Security Analysis & Portfolio Management
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Lecture 18
Interest Rate Risk - II

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Welcome back. So, let us continue from where we left off.

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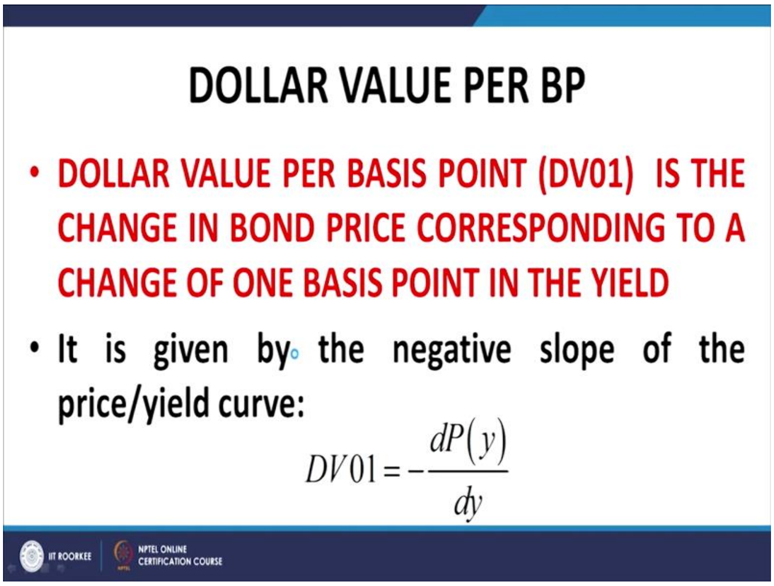

I explained the concept of dollar value per basis point as a measure of the change in the, or the change in the price of a bond, change in the value of a bond corresponding to single basis point change in the YTM. In other words, if the YTM changes by 1 basis point, the corresponding change in the value of the bond or the bond portfolio, as the case may be represents DV01.

I reiterate that there is a negative sign attached to it. So, DV01 would return a positive figure. Because invariably, the YTM and the price of a bond are inversely related, let us move to duration.

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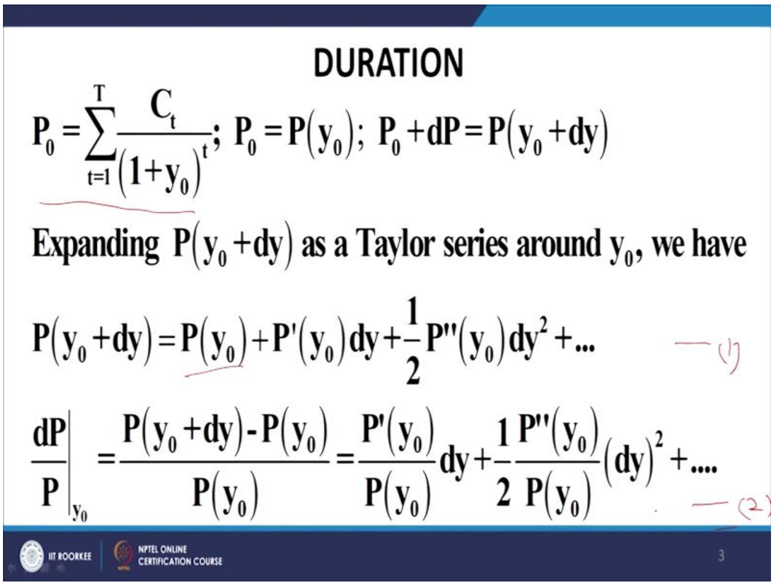

We start with the standard expression of the discounted cash flow price and that is given right at the top left hand corner of your slide and we define P0 as the price corresponding to yield of y0, we also define. Now we assume that the yield changes to y0 plus dy, the yield changes by an infinite decimal amount by a very small amount dy and it goes to y0 plus dy and as a result of which the price changes to P0 plus dP.

So, let me repeat P0 is the original the price that we start taking the basic observations, and that is at a yield of y0. In other words, P0 corresponds to a YTM of y0 and now the YTM changes from y0 to y0 plus dy and as a result of it the price changes from P0 to P0 plus dP. Therefore, P0 plus dP is equal to P at y0 plus dy.

Now we expand P of y0 plus dy that is P0 plus dP, as a Taylor series around the point y0, I repeat, we expand P y0 plus dy which is nothing but P0 plus dP around y0 as a Taylor series, and what we get is the expression that let me call this equation 1. This is a standard Taylor series expansion. So, not much to explain here.

On simplifying, on taking the first term on the right hand side to the left hand side and dividing throughout by P0 or P of y0, what we get is the expression that is there in the last or the second equation, let me call it equation number 2. In other words, dP upon P at y0, dP upon P at y0 at the point y0 is equal to P dash y0 divided by P y0 into dy plus 1 by 2, P double dash y0 upon P y0 dy squared and we are retaining Taylor series up to only two terms second order terms in the derivative.

So, let me repeat, we expand P y0 plus dy or P plus P0 plus dP as a Taylor series and retain only terms up to second order in the derivatives and then we do a transposition take P y0 to the left hand side and divide throughout by P y0. So, P y0 plus dy minus P y0 divided by P y 0 which is nothing but dP upon P at the point y0 and which is given by the expression in the right hand corner of your slide which I called equation 2.

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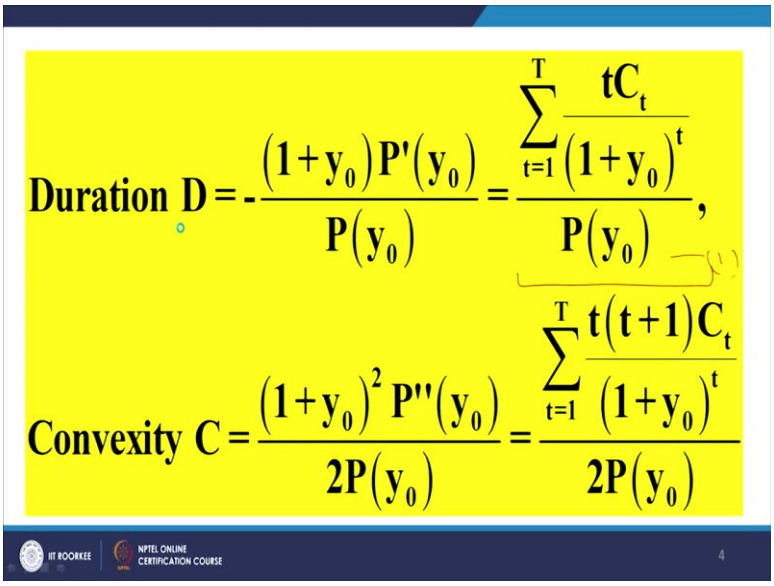

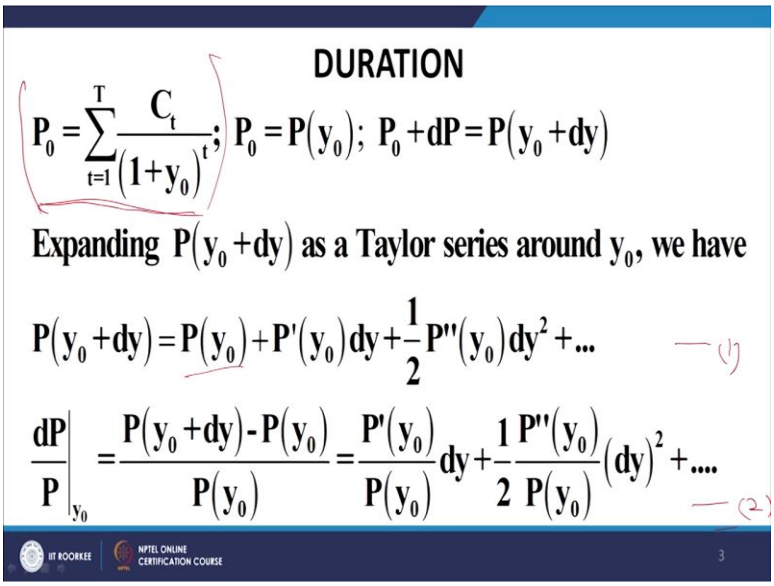

Now, we defined two quantities. explicitly we define two quantities, which are shown on your slide. We define the duration D as 1 plus y0 P dash y0 upon P y0 and we define convexity as 1 plus y0 square P double dash y0 upon 2 P y0. Of course, if we substitute the value of P dash y0 from the expression using P0 is equal to, using this expression. That is the top left hand corner that is the DCF price. If you differentiate this explicitly, with respect to y and take the derivative at y0, we get what is here in the right hand corner of the, right hand side of the first equation on the slide.

Let me repeat the numerator of this expression has been obtained by explicitly differentiating the expression here, this expression in the round brackets and then taking the value at y equal to y0, you differentiate the right hand side with respect to y and then take the value at y equal to y0, you get the numerator of the duration.

And similarly, you get the numerator of the convexity also. So, let me repeat what we have done here, we have introduced two new terms. One called the duration and the other called the convexity by the expressions that are given adjoining those two terms D is equal to minus 1 plus y0, P dash y0 divided by P y0 and C is equal to 1 plus y0 square P double dash y0 upon 2 P y0.

Now, as far as P dash y0 is concerned, we can obtain an explicit expression in terms of the cash flows and the timings, timings to those cash flows by differentiating the DCF price of the bond with respect to y and taking the value of the derivative at y0.

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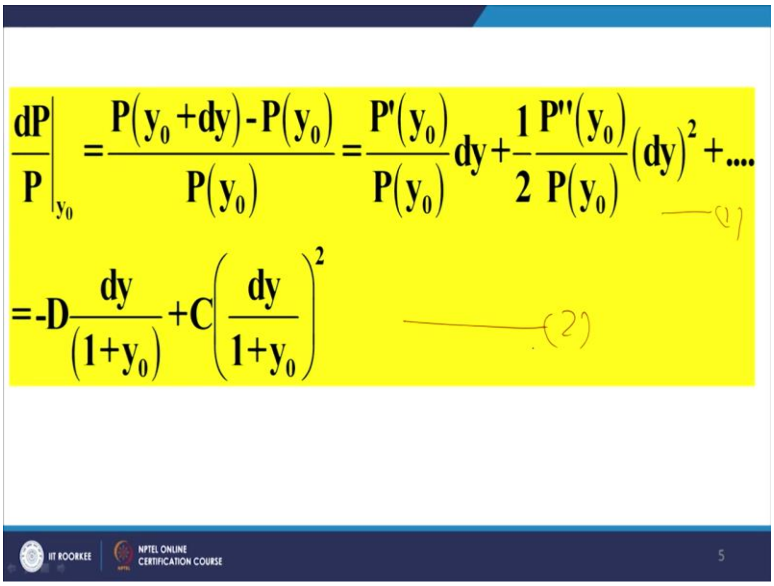

On substituting this value of C and D or D and C rather the duration and the convexity in equation number 1 here, which is brought forward from the earlier slide, which is nothing but equation number 2 here, this equation number 2 is carried forward to the slide here and we substitute the expressions for D and C in this and what we end up with is the expression that is equation number 2 in this particular slide.

So, what we get is the percentage change in price at a given point y0 along the yield price curve is given by minus d dy upon 1 plus y0 where y0 is the given point at which the percentage change is being evaluated plus C into dy upon 1 plus y0 square this is a very important formula. I repeat the percentage change in price at a given point y0 along the yield price curve is given by minus d into d where dy is some infinite decimal change in the yield divided by 1 plus y0 plus C, where C is the convexity into dy upon 1 plus y0 squared. Definition is definitely desirable, definitely appropriate at this point for the important concept of duration.

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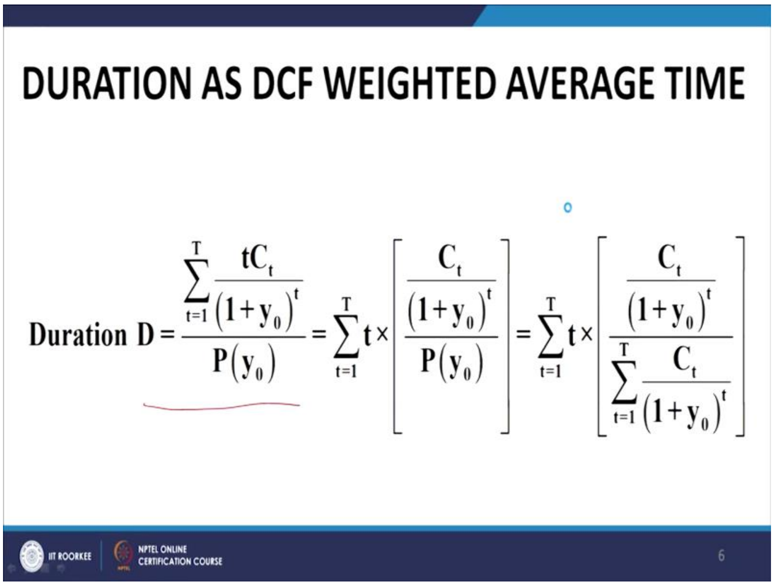

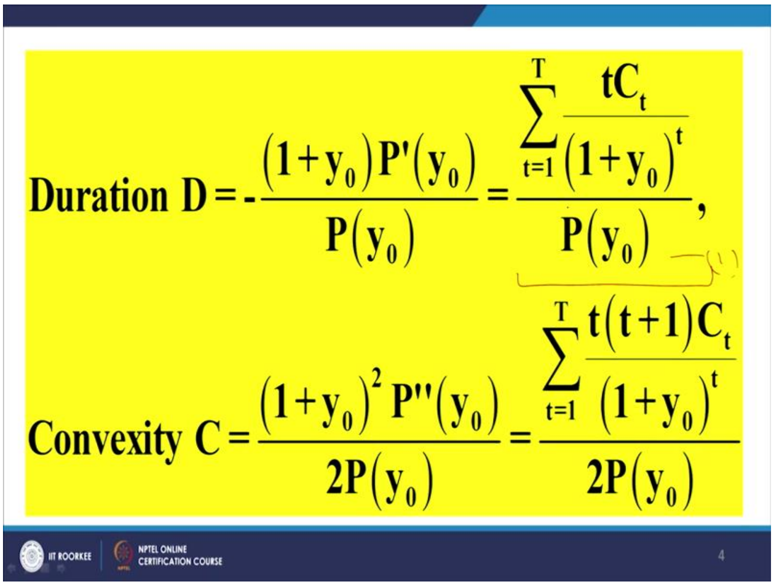

So, let us rearrange the expression for the duration. The expression here, underlined expression here is what we had carried forward from the earlier slide, let me go back to the slide to, in this particular slide and this we have obtained by differentiating the DCF formula with respect to y and then taking the value at y0. So, this is expression we start with, as the expression for the duration.

We rearrange the terms a little bit, we take the summation outside and we divide throughout by P y0 within or we take the P y0 in the denominator with each term in the numerator, instead of having an overall denominator of P y0, we are now having P y0 as a denominator corresponding to each term.

And what we find now is that this is an expression of t into a particular weight and what is that weight, that weight it is the fraction of the discounted cash flow to the total cash flows or the price of the bond. So, how can we define on the basis of this definition? How can we define or how can we provide an explicit definition of the duration that is given in the next slide.

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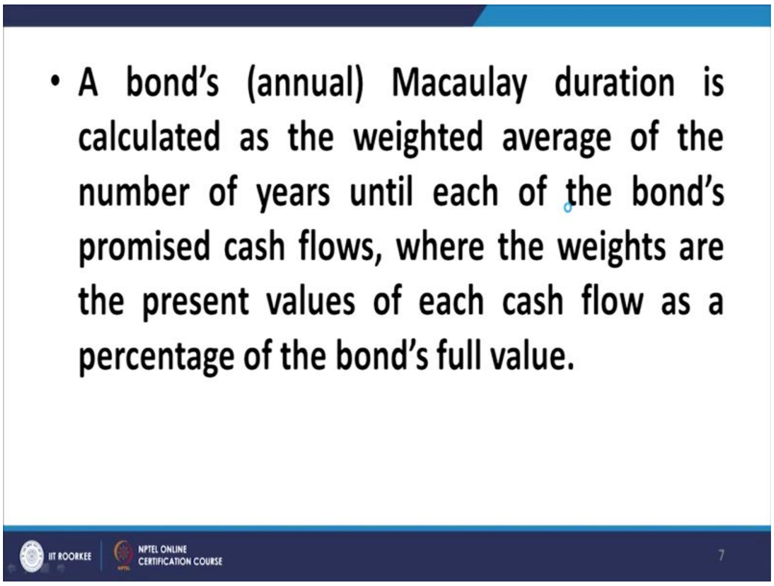

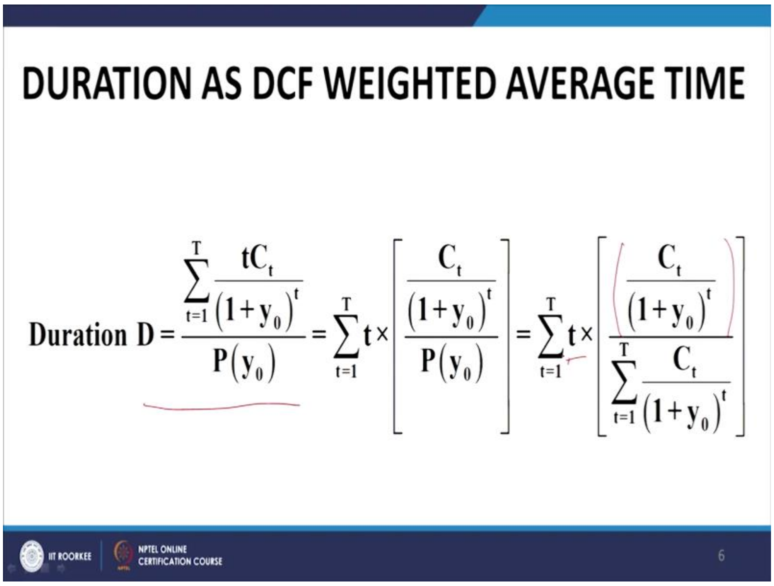

Now, bonds annual Macaulay's duration or this duration is called Macaulay's duration, there is another duration which is modified duration, which I shall come in a few minutes. So, a bond's annual Macaulay duration is calculated as the weighted average of the number of years until each of the bonds promised cash flows that is small t, where the weights the present values of each cash flow as a percentage of the bond's full value.

So, let us go back and look at this formula. This is time, time to next cash flows t equal to 1, t equal to 2, t equal to 3 and so on. This is the discounted value of the cash flow corresponding to the time t. In other words, this is the discounted value of the cash flow at time t and this discounted value is being represented by, is being divided by the summation of all discounted values, which is nothing but the full price.

So, the weight, the expression within the square brackets, constituency the weights and what are these weights, these weights are the discounted value of the cash flows, expressed as a fraction of the full price which is the denominator, which is the DCF value of the market price and what is being weight that is the time to the various cash flows.

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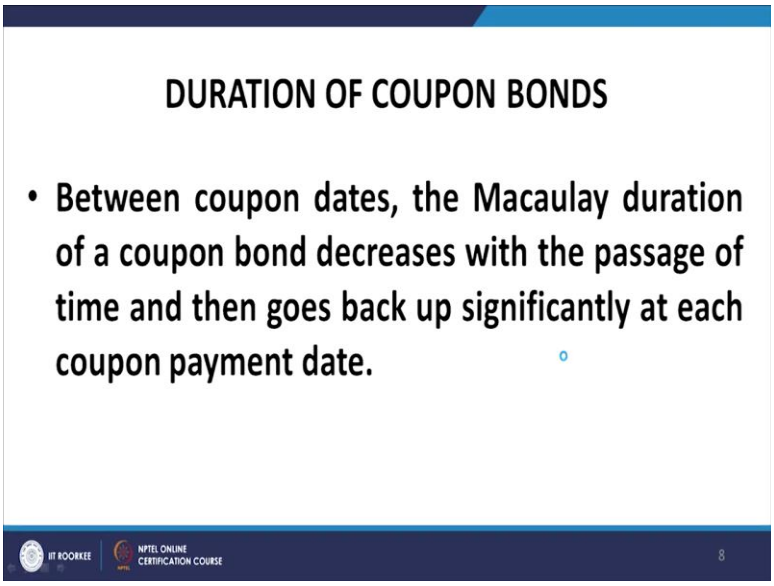

Duration of coupon bonds between coupon bonds the Macaulay's duration of a coupon bond decreases with the passage of time and then goes back up significantly at each coupon payment date. So, when you evaluate the duration between two coupon dates, the durations tends to decrease as the, as you approach the next coupon date from a particular coupon date when the next coupon date actually arises, the bond and the duration again increases significantly.

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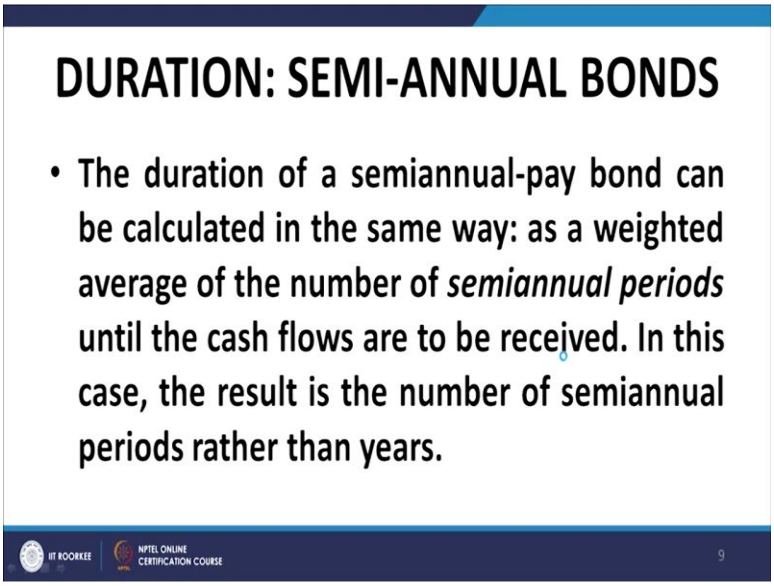

For semi-annual coupons what do we have the duration of a semi-annual bond can be calculated in the same way as a weighted average, the number of semi-annual periods until the cash flows are to be received. In other words, everything all those calculations that we have enumerated so far for annual coupon bonds, carry forward at its (())(11:38) to semi-annual coupon bonds with the difference that now the time periods, which were annual in the earlier calculations are now taken as half years.

So, the duration figures that you will return on doing these calculations will also be in terms of half year, you take the coupon payments on the basis of half years, you take the times also, in other words, the number of periods would double. If it is a five year coupon bond, the number of half years periods would be 10, the coupon payments would be half of the nominal coupon rate per annum and on that basis, you will work up the duration and that duration will be in terms of the number of half year periods, please note this particular point. So, now the interpretation of duration, this is important. Let us look at this slide for a minute.

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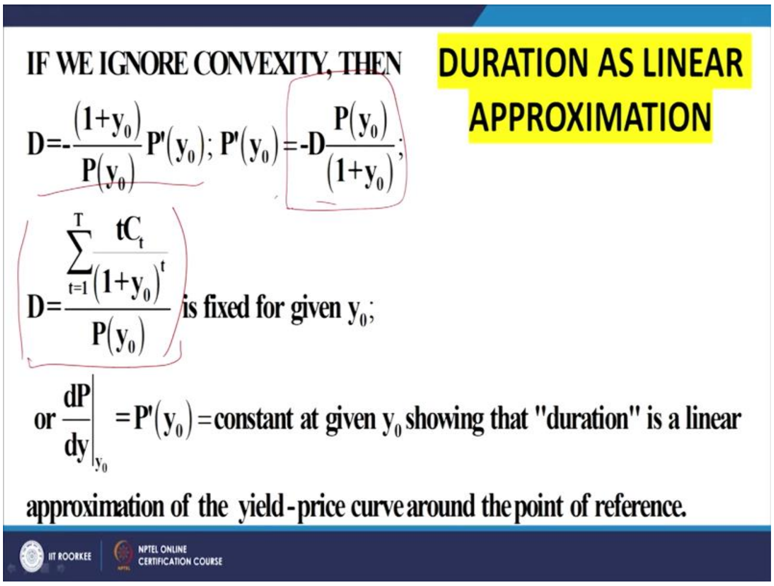

This the first term is the definition of duration and on rearranging terms, what we find is P dash y0, what is P dash y0? P dash y0 is the slope of the yield price curve at the point y0, P dash y0 is the slope of the yield price curve at y0. And what we find on rearranging this term, the rearranging the definition of duration, we find that P dash y0 is equal to minus d P y0 upon 1 plus y0.

And when we look at the expression for duration, what we infer is that using this expression for duration or observing from this expression of duration, what we find is that the duration of a particular bond or duration of a pattern of cash flows embedded in a particular bond is constant on for a particular y0, because the point, the pattern of cash flows that is the point in time at which the cash flows are occurring and the amount to those cash flows are identified in the terms of the contract and we are talking about the evaluation over the entire life of the bond, so there is no question of holding period being less or more than maturity and the market price creeping into the formula at a later, an estimate of market price rather creeping into the formula.

Of course, you have the market price in the denominator, but that is the current market price that is the market price at the point at which the duration is calculated, which is explicitly known, there is no randomness about that, no estimation. So, that is what is the actual market price and the pattern of cash flow that is the timing as well as the magnitudes are known.

So, whatever is there in this particular expression for duration within the round brackets is explicitly known and depends only on y0. For a given y0, the duration of the bond is constant. And secondly, if you look at this P y0 that is the current market price of the bonds, the price of the bond at the point y0 or a corresponding to a YTM of y0 which is also known and the denominator is also known the current y0 at which you are doing the calculations.

So, the net result of what I am trying to say is, that the entire right hand figure here, P dash y0 is equal to minus d into P y0 upon 1 plus y0, the right hand side is completely specified at a point on the yield price curve, given a point on the yield price curve, the entire quantity on the right hand side is known and is constant, what does it mean?

It means that the slope of the yield price curve is constant, it means that in the close approximation or in the closed neighbourhood rather, in the closed neighbourhood of the point at which you are doing the explicit calculation of duration or at which you are calculating the percentage change in price, the use of duration alone, the use of duration alone assumes that the yield price curve is a straight line.

Let me repeat this very significant result, around the point y0 at which along the yield price curve at which you are doing the calculation, the use of duration alone that is, if you ignore the second derivative, you ignore the convexity or whatever, if you use Taylor expansion only up to first order, then the inference is that you are assuming that around the point y0 the yield price curve in the close neighbourhood is approximately a straight line.

Or in other words, a straight line approximation is good enough to provide you accurate results about the percentage change in price corresponding to points, corresponding to small changes in the yield to maturity from the point y0 either side. In other words, duration is the straight line approximation, it is the linear approximation of the yield price curve, duration assumes a linear straight line, a linear yield price curve around the point on in close proximity of the point at which the duration is calculated for the calculation of the percentage change in price.

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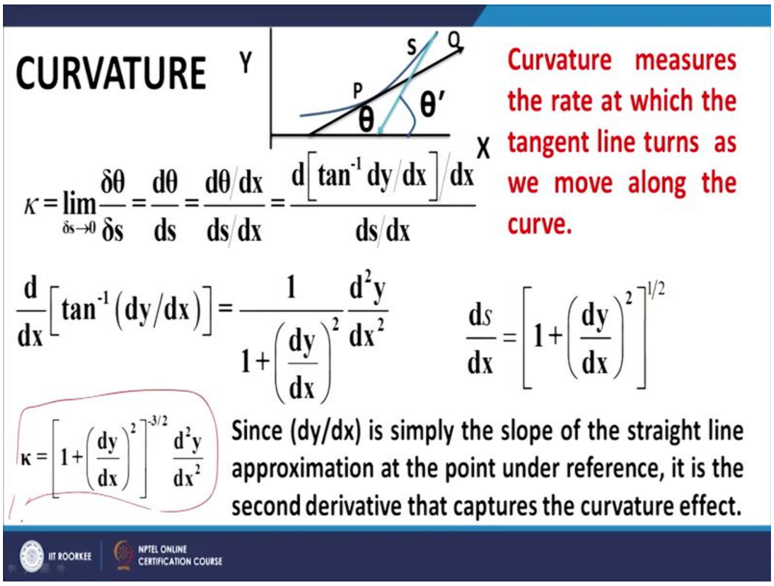

Now, this slide is slightly mathematically oriented, it is for those who are mathematically inquisitive. The bottom line is that the curvature, the definition of curvature is derived, the definition of curvature is given in the box that I am putting in the left hand corner of your slide. The important observation here is that curvature is captured by the second derivative of the curve at the point under reference.

The takeaway that I want you to carry forward I will not explain the entire process, it is there on the slide and the mathematically inclined learners could work it out, it is quite simple. But the important thing is that the message that I want to convey by the slide, the takeaway from the slide is that curvature is represented or curvature is captured by the second derivative of the curve. At the point under reference, the explicit definition of the curvature is kept, is given in the box here right at the bottom left hand corner of the slide.

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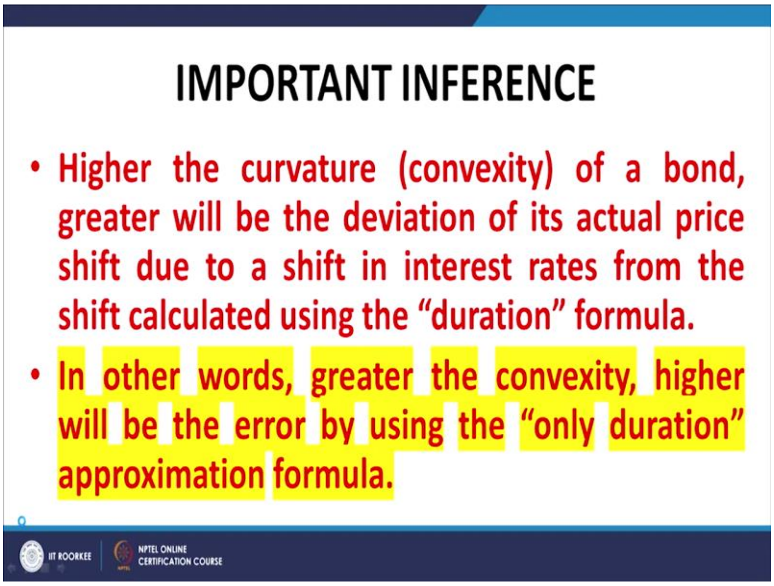

So, what is the important inference? The inference is that higher the curvature, higher the convexity of a bond, we have already talked about convexity, convexity is a form of curvature in fact. So higher the curvature, higher the convexity of a bond, greater will be the deviation of the actual price shift, actual percentage price change due to a shift in interest rates due to a change in YTM from the shift calculating using the duration formula.

See the point is, a duration assumes a straight line approximation and therefore and we know for certain that the yield curve is convex, the yield curve has a curvature. So, that being the case, when we approximate that the curve by a straight line around the point or in the neighbourhood of the point at which we are calculating this percentage price change or the price change as the case may be, as you may like.

The important point is that we are approximating it by a straight line, actually it is a curve, therefore the greater the curvature, then greater would be the deviation from the approximation, greater would be the inaccuracy of the approximation that we arrive at by using the duration alone by ignoring the convexity, by ignoring the second order derivatives in the Taylor expansion.

If we use only the duration and calculate the percentage price change, what we end up with is a value which could be significantly different from the actual change depending on what is the level of curvature of the yield price curve, if the yield price curve is highly convex, then the result that you obtained by using the duration would be misplaced, would be misleading because of the impact of curvature. Let us illustrate by an example.

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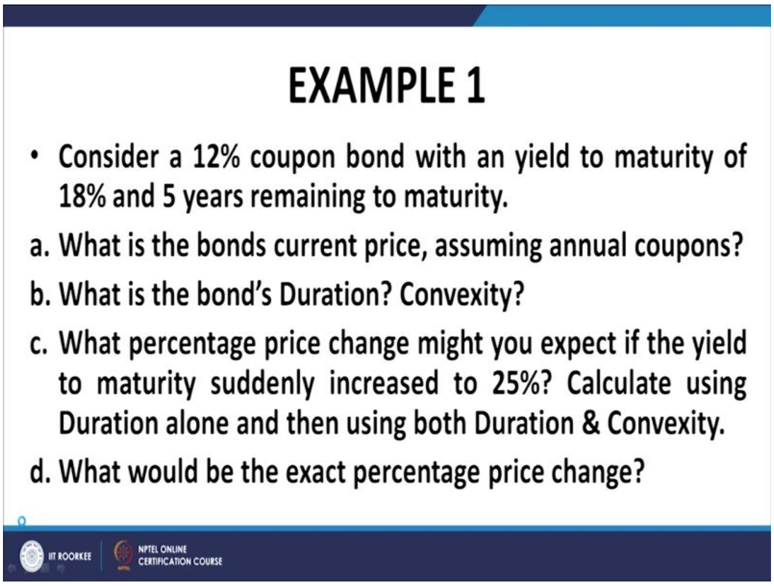

Consider a 12 percent coupon bond with a yield to maturity of 18 percent and 5 years remaining to maturity, consider a 12 percent coupon bond with a yield to maturity of 18 percent and 5 years remaining to maturity. What is the bond's current price? DCA price assuming annual coupons number two, what is the bonds duration? Number three, what is the bonds convexity?

What percentage price change might you expect if the yield to maturity suddenly increased to 25 percent? It is 18 percent, it suddenly increases to 25 percent. Now, it is important for me to emphasize that this kind of change is very unlikely, in fact close to being impossible in real life, I have magnified the chain significantly in order to make the results conspicuous, in order to make the results apparent.

Now, I repeat, you cannot have a sudden change of 7 percent in the YTM market interest rates that is somewhat absurd, but the message that I want to convey by this example gets amplified, gets magnified by using this magnified YTM change. So, what percentage price change would you expect in the yield to maturity suddenly increased to 25 percent, calculate using duration alone, then introduce the convexity correction and then calculate the actual price change or actual percentage price change. So, this is a comprehensive example, which would illustrate all the concepts that I have elucidated so far.

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TIMELINE	0	1	2	3	4	5
YTM		0.18	0.18	0.18	0.18	0.18
DISC FACTOR		0.84745763	0.71818	0.60863	0.5157889	0.4371
CASH FLOW		12	12	12	12	112
DCF	81.23697	10.1694915	8.61821	7.30357	6.1894665	48.956
tC(t)		12	24	36	48	560
DISC tC(t)	318.8557	10.1694915	17.2364	21.9107	24.757866	244.78
DURATION	3.925007					
t(t+1)C(t)		24	72	144	240	3360
DISC t(t+1)C(t)	1752.167	20.3389831	51.7093	87.6428	123.78933	1468.7
CONVEXITY	10.7843					

So, first step is calculate the current price, which is quite straightforward, we are given that the YTM is 18 percent, we are given that the coupon rate is 12 percent. So, given the coupon rate of 12 percent assuming a face value of 100, in fact, all these calculations would be independent of face value for simplicity of calculation, we are assuming that the face value is 100.

So, let the face value in fact gets cancelled out between the numerator and the denominator. So, and the results are independent of face value. Nevertheless, we use a face value of 100. So, we get coupon payments of 12 units, because the coupon rate is 12 and the coupon is paid annually. So, we get coupon payments of 12 units at the end of first year, second year, third year, fourth year and fifth year.

And in the fifth year, assuming that we have a level coupon bond redeemable at par, the redemption value is also paid back which is equal to the par value of 100. So, the cash flow at the end of the fifth year is 112, the discount rate is equal to the YTM that is 18 percent and when we do this discounting process, we arrive at the current market price of the bond, which is at 81.29.

This is the current market price of the bond 81.29. Using the same inputs we can calculate the duration, duration calculation is quite straightforward, t into Ct that is we multiply the first year's discounted cash flow with 1, we multiply the second year's discounted cash flow with 2, the third year's discounted cash flow with 3, fourth year discounted cash flow with 4 and fifth year's discounted cash flow with 5 you sum them all up and divide them by the market price, which we have arrived at a year of 81.23.

After doing all this process, what we arrive at is a value of discount duration equal to 3.925 years 3.925 years, that is the duration. Similarly, we work out the convexity and by working out the convexity we find that the convexity is equal to 10.7843 years squared. Please note the units of duration are obviously years, because it is a quantity of time which is weighted by a certain factor.

So, we get a value of weighted average time in some sense and therefore, the unit of duration as years and similarly the unit of convexity because it is t into t plus 1. Therefore, the units of convexity is year square. Now, we come to the important slide.

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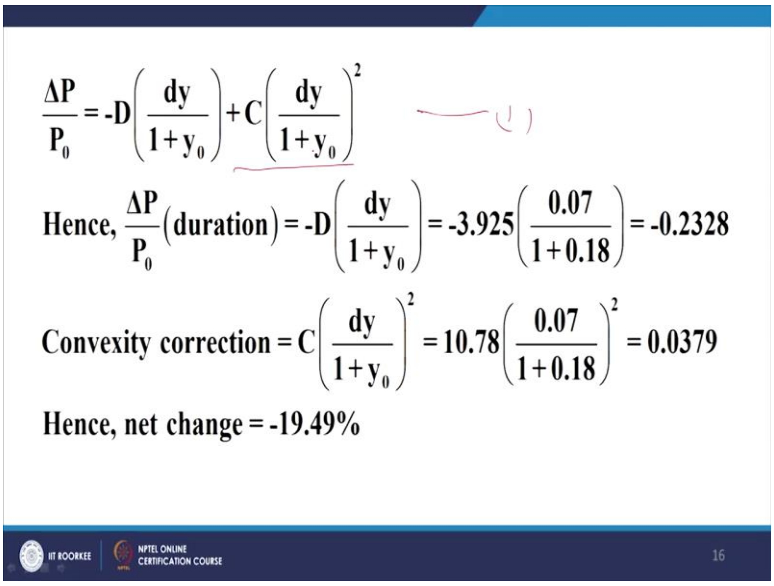

Before I talk about this slide, let us go to the next slide where the calculations are explicitly given. The percentage change in price we have already worked out the formula for this, are given by minus d into dy upon 1 plus y0 plus c into dy upon 1 plus y0 square. Using duration alone therefore, what we find if we ignore the second term, if we ignore the convexity term, the duration is 3.925.

So, a percentage change in price is given by minus 3.925 into 0.07 which is the genuine yield which is dy from 18 percent to 25 percent that is 0.07 and we divided by 1 plus y0, y0 is the point at we started our calculation which is 18 percent. So, this one simplification gives us a percentage change of price using duration alone of 23.28 percent.

If we introduce the convexity correction, what is the convexity correction, this is the second term in the first formula let us call it formula 1 and the second term is known as the convexity correction C into dy divided by 1 plus y0 whole square. So, C is given, is found out to be

10.78 So, 10.78 into 0.07 divided by 0.18 plus 1 whole square which we get is equal to 0.0379 that is 3.79 percent.

Please note that the figure that is returned by the use of duration alone is minus 23.28 percent whereas, the convexity correction is positive. So, these two figures have to be algebraically added, you do not simply add them up, you would add them up, keeping in view their signs. So, you have to add minus 23.28 plus 3.79 and that gives you a figure of minus 19.49.

So, which is the estimate of the yield percentage change in price corresponding to a 7 percent change in yield from 18 percent to 25 percent using the formula of duration and convexity. Let us go back to this slide now.

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ACTUAL REVISED PRICE						
TIMELINE	0	1	2	3	4	5
YTM		0.25	0.25	0.25	0.25	0.25
DISC FACTOR		0.8	0.64	0.512	0.4096	0.3277
CASH FLOW		12	12	12	12	112
DCF	65.03936	9.6	7.68	6.144	4.9152	36.7
ORIGINAL PRICE	81.23697					
ACTUAL % CHANGE	-0.199387					

GIVEN CHANGE IN YTM	0.07
% CHANGE IN PRICE USING DURATION	-0.232839
CONVEXITY CORRECTION	0.037951
NET % CHANGE IN PRICE	-0.194888

Now, as you can see here, the actual percentage change in price is given by 19.93 percent minus 19.93 percent that is the actual percentage change in price, how this is obtained? Well, I have worked out the price using duration, using YTM of 25 percent again and on that basis, I find the market price to be 65.039 and the earlier price was about 81.

So, the percentage change in price using these two explicit calculations of the price turned out to be minus 19.93 percent, which is the correct change in price. However, the approximation that we get by using duration alone is minus 23.28 percent. And if we do the convexity correction, we come very close very, very close to the change in price that we actually achieve which is minus 19.49 percent.

Now, please note in spite of the YTM changing by 7 percent, which I emphasize is not realistic, but in spite of YTM changing by 7 percent, by using this convexity correction, we are so very close to the actual price change. So, if the YTM changes were small, if the YTM changes were in terms of basis points, then our approximation would have been much, much more closer and good enough for all real life calculations.

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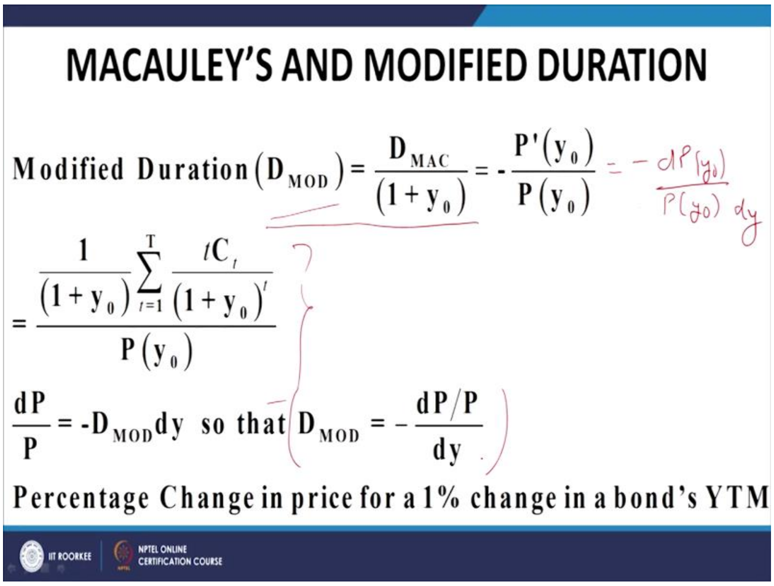

Now, we come to another related concept. Another rescaling I must say variant of duration, which we call modified duration. It is basically a variant of the Macaulay's duration, it is the rescaling of Macaulay's duration and the expression for modified duration is given by Dmac that is the Macaulay's duration rescaled by 1 plus y0.

So, the modified duration is equal to Dmac divided by 1 plus y0. So, in other words is equal to minus P dash y0 divided by P y0 minus P dash y0 divided by P y0. The relation between modified duration and Macaulay duration is important. The modified duration is given by the Macaulay's duration normalized by a factor of 1 plus y0.

In terms of the explicit values of the cash flows and so on. It is given by the expression that is here that I enclose within the curly brackets 1 upon 1 plus y0 summation TCT upon 1 plus y0 to the power t divided by the current market price. And in terms of the modified duration using this expression Dmod is equal to minus P dash y0 upon P y0, what we get is d P y0 upon P y0 is equal to minus modified duration into dy.

If you rearrange this formula simple, if you rearrange the formula Dmod is equal to minus P dash y 0 upon P y0, what we get is d P y0 upon P y0 is equal to minus of Dmod into dy, I repeat simply rearranging this formula, you can see it explicitly here minus d P y0 divided by

P y0 into dy. So, simply rearranging this formula and taking dy to the left hand side, what we get is dP y0 upon P y0 is equal to minus of Dmod into dy.

This gives us another definition, another expression for the modified duration, modified duration is the percentage change in price, modified duration is the percentage change in price per unit change in the YTM. If the YTM changes by 1 unit, the percentage change in the price of the bond or the bond portfolio as the case may be represents the modified duration.

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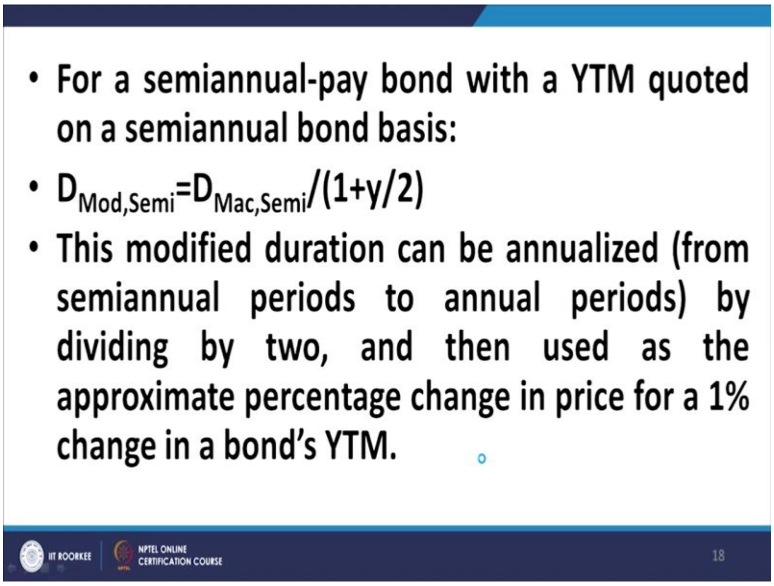

Now, for semi-annual bonds, what is the relationship? For a semi-annual pay bond with a YTM quoted on a semi-annual bond basis, D modified semi-annual is equal to D max semi-annual divided by 1 plus y upon 2. This modified duration that you will get by using the semi-annual formula needs to be annualized by multiplying by 2 and then you can use it as a measure of the approximate percentage change for a percent change in the bonds YTM.

So, this is as far as the semi-annual modified duration is concerned, because most of the bonds in general pay semi-annual coupons and therefore, this is important from a practical aspect. D modified semi-annual is equal to D max semi-annual divided by 1 plus y to the power 2, I am sorry, 1 plus y by 2, not y to the power 2, 1 plus y by 2 and then you multiply the expression by 2 to get the annual modified duration and on that basis, you can work out the percentage change in price corresponding to a given change in YTM.

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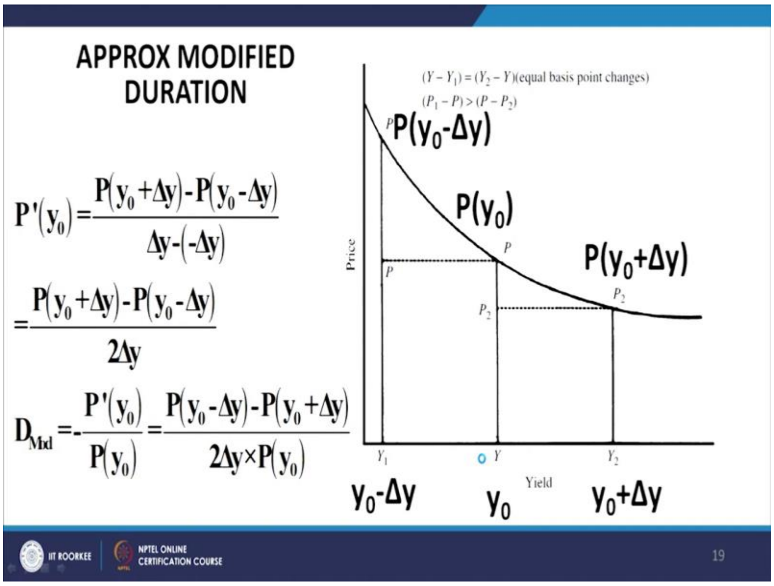

So, in order to calculate the modified duration of semi-annual coupon bond, because most of the bonds pay interest semi-annually, let me reiterate that we work out the Macaulay’s duration on the basis of semi-annual payments, on the basis of half years as the unit of time instead of having full years for it. As is the case for annual coupon bonds, we use the half years as units of time and obviously, the number of time periods should be doubled compared to the years for which the bond is alive.

And on that basis, we worked out the Macaulay’s duration on a semi-annual basis and to convert the Macaulay’s duration to modified duration, we use this formula Dmod semi is equal to Dmax semi divided by 1 plus y by 2 and then we can double this expression to arrive at the percentage change in price corresponding to a unit change in YTM. Now, we talk about the approximate modified duration formula, which is very often used in practice, but we will talk about it in the next lecture. Thank you.