Security Analysis and Portfolio Management
Professor J.P Singh
Department of Management Studies
Indian Institute of Technology, Roorkee
Lecture 41
Mean Variance Portfolio Optimization - I

Welcome back. So, today we start a new topic; we move into the realm of portfolio theory. In portfolio theory our objective is that given a set of securities given a set of assets, we try to optimize the composition of a portfolio with the desired objective function. So, that briefly is the objective that we are going to move forward towards.

We will start with the Mean Variance Portfolio Optimization model. As I mentioned at the beginning of this lecture series that when we evaluate an investment; we evaluate it on two parameters that is the expected return and the risk. The expected return gives us what to expect from making the investment and the risk tells us how likely is that expectation to be realized if the investment is risky we feel that the likelihood of realization of that expected return is subject to uncertainty, significant uncertainty.

And if the investment is risk free then that expected return is more or less certain to be realized. So, the essential the take away of what I have mentioned just now is that we evaluate the investment on two parameters; number one, expected return number two; risk. So, far the process that we have been following has been of incorporating making a subjective assessment of risk and then incorporating that subjective assessment as a part of the discount rate which we use for discounting the future cash flows to arrive at the present value which represents the intrinsic value of the asset.

So, in a nutshell the risk is encapsulated in the discount rate that we use for evaluating or for arriving at the intrinsic value of the investment. Now, we move over to a situation or a paradigm where the risk is explicitly represented in the analysis and we now use a two-dimensional framework for the evaluation of investment which is called the risk returns phase. There could be different measures of risk, there could be different measures of return so let us first acquaint ourselves with the measures of return and risk and then we will move into the deep into the mean variance optimization theory.

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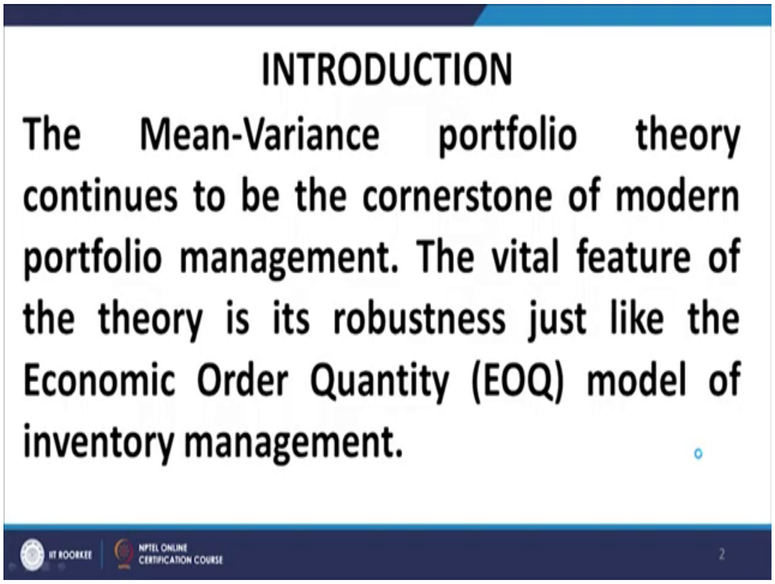

So, that is the introduction; the mean variance portfolio theory it continues to be the cornerstone of modern portfolio management. The vital feature of the theory is its robustness just like we have the economic order quantity in the case of inventory management.

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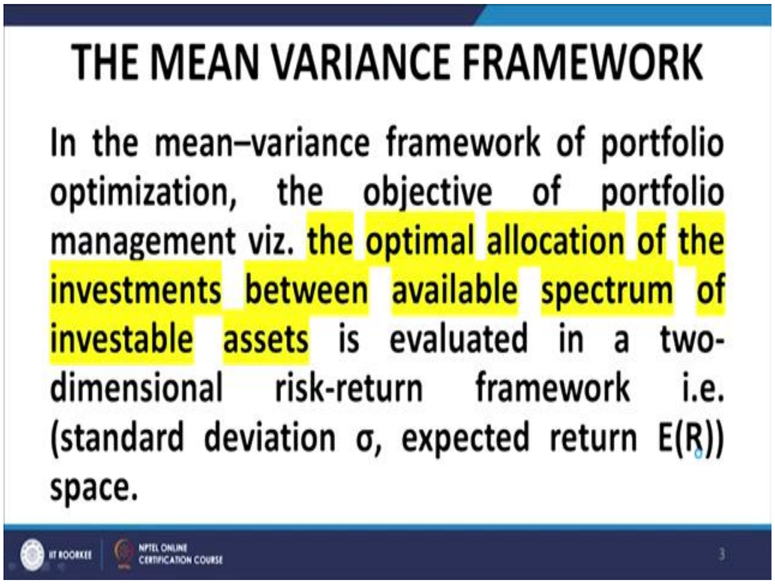

Now, in this mean variance framework of portfolio optimization the objective of portfolio management that is the optimal allocation of the investments between available spectrum of investable assets is evaluated on a two-dimensional risk return framework that is the expected return and that measures the return part and the risk is measured by the standard deviation or the variance that is why it is called the mean variance portfolio theory.

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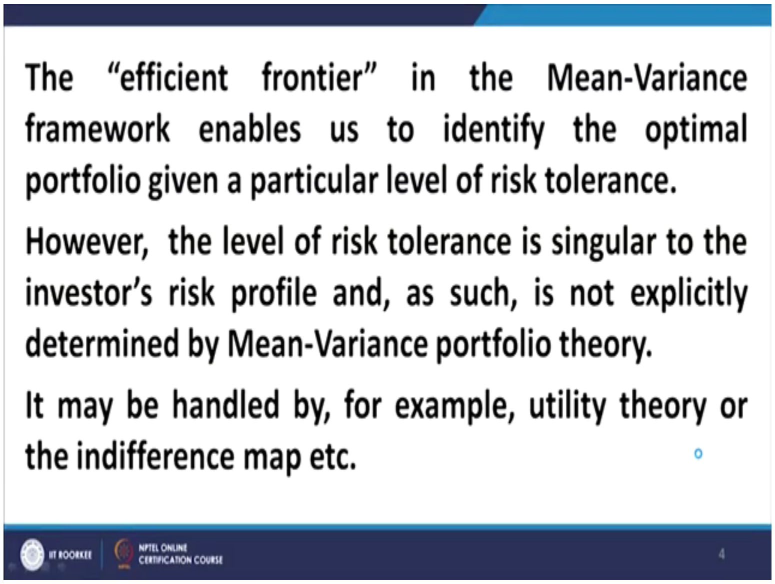

The efficient frontier in the mean variance framework enables us to identify the optimal portfolio given a particular level of risk tolerance however this is an important part that I am going to read out now. The level of risk tolerance is singular to the investor's risk profile and therefore what point he attains this optimality is dependent not only on the characteristics of the securities available in the in this set of securities but also on his own risk profile.

So, it is a two-way interaction; the interaction between the security set and the interaction of the risk written trade-off of the investor. The security analysis part or the portfolio theory part which we are going to talk in future for the rest of this set of lectures is confined or is destined to give bringing to you a frontier or a set or a set of security, set of combinations, set of combinations such that any point on that particular set of combination or any combination out of this set of combinations would be better than the other combinations given a certain level of risk tolerance.

In other words, given a certain level of risk tolerance we can identify the optimal combination relating to that risk tolerance. However in the absence of any information about the risk tolerance of the investor the risk return profile of the investor, the risk written trade-off of the investor we cannot give a singular portfolio a single portfolio which is supposed to be optimal I can only give you a set of portfolios which lie which lie on a frontier, which lie on a curve and it is the interaction between the portfolios that lie on the curve and the risk return characteristics of the investor which may be captured by the indifference map.

For example, or the utility of the utility function of the investor and then the interaction between them will give you the optimal portfolio for that particular investor. Now, please note this particular optimal portfolio for that particular investor may not be optimal for another investor who has a different risk return profile for him again we have to superpose his indifference map on the efficient frontier and then arrive at what is the optimal portfolio for the second investor. So, it is the interaction of the risk return trade-off as captured by the by the indifference map of the investor and the portfolio or the frontier of efficient combinations of securities which end up in giving you a singular optimal portfolio.

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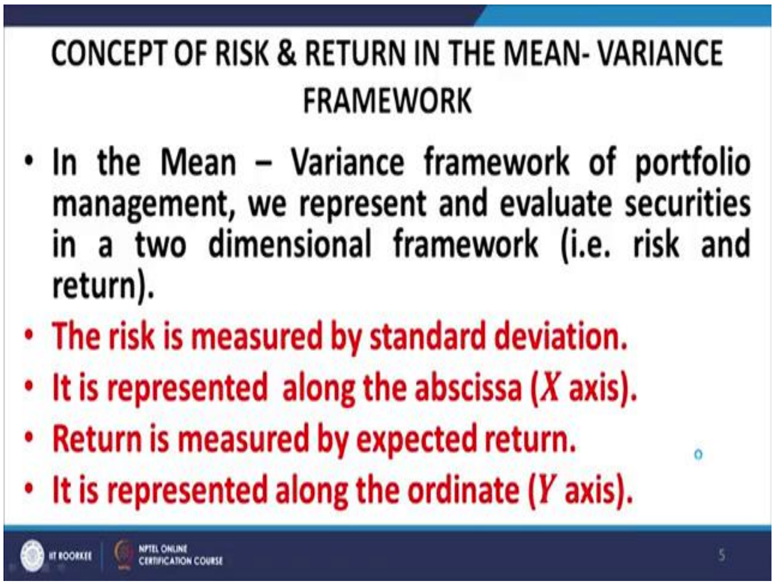

Now, we talk about concept of risk and return in the mean variance framework. In the mean variance framework of portfolio management, we represent and evaluate securities in a two-dimensional framework as I mentioned just now. The risk is represented by standard deviation and it is represented along the x axis as a matter of convention and return is measured in terms of expected return which is represented along the y axis.

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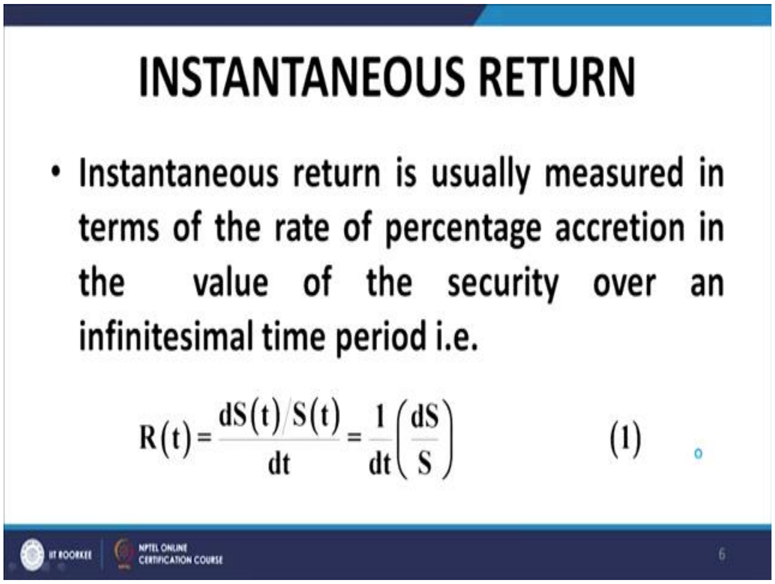

Now, instantaneous return; we define instantaneous return as in terms of the rate, rate means per unit time rate of percentage accretion in the value of the security over an infinite decimal time period. Let me repeat instantaneous return is usually measured in terms of the rate of percentage accretion in the value of the security over an infinite decimal time period, this is given in equation number 1 at the bottom of your slide.

The infinite decimal time period is given represented by dt so dividing by dt gives you the per unit time change in the percentage value of the portfolio which is given by ds upon s. Now, we talk about log return. The average log return over a period 0 to capital T now that was in the return that we talked about just now, the instantaneous return that is return for a very small for an infinite decimal time period measured over an infinite decimal time period.

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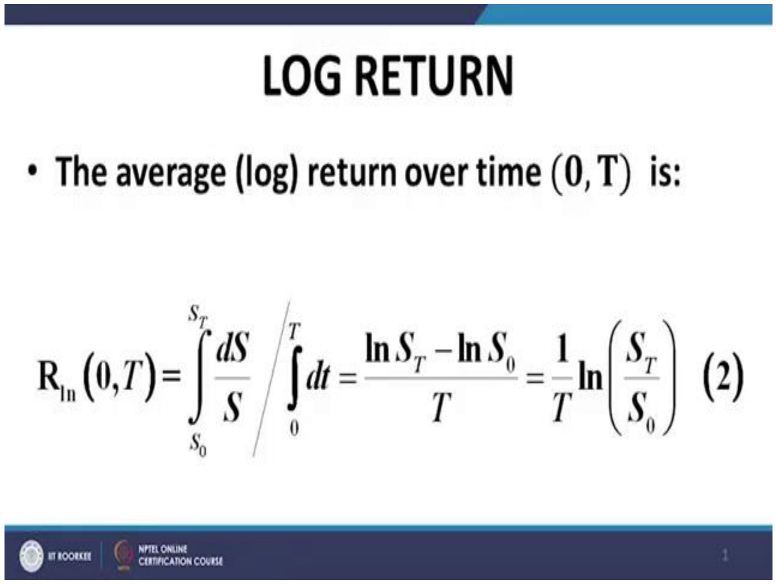

When we talk about finite time period 0 to capital T, the log return or the average log return over the time period can be worked out as shown in equation number 2 and we get T1 upon capital T log of the natural log of the ratio of the price at the maturity or price at capital T or value of capital t of the investment compared to or with reference to the initial value of the investment, the expression is given in the equation number2, right hand corner.

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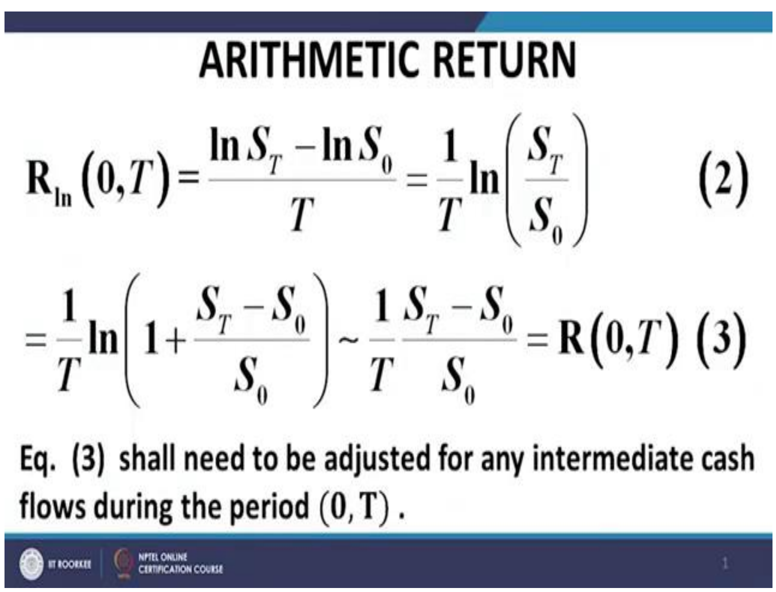

Now, we have worked out the log return formula for the log written that is 1 upon t log natural S capital t upon S0, now this can be expanded as a logarithmic series and if we retain the terms up to first order what we get is the expression for the arithmetic return. Arithmetic return is given by the ratio of the accretion of the value of the investment as a percentage of the original value of the investment ratio of the accretion in value over the original investment and measured per unit of time.

Please note this per unit of time normalization so as is usually the case we have two normalizations when we measure the return on a security or a portfolio or an investment in general. It is the normalization with respect to the initial investment which is here in formula number 3 and the normalization is also with respect to time which is also shown in formula number 3 by dividing by the time of holding of the investment. So, 1 upon St minus S0 is the accretion in investment in absolute value and this is divided by S0 to convert it to a percentage and the percentage is divided by t to convert it to a percentage per unit time. So, equation number three gives us the arithmetic return.

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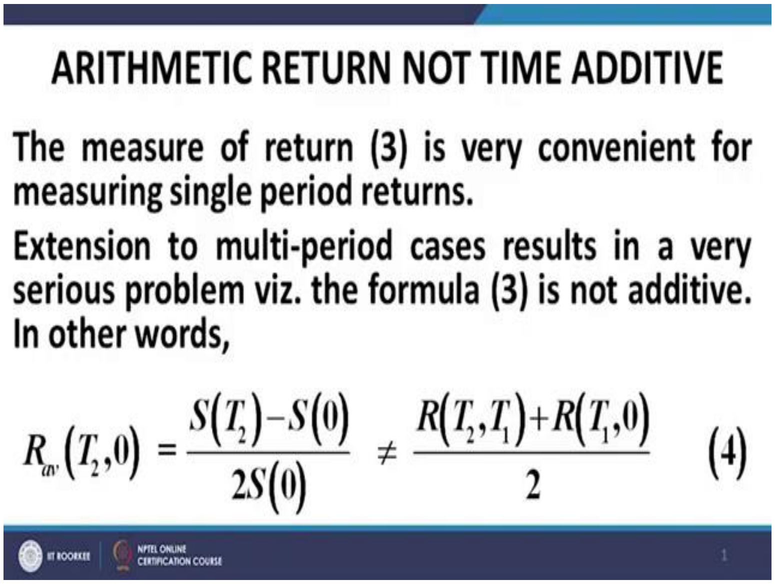

Arithmetic return is not time additive now this is a very interesting flaw when we use the arithmetic return. Suppose I am to calculate the average arithmetic return over two periods, so long as the we have a single period situation arithmetic return is good enough provided the returns are small or as is the case in practical life but the important thing is this return is good when we are using single period investment horizons.

However, when we talk about multi-period investment horizon, suppose we have to work out the average return over two periods two equal periods let us say of one year each and we want to work out the average return over two periods then the formula for the arithmetic return breaks down because it is not time additive. By not being time additive what i mean is suppose I work out the return over a period 0 to T2 then it would be equal to s T2 minus S0 divided by 2 into S 0, 2; because we are having 2 periods here; 0 to T1 and T1 to T2 both of unit time magnitude. S

o the average return over 0 to t 2 would be given by s T2 minus s 0 divided by 2 S0 but this is not equal to the arithmetic average return, arithmetic average of the arithmetic return over the period 0 to T1 and the arithmetic return over the period 0 to T2 as can be explicitly seen very conveniently, very easily. So, this is a fundamental flaw with the arithmetic return it is not time additive.

However arithmetic return happens to be portfolio additive in the sense that the return on a portfolio the return on a portfolio is a weighted average return of the returns on its component securities. I repeat the return on a portfolio is the weighted average return of its component securities. So, that is the plus point in so far as the arithmetic return is concerned the negative feature is that it is not time additive.

In other words, the average return over a number of periods is not equal to the average of the returns of each individual period. So, I repeat the average return taken over a number of periods is not equal to the average of the returns over that over those periods, so in that is what is shown in equation number 4 here.

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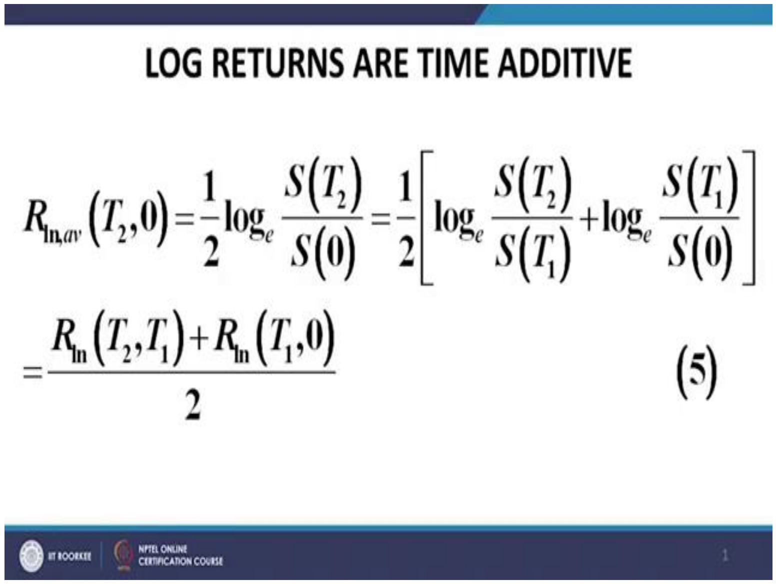

Log returns on the other hand turn out to be time additive in other words if I work out the average return between T2 and T0 it turns out to be the average of the returns over 0 to T, T1 and the return over T1 to T2. so I repeat if i work out the average return between the period T equal to 0 to t equal to T2 then that is equal to the average of the returns over the periods 0 to T1 and T1 to T2 as should normally be the case.

However unfortunately if I, if you use log returns these log returns are not portfolio additive in the sense that if we use log returns then we do not end up with the return over a portfolio being equal to the weighted average of the log returns on the individual securities comprising the portfolio that is not the case.

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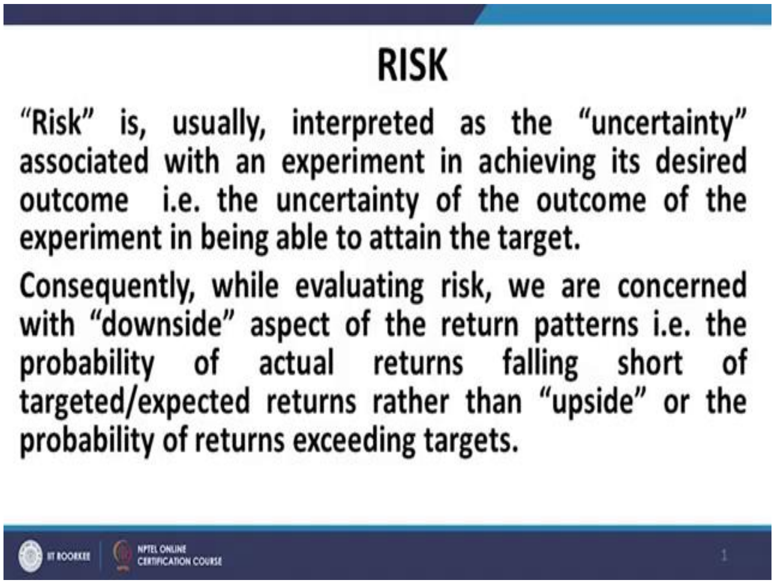

Now, we talk about risk, we have already introduced the concept of risk in the earlier lectures, I introduced the risk as related to or emanating from the uncertainty in the attainment of the future values of an investment what value the investment is likely to take in the future there is an uncertainty about it because the investment could take a spectrum of values, could take a number of values out of which what value it is going to take is not known in advance.

So, there is uncertainty in so far as the final value of the investment and that can arise only because there are chances of fluctuating in the final value of the investment. The final value of the investment can take a number of values and there is a possibility that the investment could take any of those values whatever those set of probabilities can be. So, in other words what inferred in an earlier section is that the risk depends on the level of fluctuations as well as the probabilities of those fluctuations and is therefore captured by the probability distribution of the investment values at the maturity or at the end of the holding period of the investment.

So, that being the case, the variance or the standard deviation holds promise as a measure of risk because standard deviation is a measure of dispersion it is a measure of fluctuations and it encapsulates information about both the amplitude of the fluctuations as well as the magnitude of attainment of those fluctuations both are captured in the expression for the standard deviation variance of a particular probability distribution.

So, that being the case but there is a problem with this standard deviation there is an issue which is raised by many academicians and practitioners alike in so far as the use of standard deviation is concerned as a measure of risk. The issue that they contained is that the investor is primary concerned, worried about the downside deviations of the investment or downside attainment of values of the investment and maturity.

In other words given a certain expected value the investor would be more than happy with the investment taking up values more than the expected value and it is the issue, it is the possibility of the investment ending up in values lower than the expected value or the target value of the investment that presents itself or that manifests itself as a risk in the minds of the investor.

So, the measure of risk should relate to or should encapsulate information about the downside deviations of the investment rather than all the divisions of the investment as are captured by the variance of standard deviation. So, this is this is an important issue this is an issue which is debated upon long and thorough but at the end of the day we still continue using standard deviation as a measure of risk with immense popularity.

Standard deviation unquestionably happens to be the most popular measure of risk there are reasons for this. The reason is that the contrary to the arguments propounded by the antagonists of standard deviation, the protagonists of standard deviation say that the return processes of most of the financial assets are pretty much symmetrical or the prices of the financial assets are pretty much symmetrical about the expected value at least to a significant extent.

They may not be exactly symmetrical as has been observed now with the detailed evaluation of empirical evidence but then the contention is that nevertheless standard deviation provides an adequate measure because by and large to some approximation the upside fluctuations and the downside fluctuations are more or less symmetrical about the mean position. So, that is one argument that is from propounded in favor of the standard deviation the second argument is that the returns return processes of most financial assets again approximately follow the normal distribution approximately I repeat the word I emphasize the word approximately, approximately follow the normal distribution.

And we all know that the normal distribution is entirely captured by the mean and variance of the distribution. We need only the mean and the variance of the normal distribution to completely define and identify a normal distribution. So, these are two fundamental features of standard deviation and then there is one cosmetic aspect to the use of standard deviation as well.

As you shall see as we gradually progress in discussing this theory the aesthetic value of the theory or the beauty of this theory is another contributor to the use of standard deviation as a measure of risk. So, for the purposes of this particular theory the mean variance portfolio optimization theory we measure risk in terms of the standard deviation along the x axis and we measure the return in terms of the expected return along the y axis.

And for the movement we shall be using arithmetic return because this is a single period model in essence and secondly the where we are talking about portfolios in the context of which arithmetic return is portfolio additive making it extremely convenient and useful in the context of this particular model. So, while evaluating risk we are concerned with the downside aspect that is what I emphasized a few minutes back and that is the flaw with standard deviation but there are lot of positives about standard deviation which also I have explained.

Now, the return of a portfolio, we have talked about the arithmetic return on a single security as being the appreciation in the value of the investment or the value of that particular security per unit time per unit value of the investment at the entry value of the investment that per unit value being calculated with respect to the entry value of the investment S capital, capital T where t is the investment horizon minus S naught whereas naught is the price at entry divided by s naught which is the price at entry and then divided by t to normalize it with respect to unit time.

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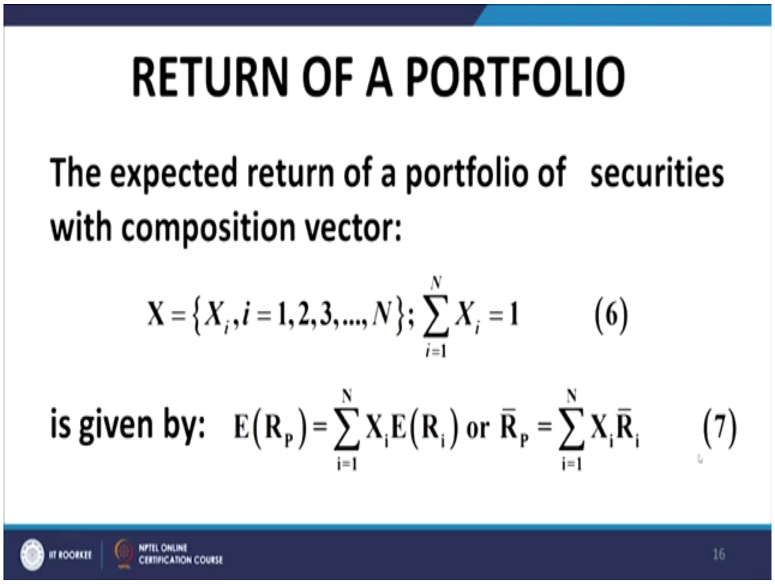

So, now the expected return on a portfolio of security as I mentioned the arithmetic return is portfolio additive in the sense that the return on the portfolio is the weighted average of the return of its constituent security, so that is what is represented in this formula, formula number 7 and that the expected return of the portfolio of securities of a number of securities is equal to the weighted average of the expected returns of the constituent securities. Weighted by what? Weighted by the proportion of money that is invested in its security.

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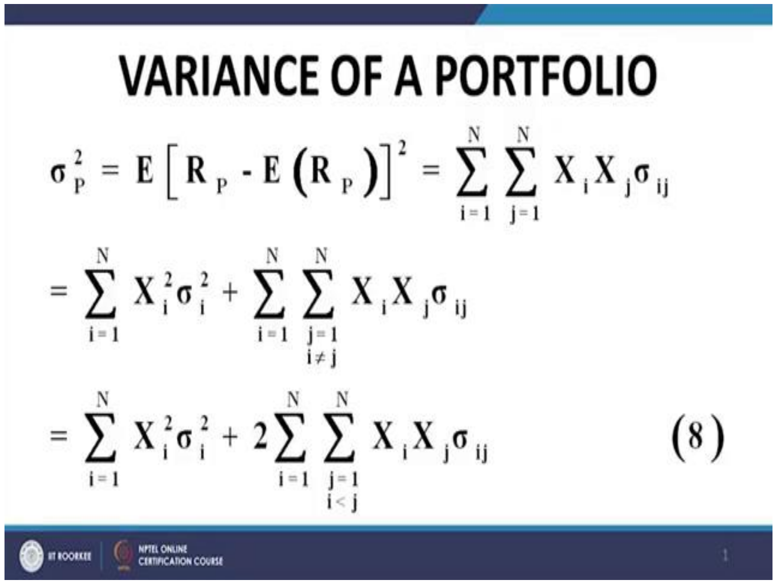

The variance of a portfolio well that this is slightly complicated but pretty much derivable from elementary algebra; the variance of a portfolio can be represented in terms of 3 expressions expression; number 1 that is the uppermost expression double summation over i and j over all the securities xi xj sigma i j; what is sigma i j? Sigma i j is the covariance between security i and security j this can be separated into two parts.

One the variance part and the other the covariance part as is done in the second equation and then we can also write it in the form of the third equation. Please note when we write it in this second equation case the i unequal to j expression has to be incorporated in the summation and in the third equation we have a factor of 2 here but then we replace i n equal to j by i less than j or j less than i as in either case it amounts to the same thing. Now, for the two-security portfolio, for the two security portfolio what do we have?

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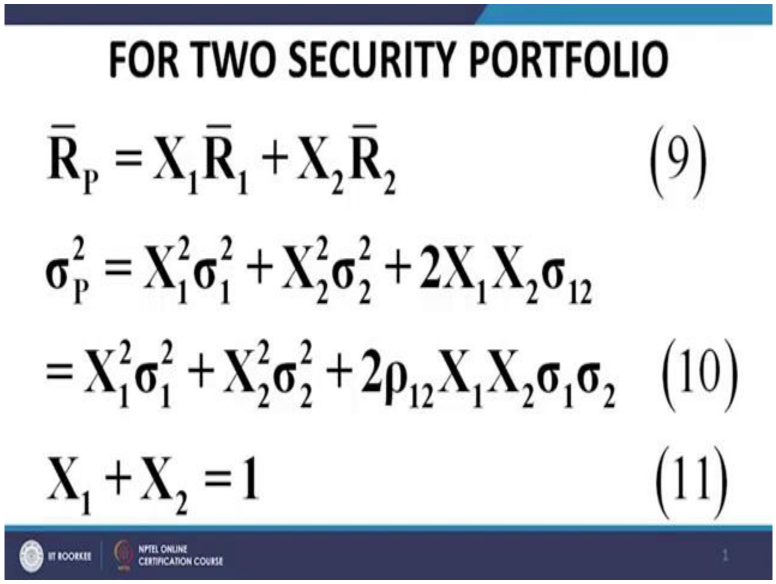

The expressions for the mean and the variance; the expected return and the variance of the two security portfolio simplifies considerably and it is given by equation number 9 and equation number 10; the expected return is given by equation number 9 and the variance of a two security portfolio is given by expression 10 where we have in the first of equation 10 we have the covariance sigma 1 2 and in the second of the equation 10 we replace the covariance sigma 1 2 by rho 1 2 sigma 1 sigma 2 where rho 1 2 is the correlation coefficient between the two securities 1 and 2.

So, this is the expression for the expected return; number 9 and variance number 10; for a two-security portfolio and obviously the weights must add up to one that is equation number 11 on the slide. Now, what are the features of this particular set of two security portfolios? Two security portfolio presents a very rich theory and we shall be discussing it in details.

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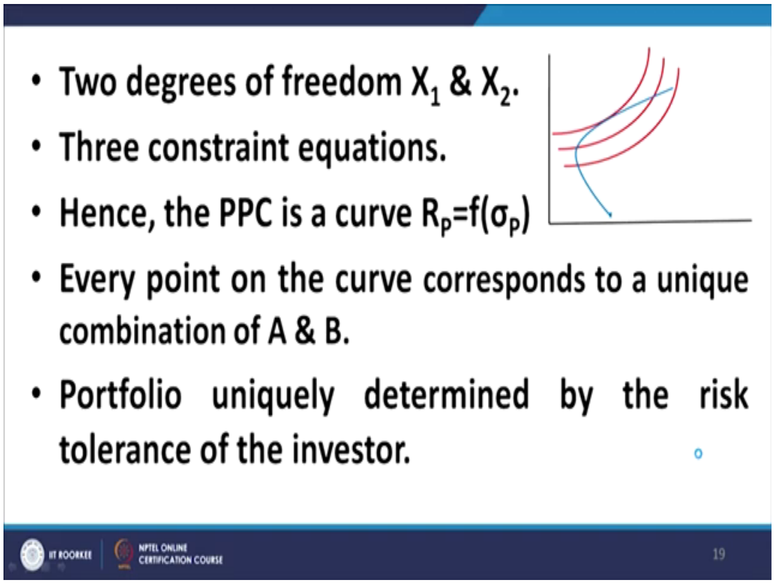

There are two degrees of freedom which are the constituents or the composition of the security the composition vector as you may call it x1 and x2 these represent the two free, free variables and we have three equations as you saw here; equation number 9, equation number 10 and equation number 11. So, you have 3 variable 3 constraint equations representing the equation number 9, 10 and 11.

In other words given, let us say you are given RP you are given a sigma square p then and the other equation is also there which requires that the sum of the weights must be equal to 1 then you need to determine x1 and x2 and that can be done by solving these equations in other words in other words what I am trying to say is that there are two unknowns x1 and x2 and there are three equations.

So, we can eliminate x1 and x2 between these three equations we can eliminate the weights between these three equations that will give us the that will give us a curve of the form RP is equal to f of sigma p in other words the expected return is a function of the standard deviation of the portfolio, the expected return is a function of the standard deviation of the portfolio. This curve as you will see later turns out to be a hyperbola for the moment let us keep this curve as it is.

And the important thing is every point on this curve represents a unique combination of x1 and x2. The specific values of x1 and x2 lead to the lead to a point on this curve which is represented by this a particular expression r p is equal to function of sigma p. In other words r p is equal to say f of sigma p represents portfolio possibilities curve every point on this portfolio possibilities curves represents a combination of the two securities given to us that is security one and security two.

So, every point on the curve corresponds to a unique combination of a and b and the portfolio is uniquely determined by the risk tolerance of the investor. If you are given the measure of risk you will have a unique value of the corresponding expected return that is in fact we will get two values as you shall see later but one of the values would be redundant and you will come up with a unique optimum value of the combination.

Now, the portfolio possibilities curve the explicit definition of the portfolio, possibilities curve for the two-security situation we define the portfolio possibilities curve I shall abbreviate it as PPC as the locus of a point in risk return space, risk return please note it is the sigma r space sigma expected return space that identifies an admissible portfolio. So, every point on the portfolio possibilities curve would identify a admissible portfolio, a possible portfolio. For a two-security portfolio with the composition vector given by x1 comma x2 where x2 is equal to 1 minus x1 because the sum of the two (have) has to be equal to 1; what do we have?

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$$E(R_P) = X_1 E(R_1) + (1 - X_1) E(R_2) \quad or$$

$$\bar{R}_P = X_1 \bar{R}_1 + (1 - X_1) \bar{R}_2 \qquad (12)$$

$$\sigma_P^2 = X_1^2 \sigma_1^2 + (1 - X_1)^2 \sigma_2^2 + 2X_1(1 - X_1)\rho\sigma_1\sigma_2 \qquad (13)$$

Eliminating X_1 between eqs. (12) & (13), we obtain the equation for the PPC for the two security case:

We have e r p that is the expected return of the portfolio is equal to this expression equation number 12, we have already seen that, seen that in equation number 9. If we substitute x 2 equal to 1 minus x 1 we get equation number 12. And similarly, if we substitute x2 equal to 1 minus x 1 in the expression for the variance we get the next equation.

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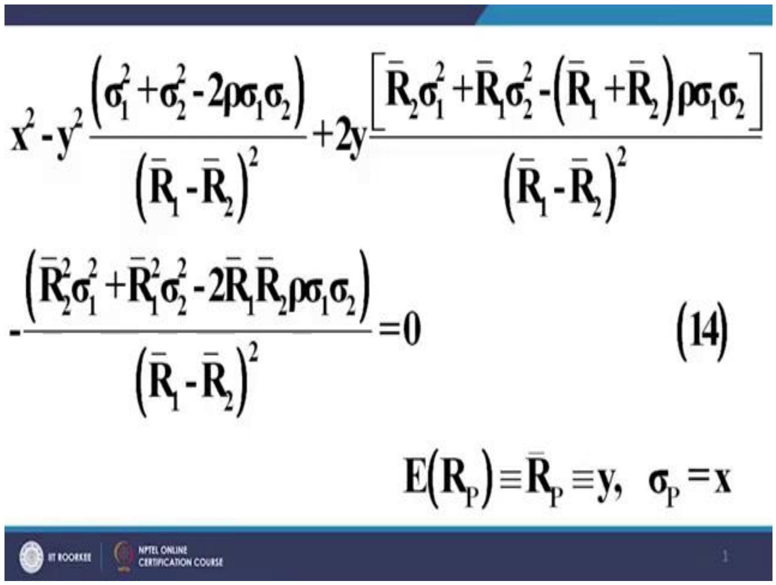

Now, if we eliminate x 1 between these two equations and that you have on the slide what we get is a very interesting result that is the expression here equation number 14, where I have abbreviated where I have substituted ERP is equal to RP is equal to y, I will use RP or r as an abbreviated version of ERP or the expected return to reduce unnecessary proliferation of symbols and sigma p is equal to x.

In other words, the portfolio standard deviation is the x variable and the portfolio expected return is the y variable but I shall be abbreviating the expected return ERP simply by RP for simplicity for abbreviating. So, this is the expression that we get, how do we get this expression? We simply eliminate x1 and x2, we simply eliminate x1 and x2 between the three equations 9, 10 and 11.

We eliminate x2 by writing it as 1 minus x 1 and then the results that we get the two equations that we get we eliminate x 1 between those two equations and we get this expression; what are the properties? They are very interesting very important as well. What are the, what is the structure of this curve, particular curve? This is that functional curve which I talked about a couple of slides back r p is equal to f of sigma p this is the explicit representation of that curve r.

You can see here it is a relationship between r p which is captured by y and sigma p which is k which is represented by x so this is this explicit curve explicit equation of the curve which represents the functional relationship between the standard deviation and the expected return for a two security portfolio, two security risky portfolio let me rewrite it both the securities that are incorporated in this portfolio are risky security they have non-zero standard deviation.

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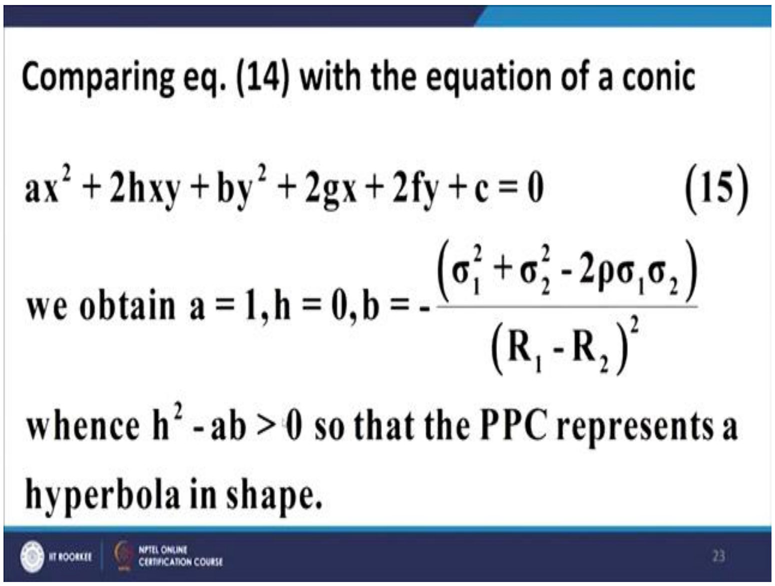

Now, if you compare the equation of this particular equation, equation number 14 with the general equation of a conic which is given as equation number 15, we identify a as 1, we identify h as 0 and we identify b as this extended expression and this gives us s square minus a b is greater than 0; what does that imply? That implies that the curve that we have traced out or that the curve that we have obtained by eliminating x1 and x2 between the expressions for the expected return and the standard deviation is a hyperbola.

I repeat this is very important the curve that we obtained by eliminating x1 and x2 between equations number 9, 10 and 11 that is the expression for the expected return of the portfolio, the expression for the variance of the portfolio and the sum of the weights these three equations when you eliminate x1 and x2 the composition vectors between these three equations there are two unknowns; x1 and x2 you have three equations, so you can explicitly eliminate x1 and x2. The expression that you get for the curve which represents the functional relationship between sigma p and r p is a hyperbola.

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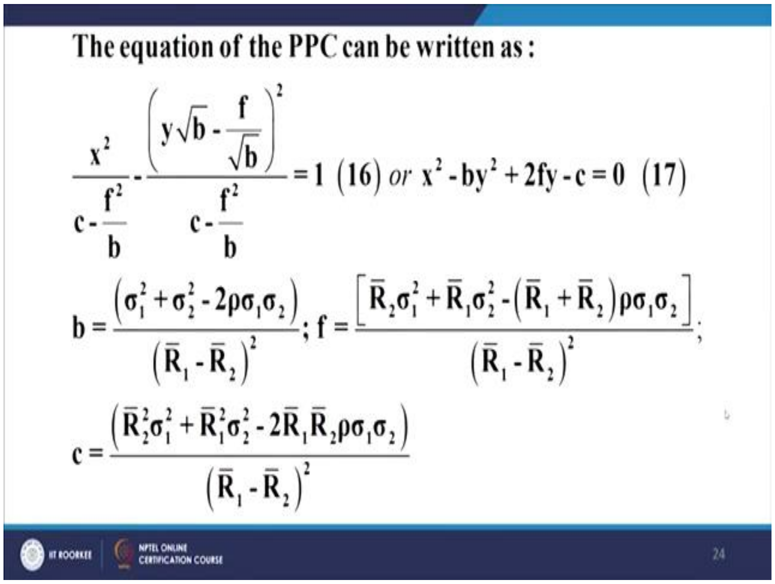

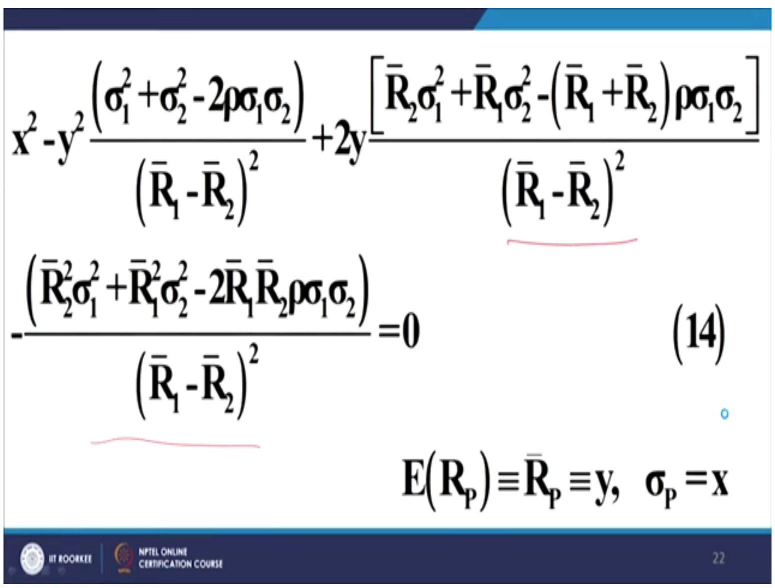

You can represent this expression for the portfolio possibilities curve that you saw in the earlier slide in in equation number 14, this can be represented in the form that is here that in the standard form x square minus s x square upon a square minus y square upon b square is equal to 1. As shown in equation number 16 or it can be represented in the form x square minus BY square plus 2 fy minus c is equal to 0 which is equation number 17, where the various constants B and CB I am sorry BF and c have the values that are shown on this slide. So, we now have explicitly shown that the curve can be represented in the standard equation of a hyperbola of the form x square upon a square minus y square upon B square is equal to 1 or it can be further shown as a quadratic in x and y as x square minus PY square plus 2 FC, plus 2 FY I am sorry minus C is equal to 0. What are the important features of this portfolio possibilities curve?

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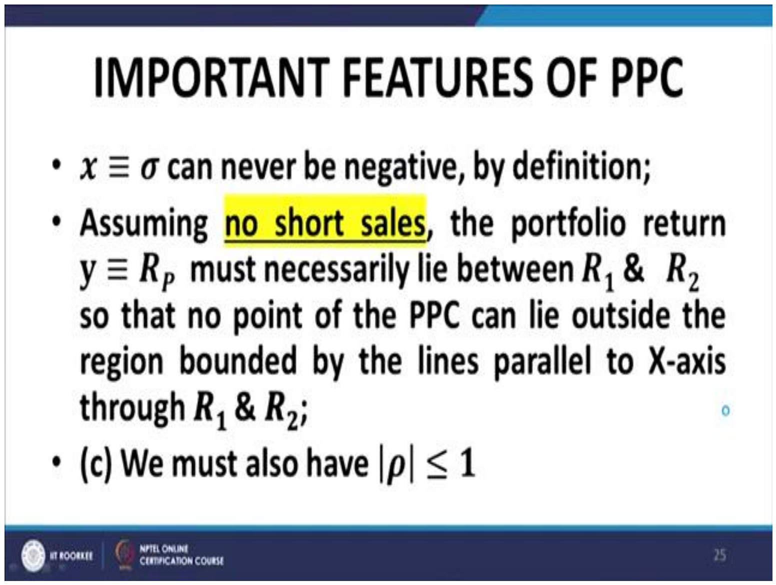

Number 1; since x represents sigma, sigma is the standard deviation and by the definition of standard deviation sigma cannot be negative, the standard deviation cannot be negative by the very definition of standard deviation and therefore the portfolio possibilities curve is the section of the hyperbola that lies on the right half plane.

And it there will be no point on the portfolio possibilities curve that lies to the left of the y axis. The entire portfolio possibilities curve will lie on the half plane that is to the right of the y axis. Number 2; assuming no short sales we shall address the issue of short sales very soon but assuming for the moment that there are no short sales the portfolio return the portfolio return please note must necessarily lie between the returns on the individual securities.

The portfolio return must lie between the returns on the individual securities because x1 and x2 are both positive because we are assuming no short sales. So, because there are no short sales x 1 and x 2 are positive and that implies that the portfolio return must lie between R 1 and R 2 so that there is no point of the portfolio possibilities curve that can lie outside the region bounded by the lines straight lines parallel to the x axis through r 1 and r 2.

But please note this important fact this will hold only if there are no short sales allowed in either security a or security b. We must also have rho the mode of rho must be less than equal to 1 in other words the correlation coefficient must lie between minus 1 and plus 1. The correlation coefficient by definition by the very formulation is of the correlation coefficient must necessarily lie between minus 1 and plus 1. So, now we will move to the perfectly correlated assets rho equal to plus 1 after the break. Thank you.