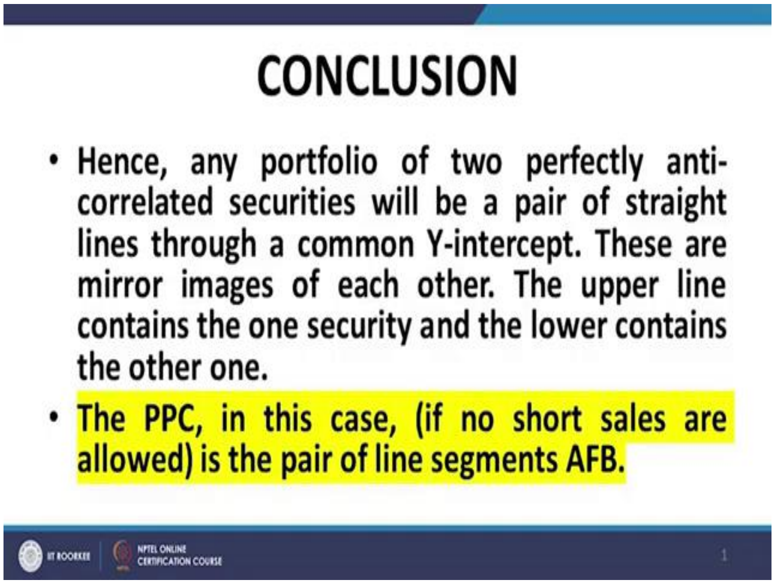

So, to conclude any portfolio of two perfectly anti correlated securities will be A pair of straight lines through a common Y intercept will be a pair of straight lines through a common Y-intercept these are mirror images of each other, the upper line contains the one security and the lower line contains the other security. The portfolio possibilities curve if no short sales are allowed is the pair of line segments AF and FB.

If short sales are allowed, what happens? These two lines get extended indefinitely as you can see beyond B; B is long A is short and beyond A; A is long B is short. So, if short sales are not allowed then the portfolio possibilities curve is confined to A F B and if the short sales are allowed then both FA and FB get extended indefinitely with points beyond A and B representing short sales in the other security. F A; beyond F A; B is short and beyond FB; A is short. Portfolio possibilities curve for general rho no short sales case.

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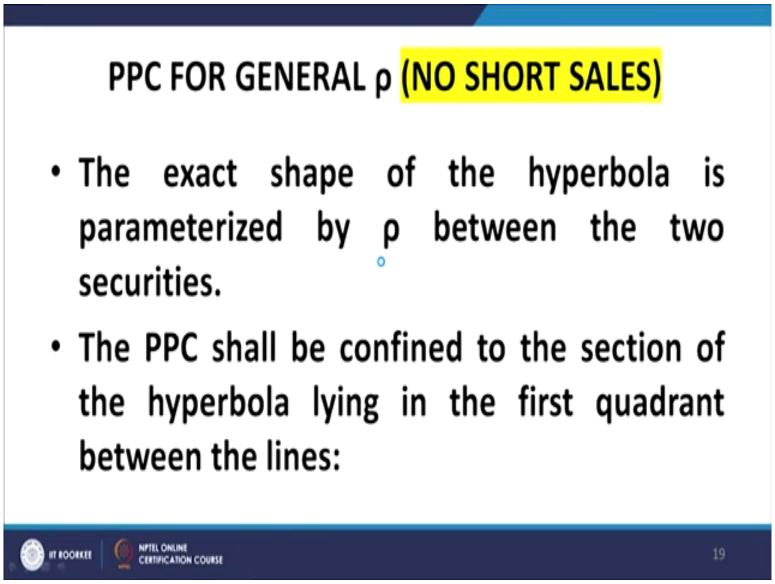

Now, the exact shape of the hyperbola we all know that the for A general rho which does not fall into this category of either being perfectly correlated or perfectly anti-correlated that is for any value of rho between minus 1 and plus 1 excluding minus 1 and plus 1 the exact shape of the portfolio possibilities curve is A hyperbola in the right half plane and it is parameterized by rho its exact shape will be determined by the correlation coefficient between the two securities.

It will be confined to the section of the hyperbola lying in the first quadrant between the lines if there is no short sales if there are short sales then the entire section of the hyperbola in the right hand plane becomes the portfolio possibilities curve but if there is no short sales then the section of the hyperbola between A and B will constitute the portfolio possibilities curve. And therefore, and furthermore the portfolio possibilities curve for any rho that lies between minus 1 and plus 1 these two extreme values excluded will lie in the triangle which is bound by the vertices given by the points at the bottom of your slide; A, B and F.

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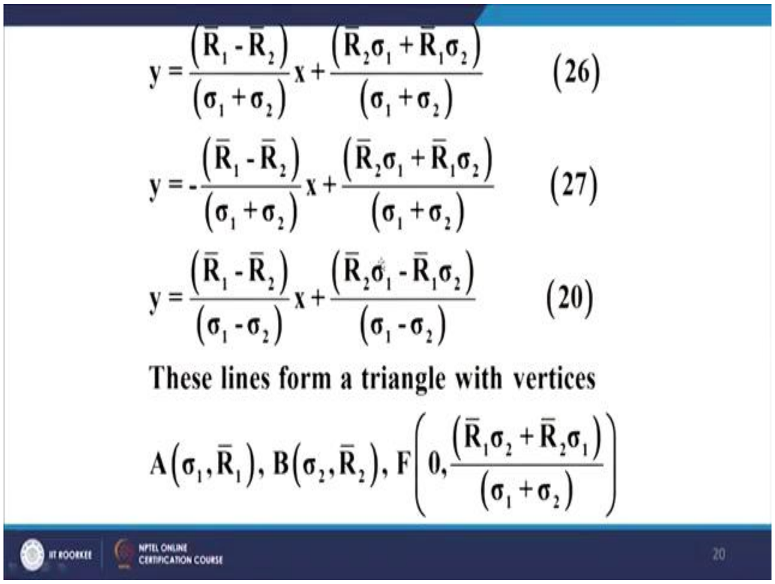

The entire portfolio possibilities curve if no short sales are allowed and A for A general rho between minus 1 and plus 1 excluding minus 1 and plus 1 will lie between the triangle or will lie within the triangle which is bounded by the three vertices A, F and B.

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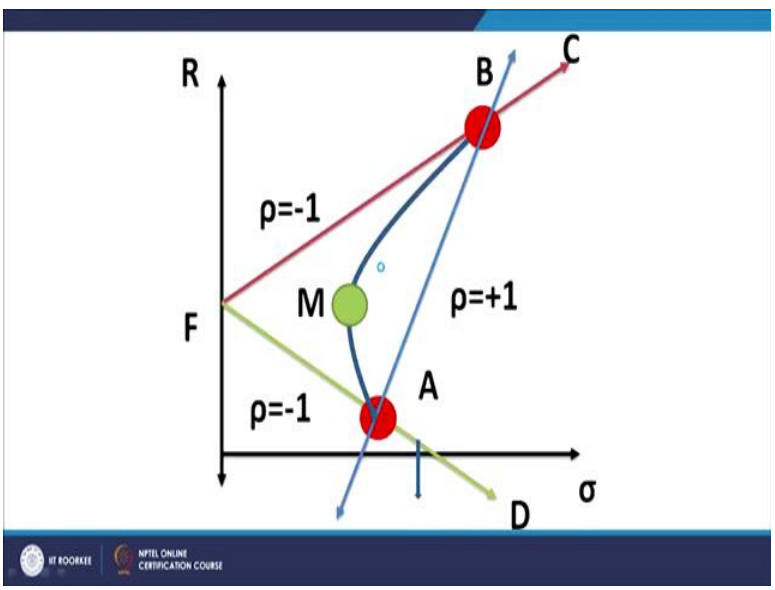

This is what, how it would look like. Now, if you look at this A B; A B represents the straight line represents the situation where the two securities are perfectly correlated. The straight line joining A and B is the portfolio possibilities curve or the portfolio possibilities line when rho is equal to plus 1 and no short sales are allowed. And when rho is equal to minus 1 and no short sales are allowed we end up with AF and PF these are the two line segments which represent portfolios for anti-correlated securities with no short sales allowed.

And if no short sales are allowed and rho lies between minus 1 and plus 1 these two extreme values excluded then we will have some curve of the type hyperbolic curve of the type A M B; A M B where the exact structure of A M B the extent of bulge of a AMB shall depend on the value of rho. The closer the rho value is to minus 1, the closer would M B to F and the closer is the rho value towards plus 1 the closer would be M to the curve to the straight line A B.

Now, condition for a risk-free portfolio of risky assets let us explicitly derive the condition when two securities which are risky securities could be combined to form a risk-free asset. So, how do we do it? For this purpose what should happen; the portfolio possibilities curve the hyperbola of equation 14 must intersect the Y axis at a real point.

So, that is the condition in that we have at one or more real points in fact because the intersection of the portfolio possibilities curve with the Y axis and gives us the points at which the risk is 0, sigma is 0, so sigma being sigma being a measure of risk in this particular framework we assume that sigma equal to 0 implies the risk is 0 and therefore we need to work out the points of intersection of the portfolio possibilities curve with the with the Y axis.

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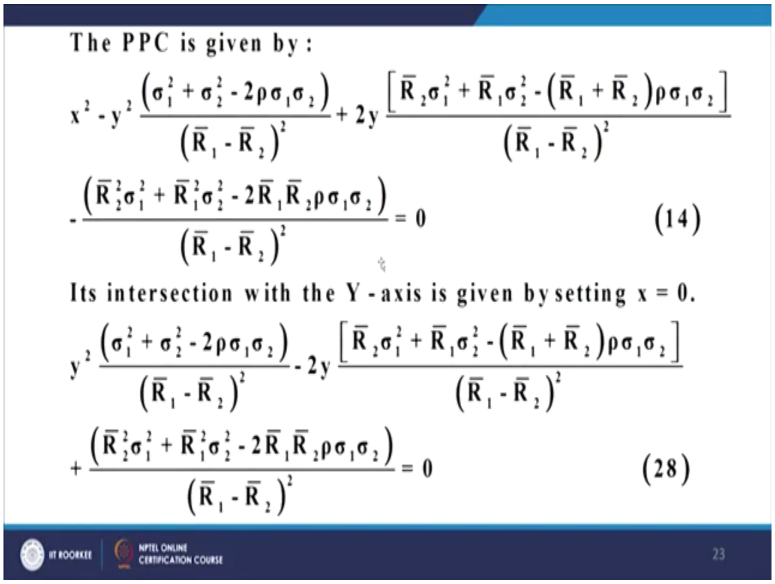

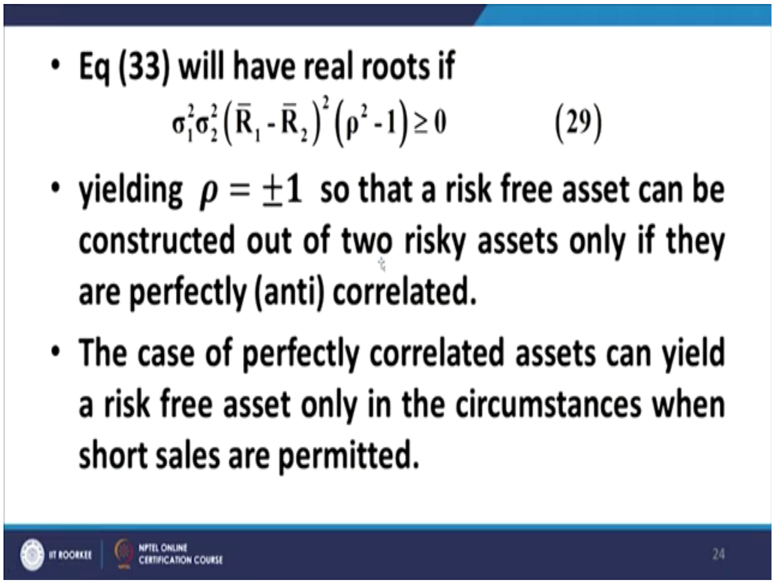

So, let us put Y X equal to 0 in equation number 14; what we get is equation number 28 and the roots of equation number 28 this is a quadratic equation and we want that the points of intersection must be real and therefore we want that the roots of this quadratic equation must be real. In other words B square minus 4 ac must be greater than 0 the discriminant of this equation must be greater than 0.

And we so when we solve this with an extended set of calculations what we end up with is the condition given in equation number 29. Now, clearly you can see this that equation number 29 mandates that rho must be equal to plus 1 or minus 1 there is no other value of rho which will satisfy which will result in A solution of equation number 29, except rho equal to plus minus 1.

So, that means what? That means given two risky securities what is the inference given two risky securities A and B we can form a risk-free combination of securities A and B only in the event that the two securities are either perfectly correlated or perfectly anti-correlated. If they are perfectly correlated then the risk-free combination sale shall comprise of one security long and the other security short and if they are perfectly anti correlated the risk-free combination shall comprise of both the securities being long.

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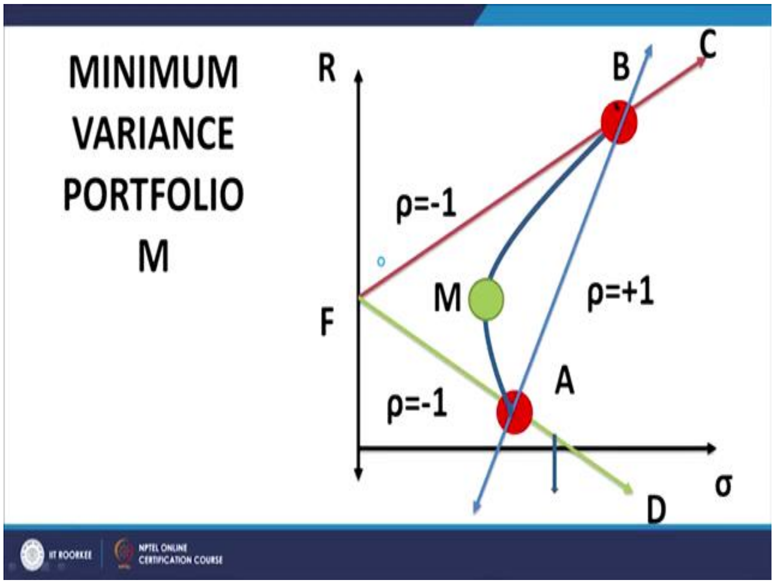

Now, we talk about the minimum variance portfolio. If you look at this figure there is a point on the on the curve A M B the point M at which the variance of the portfolio turns out to be minimum. In other words what is the objective now; the objective is get that given to securities with A given value of rho which is not plus minus 1 because we already dealt with the case of rho equal to plus 1 and rho equal to minus 1.

Now, we are looking at a general case where we are given some value of rho at and we are given A pair of risky securities with some value of the correlation coefficient between them. We want to combine the two securities in such a way I have a such a composition such a portfolio of these two securities, risky securities given that value of rho over which we have no discussion now and we want to formulate a frame a portfolio which has the minimum risk, minimum risk as epitomized by the minimum variance or minimum standard deviation.

So, we now I repeat we are now given two securities, we are given the correlation between them, we want to form a combination of the two securities in such a way that they give us the minimum variance. Irrespective; I repeat irrespective of the return on the portfolio we are not concerned with the return of the portfolio, we are not concerned with the return of the minimum variance portfolio our objective is solely confined to the working of A portfolio to the looking, to the formulating of A portfolio that has the minimum risk. In other words it has the minimum variance, so let us do that.

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$$\sigma_P^2 = X_1^2\sigma_1^2 + \left(1-X_1\right)^2 \sigma_2^2 + 2X_1\left(1-X_1\right)\rho\sigma_1\sigma_2$$

$$0 = \frac{d\sigma_P^2}{dX_1} = 2X_1\sigma_1^2 - 2\left(1-X_1\right)\sigma_2^2 + 2\left(1-2X_1\right)\rho\sigma_1\sigma_2$$

$$X_1 = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_2^2 + \sigma_1^2 - 2\rho\sigma_1\sigma_2}; X_2 = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_2^2 + \sigma_1^2 - 2\rho\sigma_1\sigma_2}$$

We start with this expression for the variance of the combination of security 1 and security 2 which is given on your slide and we start from this and then we differentiate this expression with respect to X 1 the only degree of freedom that we have; why are we differentiating with respect to X1? Because that is the only degree of freedom that is the only variable over which we have control.

What are we trying to do? We are trying to construct A portfolio of the two securities in an appropriate combination; that means what? That means we are determining and X2 but we know that X 1 plus X 2 has to be equal to 1 so if we know X1 we the X2 gets automatically determined and therefore X1 remains the only free variable which over which we have control which we can determine or which will enable us to construct A portfolio with the minimum variance.

So, we differentiate this expression with respect to X1 and when we equate this to 0 we differentiate this expression with respect to X1, we equate it to 0, the value of X1 we get is given as sigma 1 square minus rho sigma 1 sigma 2 divided by sigma 1 square plus sigma 1 square minus 2 rho sigma 1 sigma 2 and the value of X2 can immediately be obtained by 1 minus X1 and it turns out to be the expression in the right hand corner of this particular slide.

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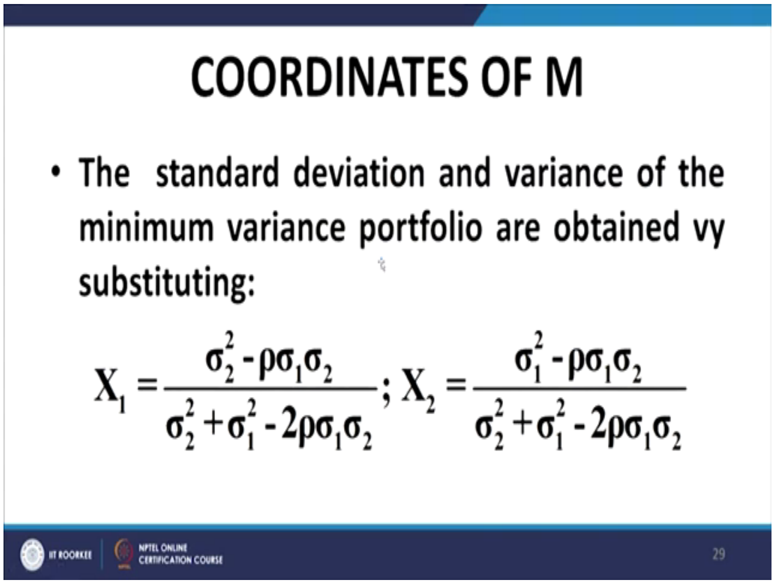

What are the coordinates of M now; we have got the composition of M we have got the composition vector of M; what is the composition of the minimum variance portfolio? Where does the point M lie on the risk return space that is the next question; where does this point corresponding to the minimum variance lie in the risk written space line in the sigma e R space that is easy to determine.

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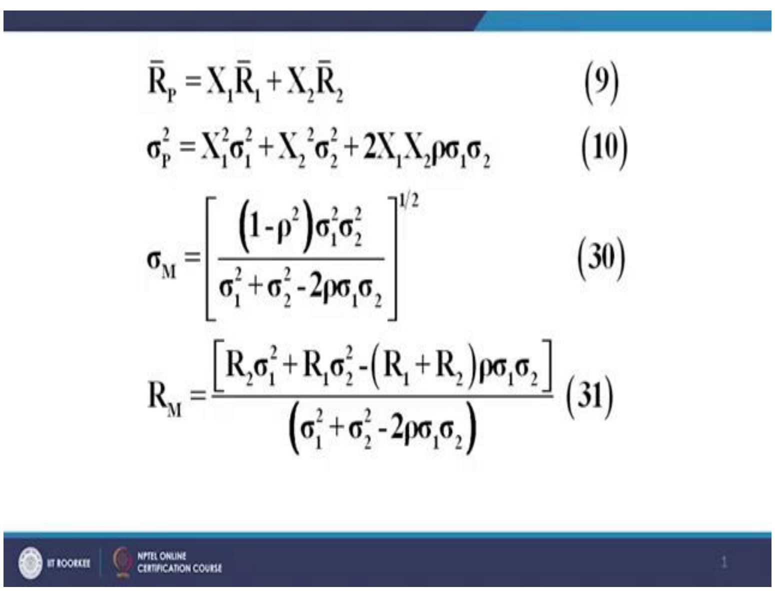

We simply put X 1 and X 2 in the expressions for the portfolio return and the portfolio variance and what we get is the expression that is given in the equations 30 and 31. So, these are the coordinates in of the these are the coordinates of the minimum variance portfolio in the risk return space in the sigma R space.

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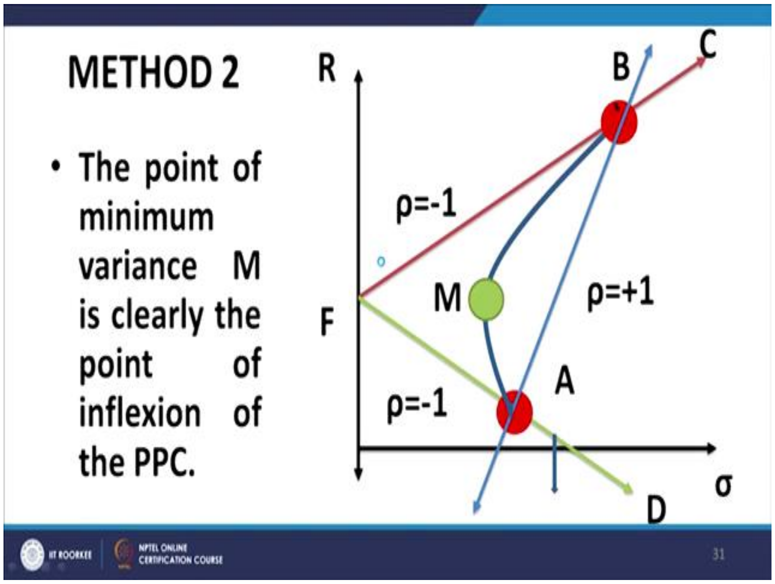

There is another approach you could do in order to arrive at the minimum variance portfolio you can see from here that the point M that is the point of minimum variance represents the point of inflection of the hyperbola represented by the curve B M A or A M B I repeat point M represents the point of inflection of the hyperbola A M B. So, using this particular property we can also work out the value or the coordinates of the minimum variance portfolio directly without going through the process of calculating the composition of the minimum variance portfolio.

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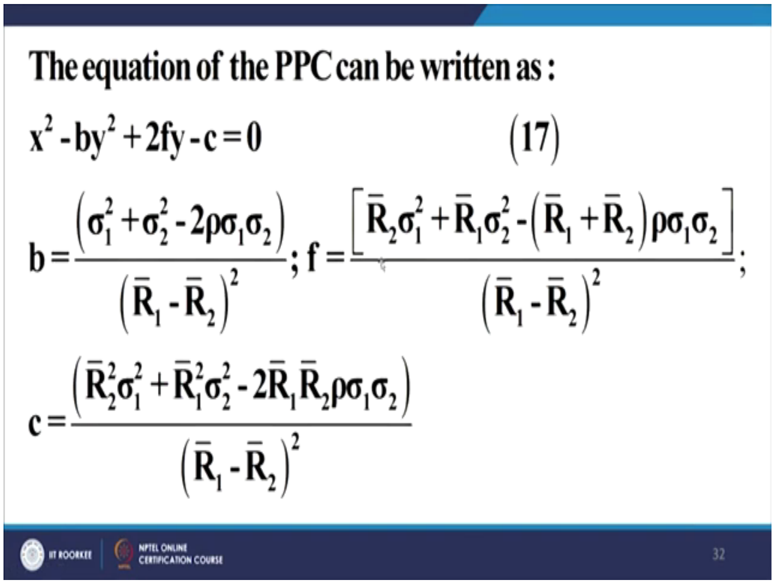

What we do is we let we assume that the equation of the portfolio possibilities curve can be represented as X square minus B Y square plus F Y minus C this was equation 17 you may recall where the M B F and C have the respective values shown in the slide.

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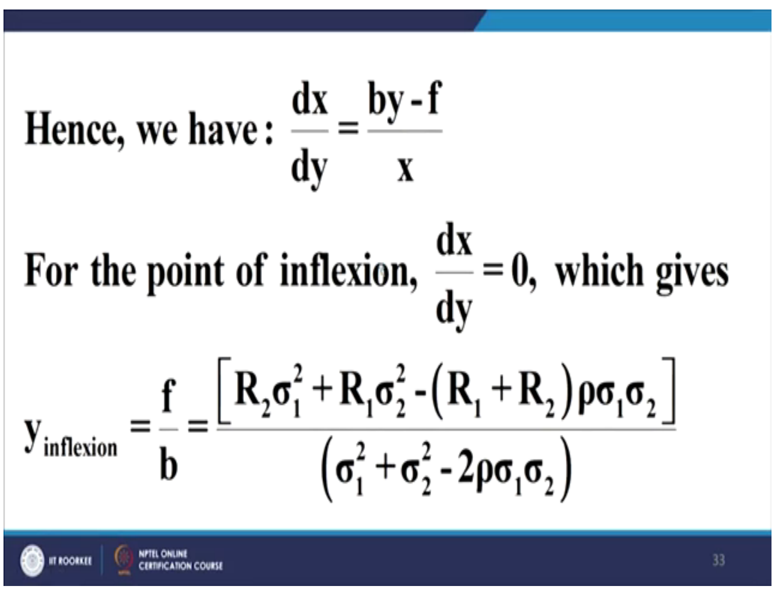

$$x_{inflexion} = \pm\sqrt{\left(by^2 - 2fy + c\right)}$$

$$= \pm\sqrt{\left[b\left(\frac{f}{b}\right)^2 - 2f\left(\frac{f}{b}\right) + c\right]} = \pm\sqrt{\left(c - \frac{f^2}{b}\right)}$$

$$= \pm\left[\frac{\left(1 - \rho^2\right)\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\right]^{1/2}$$

Now, for inflection what do we have; we have D X upon D Y is equal to 0, dx because at the point of inflection the curve or the straight line of through the point of inflection will be parallel to the Y axis. I repeat the straight line through the point of inflection to the hyperbola will be parallel to the Y axis as is shown in the diagram and therefore dx by dy at that particular point must be 0 which will give you Y inflection is equal to F upon B from the first equation that is there on the slide D X by D Y is equal to B Y minus F upon X is equal to 0.

If you equate this to 0 you get Y is equal to ah F upon B. Substituting the value of F and B we get directly the value of Y or the Y coordinate of the point M. And using this using the expression for the portfolio possibilities curve equation for the portfolio possibilities curve we can work out the value of X inflection or the X coordinate of the point of inflection and you find that it coincides with the value that we have obtained earlier through the first method.

Now, risk-free combinations of two securities must be the minimum variance this is just you may take it as an excise risk-free combinations of two securities must be minimum variance portfolio of the two assets with o equal to plus minus 1 because you cannot form risk-free combination otherwise then when rho is equal to plus 1 or rho is equal to minus 1 it is only in these two situations that given two risky securities you can form a risk free combinations.

So, let us try to vindicate that situation and now the risk-free combination must obviously be the minimum variance combination because variance cannot be negative and therefore the minimum variance combination if you have a risk-free combination then that combination must be the minimum variance combination. You cannot have variance less than the minimum where you cannot have variance less than 0.

So, that is what we are trying to establish that given rho equal to plus minus 1 if we work out; what are we trying to do? Given row equal to plus minus 1 if we work out the minimum variance portfolio it turns out to coincide with the risk-free portfolio that is what we are going to establish. Let me repeat given rho equal to plus minus 1 if we work out the minimum variance portfolio it coincides with the risk-free portfolio that is what we are going to establish.

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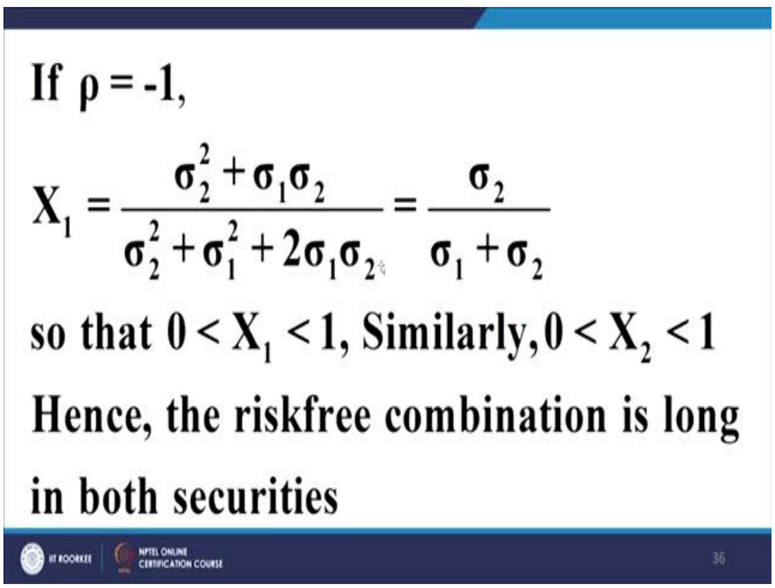

So, if rho is equal to minus 1, the minimum variance composition if you gives us X 1 is equal to sigma 2 upon sigma 1 plus sigma 2 and X2 is equal to sigma 1 upon sigma 1 plus sigma 2 and if rho is equal to plus 1 we get X 1 is equal to sigma 1, X1 is equal to sigma 2 upon sigma 2 minus sigma 1 and X 2 is equal to sigma 1 upon sigma 1 minus sigma 2.

You will recall that these were exactly the results that we obtained when we worked out the risk-free combination of the risk-free combination in the situation where rho was equal to plus 1 or rho was equal to minus 1 explicitly by solving the expressions for the expected return and variance. The portfolio possibilities curve is short sales allowed; now what happens? You know what will happen is let me go back.

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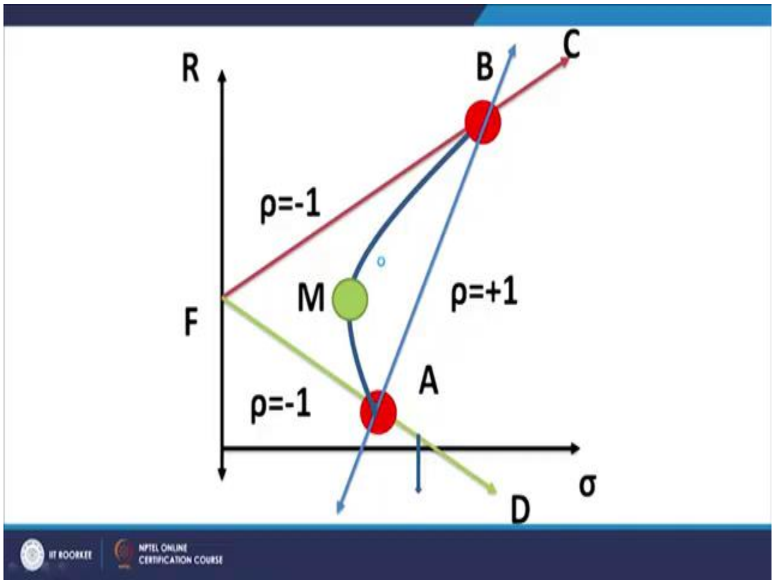

Now, let us look at this figure; now the question is when I was discussing this figure I explained that if there are no short sales allowed, if there are no short sales allowed then the portfolio possibilities curve for any value of rho other than minus 1 and plus 1 or lying between minus 1 and plus 1 rather will be a hyperbola that or the section of the hyperbola that lies between A and B and so that is the situation when there are no short sales allowed of security A or B.

What happens if this condition is relaxed? If now short sales are allowed; if short sales are allowed then the portfolio possibilities curve continues to be a hyperbola, continues to be the same hyperbola A and B but it can now be extended beyond B on the one side and beyond A on the other side. The hyperbola can be extended beyond B and beyond A. The arc of the hyperbola that that will go beyond B will represent B long and A short and the arc of the hyperbola beyond A would represent B short and A long.

So, if the short selling is not allowed then you get confined to the arc of the hyperbola A B or AMB rather and if the short sales are allowed then the both the legs of the hyperbola get extended indefinitely the on the one side B becomes long and A becomes short that is the upper side and on the other side A becomes long and B becomes short, so that is the situation when short sales are allowed so we will continue from here in the next lecture. Thank you.